Real time evolution and a traveling excitation in SU(2) pure gauge theory on a quantum computer †

Sarmed A Rahman, Randy Lewis, Emanuele Mendicelli* and Sarah Powell

Department of Physics and Astronomy, York University, Toronto, Canada.

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* Speaker

† Based on: [S. A. Rahman, R. Lewis, E. Mendicelli, and S. Powell, "Self-mitigating Trotter circuits for SU(2) lattice gauge theory on a quantum computer", arXiv:2205.09247]

Motivation

Quantum computers offer the following possibilities:

- The storage of the state vectors scales polynomially rather than exponentially
- To use the Hamiltonian formulation:
  - Access to the real-time dynamics
  - No sign problem
  - Study with non-zero baryon density

✓ The first goal is to use a quantum computer to study the real-time dynamics of an SU(2) pure gauge theory in its Hamiltonian formulation.

Challenges in the use of quantum computers:

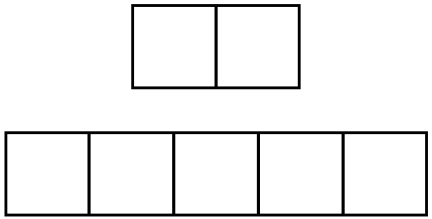
- Few qubits
- Errors in reading the qubits
- Noisy gates due to imperfect realization
- Low computation resources

✓ The second goal is to explore the available error mitigation techniques to reduce the hardware error.

SU(2) pure gauge lattice theory

$$\hat{H} = \frac{g^2}{2}\left(\sum_{i=\text{links}} \hat{E}_i^2 - 2x \sum_{i=\text{plaquettes}} \hat{\square}_i\right)$$

We have considered a single-row lattice made of plaquettes:



2 and 5 plaquettes with closed boundary conditions

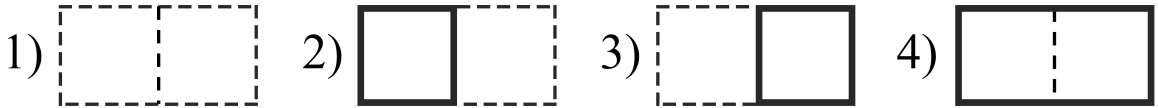
g is the gauge coupling and $x \equiv 2/g^4$.

$\hat{E}_i^2$ is the chromoelectric field for the ith lattice link.
(Returns the electric energy stored on the lattice)

$\hat{\square}_i$ is the plaquette operator trace of the product of four gauge link operators of the ith plaquette.
(Adds or subtracts energy flux on the ith plaquette)

States of the theory on 2 plaquettes

- The states are obtained applying the plaquette operators on the vacuum state (1). Fixing the maximum energy flux to $j_{max} = 1/2$, there are 4 states:



- The states can be represented using two qubits, one for each plaquette:
  1) $\longrightarrow |0\rangle|0\rangle$, 2) $\longrightarrow |0\rangle|1\rangle$, 3) $\longrightarrow |1\rangle|0\rangle$, 4) $\longrightarrow |1\rangle|1\rangle$

- The Hamiltonian representation can be obtained from the operators, and rewritten in gates as:

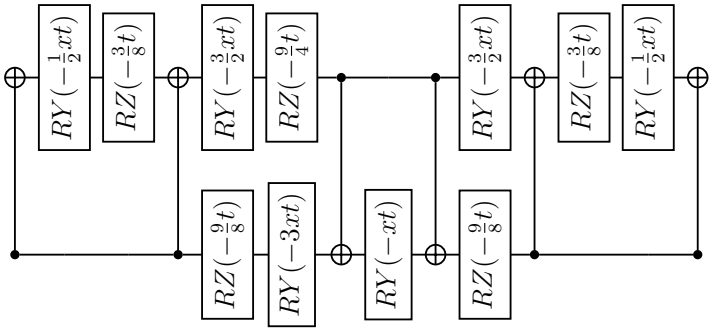
$$\frac{2}{g^2}H = \begin{pmatrix} 0 & -2x & -2x & 0 \\ -2x & 3 & 0 & -x \\ -2x & 0 & 3 & -x \\ 0 & -x & -x & \frac{9}{2} \end{pmatrix} = \frac{3}{8}(7 - 3Z_0 - Z_0Z_1 - 3Z_1) - \frac{x}{2}(3 + Z_1)X_0 - \frac{x}{2}(3 + Z_0)X_1$$

Time evolution circuit 2 plaquettes case

The time evolution operator $\exp(-iHt)$ can be approximated for small time-step dt using the second order Suzuki-Trotter expansion*:

$$e^{-iHt} = e^{-i\sum_{j=1}^{m} H_j t} = \left(\prod_{j=1}^{m} e^{-iH_j\,dt/2} \prod_{j=m}^{1} e^{-iH_j\,dt/2}\right)^{N_t} + O\left(m^2 t\,dt\right)$$

The corresponding circuit of a single Suzuki-Trotter step for the 2-plaquettes lattice is:

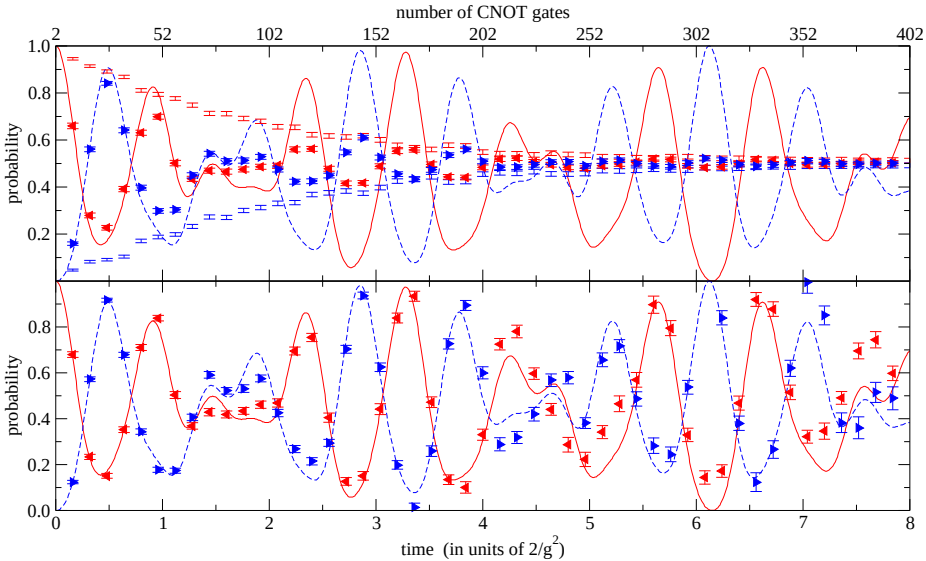


The code makes some optimization:

- The edge CNOTs cancel out, leaving 4 CNOTs per Trotter step.
- The edge rotation gates are combined in one gate.

*[Naomichi Hatano and Masuo Suzuki, Lect. Notes Phys. **679**, 37 (2005), 2005, pp. 37–68., doi:10.1007/11526216-2]

Results time evolution 2 plaquettes case



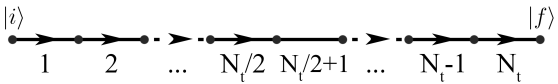
- Upper panel "raw data" after measurement error mitigation and randomized compiling (Pauli-twirling)
- Lower panel: final data, using our approach called "Self-mitigation"

- $x = 2.0$; dt=0.08;
- red solid (blue dashed) curve exact probability left (right) plaquette is on $j = 1/2$,(on $j = 1/2$)
- red and blue triangles (without symbols) are physics data (mitigation data) from the `ibmq_lagos`

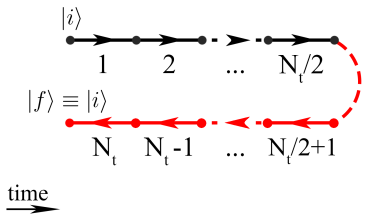
Self-mitigation [1/2]

- The idea of using a circuit with a known result for estimating the hardware errors was already presented in *[M. Urbanek, B. Nachman, V. R. Pascuzzi, A. He, C. W. Bauer and W. A. de Jong, Phys. Rev. Lett. **127**, no.27, 270502 (2021), doi: 10.1103/PhysRevLett.127.270502].
- Self-mitigation uses an error estimation circuit identical to the physics circuit:

Physics circuit: $|f\rangle = \left(e^{-iHdt}\right)^{N_t} |i\rangle$

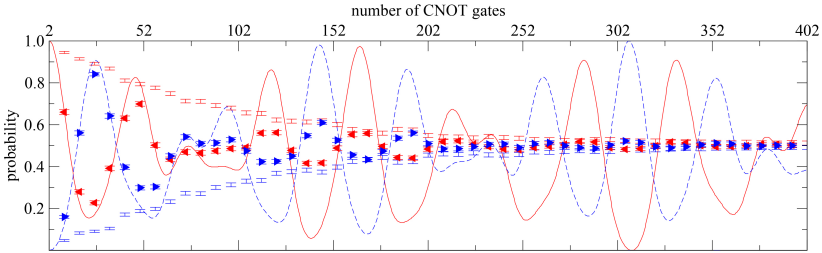


Error estimation circuit: $|f\rangle = \left(e^{iHdt}\right)^{N_t/2} \left(e^{-iHdt}\right)^{N_t/2} |i\rangle = |i\rangle$

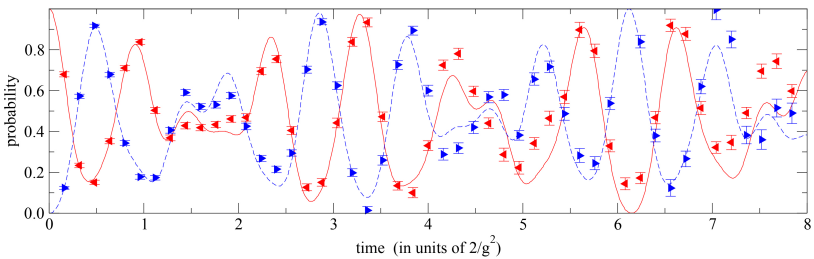


- Our error estimation circuit has the same gates, in the same order and the identical or opposite variables inside the rotations gates, and therefore it closely reproduces the physics circuit noise.
- The hardware error can be estimated in how far the final state of the error estimation circuit is measured from the initial state.

Self-mitigation [2/2]

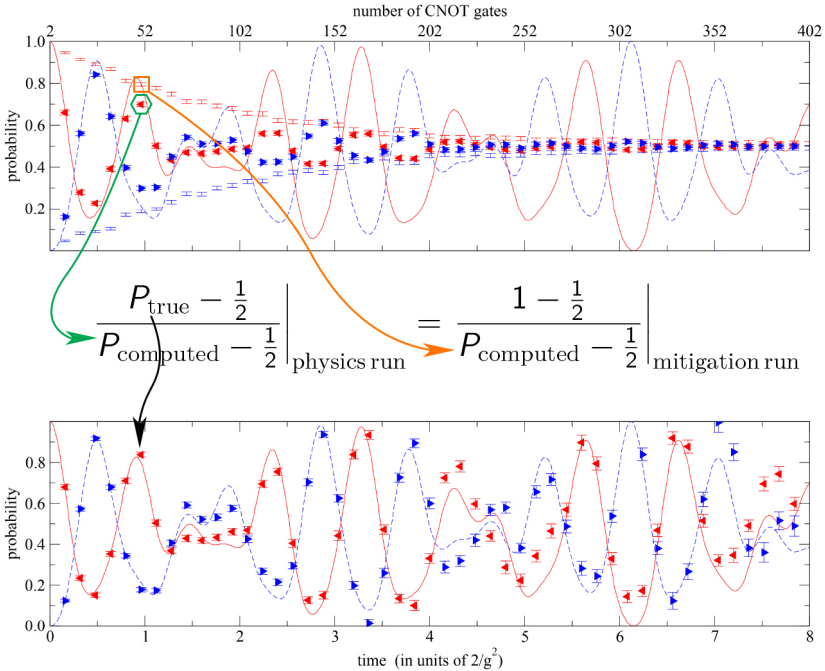


$$\left.\frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\right|_{\text{physics run}} = \left.\frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\right|_{\text{mitigation run}}$$



- On a perfect hardware the mitigation circuit returns a probability of 1 for the left plaquette and 0 for the right one.
- red error bar without symbols mitigation data left plaquette $j = 1/2$ from the `ibmq_lagos`.
- blue error bar without symbols mitigation data right plaquette $j = 1/2$ from the `ibmq_lagos`.
- red (blue) triangles left (right) plaquette $j = 1/2$ data from the `ibmq_lagos`.

Self-mitigation [2/2]



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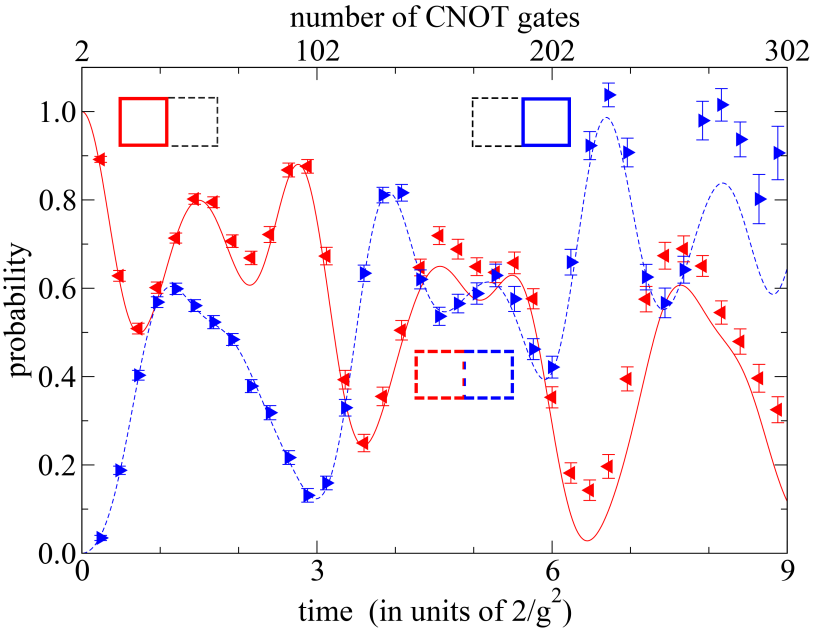
Measurement protocol

- On the hardware `ibmq_lagos` 300 circuits with 10^4 hits can be submitted as a single list.
- Submit the 2^n circuits for the mitigation of measurement error.
  (4 circuit for the 2 plaquettes case)
- Submit the randomized compiling circuits for the error estimation circuits and the physics circuits (148 error estimation circuits and 148 physics circuits)
- Collect all the measurements
- Apply the measurement-error calibration matrix to the self-mitigation and physics results
- Use the self-mitigation equation to mitigate the hardware error on the physics result:

$$\left.\frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\right|_{\text{physics run}} = \left.\frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\right|_{\text{mitigation run}}$$

- Calculate the error of the error mitigation and physics results as the sum in quadrature of the statistical error from the 10^4 hits and the 1480 bootstrap samples.

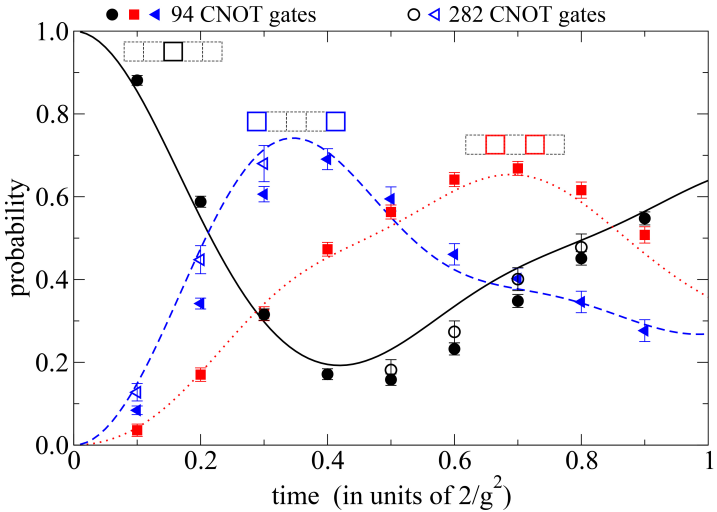
A traveling excitation



- Initial state: left plaquette on.
- $x = 0.8$, $dt = 0.12$, 2 Trotter steps.
- red solid curve exact probability left plaquette on $j = 1/2$.
- blue dashed curve exact probability right plaquette on $j = 1/2$.
- red and blue triangles are calculations on the `ibmq_lagos`.

- For $x < 1$ the dominant states are the low energy ones and these are the single-plaquette states, therefore traveling excitation across the lattice are visible.

Going toward a larger lattice: 5 plaquettes



- Initial state: center plaquette on
- $x = 2.0$, 4 Trotter steps, various time step sizes;
- full symbols are calculations on the `ibmq_lagos` after self-mitigation.
- open symbols are obtained using zero-noise extrapolation with 3 CNOTs every CNOT.

- 5 plaquettes is the largest lattice implementable on a 7-qubits hardware without swaps due to qubits connectivity.
- zero-noise extrapolation consist of creating new circuits with extra CNOT gates and then fitting the result to extract the zero noise limit.

Take-home message from our work

- The IBM gate based quantum computer can be successfully used to study the real-time evolution of a non-Abelian lattice gauge theory.
- The real-time evolution study was extended to a time range much larger than previous study on non-Abelian gauge theories.
- As an example of real-time dynamical process: local excitation moving across the lattice was observed.
- Measurement mitigation and randomized compiling were important error mitigation tools in our study, but our approach Self-mitigation that uses the same physics circuit as it own noise-mitigation circuit, was the tool for achieving the extra mile.
- Self-mitigation can be successfully combined with zero-noise extrapolation if extra error mitigation is needed.
- Self-mitigation was successfully used recently:

  [Y. Y. Atas, J. F. Haase, J. Zhang, V. Wei, S. M.-L. Pfaendler, R. Lewis and C. Muschik, Real-time evolution of SU(3) hadrons on a quantum computer, arXiv:2207.03473]

  see also: [R. C. Farrell,I. A. Chernyshev, S. J. M. Powell, N. A. Zemlevskiy, M. Illa and M. J. Savage,Preparations for Quantum Simulations of Quantum Chromodynamics in 1+1 Dimensions: (I) Axial Gauge, arXiv:2207.01731]

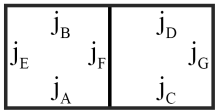
Thank you for your time!

Backup Slides

SU(2) pure gauge lattice theory, more about the operators

Multiplicities used for the basis states, $E \equiv 2j_E + 1$.

$$\hat{H} = \frac{g^2}{2}\left(\sum_{i=\text{links}} \hat{E}_i^2 - 2x \sum_{i=\text{plaquettes}} \hat{\square}_i\right)$$



$$= \left| E_A^B F_C^D G \right\rangle$$

g is the coupling constant and $x \equiv 2/g^4$.

$\hat{E}_i^2$ is the chromoelectric field for the ith lattice link.

$\langle\psi_{\text{final}}| \sum_i \hat{E}_i^2 |\psi_{\text{initial}}\rangle = \sum_{i=A}^{L} j_i(j_i+1)\, \delta_{\text{final,initial}}$

$$|\psi\rangle = |j_E\rangle\, |j_A\rangle\, |j_B\rangle\, |j_F\rangle\, |j_C\rangle\, |j_D\rangle\, |j_G\rangle$$

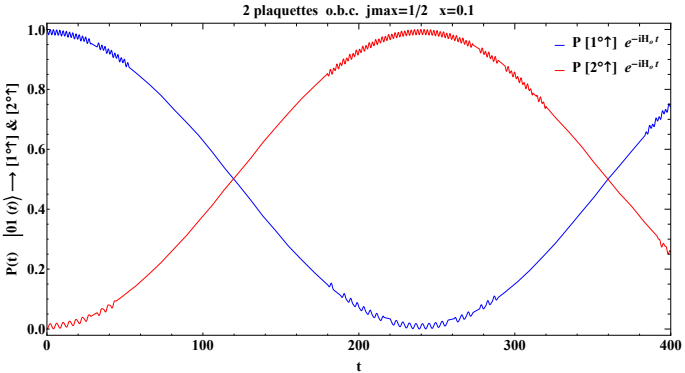
$\hat{\square}_i$ is the plaquette operator trace of the product of four gauge link operators of the ith plaquette.

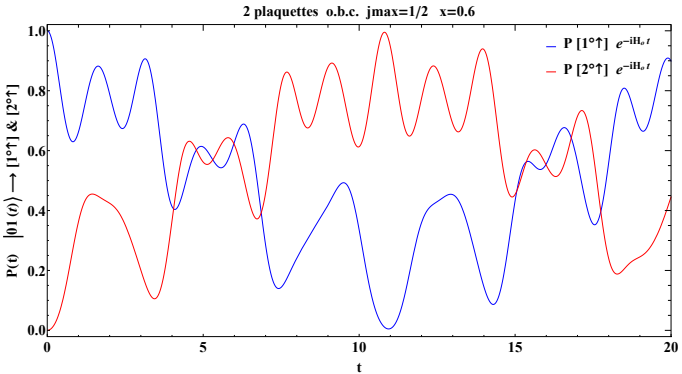
$$\begin{aligned}\langle\psi_{\text{final}}|\, \square_1\, |\psi_{\text{initial}}\rangle \;=\; & (-1)^{j_A+j_B+j_C+j_D+2J_E+2J_F+2j_I+2j_J} \\ & \sqrt{2j_E+1}\sqrt{2J_E+1}\sqrt{2j_J+1}\sqrt{2J_J+1}\sqrt{2j_F+1}\sqrt{2J_F+1}\sqrt{2j_I+1}\sqrt{2J_I+1} \\ & \begin{Bmatrix} j_A & j_E & j_I \\ \frac{1}{2} & J_I & J_E \end{Bmatrix} \begin{Bmatrix} j_B & j_F & j_I \\ \frac{1}{2} & J_I & J_F \end{Bmatrix} \begin{Bmatrix} j_C & j_E & j_J \\ \frac{1}{2} & J_J & J_E \end{Bmatrix} \begin{Bmatrix} j_D & j_F & j_J \\ \frac{1}{2} & J_J & J_F \end{Bmatrix}\end{aligned}$$

where j_i and J_i are the links in $|\psi_{\text{initial}}\rangle$ and $|\psi_{\text{final}}\rangle$, respectively.

The two different regimes $x < 1$ and $x > 1$ for the time evolution [1/2]

- For small x, the chromomagnetic contribution is negligible, therefore at low energy the dominant states are weakly coupled chromoelectric eigenstates. The single-plaquette states move across the lattice.

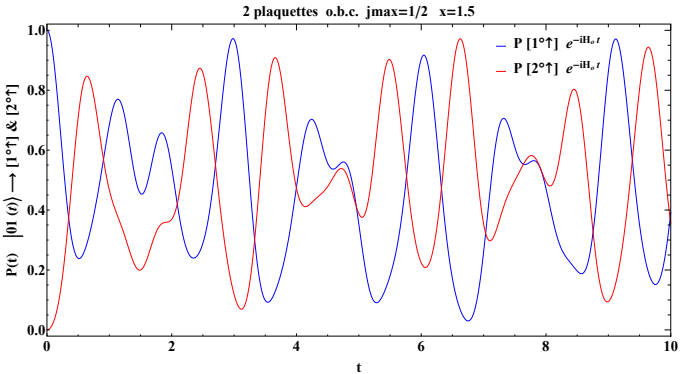


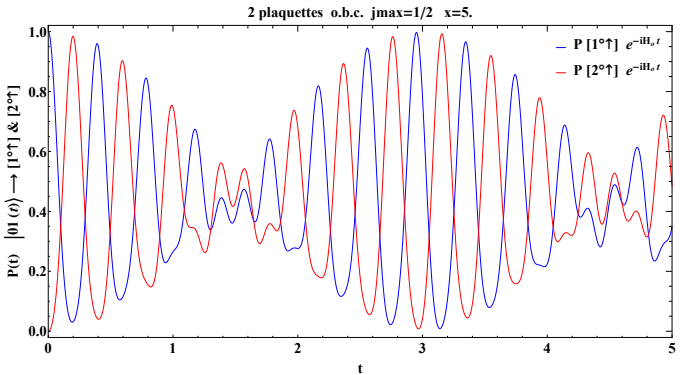


- The single-plaquette propagation time is larger for small x, diverging for $x = 0$, where the Hamiltonian is diagonal, containing only the chromoelectric term, and the single-plaquette are eigenstate, therefore are constant in time.
- In the right figure, increasing x let higher frequencies to appear as oscillations superimposed to the single-plaquette transition.

The two different regimes $x < 1$ and $x > 1$ for the time evolution [2/2]

- For x larger than 1 at larger energy, the chromomagnetic contribution dominates mixing the single-plaquette states. This is evident by the presence of many higher frequencies superimposed to the single-plaquettes transitions.





- Larger is x more high frequencies are present as evident moving from left figure at $x = 1.5$ to right figure at $x = 5.0$

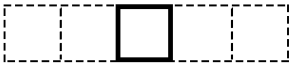
The 5 plaquettes case [1/2]

From the N plaquettes Hamiltonian written using gates, the 5 plaquettes case is obtained with N=5:

$$
\begin{aligned}
H &= \frac{g^2}{2}(h_E + h_B)\,, \\
h_E &= \frac{3}{8}(3N+1) - \frac{9}{8}(Z_0 + Z_{N-1}) - \frac{3}{4}\sum_{n=1}^{N-2} Z_n \\
&\quad -\frac{3}{8}\sum_{n=0}^{N-2} Z_n Z_{n+1}\,, \\
h_B &= -\frac{x}{2}(3+Z_1)X_0 - \frac{x}{2}(3+Z_{N-2})X_{N-1} \\
&\quad -\frac{x}{8}\sum_{n=1}^{N-2}(9 + 3Z_{n-1} + 3Z_{n+1} + Z_{n-1}Z_{n+1})X_n\,.
\end{aligned}
$$

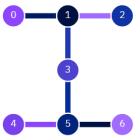
5 plaquettes lattice:

Initial state used:



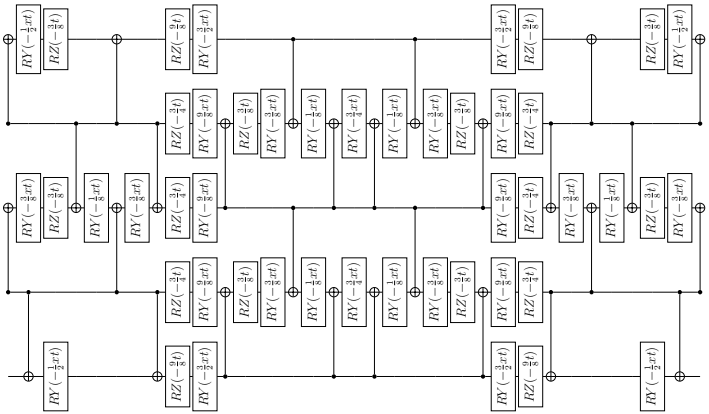
Dashed line $j = 0$ solid tick line $j = 1/2$

- A 7 qubits hardware such as `ibmq_lagos` has only 5 qubits chain with nearest-neighbor connectivity, therefore 5 plaquettes is the current largest lattice.



The 5 plaquettes case [2/2]

- The corresponding time evolution circuit of a single Suzuki-Trotter step for the 5-plaquettes lattice is:



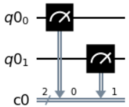
The code makes some optimization:

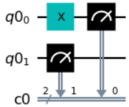
- The edge CNOTs cancel out, leaving 22 CNOTs per Trotter step.
- The edge rotation gates are combined in one gate.

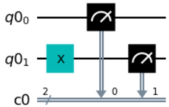
Mitigation of measurement error

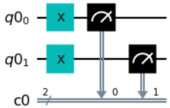
Measurement error mitigation is the procedure needed to correct the hardware errors in measuring the qubits state. Working with n qubits there are 2^n possible states. Each state should be recreated with a circuit and measured to construct the $2^n \times 2^n$ calibration matrix, whose entries are the probabilities that a particular states once is measured has a superposition with another state. This matrix is then applied to the measurements of the physics circuit.

- In the case of 2 plaquettes lattice we use 2 qubits, therefore there are 4 mitigation circuits:

1) 

2) 

3) 

4) 

A calibration matrix looks like :

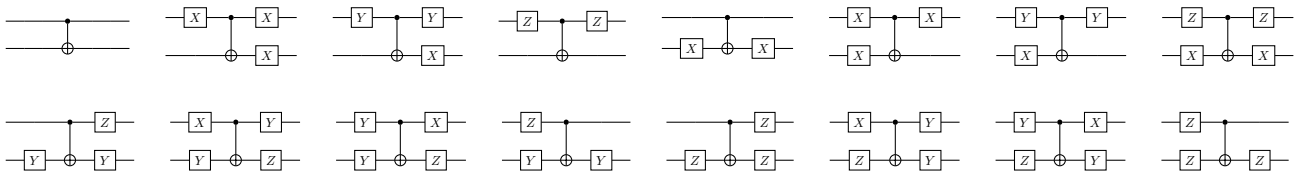
$$\begin{pmatrix} 0.9865 & 0.0131 & 0.0064 & 0.0003 \\ 0.0084 & 0.9817 & 0.0002 & 0.008 \\ 0.0049 & 0.0001 & 0.9832 & 0.0125 \\ 0.0002 & 0.0051 & 0.0102 & 0.9792 \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

- In case of no hardware errors for example on the simulator, the matrix is diagonal meaning that the states have no superposition with each other.
- Normally the hardware makes errors and the matrix is not diagonal.

Randomized compiling* (Pauli-twirling)

- Randomized compiling is a technique to transform the CNOT coherent noise into incoherent noise. This is done by "surrounding" a CNOT gate by Pauli or identity gates creating a circuit equivalent to a single CNOT gate.

- It can be shown that there are 16 possible options:



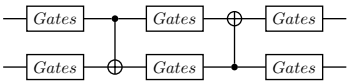
- The circuit should be runned a sufficient number of times to randomly access all the combinations for the "surrounding" gates of each CNOT gate in the physics circuit.

*[Joel J. Wallman and Joseph Emerson, Phys. Rev. A **94** 052325 (2016), doi: 10.1103/PhysRevA.94.052325

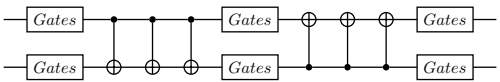
Zero-noise extrapolation *

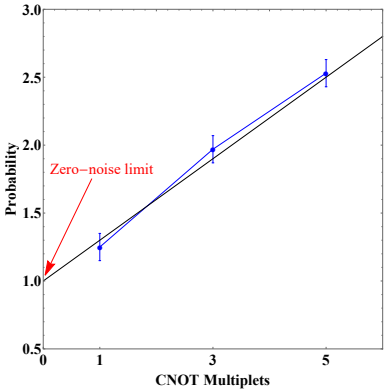
The method consists in study the circuit noise by creating copies of the original circuits where the noise is artificially increase by replacing each CNOT gate by odd multiplet (triplet, quintet etc.) of CNOTS. The zero-noise limit is extracted by fitting the circuits measurements with a function of the CNOT multiplets.

The original circuit:



A copy of the original circuit with each CNOT gates replaced by a CNOT triplet:





A possible linear extrapolation of the zero-noise limit as the line intercept with the y-axis.

*[Y. Li and S.C. Benjamin, "Efficient Variational Quantum Simulator Incorporating Active Error Minimization", Phys. Rev X **7** 021050 (2017), doi:10.1103/PhysRevX.7.021050]