Digitizing $\mathrm{SU}(2)$ gauge fields and what to look out for when doing so

Tobias Hartung [1 2] Timo Jakobs [3 4] Karl Jansen [5] Johann Ostmeyer [6] Carsten Urbach [3 4]

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[1]Department of Mathematical Sciences, University of Bath

[2]Computation-based Science and Technology Research Center, The Cyprus Institute

[3]Helmholtz-Institut für Strahlen- und Kernphysik, University of Bonn

[4]Bethe Center for Theoretical Physics, University of Bonn

[5]NIC, DESY Zeuthen

[6]Department of Mathematical Sciences, University of Liverpool

Action and Observables

Pure $\mathrm{SU}(2)$ lattice gauge action

$$S = -\frac{\beta}{2} \sum_{n \quad in \Lambda} \sum_{\mu < \nu} \mathrm{Tr}\left[P_{\mu\nu}(n) \right]$$

with

$$P_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu^\dagger(n+\hat{\nu})\, U_\nu^\dagger(n)$$

on a hypercubic lattice of length L

$$\Lambda = \left\{ (n_0, \ldots, n_{d-1}) \in \mathbb{N}_0^4 \middle| 0 \leq n_\mu \leq L-1 \right\}$$

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$$\mathbb{P}(\mathcal{U}) \propto \exp\left(-S(\mathcal{U})\right)$$

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$\Rightarrow$ To test discretizations we restrict U_μ to finite subsets of $\mathrm{SU}(2)$

Off The Shelf Solutions

Finite Subgroups

- binary tetrahedral, octahedral and icosahedral groups $\overline{T}$, $\overline{O}$ and $\overline{I}$ $(24, 48, 120$ elements respectively)

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Asymptotically Dense Partitionings

- Make use of isomorphy between $\mathrm{SU}(2)$ and S_3

$$x \in S_3 \;\Leftrightarrow\; \begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix} \in \mathrm{SU}(2)$$

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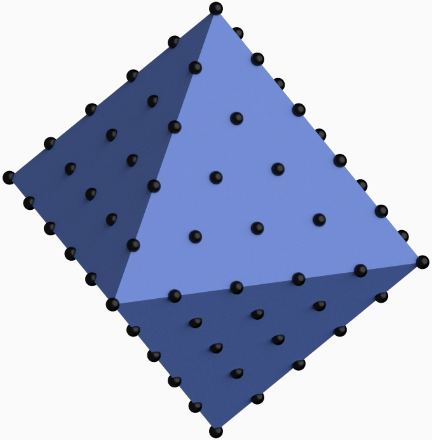
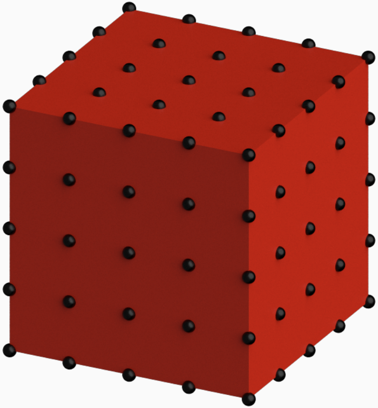
  $\Rightarrow$ Approaches can be generalized for other gauge groups

Geodesic Polyhedra





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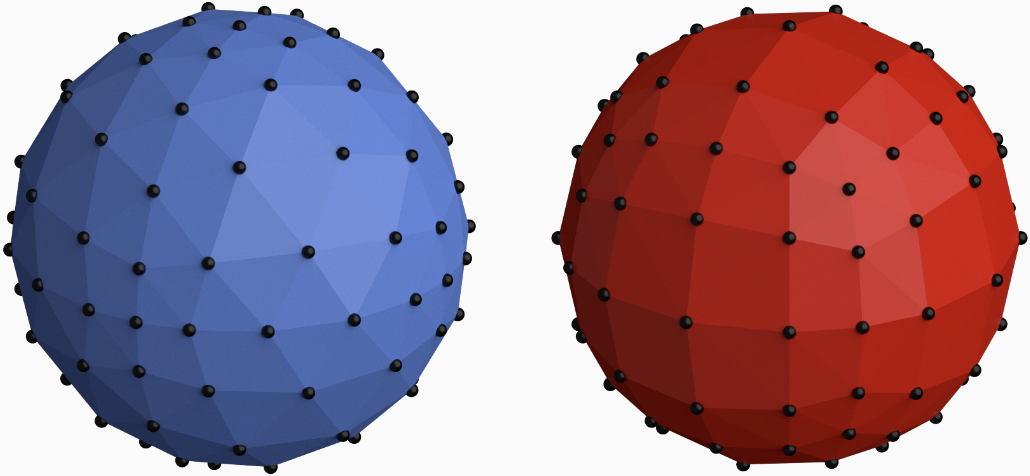

Geodesic Polyhedra

Linear Lattices

$$L_m(k) := \left\{ \frac{1}{M}\left(s_0 j_0, \ldots, s_k j_k\right) \middle| \sum_{i=0}^{k} j_i = m,\ \forall i \in \{0, \ldots, k\} :\ s_i \in \{\pm 1\},\ j_i \in \mathbb{N} \right\}, \quad M := \sqrt{\sum_{i=0}^{k} j_i^2}$$

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- Available with $8, 32, 88, 192, 360, 608 \ldots$ elements (for S_3)

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Volleyball Lattices

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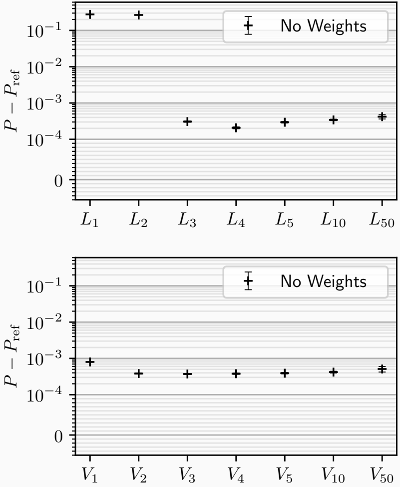
- Created from subdividing the k-dimensional cube
- Available with $16, 80, 240, 544, 1040, \ldots$ elements (for S_3)

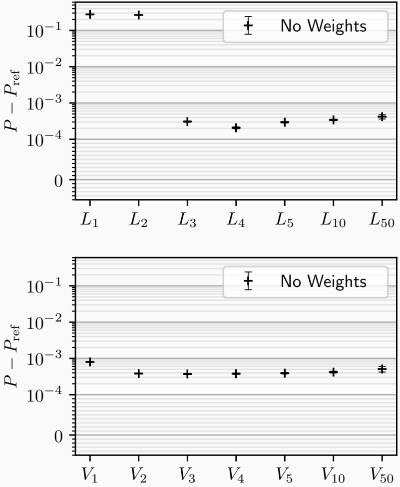
Geodesic Polyhedra - Weights



8^4 lattice at $\beta = 3$

Geodesic Polyhedra - Weights

- Systematic deviation caused by anisotropic distribution of points



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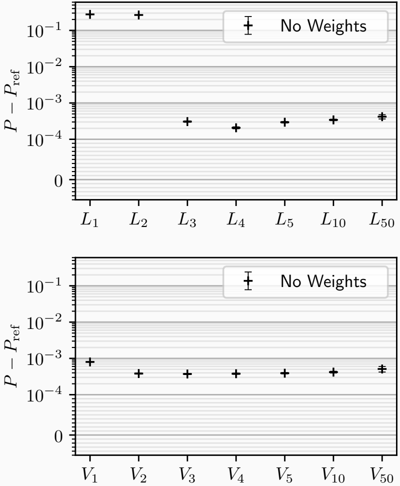
- Modification of Metropolis step:

$$\Delta S \qquad \rightarrow \qquad \Delta S' = \frac{w_{\text{new}}}{w_{\text{old}}} \Delta S$$

- w_{old} / w_{new} are proportional to the (estimated) Voronoi cell volumes surrounding the current / newly proposed link.



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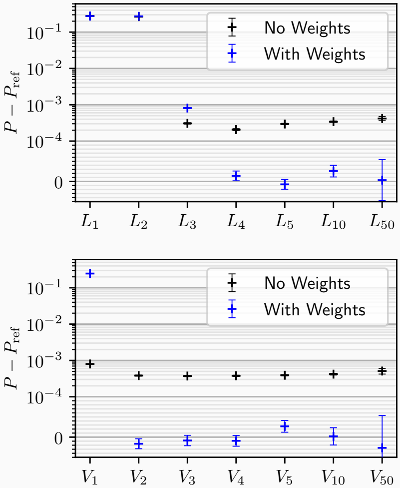

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Fibonacci Lattices

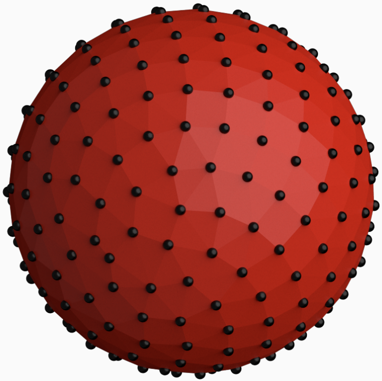
- 2D Fibonacci lattice in unit square:

$$\Lambda_n^2 = \{\tilde{t}_m | 0 \leq m < n,\ m \in \mathbb{N}\}$$

$$\text{with} \qquad \tilde{t}_m = (x_m, y_m)^t = \left(\frac{m}{\tau} \mod 1, \frac{m}{n}\right)^t,$$

$$\tau = (1+\sqrt{5})/2\,.$$

- Can be mapped from $[0,1)^2$ onto other manifolds such that volume is preserved



Fibonacci Lattice with $n = 256$ on S_2

Fibonacci Lattices

- Generalization for higher dimensions

$$\Lambda_n^k = \{t_m | 0 \leq m < n,\ m \in \mathbb{N}\}$$

$$t_m = \begin{pmatrix} t_m^1 \\ t_m^2 \\ \vdots \\ t_m^k \end{pmatrix} = \begin{pmatrix} \frac{m}{n} \\ a_1\, m \mod 1 \\ \vdots \\ a_{k-1}\, m \mod 1 \end{pmatrix}$$

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$$\frac{a_i}{a_j} \notin \mathbb{Q} \quad \text{for} \quad i \neq j$$

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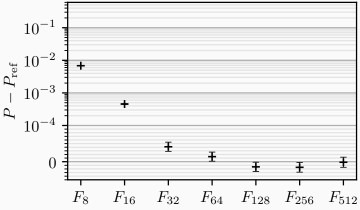
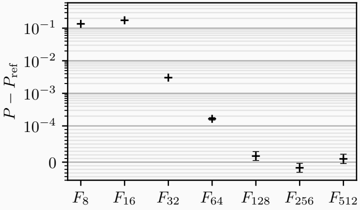


8^4 lattice at $\beta = 1.0$



8^4 lattice at $\beta = 3.0$

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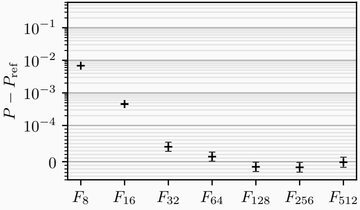
- Deviations due to the “chaotic” nature get smaller for larger sets



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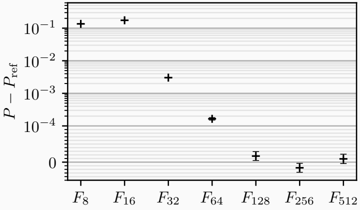

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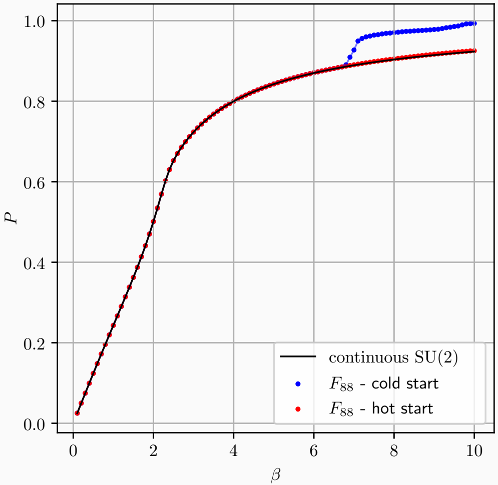
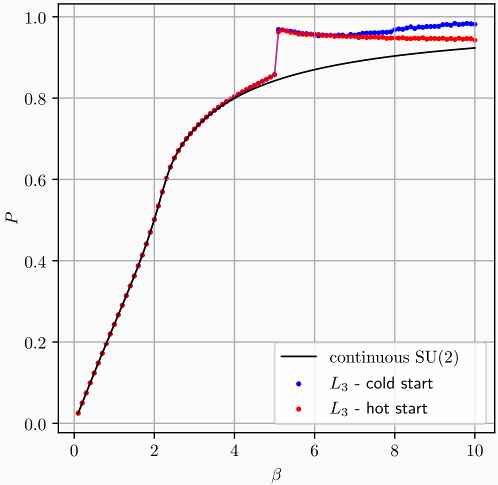
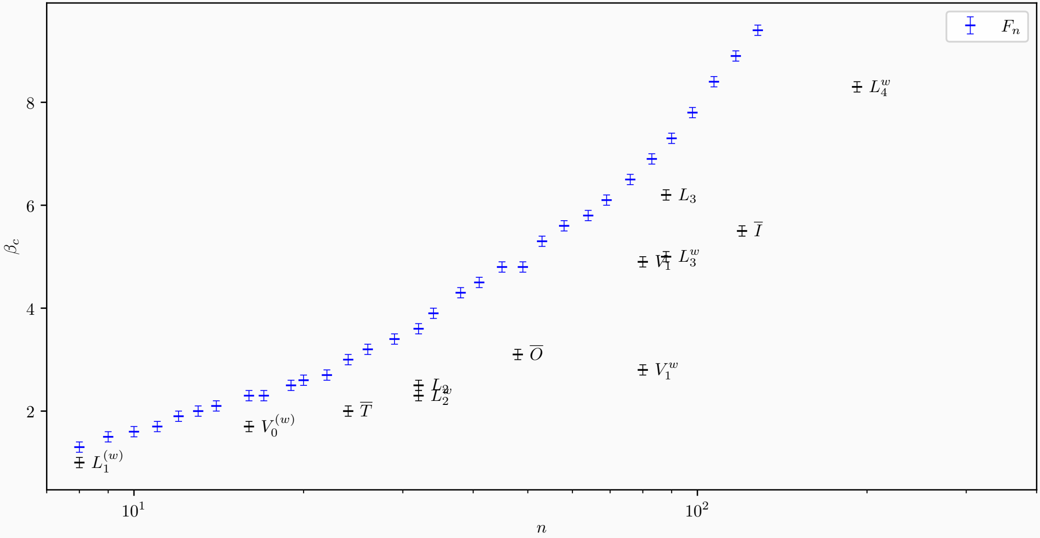
Phase Transitions





Phase Transitions I

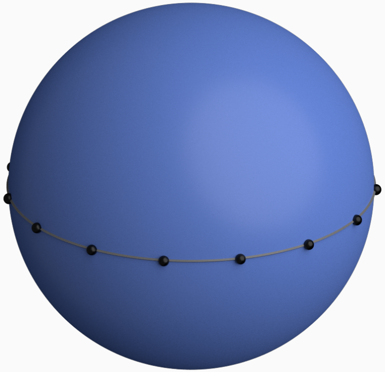


What can we hope for

- Prediction for β_c from Petcher and Weingarten 1980:

$$\beta_c \approx \frac{\ln\left(1+\sqrt{2}\right)}{1-\cos\left(2\pi/\tilde{N}\right)}$$

with $\tilde{N} = \#$ of steps along equator



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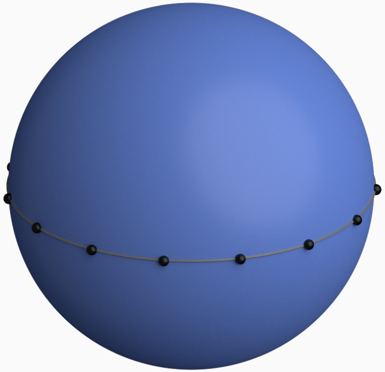

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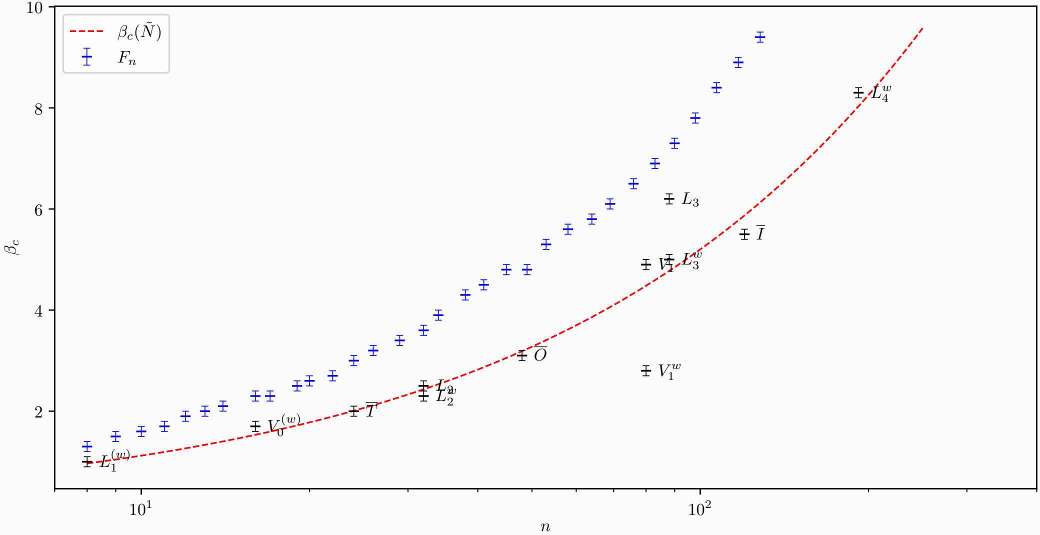

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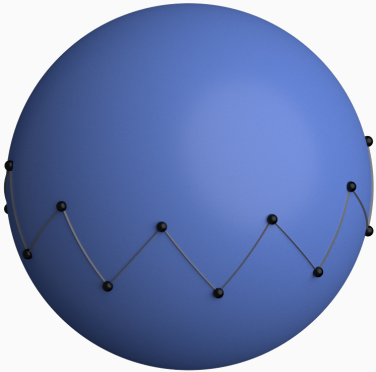
- Correct for "zick-zack" path by factor $\sqrt{2/3}$
(Ratio of side length and height of a tetrahedron)



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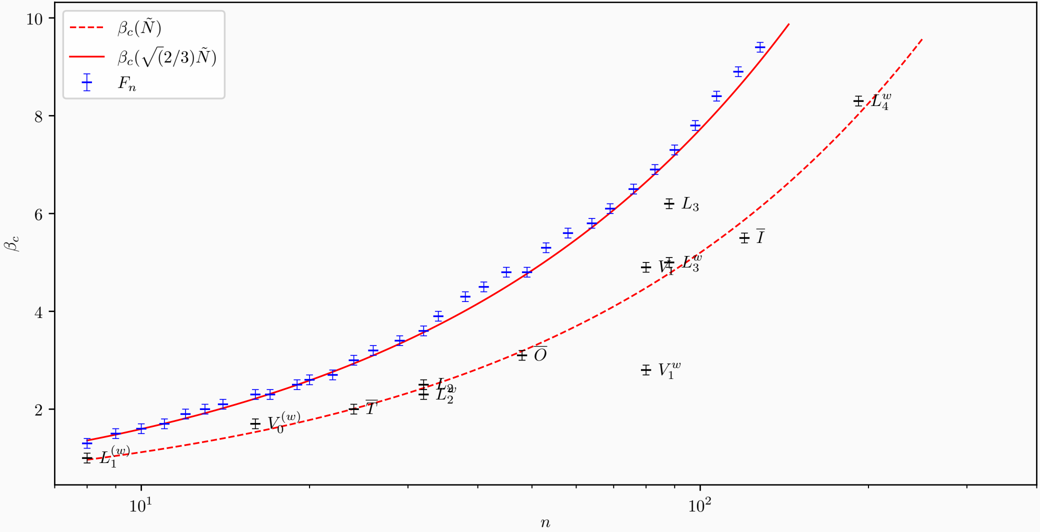

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The End - Thanks for listening

Paper can be found at:
`https://doi.org/10.1140/epjc/s10052-022-10192-5`

References

Petcher, D. and D. H. Weingarten (1980). "Monte Carlo Calculations and a Model of the Phase Structure for Gauge Theories on Discrete Subgroups of SU(2)". In: *Phys. Rev. D* 22, p. 2465. DOI: 10.1103/PhysRevD.22.2465.