Error Reduction using Machine Learning on Ising Worm Simulation

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Ising Dual representation

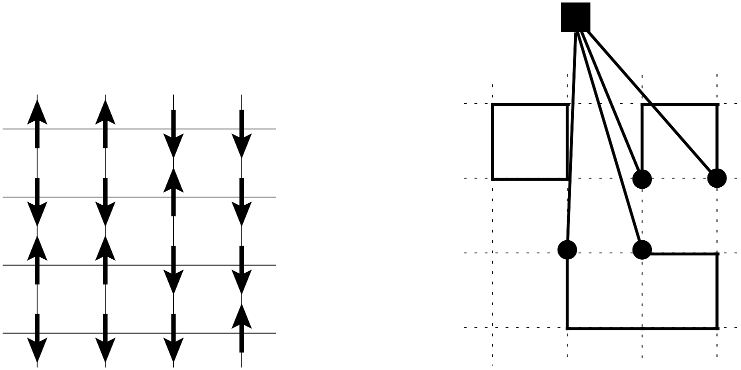
- Ising model partition function

$$Z_{\text{Ising}} = \sum_{\{s\}} e^{-\beta H(\{\sigma\})}, H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i \quad (1)$$

$$= (2\cosh(\beta h))^N \cosh(\beta J)^{N_{nn}}$$
$$\times \sum_{\substack{\{n_b, m_i\} \\ \partial\{n_b\}=\{m_i\}}} \tanh(\beta J)^{\sum_b n_b} \tanh(\beta h)^{\sum_i m_i}, \quad (2)$$

- a dual representation obtained by introducing the bond variables (nearest neighbor pairs) and integrating out the spins.
- N is the number of lattice sites, N_{nn} the number of bonds.
- $m_i \in \{0, 1\}$ for each site x.

Ising Model configurations



(a) Spin (b) Worm

- Simulation at finite external field with a ghost (master) site.
- Configuration is updated by worm evolving and the averaged worm length (G_2) until worm makes closed loop gives 2-point correration function.
- The number of monomers(M) is measured only on the closed loop configuration.

Observables in Ising Dual representation

- The number of monomers $M = \sum_i m_i$.
- The moments are

$$\langle \sigma^n \rangle = \frac{1}{(N\beta)^n} \frac{1}{Z} \frac{\partial^n Z}{\partial h^n} \quad (3)$$

$$\langle \sigma \rangle = \tanh(\beta h) + \frac{\langle M \rangle}{\sinh(\beta h) \cosh(\beta h)} \quad (4)$$

$$\langle \sigma^2 \rangle = f_2(\langle M^2 \rangle, \langle M \rangle) \quad (5)$$

$$\langle \sigma^3 \rangle = f_3(\langle M^3 \rangle, \langle M^2 \rangle, \langle M \rangle) \cdots \quad (6)$$

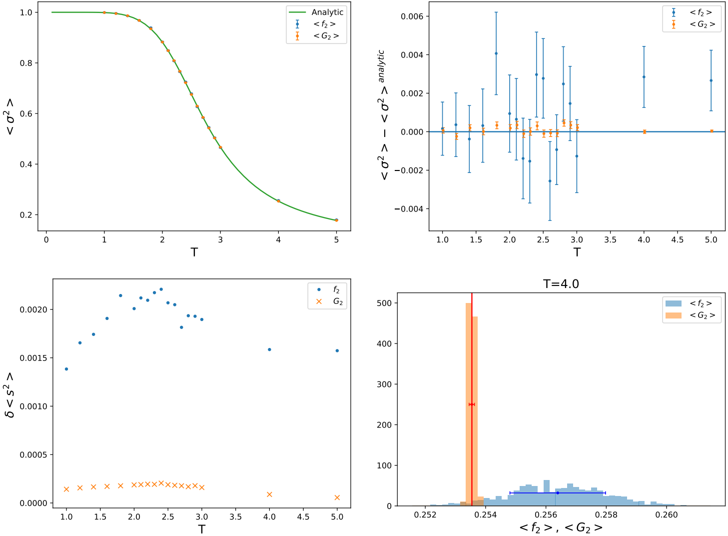
$$\langle \sigma^4 \rangle = f_4(\langle M^4 \rangle, \langle M^3 \rangle, \langle M^2 \rangle, \langle M \rangle) \cdots \quad (7)$$

$$(8)$$

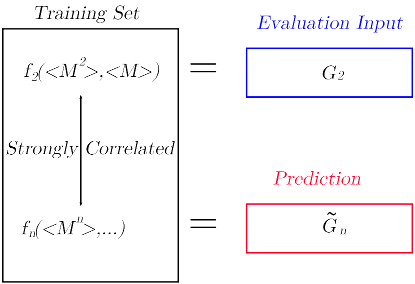
- G_2 is the averaged worm length.
- $\langle G_2 \rangle = \langle \sigma^2 \rangle$, $G_2 \neq \sigma^2$
- Any higher moments can be obtained from M.

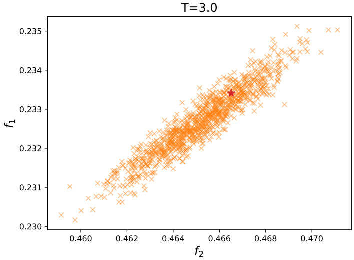
Comparison between G_2 and $f_2(\langle M^2\rangle, \langle M\rangle)$

- 2-dim, 4×4 lattice, external field $h = 0.1$, 1,000,000 measurements.



Strategy for applying machine learning

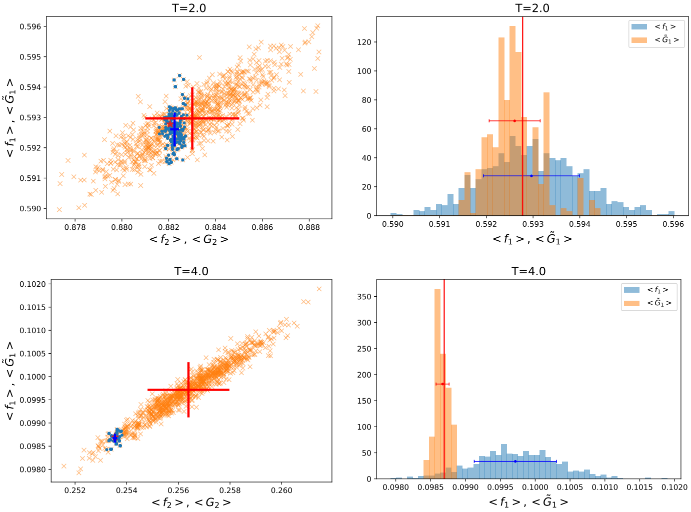




- Training the correlation between f_2 and f_n.
- Use G_2 as input data
- $\tilde{G}_n$ are machine learning prediction.
- This correlation depends on T, h, V.

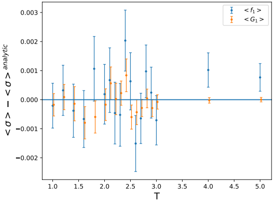
Machine learning results of $\tilde{G}_1$

- The correlation becomes stronger as the temperature increases.

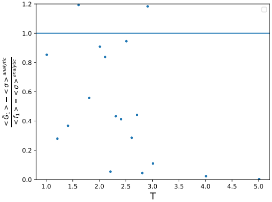


Magnetization

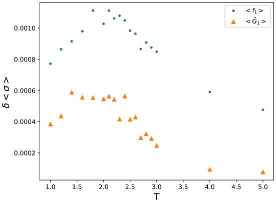
- f_1 : the number of monomers
- $\tilde{G}_1$: machine learning output from the worm estimate G_2 as input.



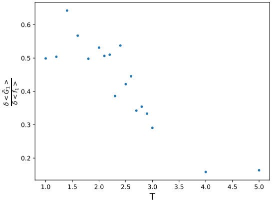
(c) $\langle\sigma\rangle - \langle\sigma\rangle^{\text{analytic}}$



(d) Deviation $\frac{\langle\tilde{G}_1\rangle - \langle\sigma\rangle^{\text{analytic}}}{\langle f_1\rangle - \langle\sigma\rangle^{\text{analytic}}}$



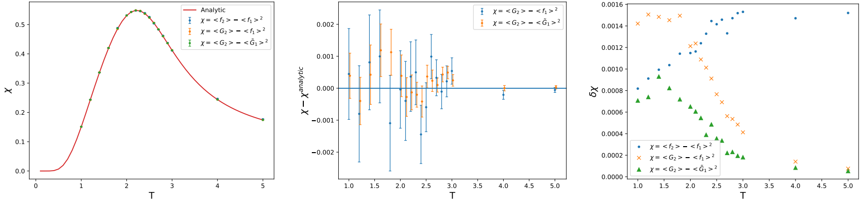
(e) Statistical Error

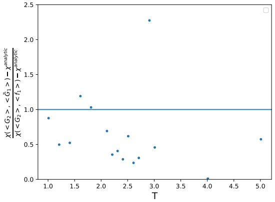


(f) Stat. Error Reduction $\frac{\delta\langle\tilde{G}_1\rangle}{\delta\langle f_1\rangle}$

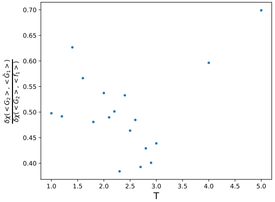
Susceptibility

- $\chi = \langle \sigma^2 \rangle - \langle \sigma \rangle^2 = \langle f_2 \rangle - \langle f_1 \rangle^2 = \langle G_2 \rangle - \langle f_1 \rangle^2 = \langle G_2 \rangle - \langle \tilde{G}_1 \rangle^2$





(g) Deviation $\frac{\chi(G_2,\tilde{G}_1)-\chi^{\text{analytic}}}{\chi(G_2,f_1)-\chi^{\text{analytic}}}$



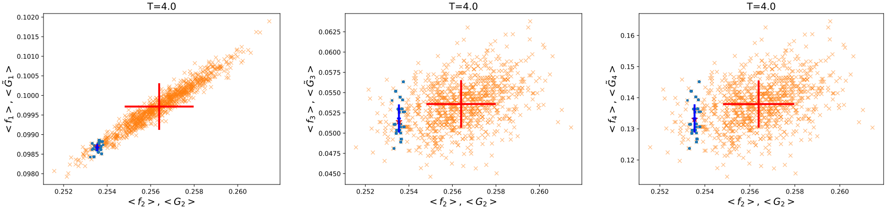
(h) Stat. Error $\frac{\delta\chi(G_2,\tilde{G}_1)}{\delta\chi(G_2,f_1)}$

Higher moments and Kurtosis B_4

- Kurtosis is a function of $\langle\sigma\rangle$, $\langle\sigma^2\rangle$, $\langle\sigma^3\rangle$ and $\langle\sigma^4\rangle$.

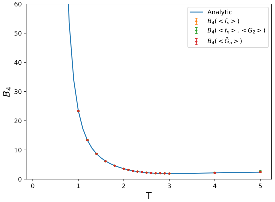
$$B_4 = \frac{\langle\left(\sigma - \langle\sigma\rangle\right)^4\rangle}{\langle(\sigma - \langle\sigma\rangle)^2\rangle^2} \tag{9}$$

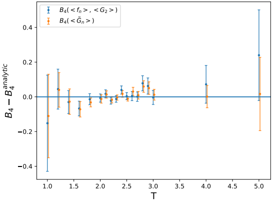
- The correlation is not as strong as f_1 and f_2.
- Therefore, the error reduction is insignificant.

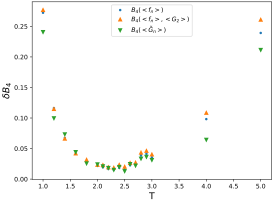


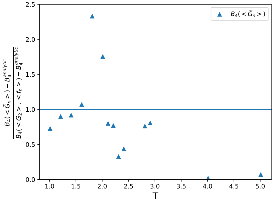
Kurtosis B_4

- Statistical error reduction: $40 \sim 90\%$

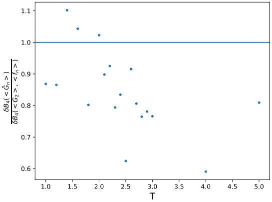








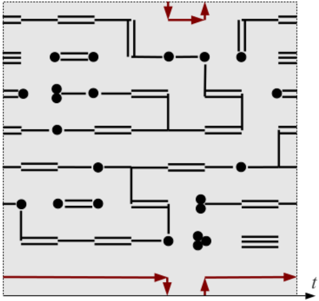
(i) Deviation $\frac{\chi(G_2,\tilde{G}_1)-\chi^{\text{analytic}}}{\chi(G_2,f_1)-\chi^{\text{analytic}}}$

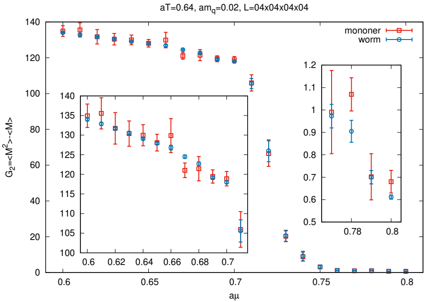


(j) Stat. Error $\frac{\delta\chi(G_2,\tilde{G}_1)}{\delta\chi(G_2,f_1)}$

Applying the same method to the strong coupling QCD

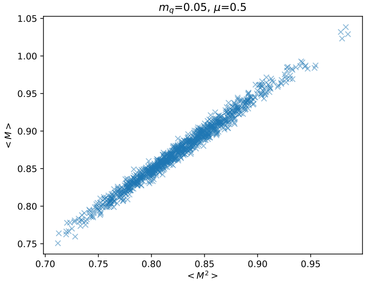
- We also can measure the same observable in two different methods.
  - One is to use the number of monomers M. (Cannot use in the chiral limit)
  - The other is to use the worm estimator (counting the number of steps during worm evolution). $G_2(x_1, x_2)$ is a 2pt function, sampled by the worm algorithm.
- Measuring the observables in terms of $\langle M \rangle$, $\langle M^2 \rangle$, $\langle M^3 \rangle$, $\langle M^4 \rangle$, $\cdots$ is very easy but noisy.
- Worm estimator provides clean signal but hard to implement.
- Using worm estimator, we can measure the chiral susceptibility in the chiral limit($m_q = 0$)

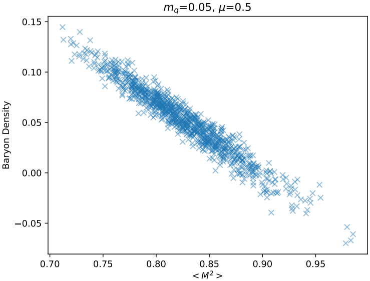




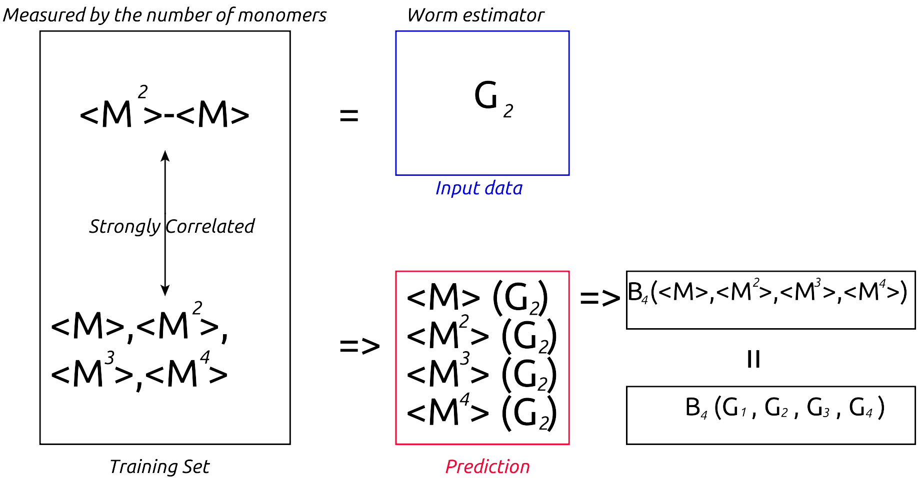
Correlations between observables

- Correlations between chiral baryon observables.
- Baryons compete with monomers in a finite lattice. It gives negative correleation.





Precise measurement of four-point function using Machine Learning

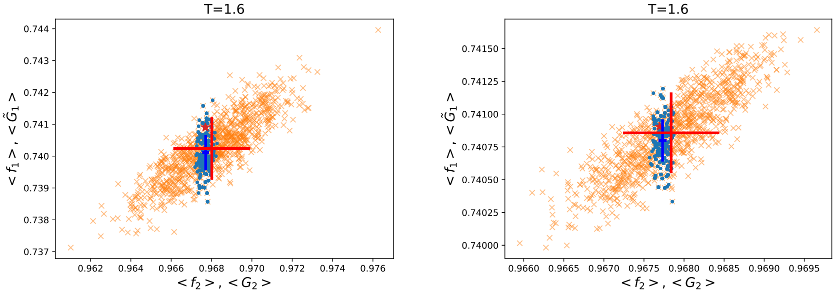


Conclusion

- We use the decision tree method to train the correlations between the higher moments and the two point function and use the accurate data of these observable as a input data.
- For the first moment and susceptibility, the errors are reduced about 30% (low T) and 90% (high T).
- For the higher moments error reductions are about 10% (low T) and 40% (high T).
- Strong correlations of the training set gives more accurate prediction.
- We have confirmed that the method works well on 2-dim Ising Model for several volumes and external fields.
- We plan to apply this method to strong coupling QCD data.

Back up

- $\langle f_n \rangle$ covers the analytic answer but $\langle \tilde{G}_n \rangle$ doesn't.
- When the correlation is not enoughly strong, it happens rarely.
- Near the critial point, it needs more statistics.



(a) 1 Million statistics

(b) 10 Million statistics