Symmetry Breaking in an Extended-O(2) Model

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Outline

Outline

1 **Motivation**

2 The Extended-O(2) Model
  - Previous Work (PRD 104 (5), 054505)
  - Phase Diagram

3 Phase Diagram

4 Summary & Outlook

Motivation

- Ultimately, we want to do quantum simulation of lattice QCD
- First, we must be able to do this with simpler Abelian models
- Given the limited number of qubits available, it is important to optimize the discretization procedure
- One can make a $\mathbb{Z}_q$ approximation of a continuous $U(1)$ symmetry
- To optimize such a $\mathbb{Z}_q$ approximation, it is useful to build a continuous family of models that interpolate among the various possibilities
- This brings us to our extended-$O(2)$ model

Outline

The Extended-O(2) Model

- We consider an extended-O(2) model in 2D with action

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$

- When $\gamma = 0$, this is the classic XY model, with a BKT transition
- When $\gamma > 0$, the second term breaks periodicity and we must choose $\varphi \in [\varphi_0, \varphi_0 + 2\pi)$ for some choice φ_0
- When $\gamma \to \infty$, the continuous angle φ is forced into the discrete values

$$\varphi_0 \leq \varphi_{x,k} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$$

  - For $q \in \mathbb{Z}$, this is the ordinary q-state clock model with $\mathbb{Z}_q$ symmetry
  - For $q \notin \mathbb{Z}$, this defines an interpolation of the clock model for noninteger q
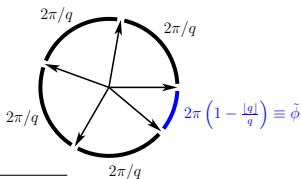
Previous Work: The $\gamma \to \infty$ limit[1]

- In the limit $\gamma \to \infty$, we can replace the action with

$$S_{\text{ext-}q} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)$$

- We directly restrict the previously continuous angles to the discrete values
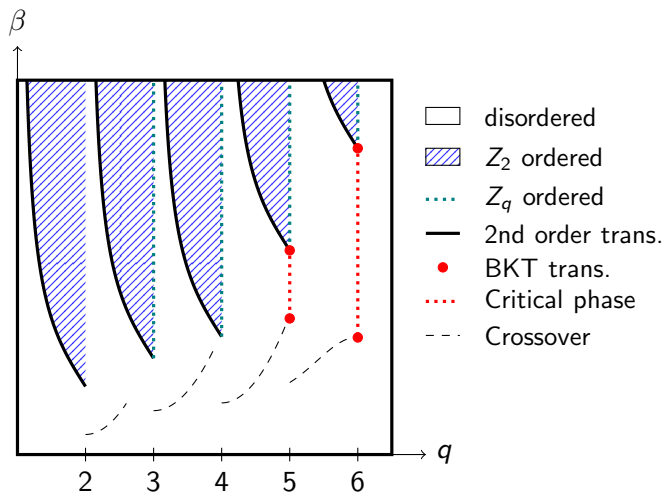
$$\varphi_0 \leq \varphi_{x,k} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$$

- We choose $\varphi_0 = 0$, i.e. $\varphi \in [0, 2\pi)$, but we also investigate $\varphi_0 = -\pi$
- For $q \notin \mathbb{Z}$, divergence from ordinary clock model behavior is driven by the introduction of a "small angle":
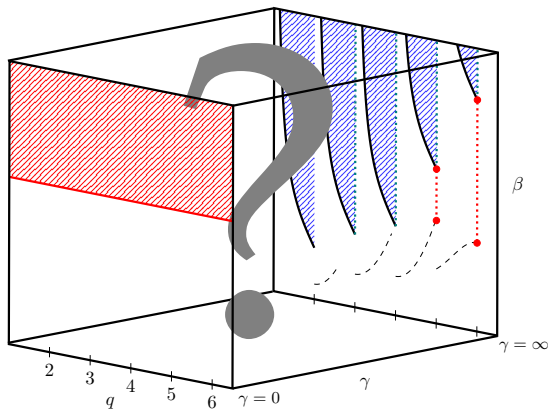

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[1]PRD 104 (5), 054505

Previous Work: The $\gamma \to \infty$ limit[2]

- ☐ disordered
- ▨ Z_2 ordered
- ⋯ Z_q ordered
- — 2nd order trans.
- ● BKT trans.
- ⋯ Critical phase
- ‐ ‐ Crossover

[2]PRD 104 (5), 054505

Phase Diagram

$$S = -\beta \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - \gamma \sum_x \cos(q\varphi_x)$$

Algorithm Developments

- In the $\gamma \to \infty$ limit, the DOF can be treated as discrete
  - Which means we could use an MCMC *heatbath* algorithm
  - We could use a TRG method for large volumes
- The model is more difficult to study at finite γ
- For finite γ, the DOF are continuous
  - MCMC heatbath is not an option, so we're left with the Metropolis, which suffers from low acceptance rates and leads to large autocorrelations in this model
  - Furthermore, our TRG method was only designed for the $\gamma \to \infty$ limit
- We needed to make some algorithmic developments
  - We implemented a *biased Metropolis heatbath algorithm*[3] (BMHA) which is designed to approach heatbath acceptance rates
  - To explore large volumes, Ryo Sakai implemented a Gaussian quadrature method
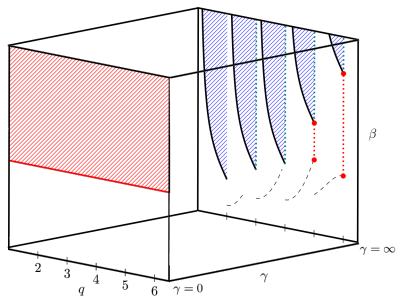
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[3] A. Bazavov and B. A. Berg, PRD 71, 114506 (2005)

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1. Motivation

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   - Previous Work (PRD 104 (5), 054505)
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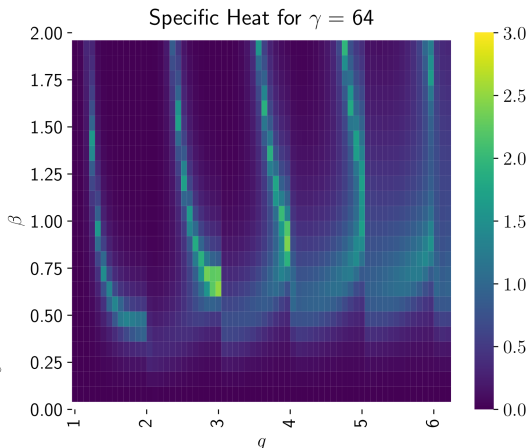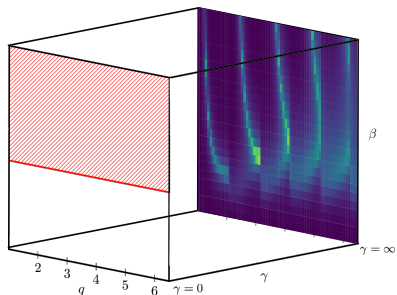
3. Phase Diagram

4. Summary & Outlook

Phase Diagram

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
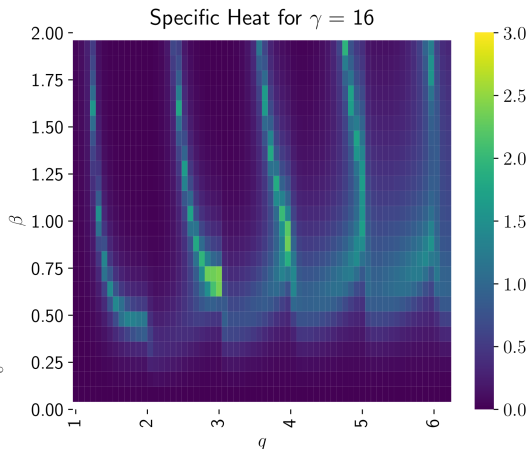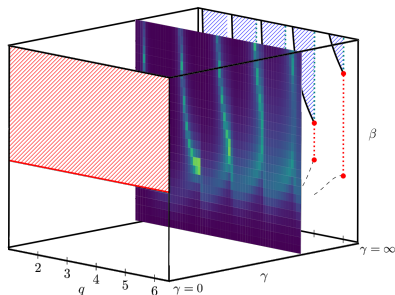
Specific Heat from TRG with $L = 1024$ and $\gamma = 64$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
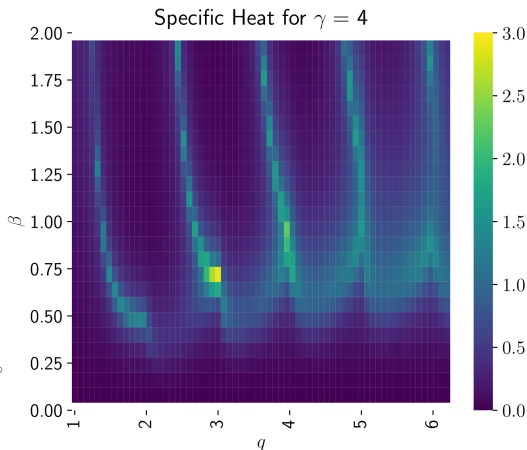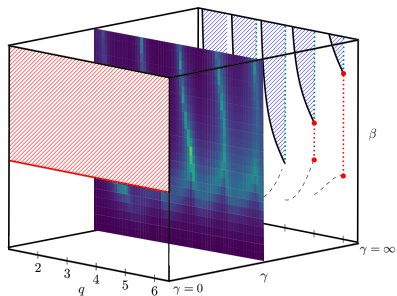

Specific Heat for $\gamma = 64$

Specific Heat from TRG with $L = 1024$ and $\gamma = 16$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
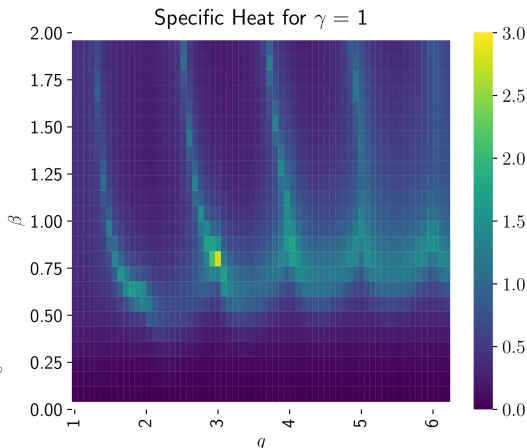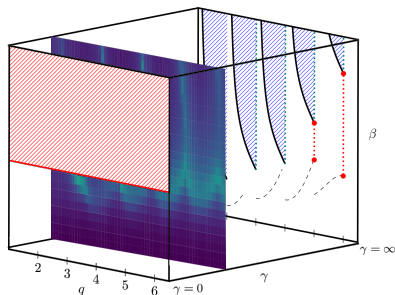

Specific Heat for $\gamma = 16$

Specific Heat from TRG with $L = 1024$ and $\gamma = 4$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
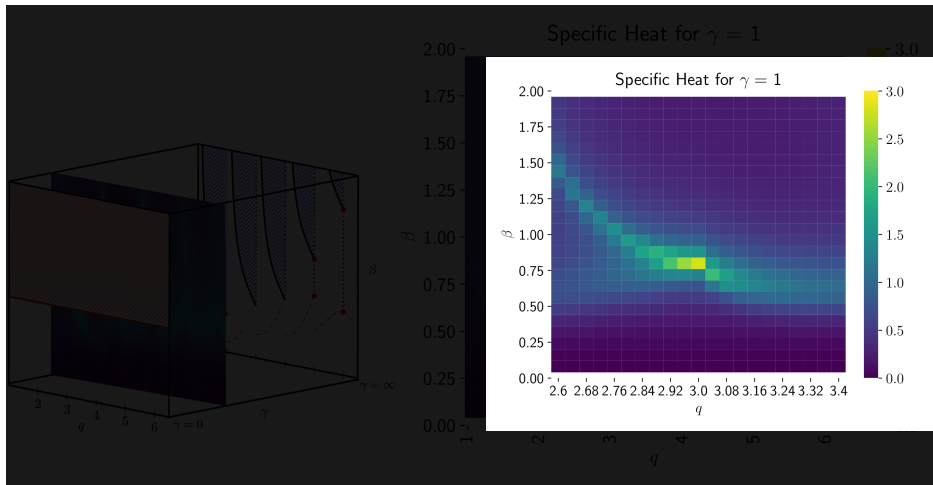
Specific Heat from TRG with $L = 1024$ and $\gamma = 1$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
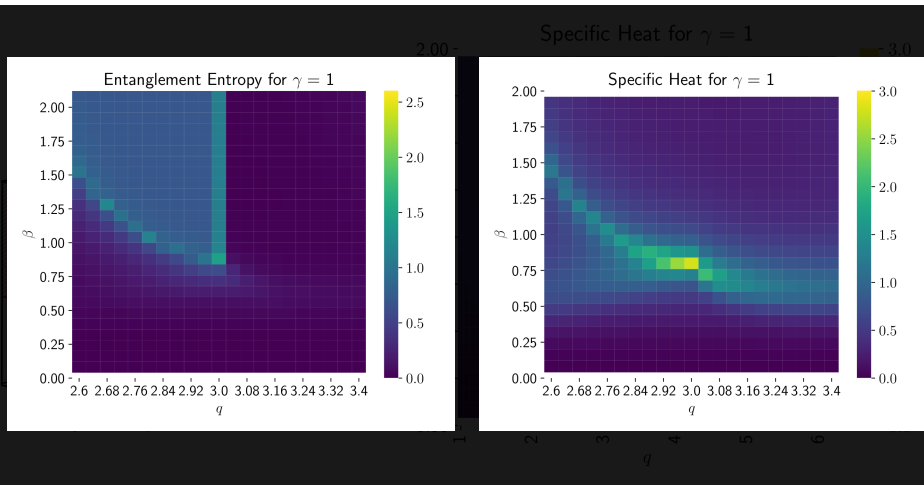
Specific Heat from TRG with $L = 1024$ and $\gamma = 1$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$
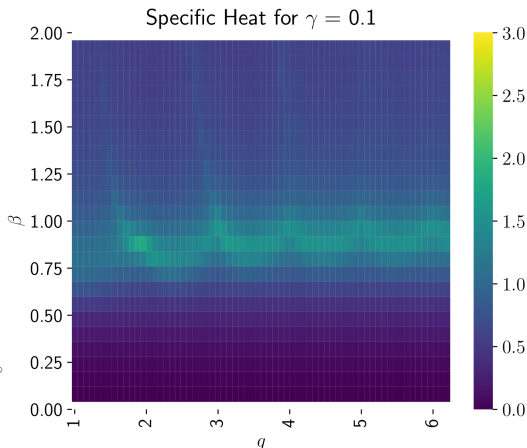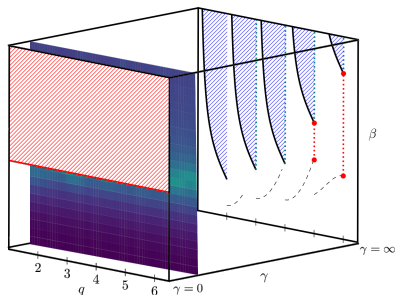
Specific Heat from TRG with $L = 1024$ and $\gamma = 1$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$

Specific Heat from TRG with $L = 1024$ and $\gamma = 0.1$

$$S_{\text{ext-}O(2)} = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x) - \gamma \sum_x \cos(q\varphi_x)$$

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Summary & Outlook

1 We looked at an extended O(2) model with parameters β, γ, and q

$$S = -\beta \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - \gamma \sum_x \cos(q\varphi_x)$$

2 Previously, we established the $\gamma = \infty$ slice of the 3D phase diagram
- When $q \in \mathbb{Z}$, we recover the classic q-state clock model which has a single second-order phase transition for $q = 2, 3, 4$ and two BKT transitions for $q \geq 5$
- When $q \notin \mathbb{Z}$, we get a crossover and a second-order phase transition

3 We are currently exploring the finite γ region of the phase diagram
- Finite size scaling
- Ryo Sakai is studying the model on large lattices using tensor methods

Connections to Quantum Simulation

1. For analog simulation, need to discretize space and truncate the fields
   1. Can approximate U(1) by $\mathbb{Z}_q$
   2. Need to optimize the approximation
   3. It is useful to have a continuous family of models that interpolate among the different q
2. The extended-O(2) model shows interesting behavior already on very small lattices making it a good test case for analog simulation
3. Quantum simulation of similar models with a continuously tunable parameter have been done with Rydberg atoms (Bernien et. al. Nature 551, 579-584 (2017), Keesling et. al. Nature 568, 207 (2019))
   - The resulting phase diagram (right) shows similarities to the phase diagram of the extended-O(2) model at finite γ. Coincidence?
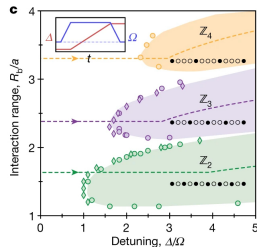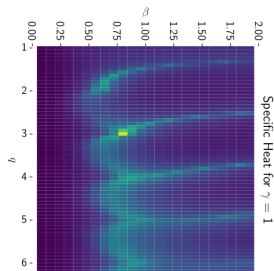
Thank you!

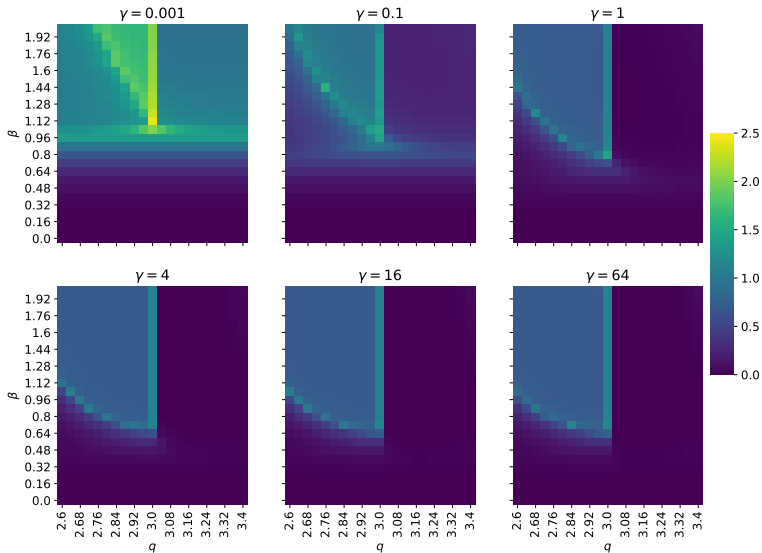Additional Slides:
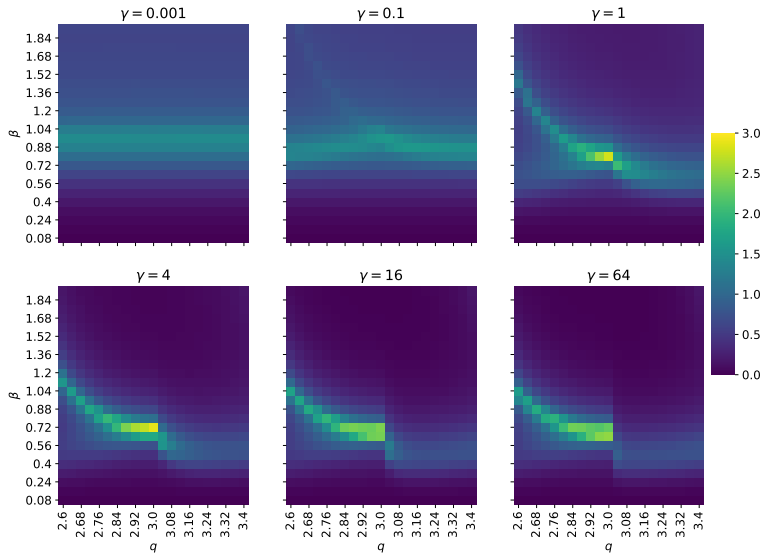
Entanglement Entropy from TRG with $L = 1024$
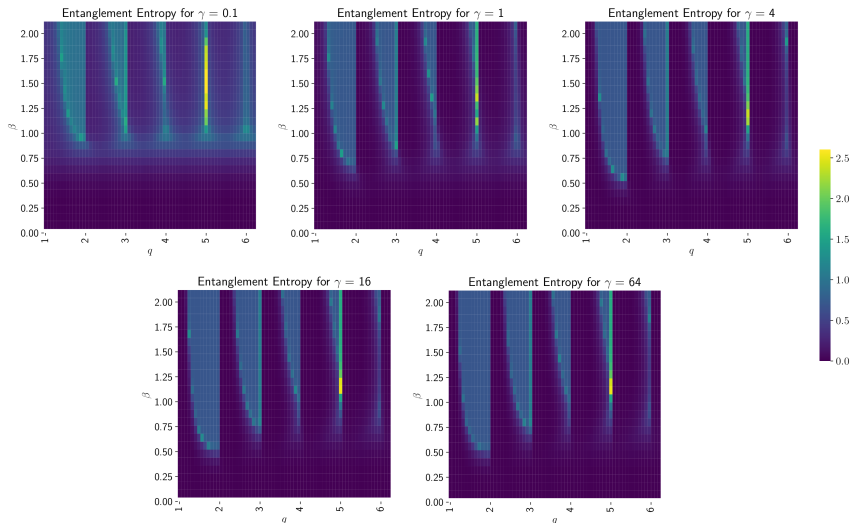
Entanglement Entropy from TRG with $L = 1024$

Entanglement Entropy near $q = 3$

Specific Heat from TRG with $L = 1024$

Specific Heat near $q = 3$

Previous Work: The $\gamma \to 0$ limit for $q \in \mathbb{Z}$

- Previous work on this model by others includes work by Jose, Kadanoff, Kirkpatrick, and Nelson in 1977 (PRB 16 3)
    - ▸ They considered the same model but with integer q and $\gamma \to 0$ as a symmetry-breaking perturbation to the O(2) model
    - ▸ They study it using two schemes—Migdal approximation and a generalized Villain model and spin-wave expansion
    - ▸ Key finding is that any $\gamma > 0$ perturbation will force the system away from O(2) behavior if β is sufficiently large
- Previous work on this model by others includes work by N. Butt, X.-Y. Jin, J. C. Osborn, and Z. H. Saleem in 2022 (arXiv:2205.03548)
    - ▸ They considered the same model also with integer q and $\gamma \to 0$ as a symmetry-breaking perturbation to the O(2) model
    - ▸ They study it using a tensor formulation
    - ▸ A key finding is that even a small perturbation results in an additional phase transition which has a non-zero critical temperature even in the limit $\gamma \to 0$

Angle Histograms

In the Extended-O(2) model, each spin variable can be represented by an angle $\varphi \in [0, 2\pi)$. Histograms of this angle over many configurations and over all sites in a configuration can help to illustrate what is happening when q and γ are varied.
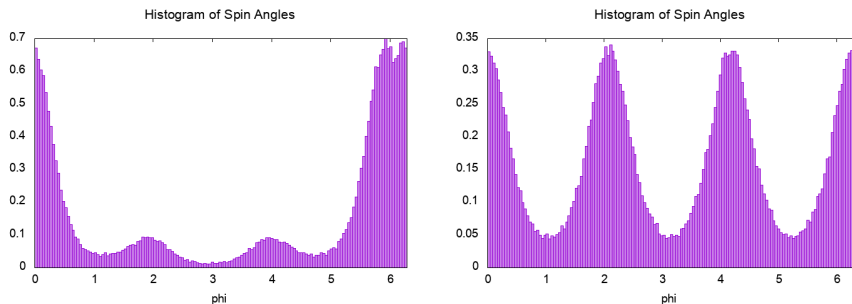

Figure: (LEFT) An example angles histogram for the case $q = 3.2$, $\gamma = 1.0$, and $\beta = 0.8$. (RIGHT) An example angles histogram for the case $q = 3.0$, $\gamma = 1.0$, and $\beta = 0.8$.
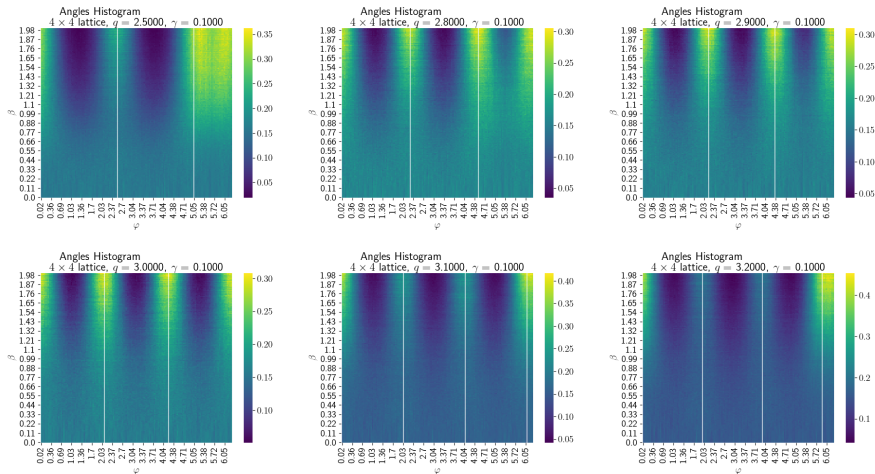
Angle Histograms at $\gamma = 0.1$

Figure: Heatmaps of the angle histograms for several q. Brighter colors correspond to higher peaks in the angle histogram. The vertical white lines were added to indicate the preferred angles (i.e. $2\pi k/q$ for $k = 0, 1, \ldots, \lfloor q \rfloor$) for that value of q.
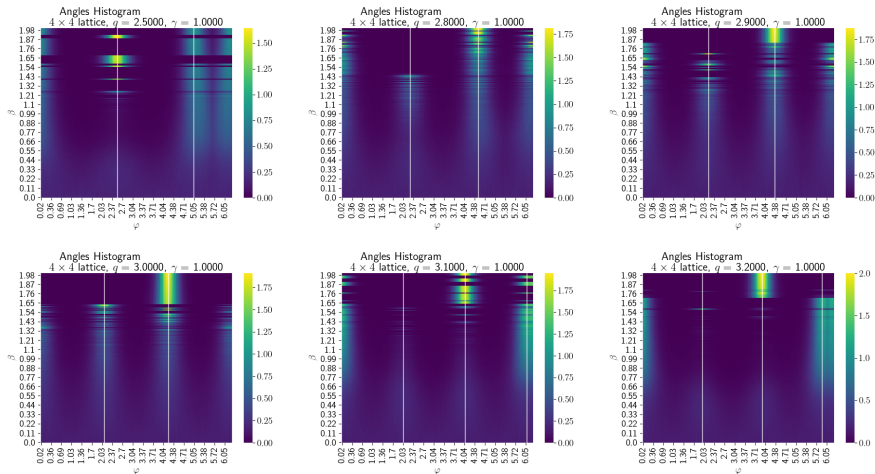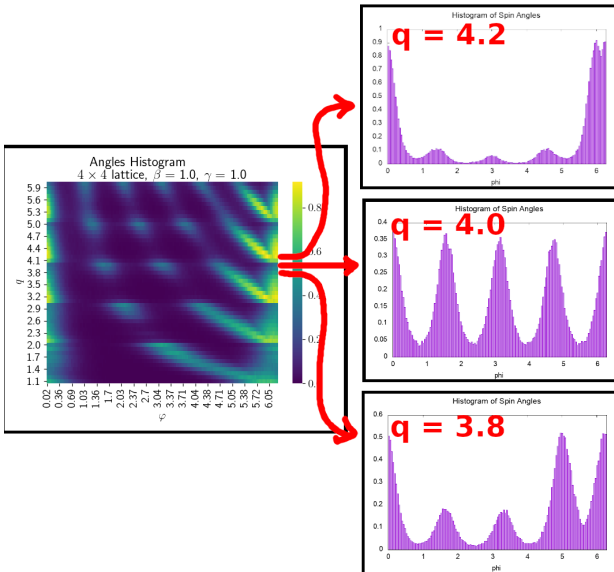
Angle Histograms at $\gamma = 1$

Figure: Heatmaps of the angle histograms for several q with $\gamma = 1$ and $\beta \in [0, 2]$. At large β, artifacts develop due to freezing/insufficient statistics, thus, one should ignore the upper parts i.e. $\beta \gtrsim 1.2$ of these heatmaps.
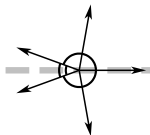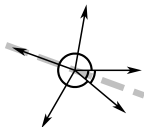
Angle Histograms along a line of Constant β

Choice of φ_0

- Choice of φ_0 can change the DOF in the model
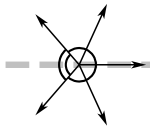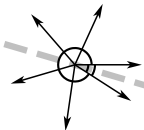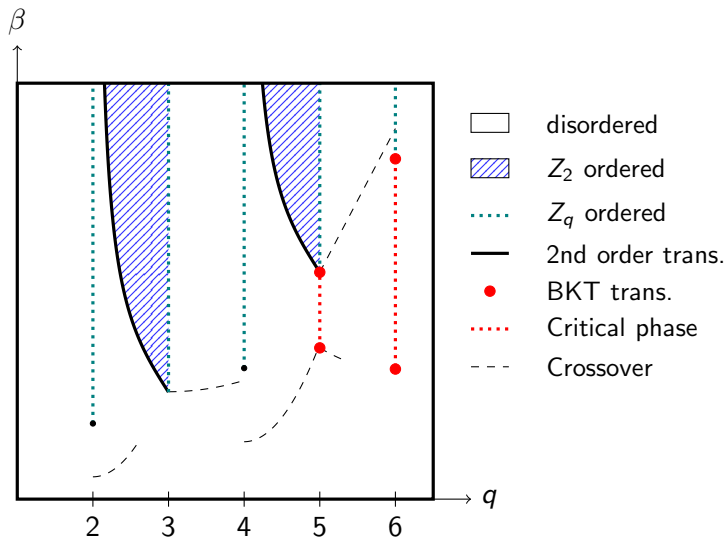- We choose $\varphi_0 = 0$, i.e. $\varphi \in [0, 2\pi)$, but we also investigate $\varphi_0 = -\pi$

Phase diagram for $\gamma = \infty$ and $\varphi_0 = -\pi$

Legend:
- □ disordered
- ▨ Z_2 ordered
- ⋯ Z_q ordered
- — 2nd order trans.
- • BKT trans.
- ⋯ Critical phase
- -- Crossover

Placement of β

- One can define the model as

$$S = -\beta \sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - \gamma \sum_x \cos(q\varphi_x)$$

  where β is multiplying the first term like a field-theoretic coupling. Then the Boltzmann factor is e^{-S}

- Alternatively, one can factor β out front and define the model as

$$S = -\sum_{x,\mu} \cos\left(\varphi_{x+\hat{\mu}} - \varphi_x\right) - \gamma' \sum_x \cos(q\varphi_x)$$

  with Boltzmann factor $e^{-\beta S}$, where β is the inverse temperature

- The two definitions are related by $\gamma' = \gamma/\beta$
- We have used both definitions, however, the Monte Carlo results shown in these slides are from the definition with β factored out front
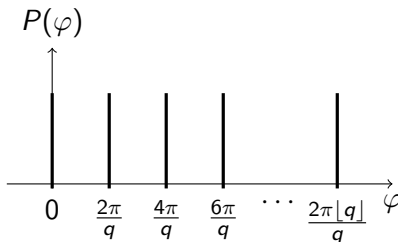
The Need to Shift the Angles: A Subtlety

- In the ordinary clock model, we have the energy function

$$S = -\beta \sum_{\langle x,y \rangle} \cos(\varphi_x - \varphi_y)$$

- The angles $\varphi_x^{(k)}$ are selected discretely as $\varphi_0 \leq \varphi_x^{(k)} = \frac{2\pi k}{q} < \varphi_0 + 2\pi$
- When $\beta = 0$ and with $\varphi_0 = 0$, the spins are selected uniformly from a "Dirac comb"

$$P_{q,\varphi_0=0}^{clock}(\varphi) \sim \sum_{k=0}^{\lfloor q \rfloor} \delta\left(\varphi - \frac{2\pi k}{q}\right)$$

The Need to Shift the Angles: A Subtlety

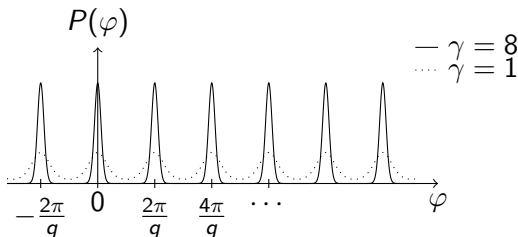- In the Extended-O(2) model, we have the energy function

$$S = -\beta \sum_{\langle x,y \rangle} \cos(\varphi_x - \varphi_y) - \gamma \sum_x \cos(q\varphi_x)$$

- The angles φ_x are now selected continuously in

$$\varphi_0 \leq \varphi \in \mathbb{R} < \varphi_0 + 2\pi$$

- When $\beta = 0$ and with $\varphi_0 = 0$, the spins are selected from a distribution

$$P_{q,\varphi_0}^{extO2}(\varphi) \sim e^{\gamma \cos(q\varphi)}$$
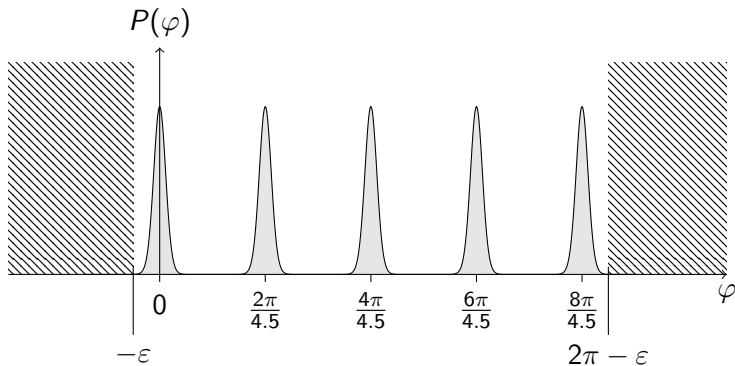
The Need to Shift the Angles: A Subtlety

Figure: To recover the Dirac comb of the clock model distribution in the $\gamma \to \infty$ limit, the angle domain must be shifted by some ε so that the histogram includes all relevant peaks.

The Need to Shift the Angles: A Subtlety

- To match the clock model in the $\gamma \to \infty$ limit, it should be sufficient to choose ε such that

$$P_{q,\varphi_0}^{extO2}(\varphi) \xrightarrow[\gamma \to \infty]{} P_{q,\varphi_0}^{clock}(\varphi)$$

  where for the clock model, angles are selected from $[\varphi_0, \varphi_0 + 2\pi)$, but for the Extended-O(2) model, they are selected from $[\varphi_0 - \varepsilon, \varphi_0 - \varepsilon + 2\pi)$

- In our case, we use $\varphi_0 = 0$, and choose

$$\varepsilon = \pi \left(1 - \frac{\lfloor q \rfloor}{q}\right)$$

  so that the $\lceil q \rceil$ peaks of the distribution $P_{q,\varphi_0}^{extO2}(\varphi)$ are centered in the domain $[-\varepsilon, 2\pi - \varepsilon)$