D_n Gauge Theory on a Quantum Annealer

Michael Fromm, Lattice 2022,
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*with **Owe Philipsen** and **Christopher Winterowd**,*
based on arXiv:2206.14679

CRC-TR 211
Strong-interaction matter
under extreme conditions

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Gauge Theory Digitization*

Recapitulation: Basic Ingredients for Digitization of Gauge Theory (for general Lie group G)

Kogut-Susskind[1] Hamiltonian H_{KS} on Hilbert Space $\mathscr{H} = \bigotimes_{\ell} \mathscr{H}_{\ell}$,

$$H_{KS} = \lambda_E \sum_{x,\mu,\alpha} (E^\alpha(x,\mu))^2 - \lambda_B \sum_p \mathrm{Tr}(U_p + U_p^\dagger)\ , \ \hat{U}^j_{mn} = \int \mathrm{dg}\, D^j_{mn}(g)\,|\,g\rangle\langle g\,|$$

with quadratic Casimir components $(E^\alpha)^2$ and link operators $\hat{U}^j_{mn}$ building the plaquette p where $g \in G$.

Gauge Invariance

$$\theta^R_g\,|\,h\rangle = |\,hg^{-1}\rangle \text{ and } \theta^L_g\,|\,h\rangle = |\,g^{-1}h\rangle \text{ have } \theta^L_g\,\hat{U}^j_{mn}\,\theta^{L,\dagger}_g = D^j_{mk}(g)U^j_{kn} \text{ and } \theta^R_g\,\hat{U}^j_{mn}\,\theta^{R,\dagger}_g = U^j_{mk}D^j_{kn}(g)$$

Now define[2] gauge transformation via $\hat{\Theta}_g(x) = \prod_{i,o} \hat{\theta}^{L,o}_g(x)\,\hat{\theta}^{R,\dagger,i}_g(x)$ and **Gauss's law** $\hat{\Theta}_g(x)\,|\,\psi\rangle = |\,\psi\rangle$.

* See also contrib. by Ch. Kane, T. Jakobs and C. Urbach in **Algo III** on Tue, Aug 9

[1] Kogut, John and Susskind, Leonard, Phys. Rev. D 11 (1975)
[2] see e.g. Bender, J. et al., New J. Phys. 20, 093001 (2018)

Gauge Theory Digitization

One can approximate[1] the infinite dimensional $\mathscr{H}_\ell$ …

- … in the **group element basis**, i.e. given the continuous gauge group G with states “labeled” by $\{|g\rangle\}_{g\in G}$, one could choose a suitable finite subset[2] / discrete subgroup G'[3], i.e. make $\mathscr{H}_\ell$ finite dimensional.

- … in the **representation basis,** i.e. change of basis $\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}}D^j_{mn}(g)$ with representation states $\{|jmn\rangle\}$ (if G is a compact Lie group), where $j = 0,\frac{1}{2},1\ldots$ labels the irreducible representation and m, n label multiplicity within that irrep. One then truncates at a suitable j_{max}.

Other approaches[4,5], involving a change in *dof…*

- **Quantum Link Models**

- **Loop-String-Hadron Formulation**

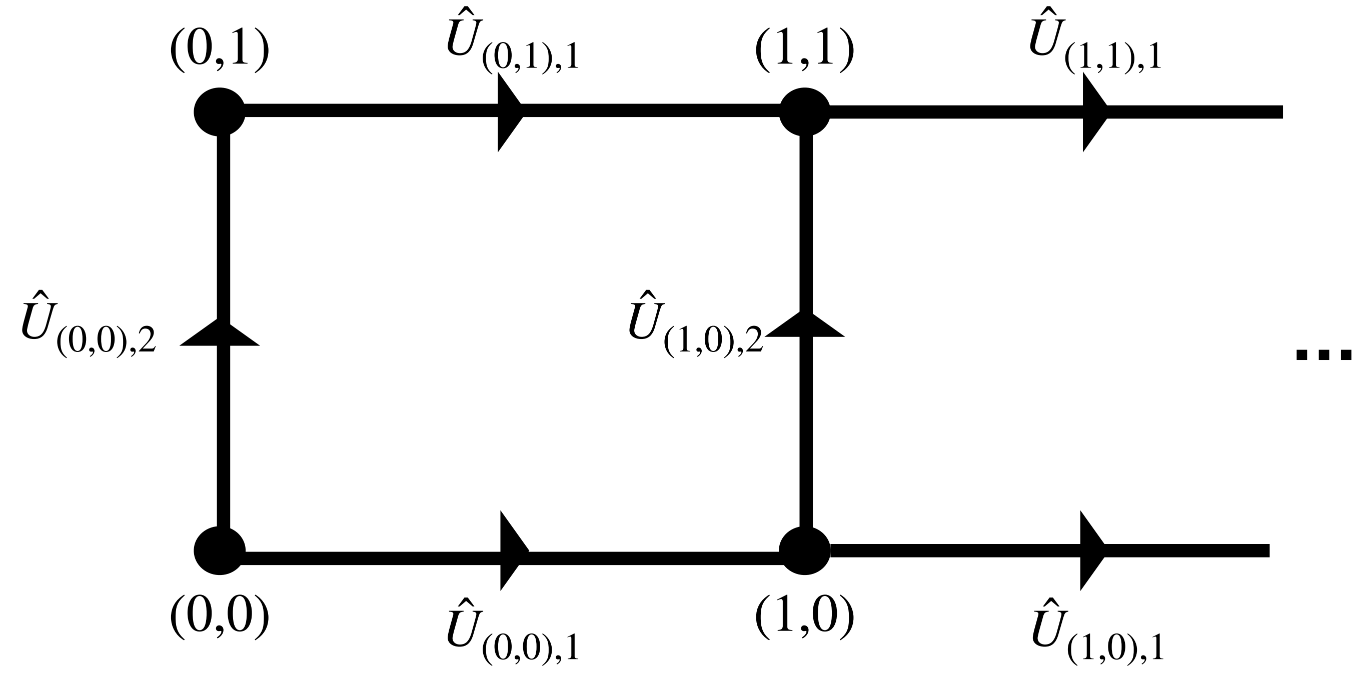
[1] Zohar, Erez, Phil. Trans. R. Soc. A.380 20210069 (2022)
[2] Hartung, T. et al., arXiv: 2201.09625 [hep-lat], see also talk by Timo Jakobs
[3] Alam, M. et al., Phys.Rev.D.105 (2022)
[4] Gustafson, E. et al., Snowmass 2021 LOI TF10-07
[5] Davoudi, Z. et al., Phys. Rev. D 104, 074505 (2021)

Gauge Invariance, Hilbert Space, Hamiltonian

- Here we combine the ideas of using a **discrete group** G and the **representation basis, motivation:**
  - Training in new approaches
  - Perhaps learn something useful, since we can e.g. compare the approximation j_{max} with the full "story" of G
- Steps ?
  1. Define a system and "build" the Hilbert Space while preserving Gauge Invariance
  2. Compute the Hamiltonian
  3. Calculate the spectrum (ground-state)
  4. Compute the dynamics, i.e. time-evolve the system

Define a system …

- Let's start with the choice of $G = D_n, n = 3,4\ldots$
  - Finite, $|G| = 2n$
  - Non-Abelian, serves e.g. as truncation to $O(2)$ for $n \to \infty$ and n odd.
  - Already studied in the context of gate-based computing with excellent description[1]
  - Geometry: Let's start with a ladder of plaquettes …



$$H_{KS} = H_E + H_B$$

$$H_B = \lambda_B \sum_p \mathrm{Tr}(U_p + U_p^\dagger) \text{ and } H_E = \lambda_E \sum_x \sum_{i=1}^{d} \sum_{jmn} f_j |jmn\rangle_{x,i} | \langle jmn |_{x,i} .$$

[1] Lamm, H. et al., Phys. Rev. D 100, 034518 (2019)

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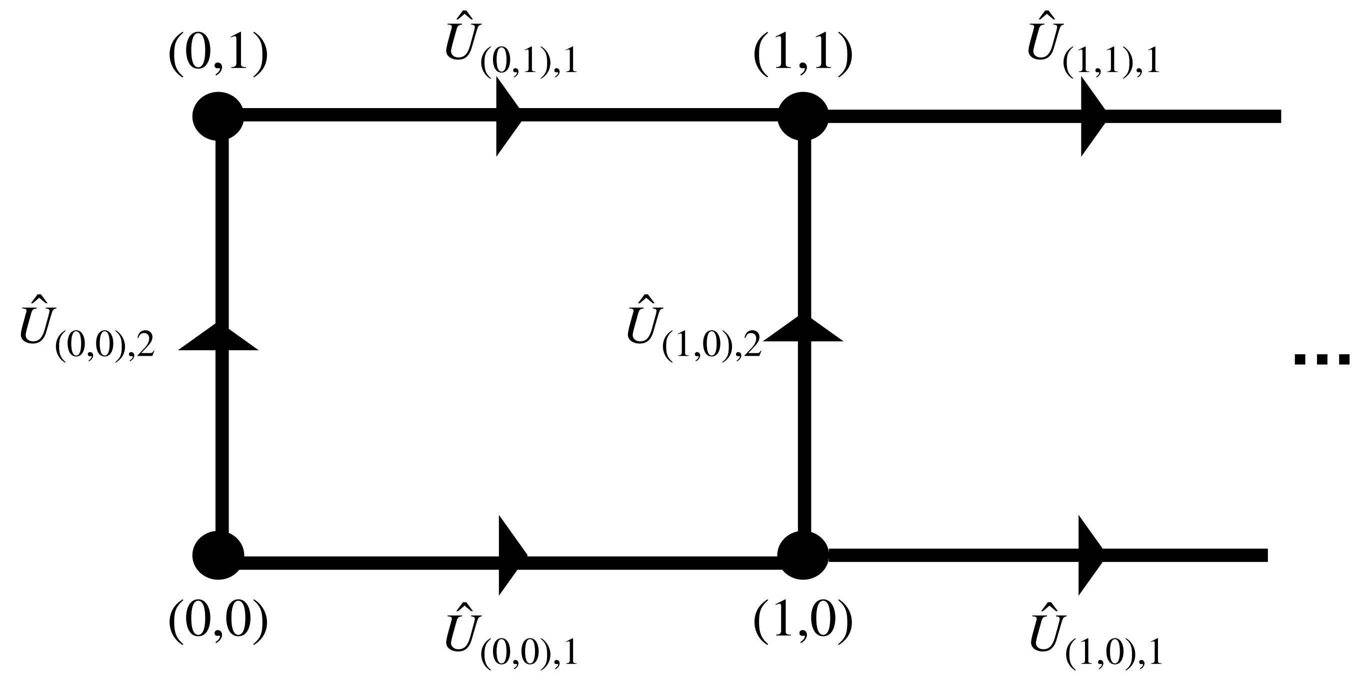
…"build" gauge-invariant Hilbert Space …

Start from the lattice with all links in the trivial rep $|000\rangle$



- Act[1] with plaquette $U_p^{(2)}$ using

$$\hat{U}^{(2)}_{m'n'}|jmn\rangle = \sum_J \sum_M \sum_N \sqrt{\frac{\text{dim}(j)}{\text{dim}(J)}} \times \langle 2m'jm|JM\rangle\langle JM|2n'jn\rangle\,|JMN\rangle$$

- Doing so in a systematic way creates an exact enumeration of $\mathcal{H}$ whose size grows exponentially with L.

- Filter those configs of $j's$ who additionally satisfy **Gauss's law,** i.e. locally

$$|00\rangle_x = \sum_{m_I} \sum_{m_A} \sum_{m_E} (-)^{f(j_A,j_E,j_I;m_I)}\,|j_A m_A\rangle \otimes |j_E m_E\rangle \otimes |j_I m_I\rangle \begin{pmatrix} j_A & j_E & j_I \\ m_A & m_E & m_I \end{pmatrix}$$

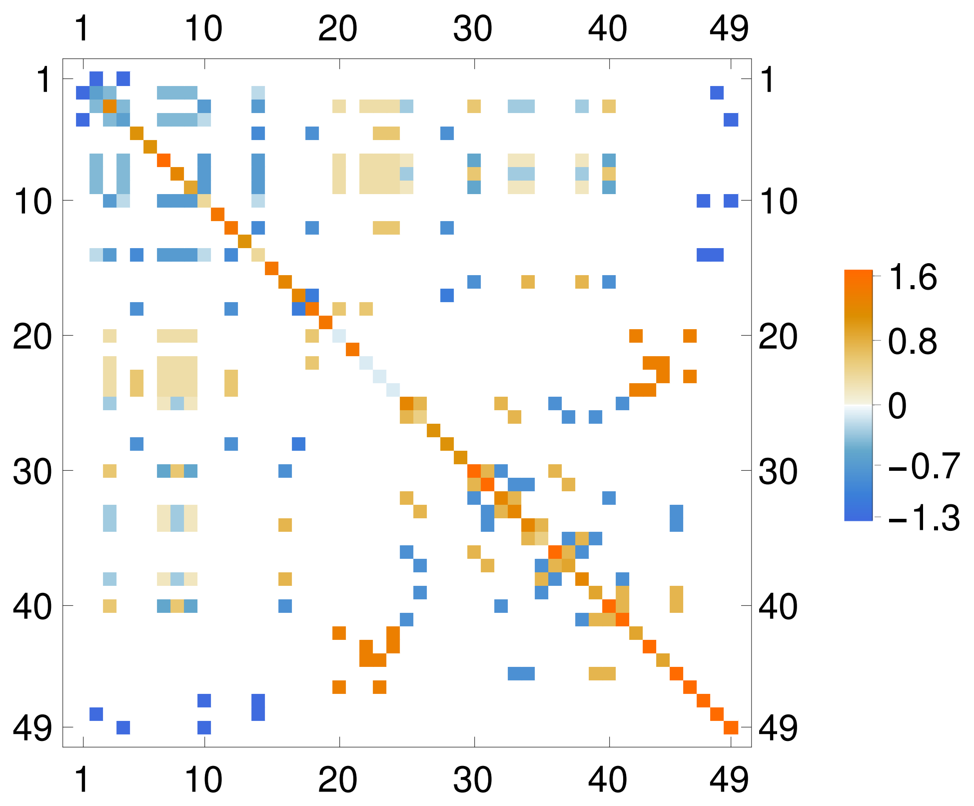
[1] Byrnes, Tim and Yamamoto, Yoshihisa, Phys. Rev. A73 (2006) 022328

Compute the Hamiltonian

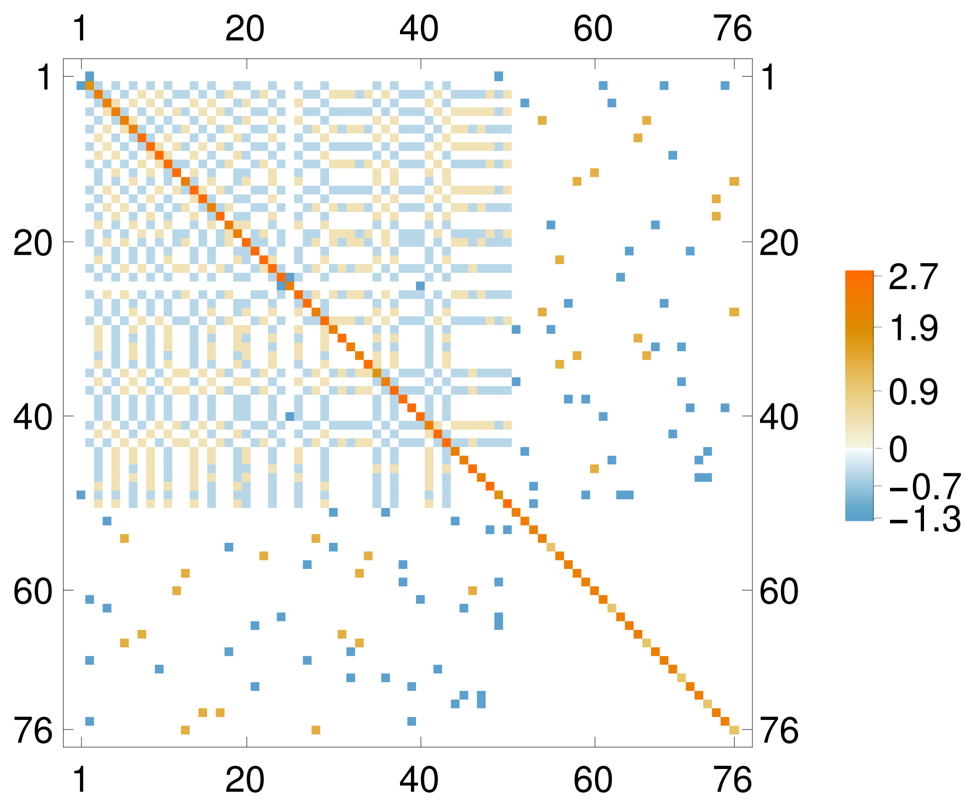
Once config space $\{j\}$ is labelled, compute $H_{ik} = \langle \psi_i | H_{KS} | \psi_k \rangle$ where $|\psi\rangle_{\{j\}} = \bigotimes_s |00\rangle_s(\{j\})$.

N	2	3	4
D_3	49	251	$O(1300)$
D_4	76	392	$O(2500)$

Table I. List of the size of the physical Hilbert space, $N_{\rm conf}$, on a ladder of size N for D_3 and D_4. The configurations are enumerated by a set of integers $\{j_i, i = 1, 2, \ldots, 3N\}$ characterizing the irrep of each link on whereby Gauss's law is satisfied at each site.



D_3



D_4

Spectrum ?

- Ground-state calculation in variational formulation

$$E_0 \leq \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}\text{, where } |\psi\rangle = \sum_i a_i |\psi\rangle_i \text{ with } a_i \in \mathbb{R}$$

- Can be cast into QUBO problem using binary variables $q_i \in \{0,1\}$ suitable for **Quantum Annealer**[1,2]

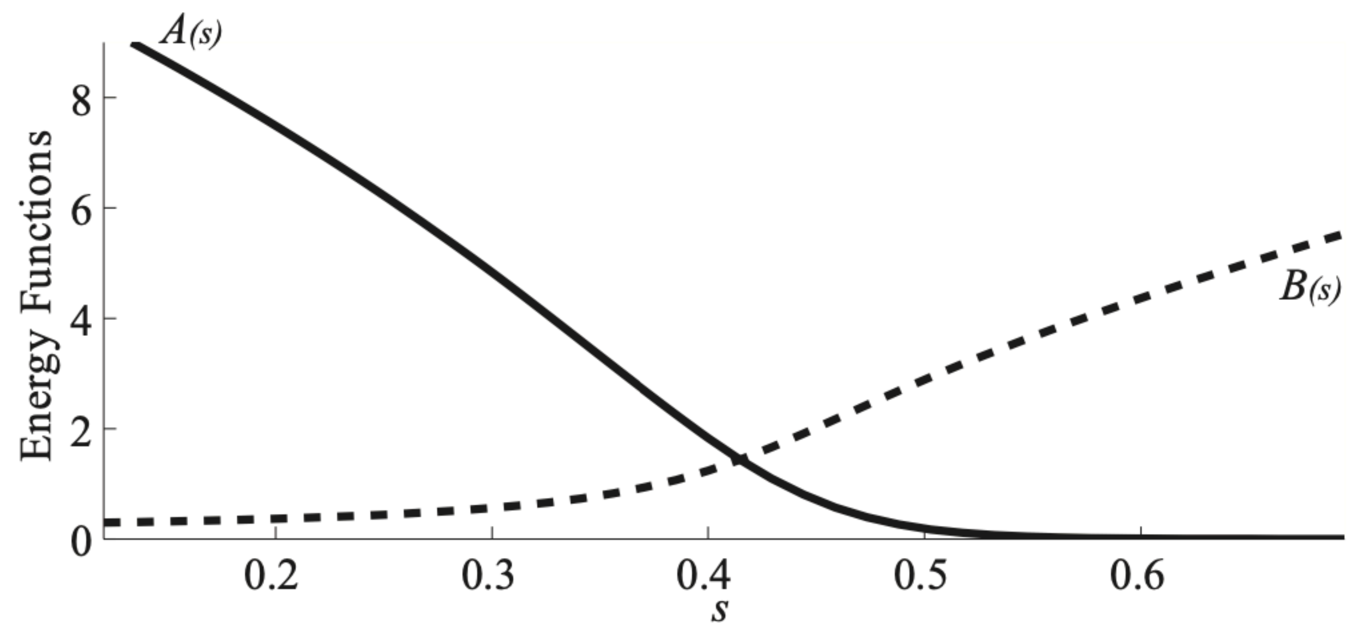
$$a_\alpha^{(z+1)} = a_\alpha^{(z)} - \frac{q_{\alpha,K}}{2^z} + \sum_{i=1}^{K-1} \frac{q_{\alpha,i}}{2^{K-i+z}},$$

$$F = \langle\psi|\hat{H}|\psi\rangle - \eta\langle\psi|\psi\rangle = \sum_{\alpha,\beta}^{N_{\text{conf}}} \sum_{i,j}^{K} Q_{\alpha\beta,ij} q_{\alpha,i} q_{\beta,j}$$

[1] Rahman, S. et al., Phys. Rev. D 104, 034501 (2021)
[2] Illa, Marc and Savage, Martin J., arXiv: 2202.12340 [quant-ph]

Ground-State Calculation via QA

Computations done on D-Wave QA in forward annealing mode[1]



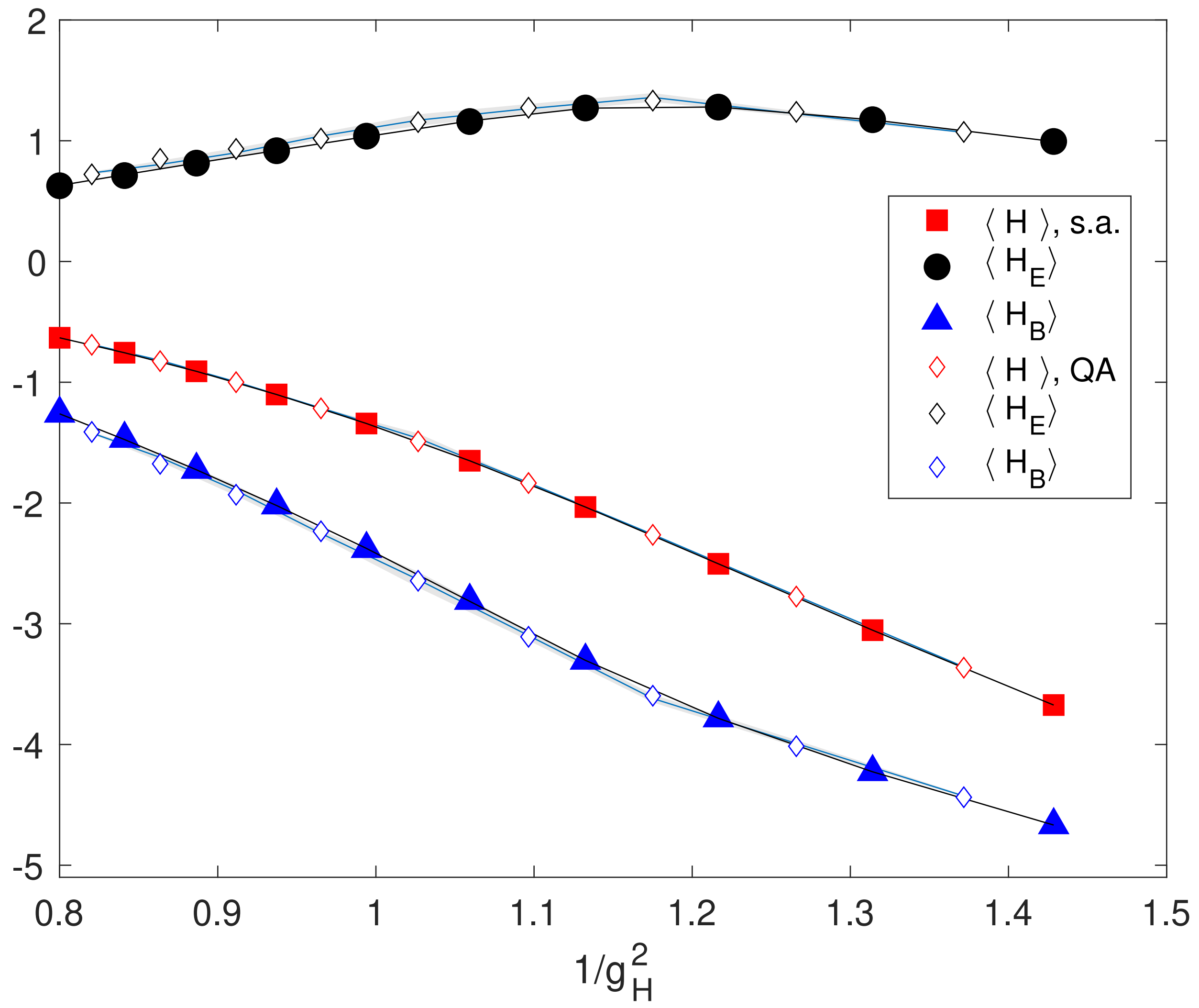
$$H_{QA} = A(s)\, H_I + B(s)\, H$$

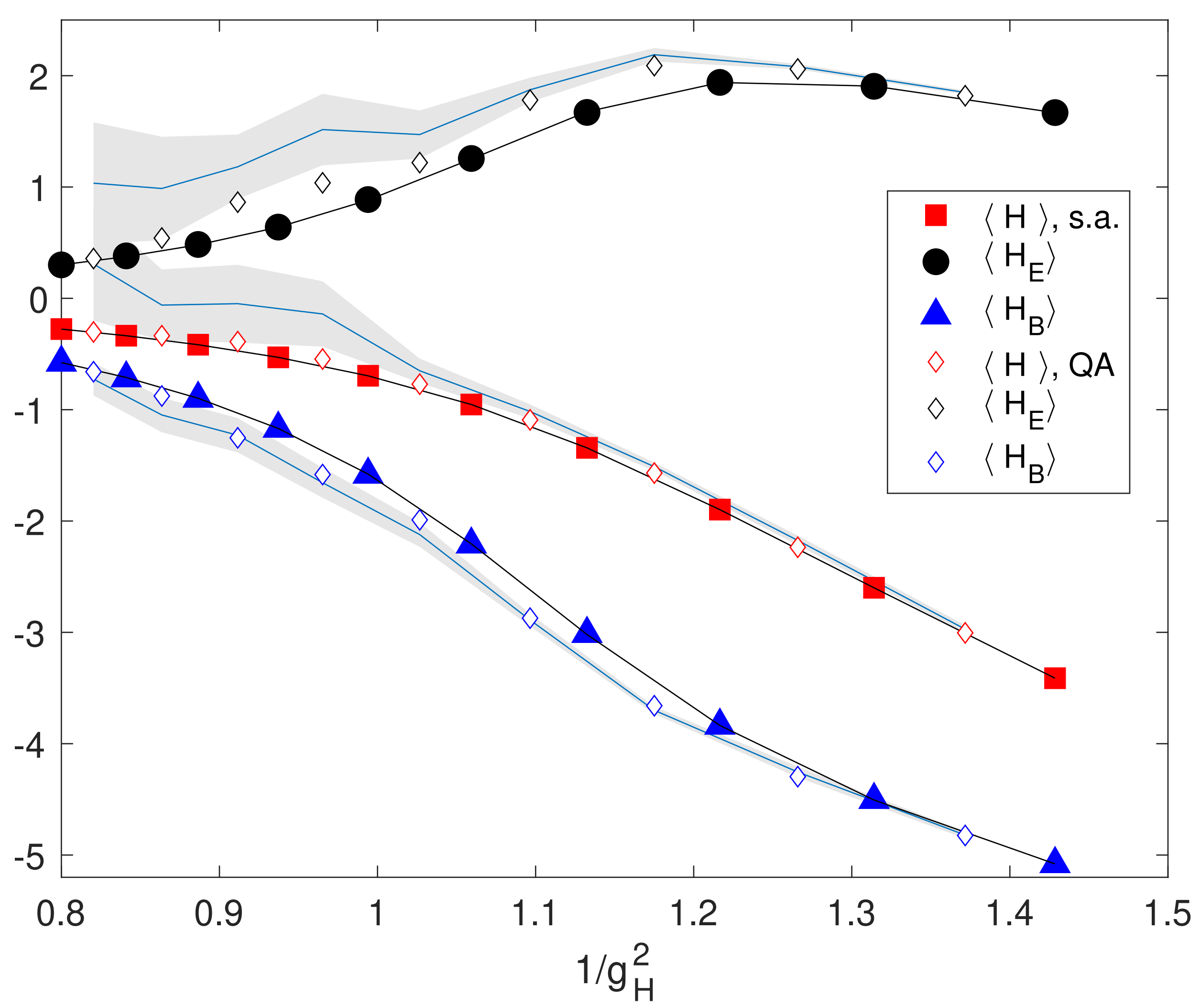
At end of annealing schedule extract ground-state coefficients a_i and hence $|\psi\rangle = \sum_i a_i |\psi\rangle_i$

Can extract

$$\langle O \rangle, O = H_B, H_E \ldots$$

Process does not always converge, results have hence uncertainty (grey band), visibly larger for the larger group D_4.





[1] Catherine C. McGeoch, Synthesis Lectures on Quantum Computing 2014 5:2, 1-93

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Time Evolution I

- Knowing H for small system allows computation of $U = \exp(-iHt)$
- Introduce[1,2,3] ancillary system of clock states $|t\rangle$, entangle with evolved state $|\psi_t\rangle \otimes |t\rangle$
- Recast time evolution as optimization problem

$$F = \sum_{t,t'=1}^{N_t} \langle t'|\langle\psi_{t'}|\hat{\mathscr{C}}|\psi_t\rangle|t\rangle - \eta\left(\sum_{t,t'=1}^{N_t}\langle t'|\langle\psi_{t'}|\psi_t\rangle|t\rangle - 1\right),$$

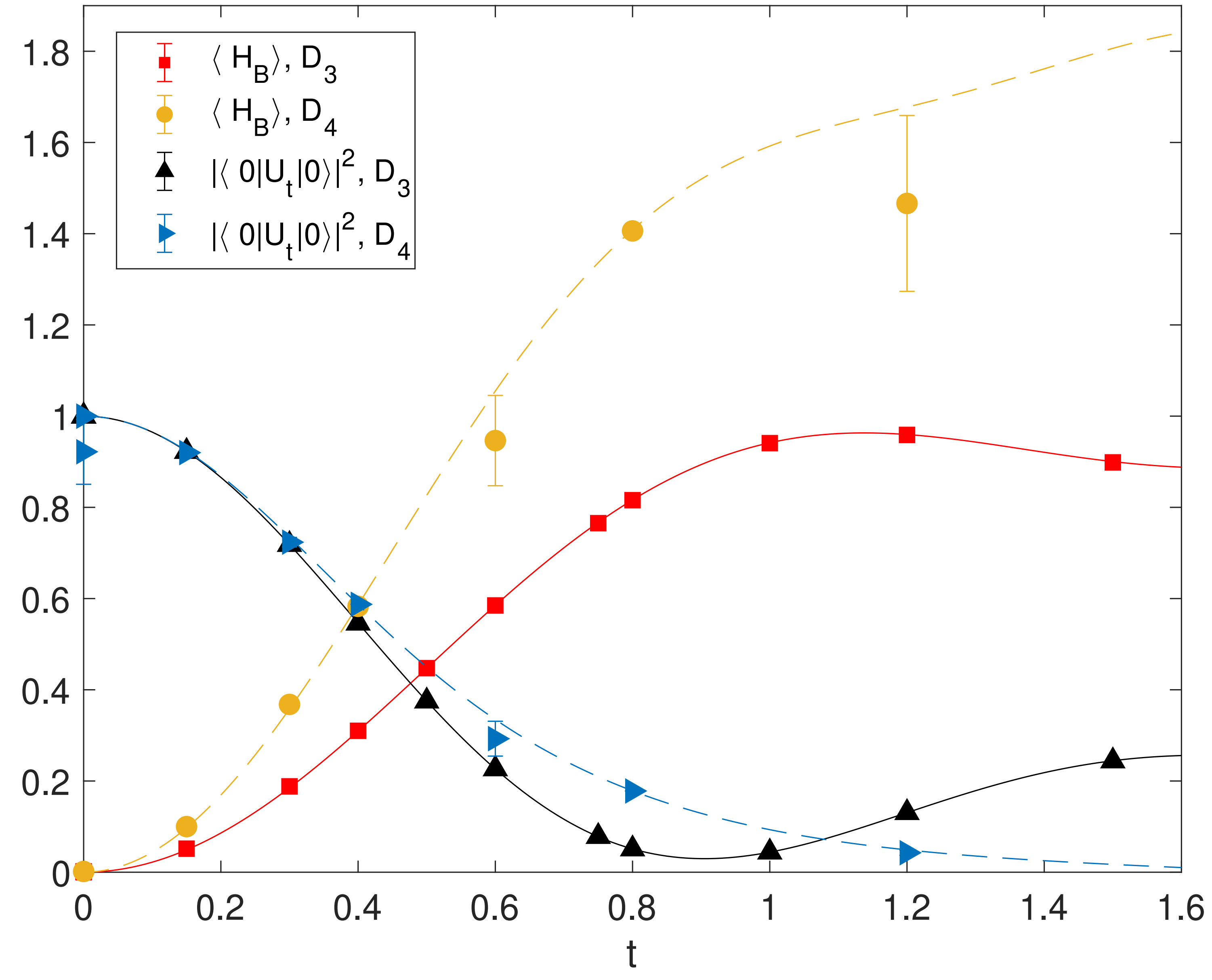
where

$$\hat{\mathscr{C}} = \hat{\mathscr{C}}_0 + \frac{1}{2}\Big(\mathbb{1}\otimes|t\rangle\langle t| + \mathbb{1}\otimes|t+1\rangle\langle t+1| - \hat{U}_{\delta t}\otimes|t+1\rangle\langle t| - \hat{U}^\dagger_{\delta t}\otimes|t\rangle\langle t+1|\Big).$$

[1] McClean, Jarrod R. et al., Proceedings of the National Academy of Sciences , Vol. 110, No. 41 (2013)
[2] Rahman, S. et al., Phys. Rev. D 104, 034501 (2021)
[3] Illa, Marc and Savage, Martin J., arXiv: 2202.12340 [quant-ph]

Time Evolution II

- Choose as initial state the trivial vacuum
- Prepare combined Hamiltonian $\mathscr{C}$ with $t = 1,\ldots,N_t$ ancillary clock states
- Q now of size $2N_tN_s \times 2N_tN_s$
- Evolve by quantum or classical annealing
- Extract time evolved $|\psi_t\rangle$ state at every time slice
- Compute $\langle O \rangle_t$ at every time slice



Conclusion and Outlook

- Interesting toy-theory to get acquainted with GT digitization
- (Hilbert Space grows exponentially (of course) with system size)
- Future Study:
  - Truncation during digitization
  - Use quantum annealer as analog quantum device

Thank you !