Toward tensor renormalization group study of three-dimensional non-Abelian gauge theory

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Abstract

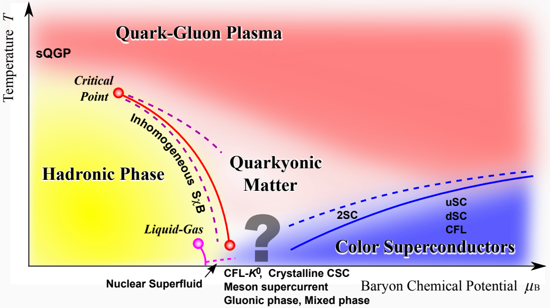
- Propose a new tensor network formulation with a trial action
- Apply the formulation to $3D$ pure SU(2) gauge theory

$\Rightarrow$ By tensor renormalization group(TRG), we obtain the result which agrees with the one obtained by weak/strong coupling expansion

It is the first TRG application to $3D$ non-Abelian gauge theory

Contents

Finite density QCD



Conjectured QCD phase diagram[Fukushima and Hatsuda, 2011]

Known that finite QCD is hard to analyze

The Monte Carlo method is difficult to apply due to the sign problem caused by the complex action

∵ We cannot simply consider the Boltzmann weight as the probability

⇒ Consider Tensor renormalization group(TRG) as an alternative method to Monte Carlo method

Tensor Renormalization Group(TRG)[Levin and Nave, 2007]

A deterministic numerical analysis method that represents the partition function as a tensor network and computes physical quantities using an appropriate information compression.

Advantages

- Free from the sign problem
- Can take large volume limit easily
  computational cost $\propto \log(\text{Lattice size})$

Disadvantages

- High computational cost for higher-dimensional theories ($\propto D^{a\cdot \mathrm{dim}+b}$)
- Whether it works or not depends on the model, TRG scheme, and initial tensor
  Needs trial & error

TRG procedure

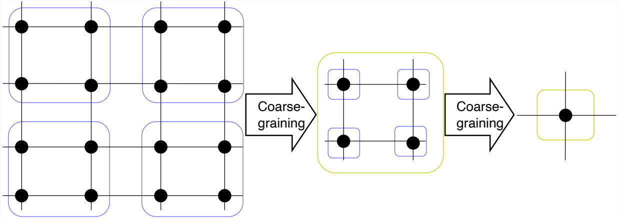
TRG algorithm

1. Tensor network representation of the partition function Z

$$Z = \sum_{\cdots, i, \cdots} \cdots T_{ijkl} T_{mnio} \cdots$$

2. Coarse-grain the tensor: Low rank approximation (of the matrix) by singular value decomposition
   $S_1 \geq S_2 \geq \cdots \geq 0$: Singular values

$$T_{ijkl} = M_{(ij),(kl)} = \sum_m U_{(ij),m} \cdot S_m \cdot V^{\dagger}_{m,(kl)} \sim \sum_{m=1}^{D} U_{(ij),m} \cdot S_m \cdot V^{\dagger}_{m,(kl)}$$



How to apply TRG to field theories

To apply TRG to field theories,
we need to make tensors of a continuous field configs

Case: Scalar field[Kadoh et al., 2019]

Gauss-Hermite quadrature

$$\int dx e^{-x^2} g(x) \sim \sum_{\alpha=1}^{K} \omega_\alpha g(y_\alpha)$$

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$$\int dx e^{-x^2} g(x) \sim \sum_{\alpha=1}^{K} \omega_\alpha g(y_\alpha)$$

Case: 2D Yang-Mills[Fukuma et al., 2021]

Generate configs chosen uniformly from the group manifold
and approximate the integral

$$\int dU f(U) \sim \frac{1}{K} \sum_{i=1}^{K} f(U_i)$$

TRG applications to field theories

Scalar field

- 4D scalar field[Akiyama et al., 2021b]

Fermion field

- 4D Nambu-Jona-Lasinio model[Akiyama et al., 2021a]

TRG applications to field theories

Scalar field

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Fermion field

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Gauge field

- 2D $U(1) + \theta$ term[Kuramashi and Yoshimura, 2020]
- 2D non-Abelian Higgs[Bazavov et al., 2019]
- 2D Yang-Mills[Fukuma et al., 2021, Hirasawa et al., 2021]
- 3D Z_2[Kuramashi and Yoshimura, 2019]
- 4D Z_2-Higgs[Akiyama and Kuramashi, 2022]
- 3D pure SU(2) [Kuwahara and Tsuchiya, 2022] arxiv:2205.08883(to be published in PTEP)

Proposal of a tensor network formulation for $3D$ SU(N) gauge theory: Introduce a trial action

The partition function

$$Z = \int \prod_{n,\mu} dU_{n,\mu} e^{-S}$$

$$S = \frac{\beta}{N} \sum_{n,\mu>\nu} \mathrm{ReTr}(1 - U_{\mu\nu}(n)), \quad U_{\mu\nu}(n) = U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}$$

Introduce a trial action while keeping Z unchanged

Add and subtract the trial action S_v from the original action

$$Z = \int \prod_{n,\mu} dU_{n,\mu} e^{-(S-S_v)-S_v}$$

$$S_v = \sum_{n,\mu} \tilde{S}_v(U_{n,\mu}), \quad \tilde{S}_v: \text{single link action}$$

$$= \sum_{n,\mu} \left(-\frac{H}{N} \mathrm{ReTr} U_{n,\mu} \right)$$

We adopt the simplest one given by the trace of a single link variable

Tensor network formulation with the trial action

$$Z = \int \prod_{n,\mu} dU_{n,\mu} e^{-(S-S_v)-S_v}$$

$$= Z_v^{3V} \langle e^{-(S-S_v)} \rangle_v, \quad \left(Z_v = \int dU e^{-\tilde{S}_v(U(n,\mu))} \right)$$

$$\langle \cdots \rangle_v = \frac{1}{Z_v^{3V}} \int \prod_{n,\mu} dU_{n,\mu} \cdots e^{-\sum n,\mu \tilde{S}(U_{n,\mu})}$$

Approximate the integral to the statistical average
under the weight of the trial action

$$\int dU_{n,\mu} g(U_{n,\mu}, U_{n',\mu'}, \ldots) \approx \frac{1}{K} \sum_{i=1}^{K} g(U_i, U_{n',\mu'}, \ldots)$$

$H = 0(\tilde{S}_v(U(n,\mu)) \propto H)$ corresponds to the random sampling method(cf.[Fukuma et al., 2021])

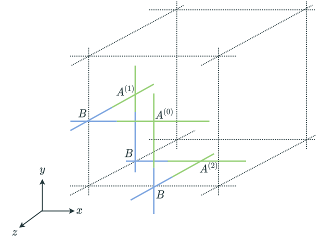
The initial tensor T

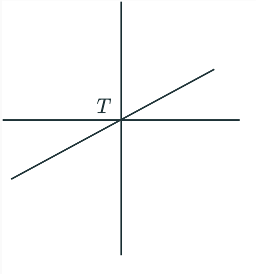
Based on the weight of the trial action $e^{-\tilde{S}_v}(U)$,
generate field configs $\{U_1, U_2, \cdots, U_K\}$ by the Monte Carlo method
We can obtain a A tensor from Z:

$$A_{ijkl} = \exp\left[\frac{\beta}{N}\mathrm{Tr}\left(U_i U_j U_k^\dagger U_l^\dagger\right) - \frac{1}{4}\left(\tilde{S}_v(U_i) + \tilde{S}_v(U_j) + \tilde{S}_v(U_k) + \tilde{S}_v(U_l)\right)\right]$$

Introduce the Kronecker delta $B_{ijkl} = \delta_{ijkl} = \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li}$
to construct a 6-rank tensor from A_{ijkl} (cf.Exact blocking formula[Xie et al., 2012])

The initial tensor $T = A \otimes A \otimes A \otimes B \otimes B \otimes B$





Tensor network representation

Tensor network representation for SU(N)
based on configs generated with the weight $e^{-\tilde{S}_v(U(n,\mu))}$

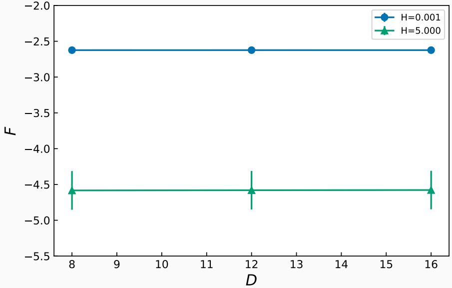
$$T = A \otimes A \otimes A \otimes B \otimes B \otimes B$$

Procedure

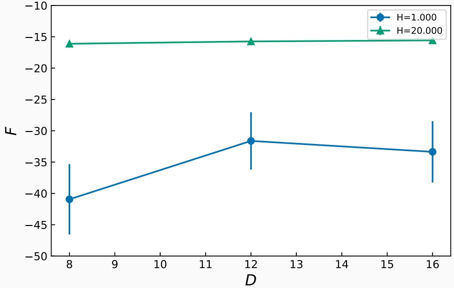
1. Generate configs $\{U_1, U_2, \cdots, U_K\}$ from $\tilde{S}_v(U(n,\mu))$ to construct the initial tensor T
2. Perform TRG
3. Repeat 1 and 2 to take the statistical average

$$\Rightarrow Z(K) = \left(e^{-\beta}\frac{Z_v}{K}\right)^{3V} \mathrm{tTr} \otimes_n T$$

D dependence(L=1024, K=12)



(a) $\beta = 1$



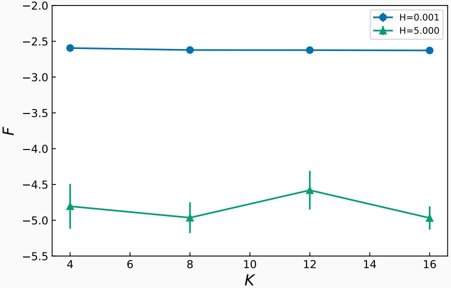
(b) $\beta = 50$

For small β, small H is efficient $\Rightarrow$ In the strong coupling regime, the uniform distribution works well

$\because H = 0$ corresponds to random sampling method[Fukuma et al., 2021]

For large β, large H is efficient $\Rightarrow$ In the weak coupling regime, tuning the variational parameter $H(> 0)$ is crutial

K dependence(L=1024, D=12)



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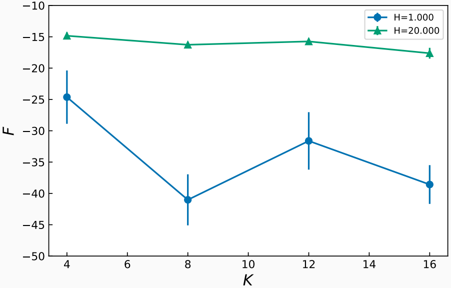

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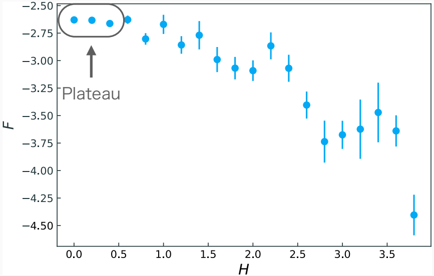
Optimize the variational parameter H

In principle the partition function is independent of the variational parameter H (if K&D are large enough)
$\Rightarrow$ Read off the optimized value H from the plateau



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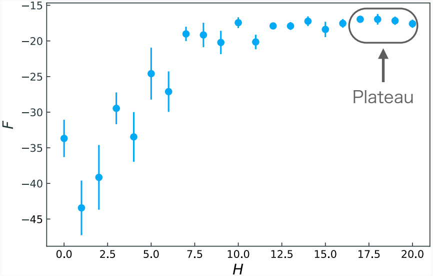
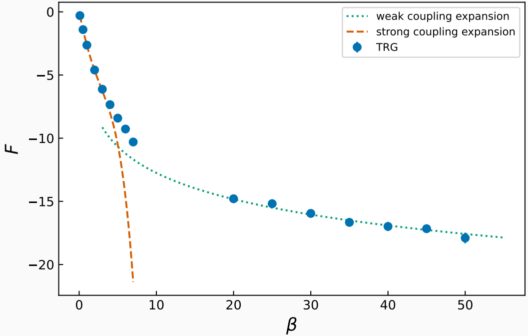


(b) $\beta = 50$

The free energy



The strong coupling expansion(small β) : $F(\beta) = -3\beta + \frac{3}{8}\beta^2 - \frac{3}{384}\beta^4 + \mathcal{O}(\beta^6)$

The weak coupling expansion(large β) : $F(\beta) = -3\log\beta + C + \mathcal{O}\left(\frac{1}{\beta}\right)$

Unfortunately, we could not find the plateau in the $7 \le \beta \le 19$ region

We expect this to be resolved by increasing K and/or improving the trial action

The TRG result agree with the one obtained by the strong/weak coupling expansion

Conclusion

Summary

- Propose a new tensor network formulation based on configs generated with the weight $e^{-\tilde{S}_v(U(n,\mu))}$
- Apply the formulation to $3D$ SU(2) gauge theory and obtain the results that agree with the one obtained by the strong/weak coupling expansion

Future work

- Calculate with large K
- Improve the trial action
- Calculate in the intermediate coupling regime
- Wilson loop
- Finite temperature transition
- θ term
- Extension to 4-dimension

Backup slides

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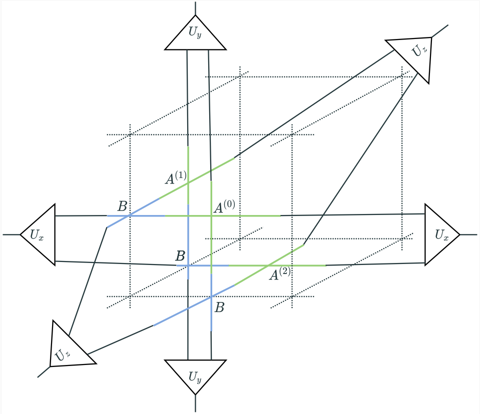
Construction of the initial tensor

$T = A^{(0)} \otimes A^{(1)} \otimes A^{(2)} \otimes B \otimes B \otimes B$

If the initial T tensor is constructed exactly, the six-rank tensor needs $O((K^2)^6)$ memory footprint.

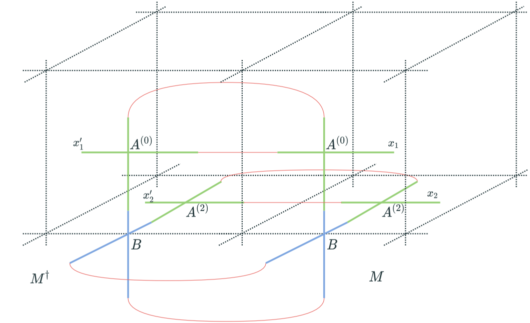
$\Rightarrow$ Install isometries[Xie et al., 2012] to truncate bond dimension from K^2 to D



Determine the isometries

Consider the indices for the x direction x_1, x_2 as the row of a matrix M
from eigen value decomposition of $MM^\dagger$ we obtain the isometry U_x for the x direction

$$MM^\dagger = U\Lambda(U)^\dagger$$



Isometries U_y and U_z are obtained in the same way