# Quantum state preparation algorithms for the Schwinger model with a theta term

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# Outline

- The model
- Lattice discretization
- Quantum Hamiltonian
- Quantum Adiabatic Evolution (QAE)
- Quantum Approximate Optimization Algorithm (QAOA)
- Rodeo Algorithm (RA)
- Conclusion

# The model

- Massive Schwinger model with a $\theta$-term:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

Schwinger, PR125 (1962),
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- Apply a chiral transformation:

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- In the temporal gauge $A_0 = 0,\ E = F^{10} = -\dot{A}^1$

$$H = \int dx \left[-i\bar{\psi}\gamma^1(\partial_1 + igA_1)\psi + m\bar{\psi}e^{i\theta\gamma_5}\psi + \frac{1}{2}E^2\right]$$

and the Gauss law $\partial_1 E = g\bar{\psi}\gamma_0\psi$

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# Lattice version

- One-dimensional lattice of size $N$

$$H = -i \sum_{n=1}^{N-1} \left( \frac{1}{2a} - (-1)^n \frac{m}{2} \sin\theta \right) \left[ \chi_n^\dagger e^{i\phi_n} \chi_{n+1} - \chi_{n+1}^\dagger e^{-i\phi_n} \chi_n \right]$$

$$+ m \cos\theta \sum_{n=1}^{N} (-1)^n \chi_n^\dagger \chi_n + \frac{g^2 a}{2} \sum_{n=1}^{N-1} L_n^2$$

Chakraborty, Honda, Izubuchi, Kikuchi,
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- Staggered fermions: $\psi(x) = (\psi_u(x), \psi_d(x))^T$

$$\chi_n = \sqrt{a}\psi_u(x_n) \quad \text{for even } n, \qquad \chi_n = \sqrt{a}\psi_d(x_n) \quad \text{for odd } n$$

- Rescaled fields

$$A^1(x_n) \to -\phi_n/(ag),\ E(x_n) \to gL_n,\ w = 1/(2a),\ \ J = ga^2/2$$

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- Gauss law: $L_n - L_{n-1} = \chi_n^\dagger\chi_n - \dfrac{1-(-1)^n}{2}$

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# Mapping to qubits

- Jordan-Wigner transformation

$$\chi_n = \left( \prod_{l<n} - iZ_l \right) \frac{X_n - iY_n}{2}$$

- $L_n = L_0 + \frac{1}{2} \sum_{l=1}^{n} \left( Z_l + (-1)^l \right), \quad \text{set } L_0 = 0 \text{ (shift in } \theta)$

- Absorb phases: $\chi_n \to \prod_{l<n} [e^{-i\phi_n}] \chi_n$

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# Mapping to qubits

- Final Hamiltonian: $H = H_{ZZ} + H_{\pm} + H_Z$

$$H_{ZZ} = \frac{J}{2} \sum_{n=2}^{N-1} \sum_{1 \le k < l \le n} Z_k Z_l$$

$$H_{\pm} = \frac{1}{2} \sum_{n=1}^{N-1} \left( w - (-1)^n \frac{m}{2} \sin\theta \right) \left[ X_n X_{n+1} + Y_n Y_{n+1} \right]$$

$$H_Z = \frac{m \cos\theta}{2} \sum_{n=1}^{N} (-1)^n Z_n - \frac{J}{2} \sum_{n=1}^{N-1} (n \bmod 2) \sum_{l=1}^{n} Z_l$$

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# Quantum Adiabatic Evolution (QAE)

- In this context also called Adiabatic State Preparation (ASP)
- Rely on the adiabatic theorem: the system will remain in the ground state if the perturbation is slow and there is a gap

Farhi, Goldstone, Gutmann, Sipser, quant-ph/0001106

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- Find a Hamiltonian $H_0$ whose ground state $|vac\rangle_0$ is known
- Construct an adiabatic Hamiltonian $H_A(t)$ that interpolates between $H_0$ and $H$, i.e. $H_A(t=0) = H_0$ and $H_A(t=T) = H$

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- The ground state $|vac\rangle$ of $H$ is then obtained through evolution:

$$|vac\rangle = \lim_{T\to\infty} \mathcal{T} \exp\left( -i \int_0^T dt\, H_A(t) \right) |vac\rangle_0$$

- In practice, $T$ is finite so the evolution is not infinitely slow

Farhi, Goldstone, Gutmann, Sipser, quant-ph/0001106

# Quantum Adiabatic Evolution (QAE)

- Let $|n(t)\rangle$ be the instantaneous eigenstates of $H_A(t)$

  $$H_A(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

- The state at time $t$

  $$|\psi(t)\rangle = \sum_n c_n(t)|n(t)\rangle$$

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- The expansion coefficients satisfy

$$\dot{c}_m(t) + \left(\frac{i}{\hbar}E_m(t) + \langle m(t)|\dot{m}(t)\rangle\right)c_m(t) = \sum_{m\neq n}\frac{\langle m(t)|\dot{H}_A(t)|n(t)\rangle}{E_m(t) - E_n(t)}c_n(t)$$

- The adiabatic approximation = neglecting r.h.s.

# Quantum Adiabatic Evolution (QAE)

- Then

$$c_m(t) = c_m(0)e^{i\theta_m(t)}e^{i\gamma_m(t)}$$

$$\theta_m(t) \equiv -\int_0^t \frac{E_m(t')}{\hbar}dt', \quad \gamma_m(t) = i\int_0^t \langle m(t') \,|\, \dot{m}(t') \rangle dt'$$

and thus $|c_m(t)|^2 = |c_m(t=0)|^2$

# Quantum Adiabatic Evolution (QAE)

- Split the Hamiltonian:

  $H = H_0 + H_\pm, \quad H_0 = H_{ZZ} + H_Z|_{m \to m_0, \theta \to 0}$

- The system can be prepared in the ground state of $H_0$ $\quad |1010\dots10\rangle$

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- Commonly used interpolating Hamiltonian

$$H_A(t) = (1 - t)H_0 + tH$$

- This amounts to evolving the parameters

$$w \to \frac{t_i}{T}w, \quad \theta \to \frac{t_i}{T}\theta, \quad m \to \left(1 - \frac{t_i}{T}\right)m_0 + \frac{t_i}{T}m$$

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- Trotter-Suzuki decomposition to represent $e^{-iH_A(t)\delta t}$ (1st, 2nd order)

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- Total time $T$, $M$ steps
- Time steps:

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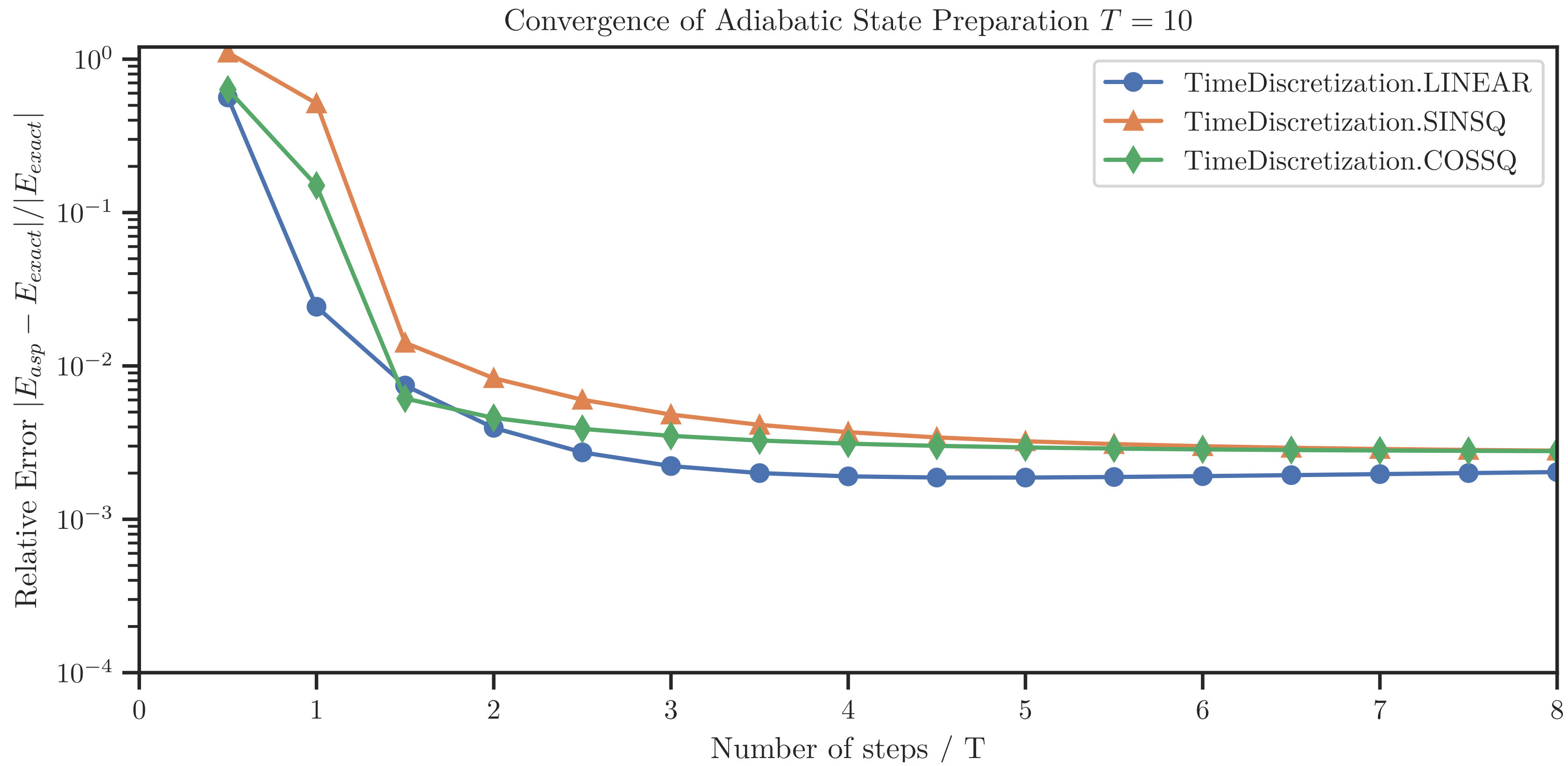
$$\delta t_n = \frac{T}{M}, \qquad \text{linear (L)}$$

$$\delta t_n = 2\frac{T}{M}\sin^2\left(\pi\frac{n}{M}\right), \qquad \text{sine (S)}$$

$$\delta t_n = 2\frac{T}{M}\cos^2\left(\pi\frac{n}{2M}\right), \qquad \text{cosine (C)}$$

# Quantum Adiabatic Evolution (QAE)



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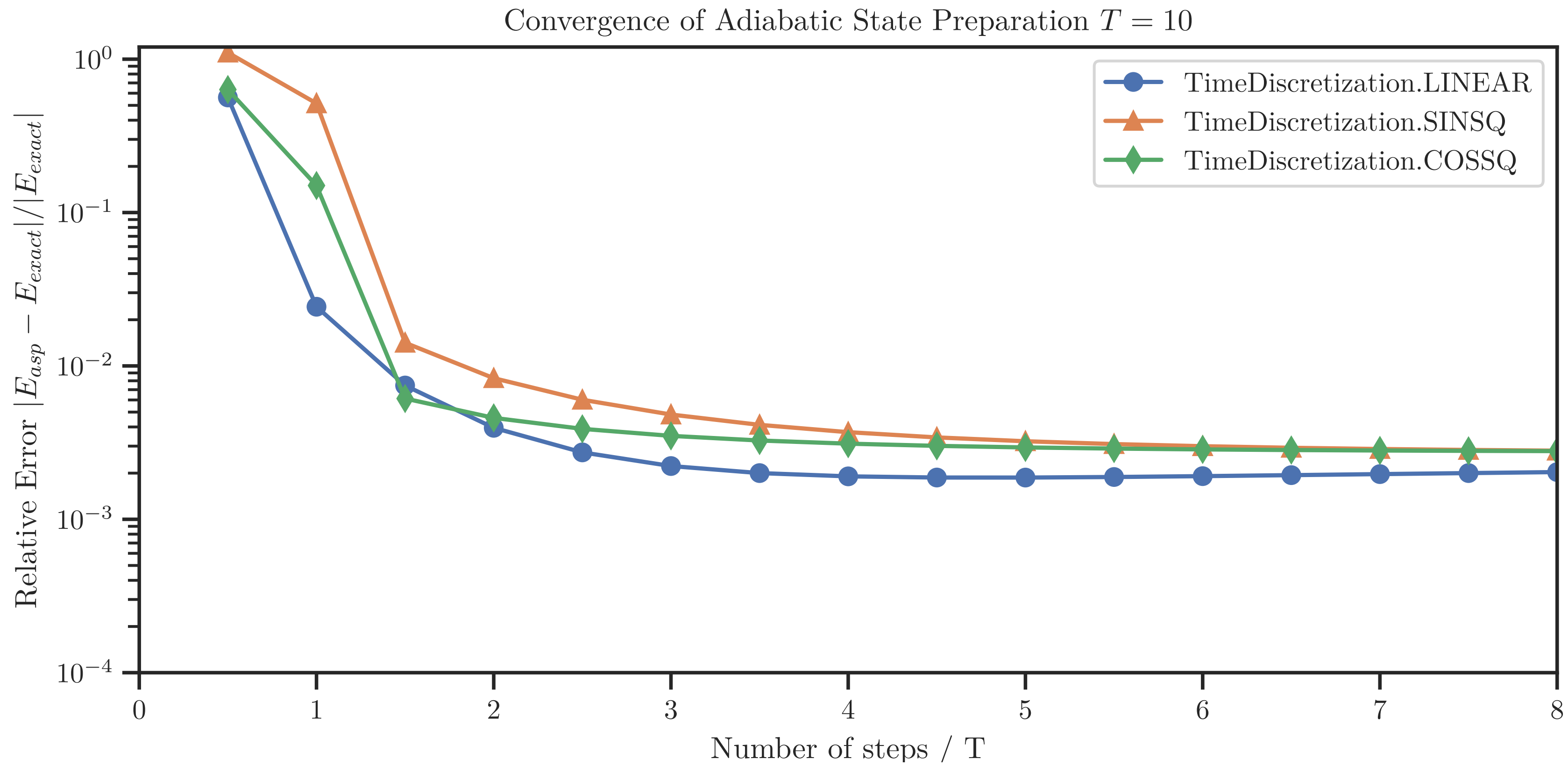


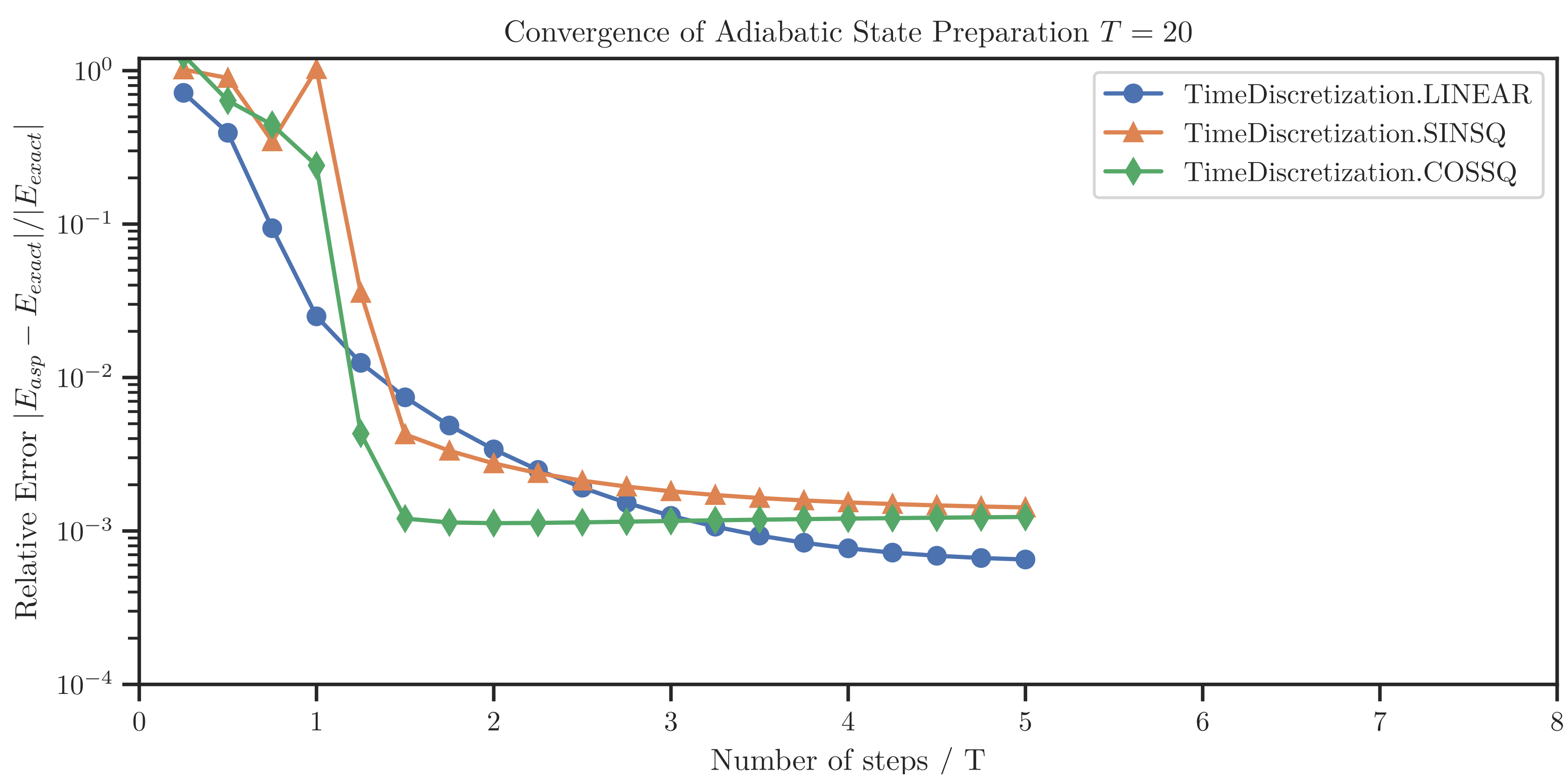

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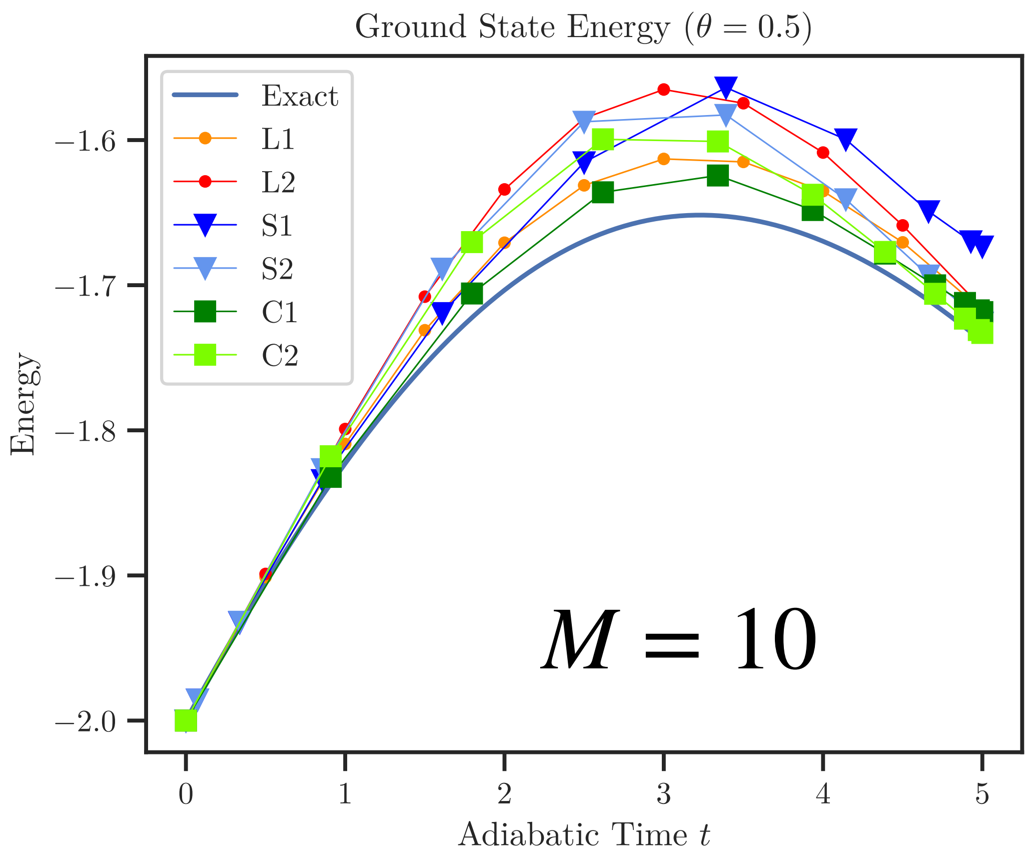

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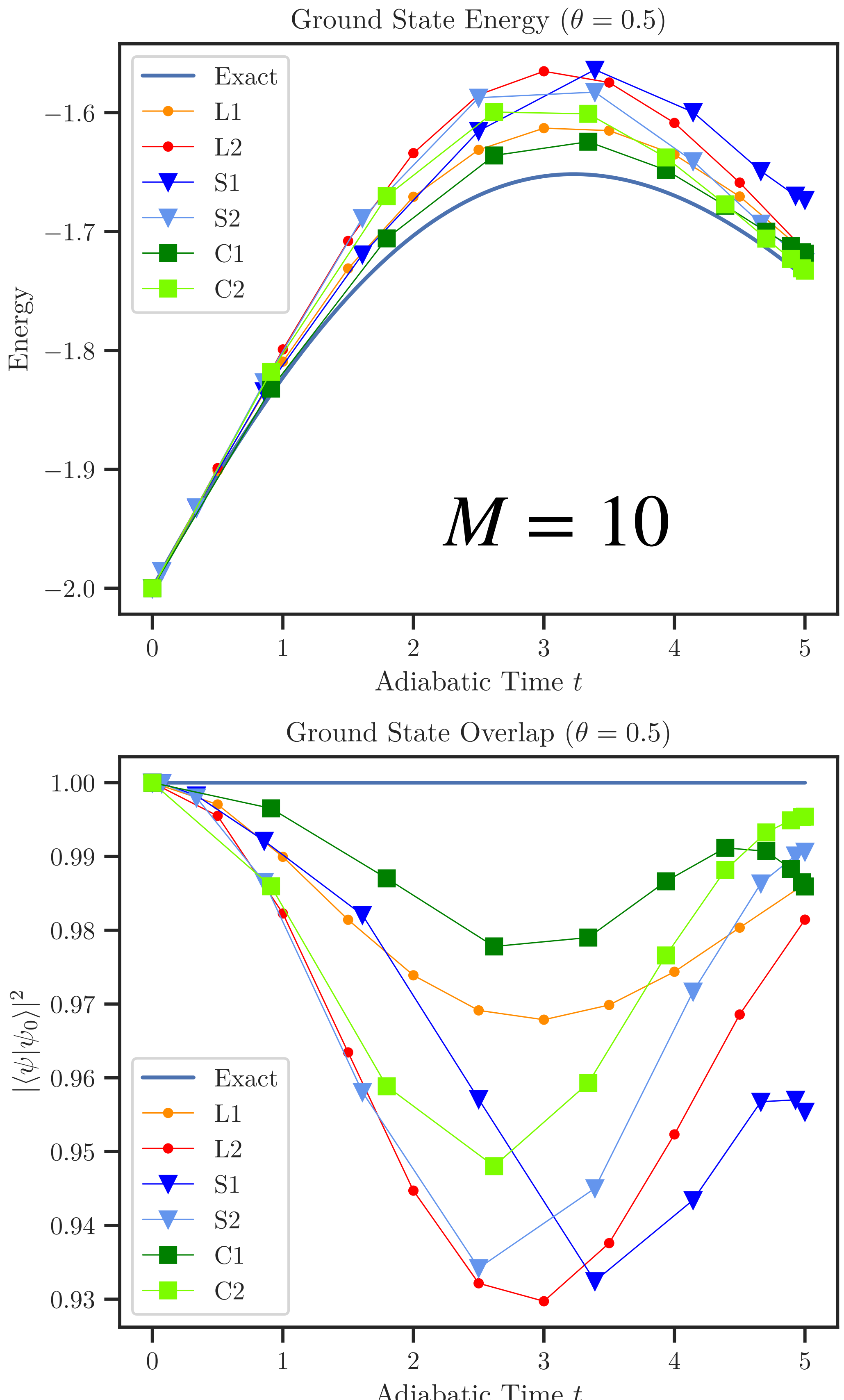

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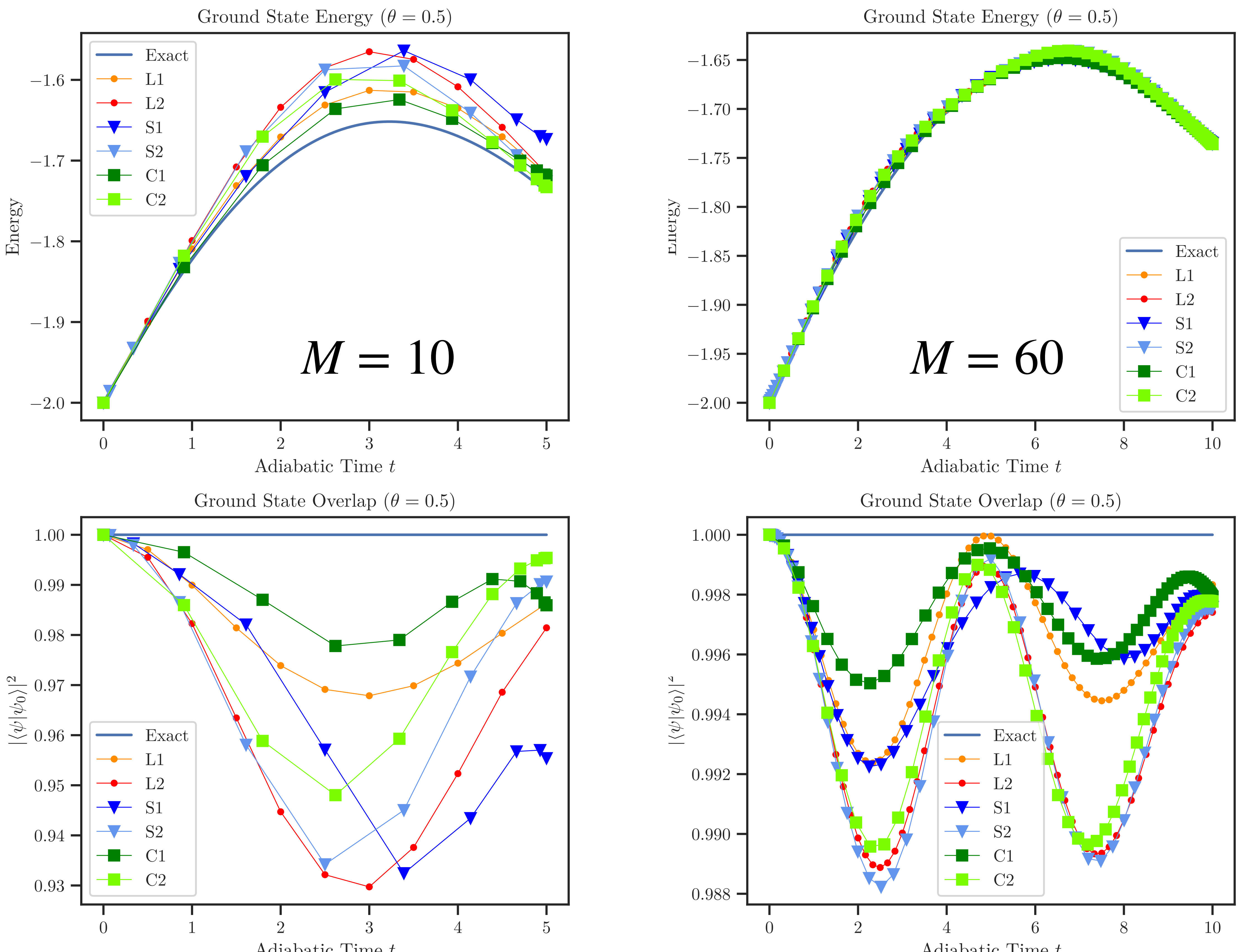
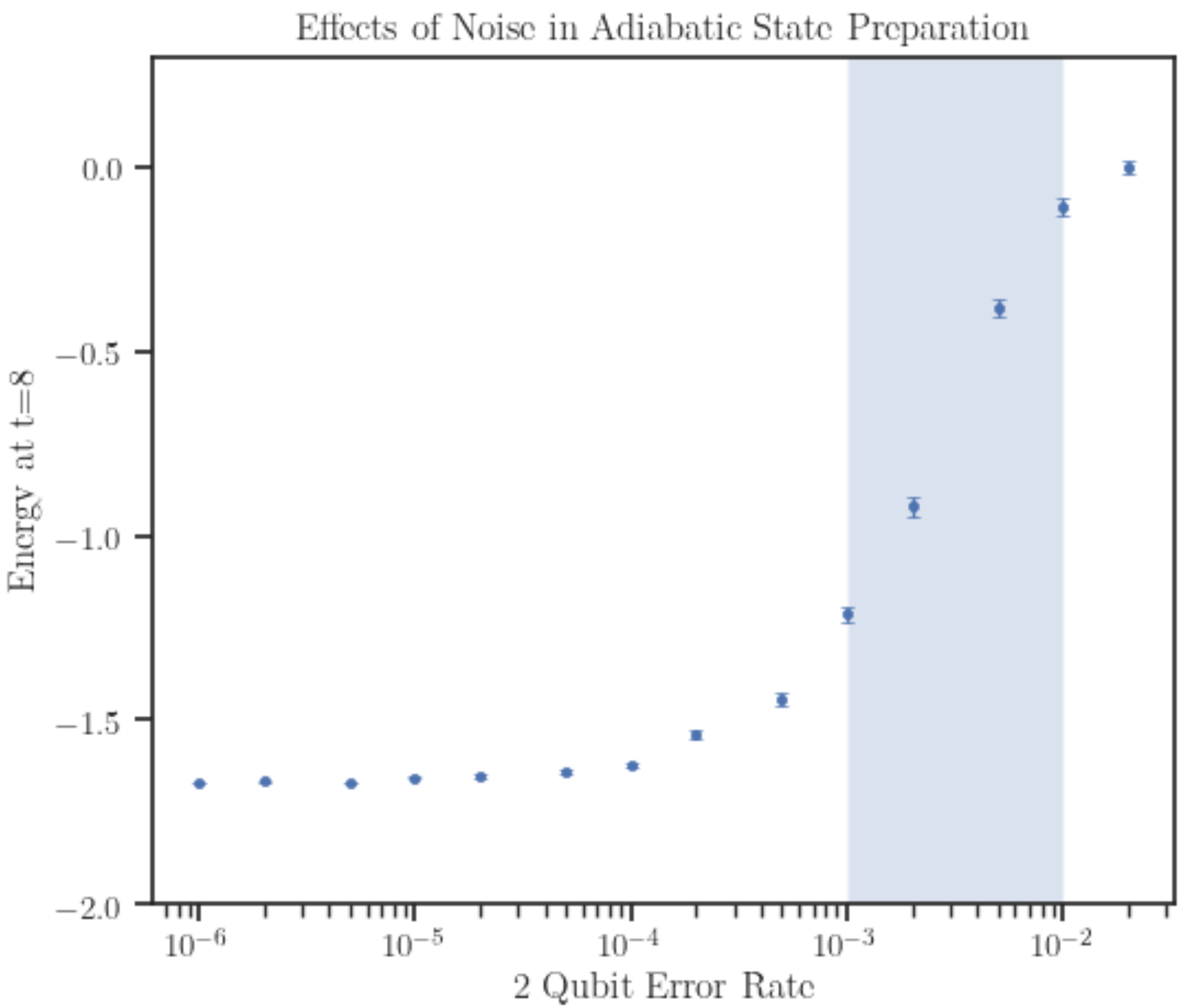


# Quantum Adiabatic Evolution (QAE) + noise



# Quantum Approximate Optimization Algorithm (QAOA)

- Idea: instead of evolving with fixed step size make the step sizes parameters for optimization

  $$|\psi_M(\overrightarrow{\beta},\vec{\gamma})\rangle = e^{-i\beta_M H_0} e^{-i\gamma_M H} \ldots e^{-i\beta_1 H_0} e^{-i\gamma_1 H} |\psi_0\rangle$$

  and rely on the variational principle to minimize

  $$\langle\psi_M(\overrightarrow{\beta},\vec{\gamma})|H|\psi_M(\overrightarrow{\beta},\vec{\gamma})\rangle \geq E_0$$

Farhi, Goldstone, Gutmann, 1411.4028

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  and rely on the variational principle to minimize
  $$\langle\psi_M(\overrightarrow{\beta}, \vec{\gamma})|H|\psi_M(\overrightarrow{\beta}, \vec{\gamma})\rangle \geq E_0$$
- Hybrid algorithm:
  energy evaluation — quantum
  minimization — classical

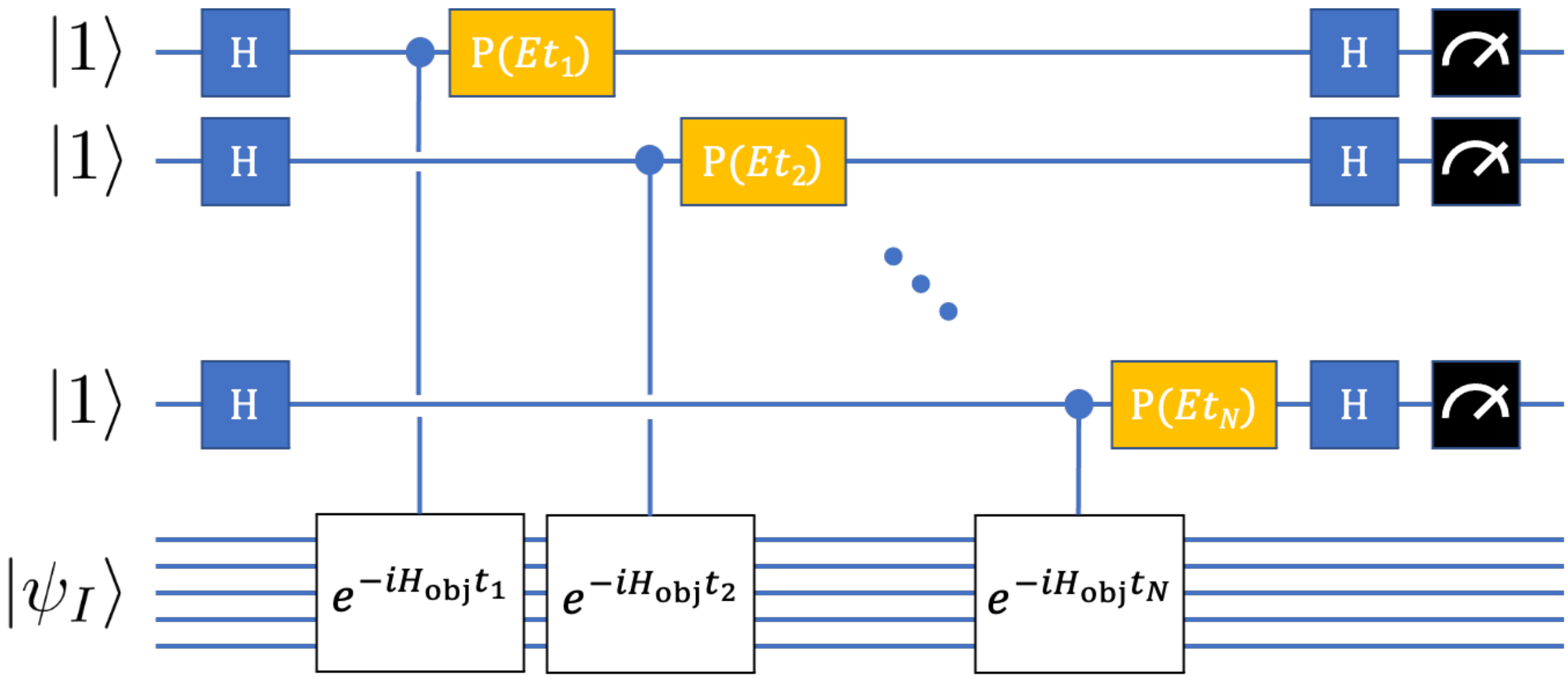
Farhi, Goldstone, Gutmann, 1411.4028

# QAE and QAOA comparison (4 qubits)

| Method | # of Steps | # of CNOT/Qubit | $E_0$ | GS Overlap |
|---|---|---|---|---|
| ASP $L1$ | 10 | 45 | $-1.7140$ | 0.9827 |
| ASP $S1$ | 10 | 45 | $-1.6751$ | 0.9599 |
| ASP $C1$ | 10 | 45 | $-1.7144$ | 0.9827 |
| ASP $L2$ | 10 | 75 | $-1.7089$ | 0.9729 |
| ASP $S2$ | 10 | 75 | $-1.7204$ | 0.9847 |
| ASP $C2$ | 10 | 75 | $-1.7260$ | 0.9880 |
| QAOA | 2 | 18 | $-1.7353$ | 0.9975 |
| QAOA | 3 | 27 | $-1.7357$ | 0.9977 |

- Exact ground state energy $E_0 = -1.7386$

# Rodeo algorithm (RA)




- Object system $H_{obj}$ coupled to $N$ ancilla qubits (rodeo arena)

Choi, Lee, Bonitati, Qian, Watkins, PRL127 (2021)

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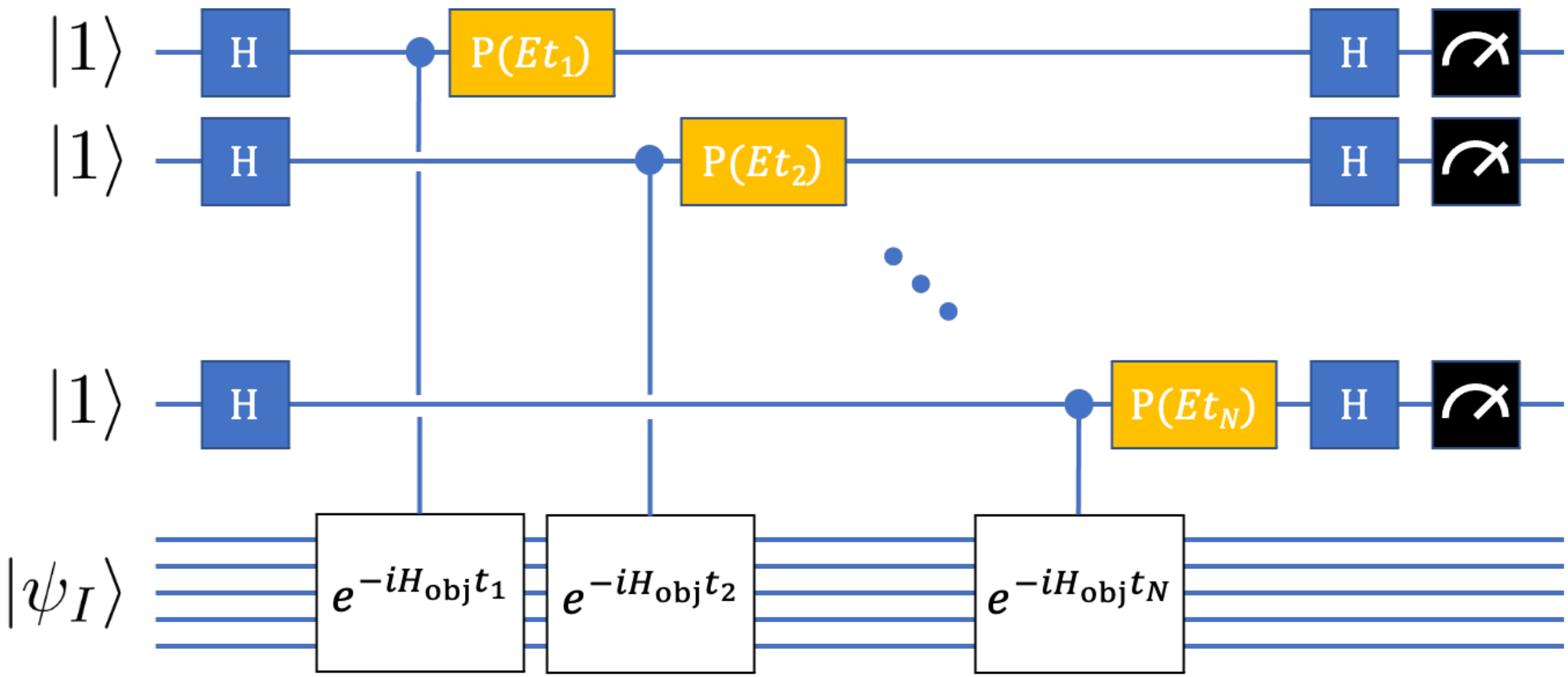


- Object system $H_{obj}$ coupled to $N$ ancilla qubits (rodeo arena)
- Controlled time evolution for time $t_n$ followed by the phase gate $|0\rangle\langle 0| + e^{iEt_n}|1\rangle\langle 1|$, $E$ — preset parameter

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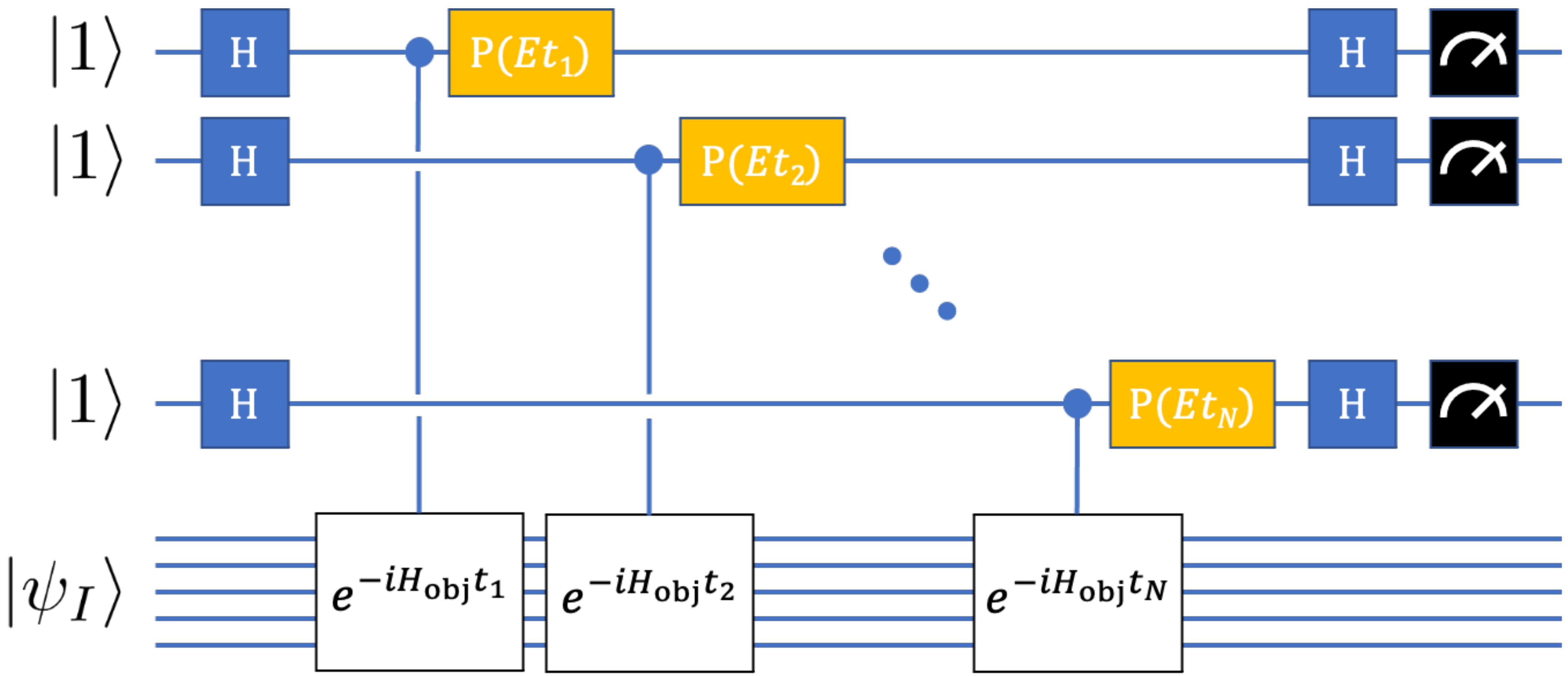


- The probability of measuring the ancilla qubit in state $|1\rangle$ is $\cos^2\left[(E_{obj} - E)t_n/2\right]$

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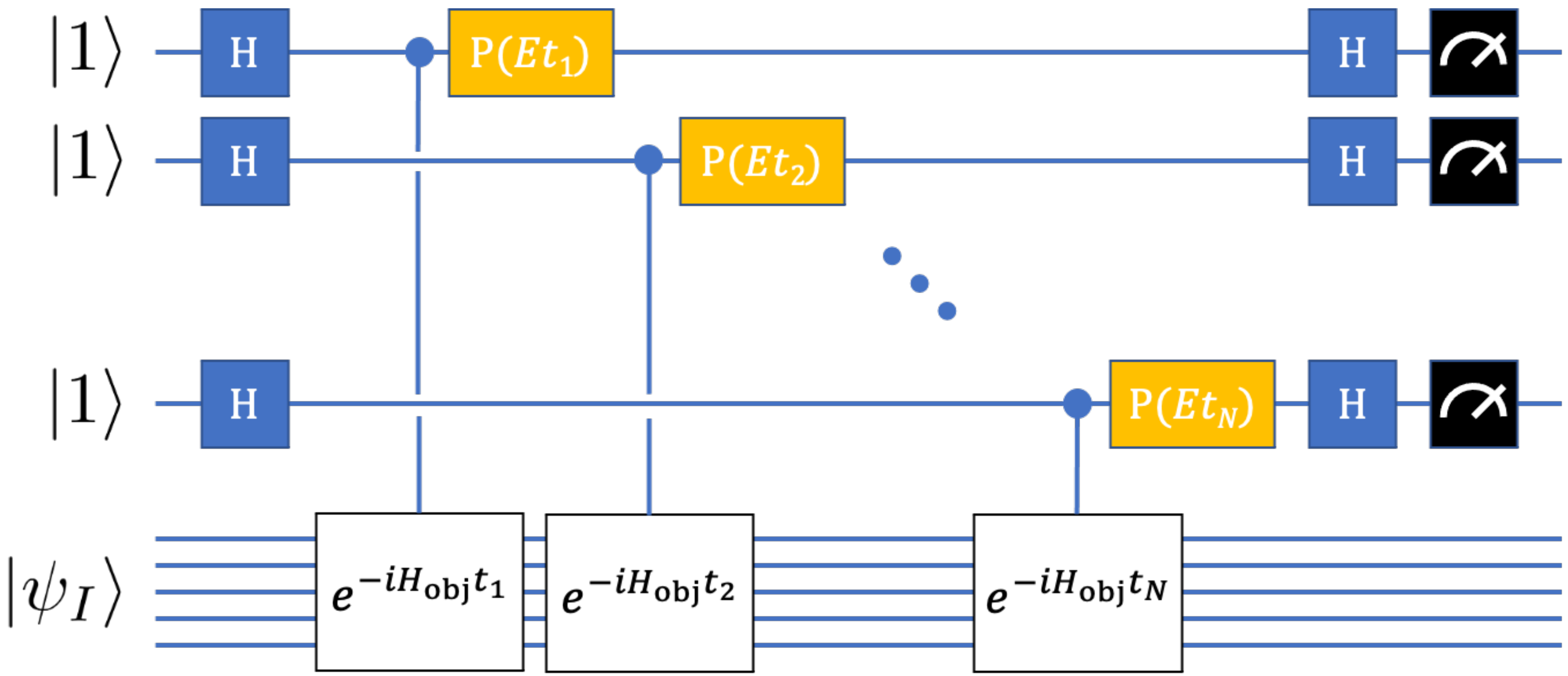


- The probability of measuring the ancilla qubit in state $|1\rangle$ is $\cos^2\left[(E_{obj} - E)t_n/2\right]$
- For $N$ ancilla qubits: $P_N = \prod_{n=1}^{N} \cos^2\left[(E_{obj} - E)t_n/2\right]$

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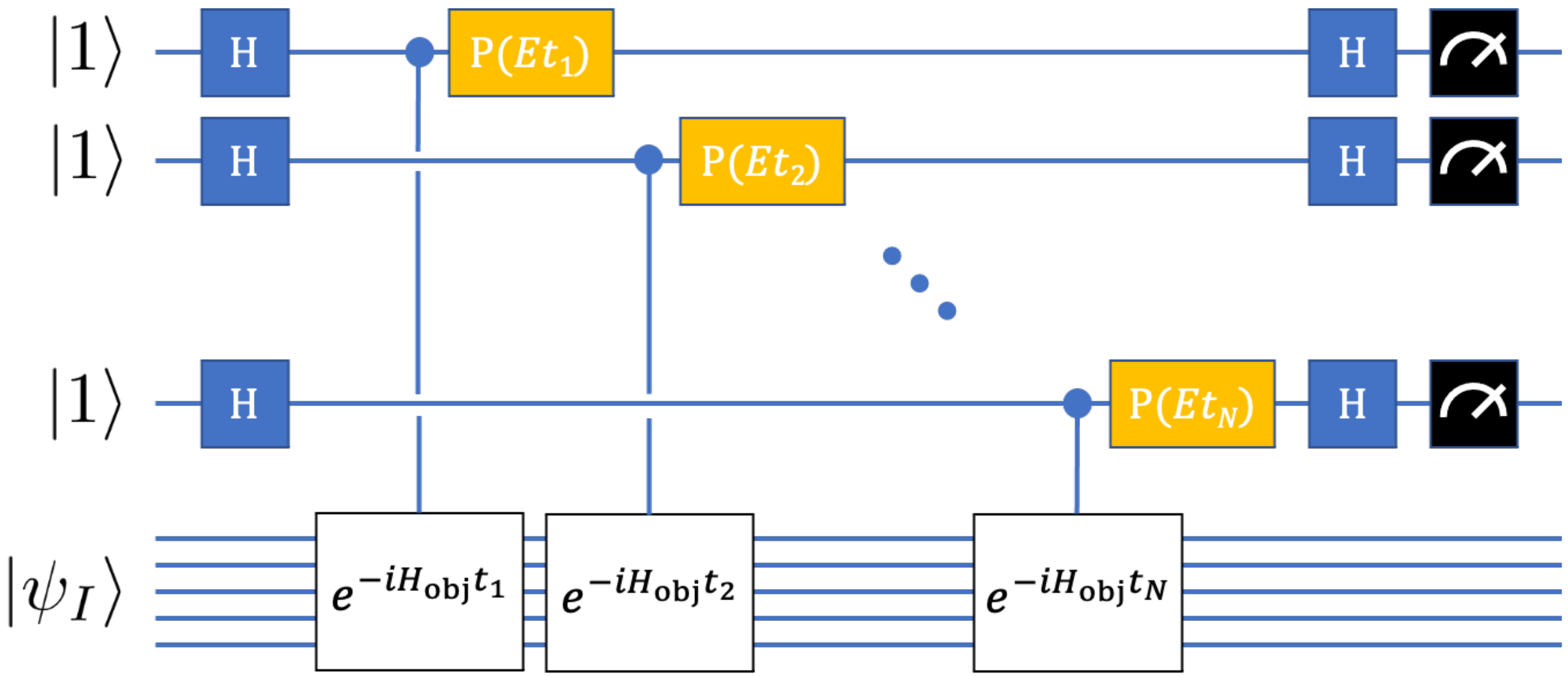


- Average over random $t_n$

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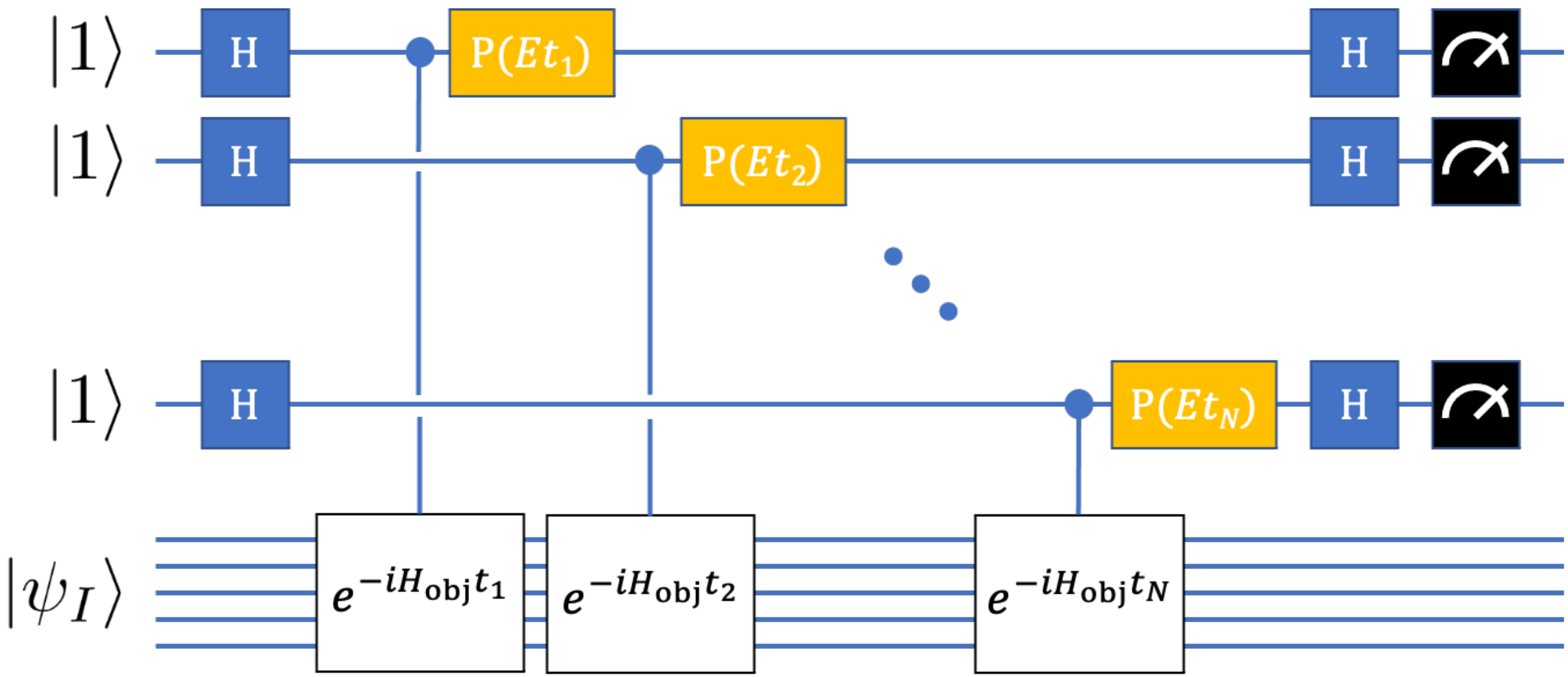


- Average over random $t_n$
- The spectral weight for $E_{obj} \neq E$ is suppressed by $1/4^N$ for $N \to \infty$
- This circuit "filters" states for a given energy

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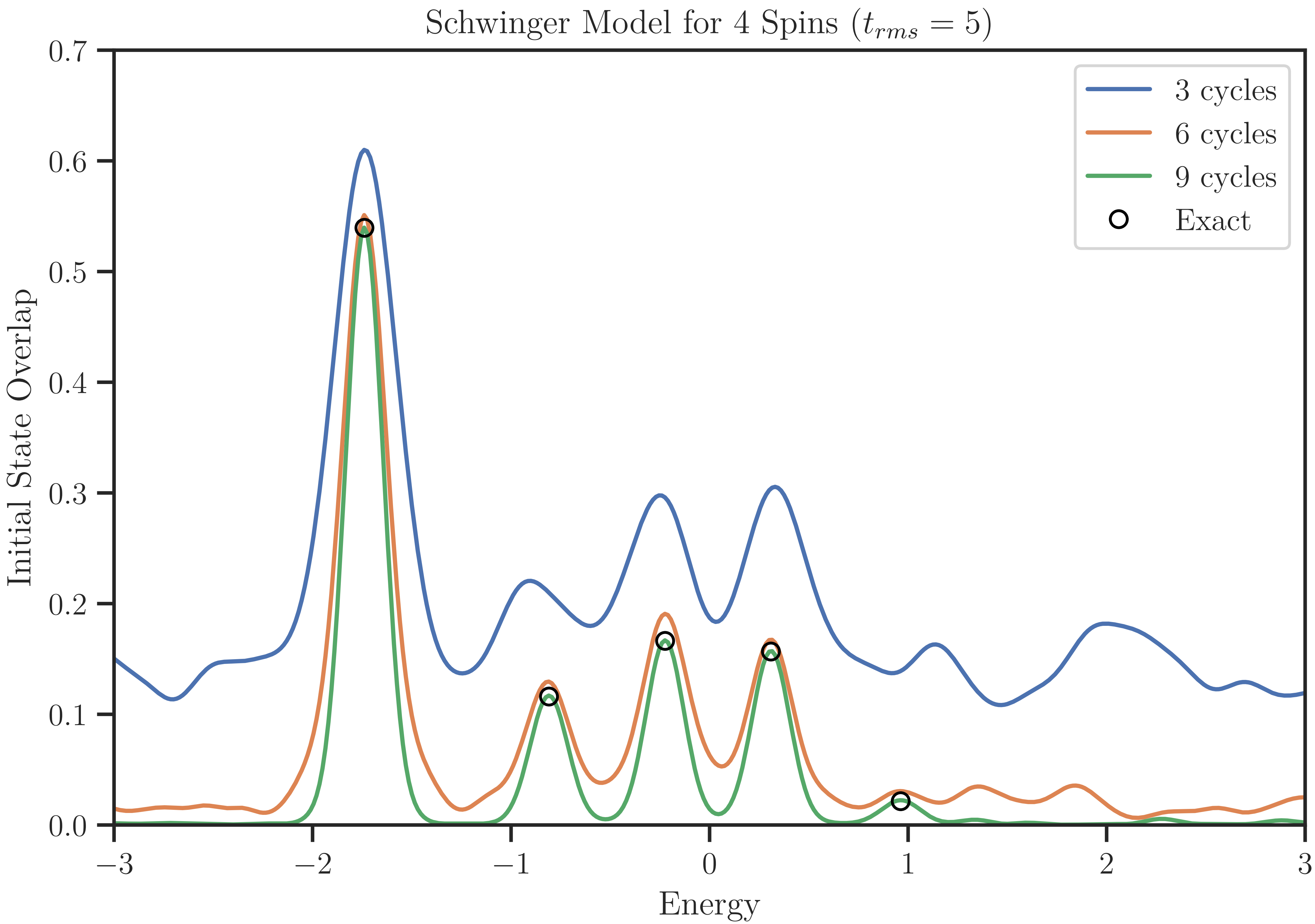

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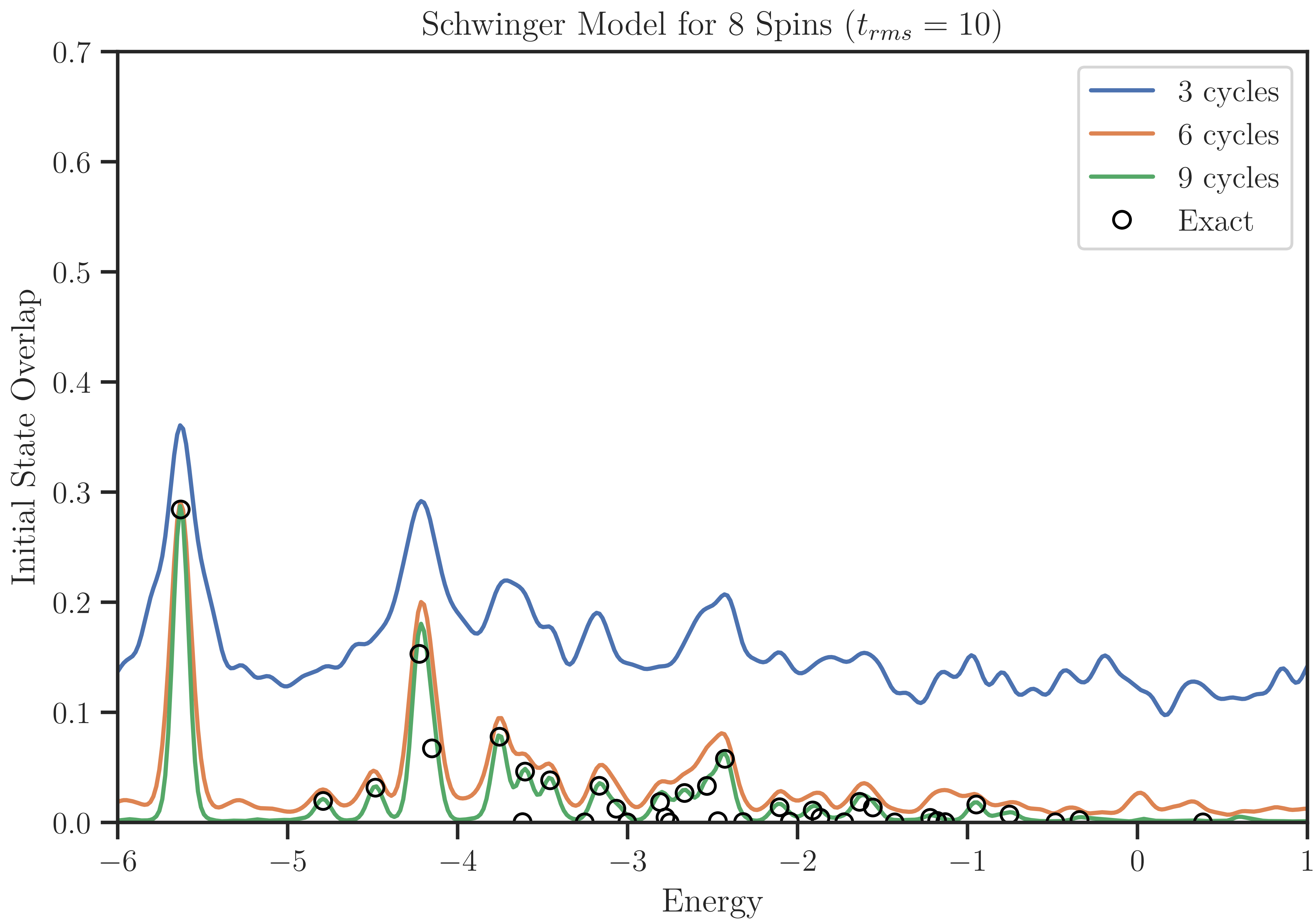

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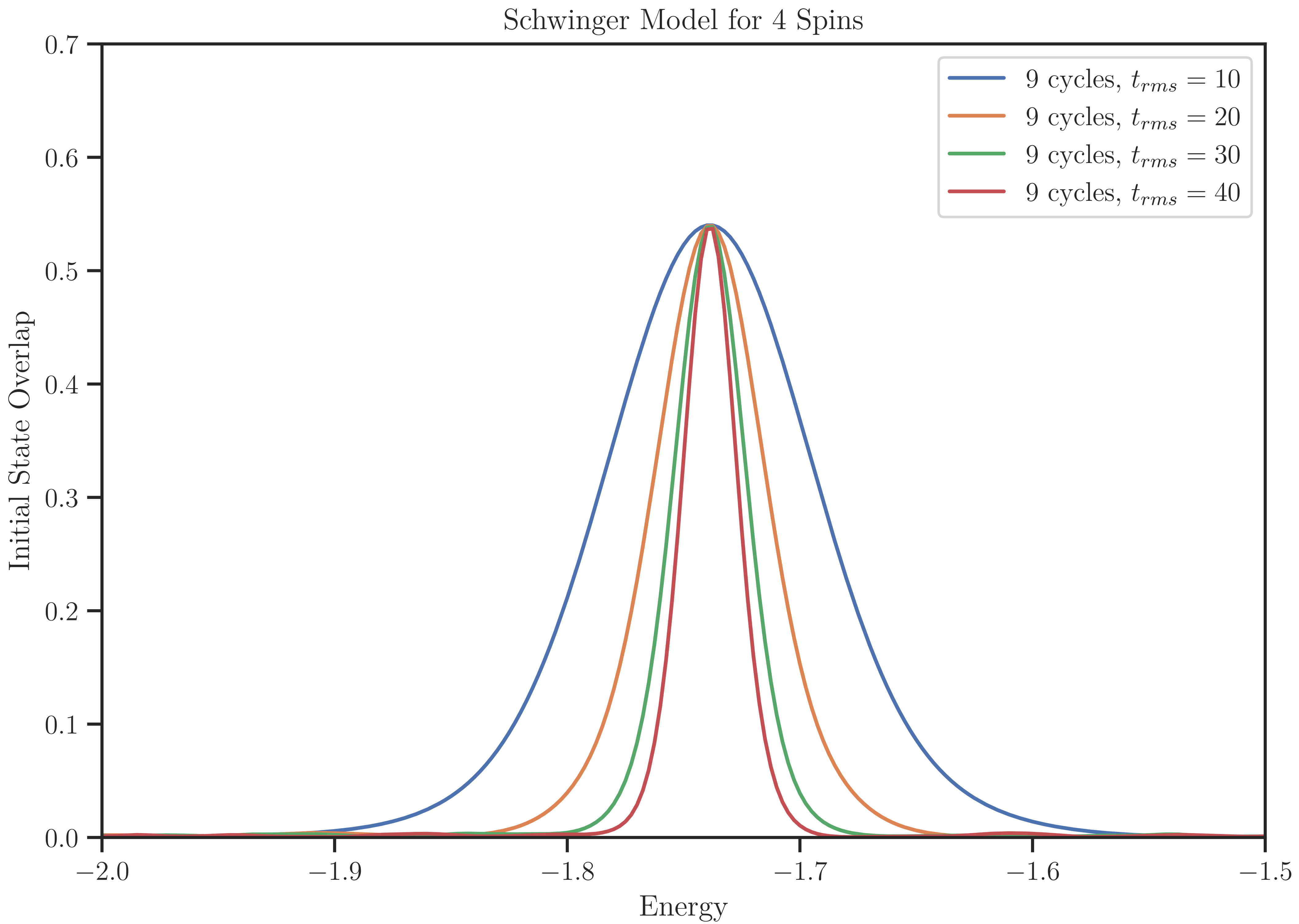

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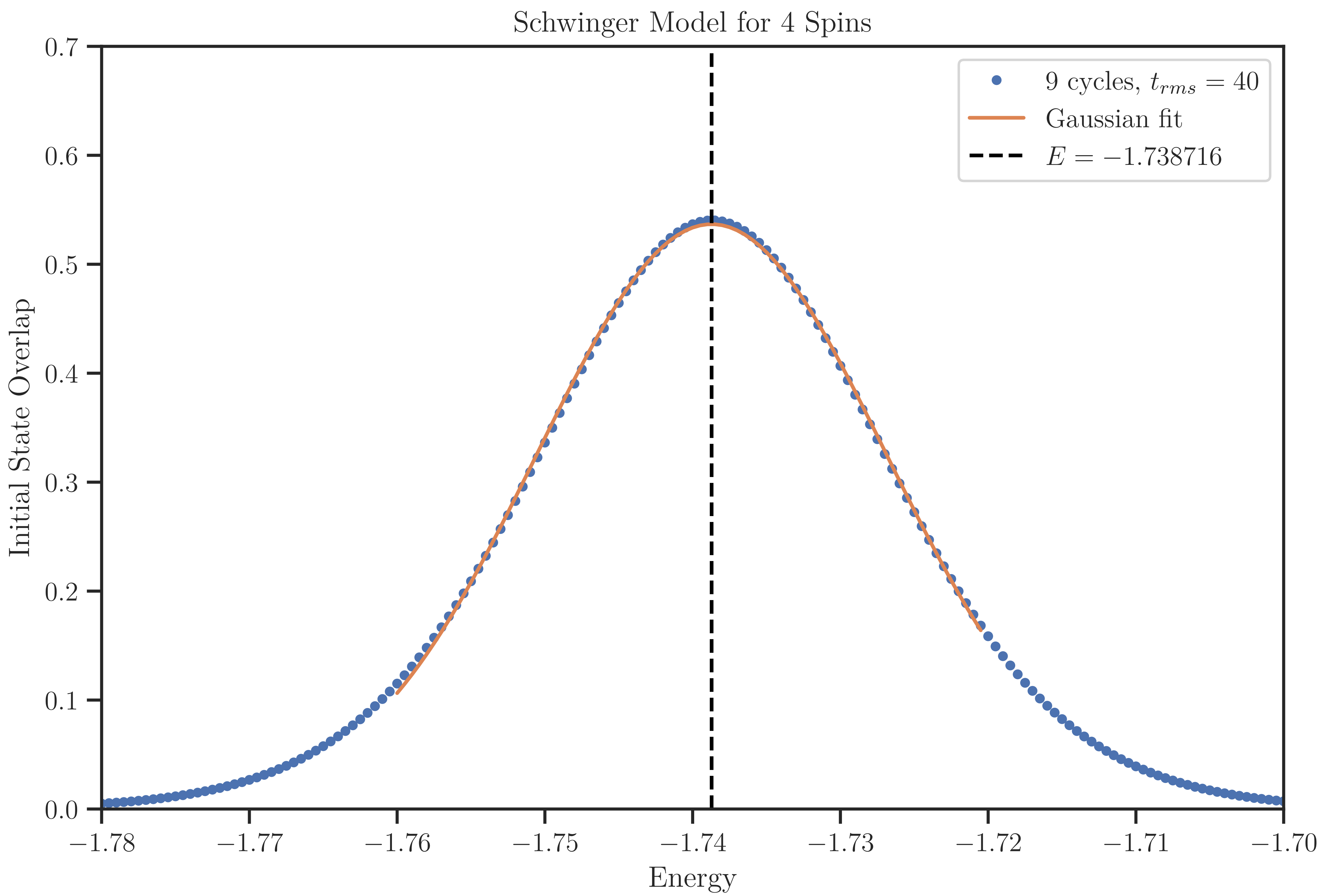

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| RA | 9 | 37 | $-1.7387$ | 0.9991 |

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# Conclusion

- The Schwinger model with a $\theta$-term can be mapped onto a quantum spin systems and the fermionic degrees of freedom are mapped onto qubits
- The Gauss law constraint is solved explicitly
- We compared three algorithms for evolving the system towards its ground state: Quantum Adiabatic Evolution (QAE), Quantum Approximate Optimization Algorithm (QAOA) and Rodeo Algorithm (RA)
- QAE and QAOA rely on finding a starting Hamiltonian whose ground state can be easily initialized and do not require ancilla qubits
- RA allows for scanning an energy range, does not require a starting Hamiltonian and requires ancilla qubits for controlled evolution
- Can chain together different algorithms — preconditioning