Circuitizing product formulas for (1+1)D SU(2) lattice gauge theories

Lessons from alternative formulations

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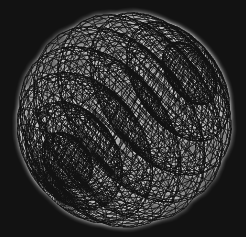
Roadmap

- This talk concerns “far-term” quantum algorithms for simulating gauge theories.
- Some base-level familiarity is assumed with respect to gate-based quantum computing, on-link gauge boson Hilbert spaces, and irreducible representations (irreps) of SU(2)
- Outline
  - Hamiltonian lattice gauge theories
  - Product formulas / Trotterization
  - Recap: Circuitized lattice Schwinger model
  - SU(2) [and SU(3)] generalizations
  - Our improvements for SU(2)
  - Takeaways

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- Non-Abelian group, e.g. SU(2)



$$\langle g|j,m,n\rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,n}(g)$$

group element basis

representation basis

$$[E^a_{L/R}, E^b_{L/R}] = i f^{abc} E^c_{L/R}$$

$$[E^a_R, U] = U T^a$$

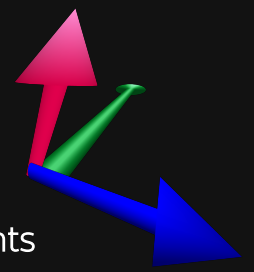
$$[E^a_L, U] = -T^a U$$

Gauge transformations:

$$\hat{U}_{n,i} \to \Omega_n \hat{U}_{n,i} \Omega^\dagger_{n+e_i}$$

"Left" and "right" electric fields to generate the independent left/right rotations.

Left and right electric fields each have 'colored' components in addition to spatial components



SU(2) "gluons" have 3 components instead of the usual 8.

Hamiltonian lattice gauge theory

Plus Gauss law constraints

U(1)

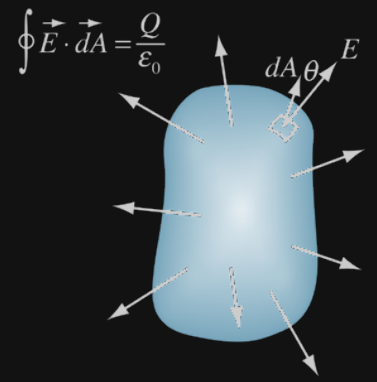
$$\nabla \cdot \mathbf{E} - \rho = 0$$

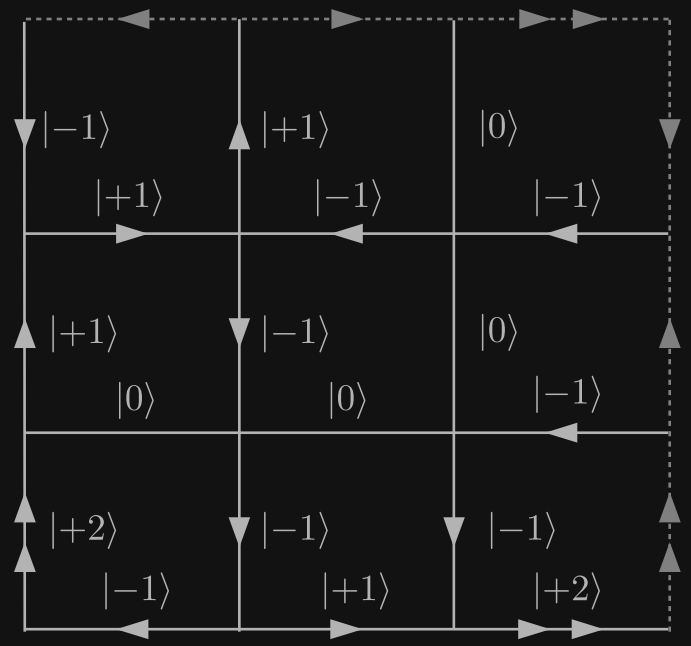
$$\hat{\mathcal{G}}_n \nearrow \qquad \rho = \psi^\dagger \psi$$

SU(N)

$$\mathbf{D} \cdot \mathbf{E}^a - \rho^a = 0$$

$$\hat{\mathcal{G}}_n^a \nearrow \qquad \rho^a = \psi^\dagger T^a \psi$$





compact U(1)
electric eigenbasis

Gauss's law ⟷ gauge invariance

Gauss's law ↔ charge conservation

gauge invariance ↔ charge conservation

Gauge theories in 1D

- Scope of this work is 1D space
- Possible in 1D to *integrate out* the gauge fields
  - 1D quantum electrodynamics + Jordan-Wigner + Gauss's law = spin model
- This trick utilized in
  - Martinez, Muschik, et al., Nature 534 516-519 (2016)
  - Banuls, Cichy, et al., Phys. Rev. X 7, 041046 (2017)
  - Atas, Zhang, et al., Nature Comm. (2021)
- Same trick does not eliminate all gauge DoFs in d>1
- Our work: Keep gauge DoFs
  - Maintains locality of Hamiltonian terms
  - Representative of d>1 gauge-matter coupling
  - Working with link operators gives insight into plaquette difficulties

Real-time evolution via product formulas

Three main steps to quantum simulation
1. Initial state preparation
2. Time evolution
3. Measurements

Trotterization: Evolve for time *t* in *s* steps,

$$e^{-i\,t\,H} = \left(e^{-i\,\frac{t}{s}\,H}\right)^s$$

Product formulas: Approximate exponential of a sum by product of exponentials

$$e^{-i\,\delta t\,\sum_k H_k} \simeq \prod_k e^{-i\,\delta t\,H_k}$$

Simplest case: Same ordering of H_k in every step
Generalizations: Higher-order Trotter; randomized; ...

2nd-order Trotterization

- Trotterization is increasingly being taken seriously as a realistic candidate for quantum simulation.
- Historically, theoretical error bounds have been empirically shown to be too pessimistic.
- Childs, Su, et al. (2021) give a 'tight' error bound in terms of nested commutators.

PHYSICAL REVIEW X **11,** 011020 (2021)

Theory of Trotter Error with Commutator Scaling

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Proposition 10: (Tight error bound for the second-order Suzuki formula). Let $H = \sum_{\gamma=1}^{\Gamma} H_\gamma$ be a Hamiltonian consisting of Γ summands and $t \geq 0$. Let $\mathcal{S}_2(t) = \prod_{\gamma=\Gamma}^{1} e^{-i(t/2)H_\gamma} \prod_{\gamma=1}^{\Gamma} e^{-i(t/2)H_\gamma}$ be the second-order Suzuki formula. Then, the additive Trotter error can be bounded as

$$\|\mathcal{S}_2(t) - e^{-itH}\| \leq \frac{t^3}{12} \sum_{\gamma_1=1}^{\Gamma} \left\| \left[\sum_{\gamma_3=\gamma_1+1}^{\Gamma} H_{\gamma_3}, \left[\sum_{\gamma_2=\gamma_1+1}^{\Gamma} H_{\gamma_2}, H_{\gamma_1} \right] \right] \right\| + \frac{t^3}{24} \sum_{\gamma_1=1}^{\Gamma} \left\| \left[H_{\gamma_1}, \left[H_{\gamma_1}, \sum_{\gamma_2=\gamma_1+1}^{\Gamma} H_{\gamma_2} \right] \right] \right\|. \quad (121)$$

← Triple sum, [A,[B,C]]

← Double sum, [A,[A,B]]

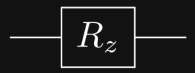
- For a 1D local LGT, nonzero [,[,]]s come from operators sharing common vertices/links
- Error bound directly tied to selection of the H_γ

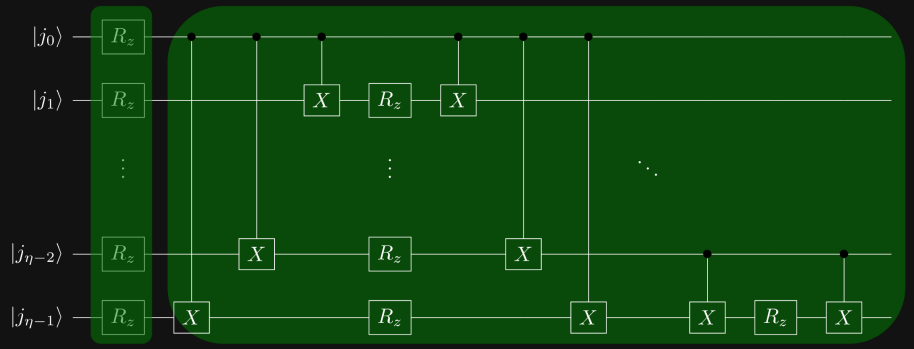
Review: Schwinger model [U(1)]

A.F. Shaw, P. Lougovski, JRS, N. Wiebe [2002.11146]

$$H = \underbrace{\mu \sum_r (-)^r \psi^\dagger(r)\psi(r)}_{H_M} + \underbrace{\sum_r E(r)^2}_{H_E} + \underbrace{x \sum_r \psi^\dagger(r)\psi(r+1)U(r) + \text{H.c.}}_{H_I}$$

$$\sum_r (-1)^r Z(r) \qquad \sum_{j=0}^{\eta-1} 2^{j-1} Z_{E,j} + \sum_{j=0}^{\eta-1}\sum_{k=0}^{\eta-1} 2^{j+k-2} Z_{E,j} Z_{E,k} + 1/4$$





- H_M: Too easy
- H_E: ZZ operators at worst – still easy
- Both: *Diagonal* AND *friendly* functions
- H_I ..?

Schwinger model: Off-diagonal operators

A.F. Shaw, P. Lougovski, JRS, N. Wiebe [2002.11146]

$$H_I \to x \sum_r \sigma^-(r)\sigma^+(r+1)U(r) + \text{H.c.}$$

1. Expand operators into Hermitian and antihermitian parts

$$\mathcal{O} = \underbrace{\frac{\mathcal{O}+\mathcal{O}^\dagger}{2}}_{\mathcal{O}^R} + i\underbrace{\frac{\mathcal{O}-\mathcal{O}^\dagger}{2i}}_{\mathcal{O}^I}$$

$$AB + \text{H.c.} = 2(A^R B^R - A^I B^I)$$

$$A \to \sigma^-(r)\sigma^+(r+1)$$

$$B \to U(r)$$

2. Decompose multiqubit *U* into terms that rotate 2-dimensional subspaces

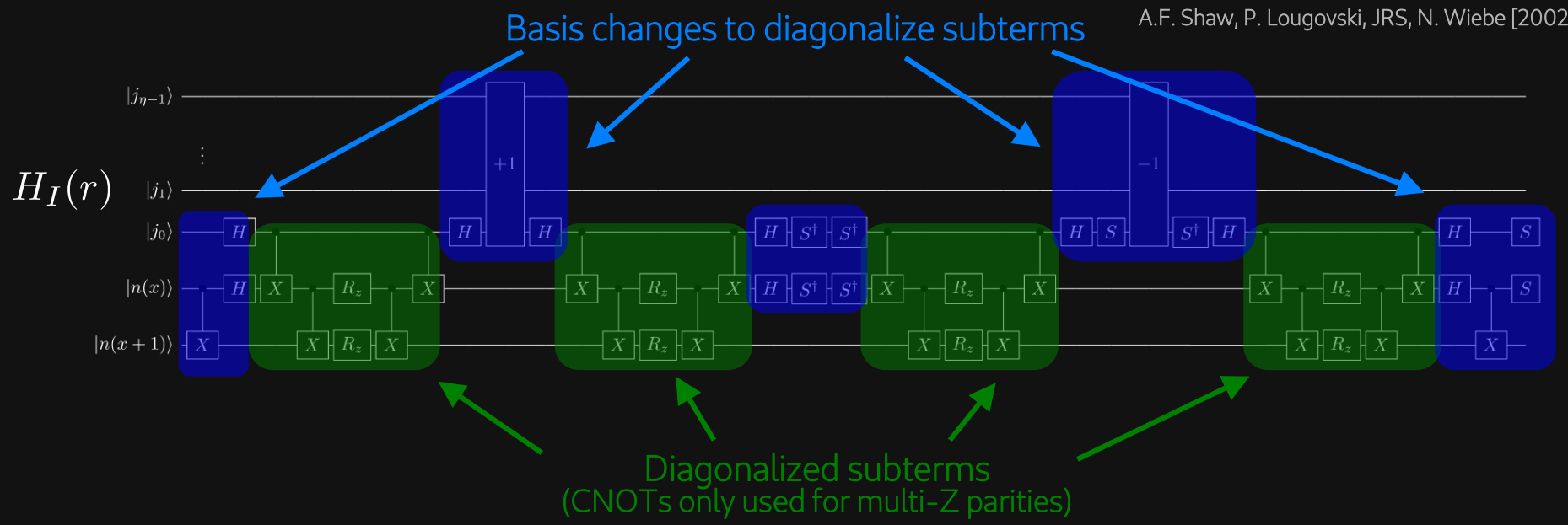
- One turns out to be *X* on the least significant bit: $X_{E,0}$
- Other: Similarity transformation of $X_{E,0}$ – just need a cyclic shift

$$U + U^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{"even"}} + U \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{"odd"}} U^\dagger$$

<u>Key point:</u> $H_I(r)$ gets divided into ν=2*2 = 4 (noncommuting) subterms. $H_I(r) = \Sigma_{j=1}^4 H_I^{(j)}(r)$

Schwinger model: Off-diagonal operators

A.F. Shaw, P. Lougovski, JRS, N. Wiebe [2002.11146]



Circuitizing diagonalized subterms in U(1): Not so bad!

Nonzero entries of U do not vary with E

Caveat: Several noncommuting $H_I^{(j)}$ enlarge the Trotter error

(Far-term) error sources: Theoretical Trotter + inexact R_z rotation angle synthesis

More Trotter steps needed for fixed error budget → Gate count rises

1D SU(2) + fermions

Toy theory of colored matter: D=1+1, SU(2), 1 fundamental fermion (staggered)

$$H = \mu \sum_r (-)^r \psi^\dagger(r)\psi(r) + \sum_r E^\alpha(r)E^\alpha(r) + x \sum_r \sum_{a,b} \psi^\dagger(r)_a \psi(r+1)_b U(r)_{ab} + \text{H.c.}$$

$$\psi(r) = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix}$$ fermionic color components

E^α : chromoelectric field

$\alpha = 1, 2, 3$ (adjoint rep)

Irrep basis formulation of link operator

$$U_{ab} \left| J, m^L, m^R \right\rangle = \sum_{j=J\pm 1/2} \sqrt{\frac{2J+1}{2j+1}} \left\langle J, m^L; \frac{1}{2}, a \middle| j, m^L + a \right\rangle \times \left\langle J, m^R; \frac{1}{2}, b \middle| j, m^R + b \right\rangle \left| j, m^L + a, m^R + b \right\rangle$$

U_{ab} ...
- changes m^L by a
- changes m^R by b
- can raise OR lower j (electric flux)

Same theory w/o gauge fields: Bañuls, Cichy, et al. 1707.06434; Atas, Zhang, et al. 2102.08920

Simulation algorithm in irrep basis

Kan & Nam, 2107.12769

Even-odd splitting of bosonic ladder operators carried over to link operators in SU(2) & SU(3).

U(1): They reduced Schwinger hopping to $\nu=2$ subterms.

(Nowadays, circuits exist for diagonalizing U(1) hopping terms exactly: $\nu\to 1$.) JRS [2105.11548]

SU(2): Hopping term ends up with $\nu=64$ subterms
- Each subterm calls for evaluating Clebsch-Gordan coefficients
- Irrational number approximation dwarfs the gate cost
- Large coefficient 64 to gate operations per link
- 2nd-order Trotter error also balloons with many nested-commutators ([A,[B,C]]) involving hopping subterms
- These commutators grow fastest as $g\to 0$!

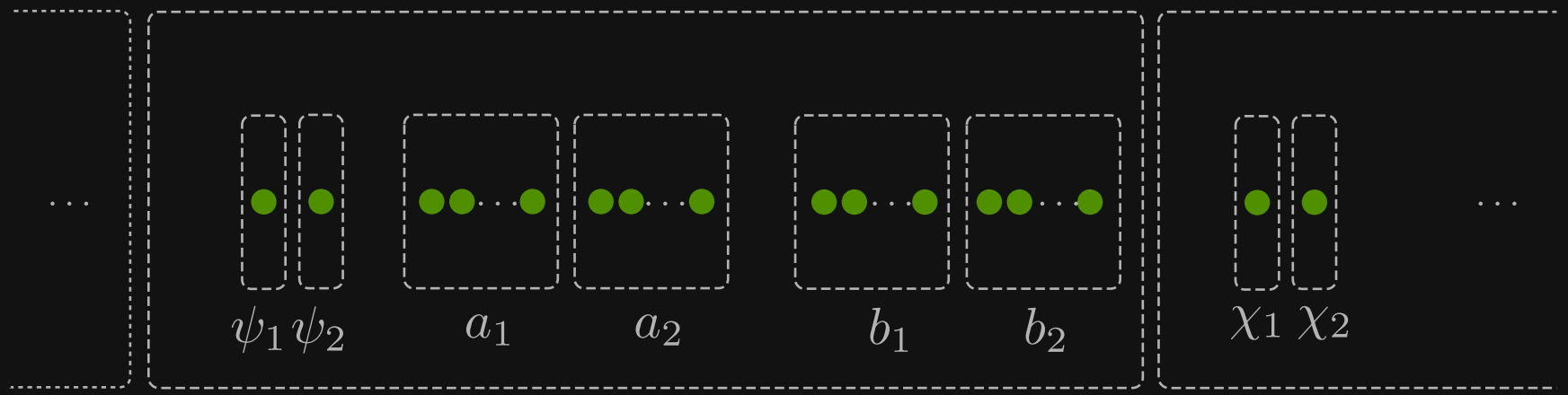
preliminary $\nu=64 \rightarrow$ 873792 distinct commutators of order x^3

2107.12769 reports ~10^{34} T gates sufficient to compute SU(2) transport coefficients

Schwinger boson (prepotential) formulation

Gauge field operators realized using harmonic oscillator doublets *

- One doublet per side of link. 1 link = 4 elementary bosonic modes



graphic credit: Z. Davoudi

* Works by M. Mathur, R. Anishetty, I. Raychowdhury

SB is similar to irrep basis in many ways.

Our belief: Few intrinsic, algorithmic differences exist in (1+1)D

Costs (Schwinger boson formulation)

Even-odd splitting is not necessary for hopping terms because

$$\left[\psi(r)_a^\dagger U(r)_{ab}\psi(r+1)_b\right]^2 = 0$$

All rotations induced by this are in 2-dimensional subspaces… Not necessary to "disentangle" even and odd couplings.

Minimum necessary subterms appears to be
8 = 4 (choices of *a* and *b*) * 2 (choices *j* raising vs. lowering)

→ 8x reduction in number of costly subterms

preliminary

ν=64 → 873792 distinct commutators of order x^3

ν=8 → 1704 distinct commutators of order x^3

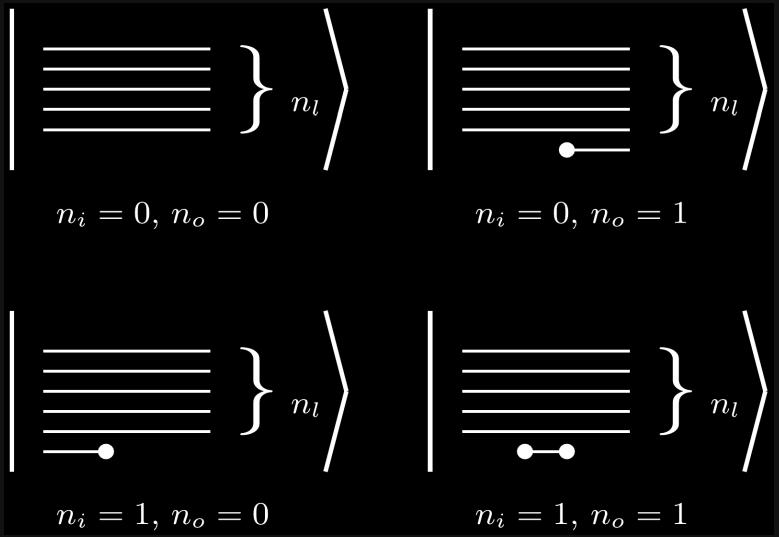
Loop-string-hadron formulation

I. Raychowdhury, JRS
Phys. Rev. D 101, 114502 (2020)

1D SU(2) LSH

- Strictly SU(2)-invariant Hilbert space
- Electric basis states characterized by one gauge-flux number n_l and two colorless quark numbers n_i, n_o
- Hamiltonian manifestly decouples *i* and *o* quark modes.

$$H_I^{(\mathrm{LSH})}(r) \sim \chi_i^\dagger(r)\chi_i(r+1)\,(\text{bosonic stuff}) + \chi_o^\dagger(r)\chi_o(r+1)\,(\text{bosonic stuff})$$



preliminary

ν=64 → 873792 distinct commutators of order x^3
ν=8 → 1704 distinct commutators of order x^3
ν=2 → 26 distinct commutators of order x^3

LSH *appears* to be 16x cheaper in T gates than SB – 'apples to apples' comparisons IP

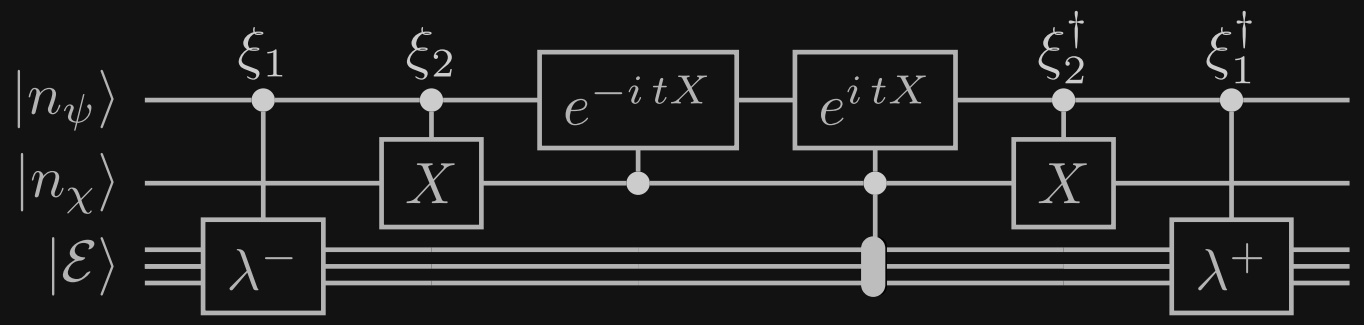
Summary: Lessons from electric basis formulations

- Electric Hamiltonians tend to be easy
- Off-diagonal interactions, i.e. hopping terms, dominate the costs for nonabelian
- Nonabelian is much harder than U(1) due to irrational functions (Clebsch-Gordans) that dramatically raise the cost of Trotter steps
- Organization of the interactions into simulatable subterms is key to picking up substantial savings in cost coefficients
- Bottlenecks are similar across the formulations, but specific costs can differ notably
- Loop-string-hadron approach has very attractive features in 1D that we hope carry over to d>1, SU(3), etc.

EXTRA SLIDES

Schwinger model hopping term: Circuit

- Shearing leads to *exact* hopping circuit
- Error in time evolution operator due to Trotterization can be shown to be reduced



A cheaper variant of this approach (cf. arXiv:2105.11548) is implementable with $4\eta^2$-4η+20 CNOTs – only two more than original approach

SU(2) gauge-matter interaction

$$H_{\text{hop}}/x = \sum_{m,n} \psi_m^\dagger \chi_n U_{mn} + \text{H.c.}$$

(Schwinger boson formulation)

$$U = \frac{1}{\sqrt{a^\dagger \cdot a + 1}} \begin{pmatrix} -a_1 b_2 + a_2^\dagger b_1^\dagger & a_1 b_1 + a_2^\dagger b_2^\dagger \\ -a_2 b_2 - a_1^\dagger b_1^\dagger & a_2 b_1 - a_1^\dagger b_2^\dagger \end{pmatrix} \frac{1}{\sqrt{a^\dagger \cdot a + 1}}$$

$$\equiv \begin{pmatrix} -A_1 b_2 + A_2^\dagger b_1^\dagger & A_1 b_1 + A_2^\dagger b_2^\dagger \\ -A_2 b_2 - A_1^\dagger b_1^\dagger & A_2 b_1 - A_1^\dagger b_2^\dagger \end{pmatrix}$$

Absorbed inverse roots into A_k

Generalizing original U(1) strategy:

- Split all harmonic oscillators into Hermitian and anti-Hermitian parts $\quad O = (O + O^\dagger) + i\frac{1}{2i}(O - O^\dagger)$
- Expand out the complete hopping term into a sum of individually Hermitian subterms
- Trotterize for each Hermitian subterm

64 subterms to simulate! [Kan & Nam, arXiv:2107.12769]
Many nonzero commutators use up the error budget
Inverse square roots are extremely expensive subroutines