Overcoming exponential volume scaling in quantum simulations of lattice gauge theories

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Lattice 2022

Date: August 9, 2022

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Scaling of Gate Count for Simulations of pure U(1) gauge theory in 2+1 Dimensions using Suzuki-Trotter methods

Main Take-Away Point 1: Naive implementation using only physical states has exponential volume scaling in gate count

Main Take-Away Point 2: Scaling can be made polynomial with carefully applied change of operator basis

(for more details: D. Grabowska, **C. Kane**, B. Nachman, C. Bauer, arXiv:[2208.03333])

Gauge Invariance and Gauss' Law

Continuum theory: Integral over electric and magnetic fields

$$H = \int d^2x[\vec{E}(x)^2 + B(x)],$$

Need to impose additional constraints

$$\underbrace{\vec{\nabla} \cdot \vec{E}(x) = 0}_{\text{constraint}}, \quad \underbrace{\nabla \cdot B(x) = 0}_{\text{constraint}}$$

Gauge Invariance and Redundancies

- **Problem:** Gauss' law not automatically satisfied for Hamiltonian formulations
  $\rightarrow$ allows for charge-violating transitions
- **Problem:** Naive basis of states is over-complete
  $\rightarrow$ requires more quantum resources than strictly necessary

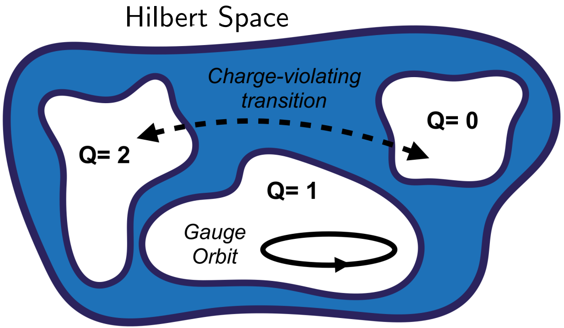
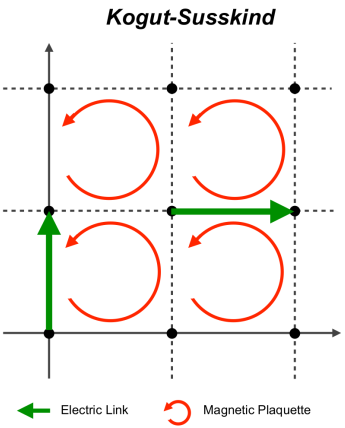


*Figure credit: D. Grabowska

Lattice U(1) Gauge Theory



Hilbert space **does allow** gauge violating transitions

*Figure credit: D. Grabowska

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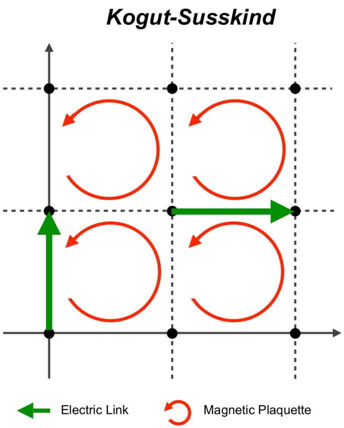

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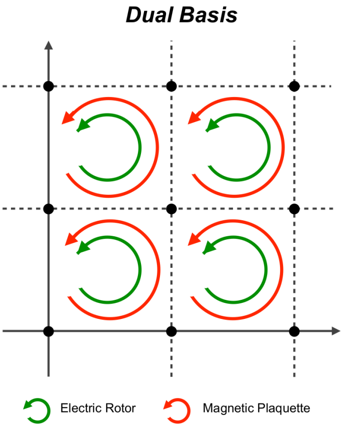

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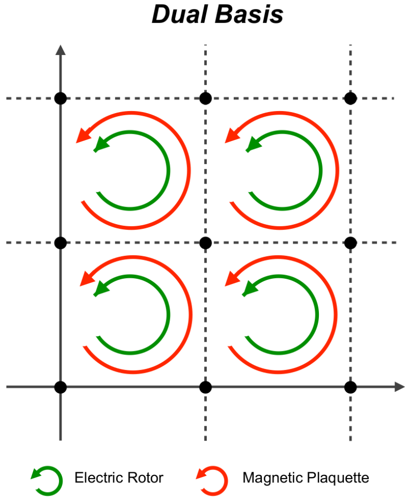
Dual Basis (Rotor) Formulation

General idea: Work with "gauge-redundancy free" formulation

- Work with plaquette variables: electric rotors and magnetic plaquettes
- Rotors R defined through $\vec{E} = \vec{\nabla} \times R$
  $\rightarrow$ Gauss' law automatically satisfied
- $[R_p, B_{p'}] = i\delta_{pp'}$
- Formulation works for all values of the gauge coupling

$$H = \underbrace{2g^2 \sum_{p=1}^{N_x N_y} (\vec{\nabla} \times R_p)^2}_{H_E} + \underbrace{\frac{1}{g^2} \sum_{p=1}^{N_x N_y} \cos(B_p)}_{H_B}$$



D. Kaplan, J. Stryker, PRD 102, 094515; J. Unmuth-Yockey, PRD 99, 074502 (2019); J. Haase et al. , Quantum 5, 393 (2021); J. Bender, E. Zohar, PRD 102, 114517 (2020); S.Drell, H. Quinn, B. Svetitsky, M. Weinstein, PRD 19, 619 (1979); C. Bauer, D. Grabowska, arXiv: 2111.08015

Global Constraints in Rotor Formulation

General idea: Locally imposed constraints automatically satisfied, but not global

Seeing the global constraint:

- Basis is over-complete: number of DOF's in rotor formulation too large *
- Product of plaquettes around closed surface must be identity
  $\rightarrow$ lattice version of $\int d^2x\, B = 0$

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Work-around: remove redundant DOF by enforcing constraint

$$R_{N_x N_y} = 0, \qquad B_{N_x N_y} = -\sum_{p=1}^{N_x N_y - 1} B_p$$

Magnetic Hamiltonian becomes (up to overall constant)

$$H_B = -\frac{1}{a\, g^2}\left[\sum_{p=1}^{N_p} \cos B_p + \cos\left(\sum_{p=1}^{N_p} B_p\right)\right], \qquad N_p \equiv N_x N_y - 1$$

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Time evolution strategy + Digitization Scheme

Suzuki-Trotter: $U(t) = \left(e^{-i\delta t H_E} e^{-i\delta t H_B}\right)^{N_{\text{steps}}} + \mathcal{O}(\delta t), \quad \delta t \equiv t/N_{\text{steps}}$

1. Implement diagonal operator $e^{-i\delta t H_B}$
2. Switch to electric basis using Fourier transform
3. Implement diagonal operator $e^{-i\delta t H_E}$
4. Switch to magnetic basis using Fourier transform

(remember $[R_p, B_{p'}] = i\delta_{pp'}$)

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Digitization of operators R_p/B_p [C. Bauer, D. Grabowska, arXiv: 2111.08015]

- Diagonal operators with evenly spaced eigenvalues
- Each lattice site represented by n_q qubits
- b_{max} function of coupling to minimize digitization errors
  $\rightarrow n_q = 3$ achieves per-mille accuracy of low-lying spectrum

$$R = \frac{r_{\text{max}}}{2^{n_q} - 1} \sum_{j=1}^{n_q} 2^j \sigma_j^z$$

$$B = \frac{b_{\text{max}}}{2^{n_q} - 1} \sum_{j=1}^{n_q} 2^j \sigma_j^z,$$

[N. Klco, M. Savage, PRA, arXiv:1808.10378]

Gate Count for Suzuki-Trotter methods

Suzuki-Trotter: $U(t) = \left(\mathsf{FT}^\dagger e^{-i\delta t H_E} \mathsf{FT} e^{-i\delta t H_B}\right)^{N_{\text{steps}}} + \mathcal{O}(\delta t), \quad \delta t \equiv t/N_{\text{steps}}$

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Electric Hamiltonian:

- Bilinear structure, $R^2 \sim \sum_{i,j=1}^{n_q} \sigma_i^z \sigma_j^z \rightarrow e^{-i\delta t H_E}$ requires $\mathcal{O}(n_q^2 N_p)$ gates

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- $B \sim \sum_{i=1}^{n_q} \sigma_i^z$

$$\exp(iH_B) \sim \exp\left(i \sum_{p=1}^{N_p} \cos B_p\right) \times \exp\left(i \cos\left(\sum_{p=1}^{N_p} B_p\right)\right)$$

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Exponential volume scaling

$$H_B \sim \underbrace{\sum_{p=1}^{N_p} \cos B_p}_{\mathcal{O}(N_p 2^{n_q})\text{ gates}} + \underbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)}_{\mathcal{O}(2^{n_q N_p})\text{ gates}}$$

Exponential volume scaling $\mathcal{O}(2^{n_q N_p})$ comes from maximally coupled term

$\rightarrow$ simulating realistic values of $N_p \sim 400$ requires $\mathcal{O}(2^{400 n_q})$ gates

Reducing degree of coupling

Requirement: perform orthonormal operator basis change such that no single term in the Hamiltonian acts on more than $\mathcal{O}(\log_2 N_p)$ qubits

Basis Change

$$B_p \to \mathcal{W}_{pp'} B_{p'}$$

$$\mathcal{W} = \begin{pmatrix} W_{d_{(1)}} & 0 & 0 & 0 \\ 0 & W_{d_{(2)}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & W_{d_{(N_s)}} \end{pmatrix}$$

W_d: "Weaved" matrix of dimension d

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- Each row of W_d has no more than $\lceil \log_2 d \rceil + 1$ non-zero entries

$\rightarrow$ **Previously local terms now require $(N_p / \log_2 N_p)^{n_q}$ gates**

Breaking of exponential volume scaling

Implementing new "Weaved" magnetic Hamiltonian requires $\mathcal{O}({N_p}^{n_q})$ gates

(n_q number of qubits used to represent each lattice site, volume independent)

Basis change example: $N_p = 16$

Properties of $\mathcal{W}$ and W_d

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Properties of $\mathcal{W}$ and W_d

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  $\rightarrow$ max number of non-zero entries in a given row is 3

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$$W_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Basis change example: $N_p = 16$

$$H_B \sim \sum_{p=1}^{N_p} \cos B_p + \cos\left(\sum_{p=1}^{N_p} B_p\right) \longrightarrow_{B_p \to \mathcal{W}_{pp'}\tilde{B}_{p'}} H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \cos \sum_{p=1}^{3} \tilde{B}_p + \cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)$$

(schematically)

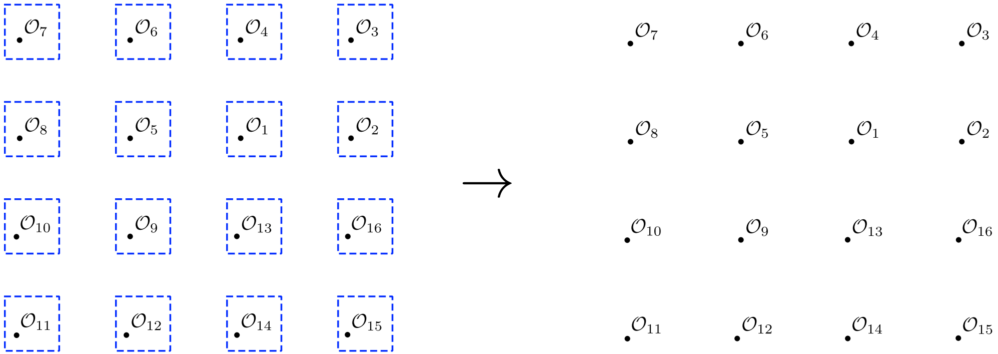
$\mathcal{O}_7$	$\mathcal{O}_6$	$\mathcal{O}_4$	$\mathcal{O}_3$
$\mathcal{O}_8$	$\mathcal{O}_5$	$\mathcal{O}_1$	$\mathcal{O}_2$
$\mathcal{O}_{10}$	$\mathcal{O}_9$	$\mathcal{O}_{13}$	$\mathcal{O}_{16}$
$\mathcal{O}_{11}$	$\mathcal{O}_{12}$	$\mathcal{O}_{14}$	$\mathcal{O}_{15}$

$\longrightarrow$

$\mathcal{O}_7$	$\mathcal{O}_6$	$\mathcal{O}_4$	$\mathcal{O}_3$
$\mathcal{O}_8$	$\mathcal{O}_5$	$\mathcal{O}_1$	$\mathcal{O}_2$
$\mathcal{O}_{10}$	$\mathcal{O}_9$	$\mathcal{O}_{13}$	$\mathcal{O}_{16}$
$\mathcal{O}_{11}$	$\mathcal{O}_{12}$	$\mathcal{O}_{14}$	$\mathcal{O}_{15}$

Basis change example: $N_p = 16$

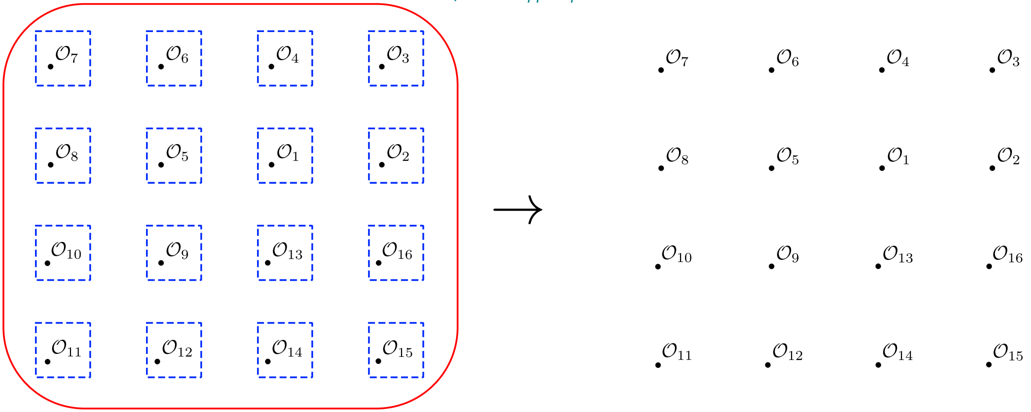
$$H_B \sim \sum_{p=1}^{N_p} \overbrace{\cos B_p} + \cos\left(\sum_{p=1}^{N_p} B_p\right) \xrightarrow[B_p \to \mathcal{W}_{pp'}\tilde{B}_{p'}]{} \overset{\text{(schematically)}}{H_B^{\text{weaved}}} \sim \sum_{p=1}^{N_p} \cos \sum_{p=1}^{3} \tilde{B}_p + \cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)$$



Basis change example: $N_p = 16$

$$H_B \sim \sum_{p=1}^{N_p} \overbrace{\cos B_p} + \overbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)} \longrightarrow \text{(schematically)}\; H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \cos \sum_{p=1}^{3} \tilde{B}_p + \cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)$$

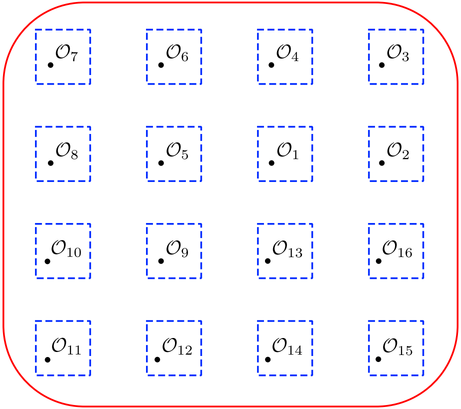
$B_p \to \mathcal{W}_{pp'} \tilde{B}_{p'}$



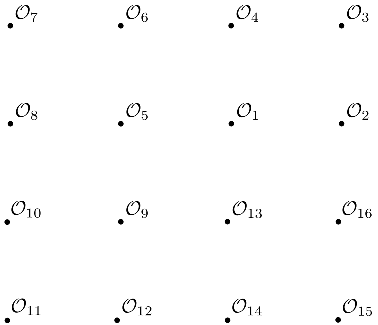
Basis change example: $N_p = 16$

$$H_B \sim \sum_{p=1}^{N_p} \overbrace{\cos B_p} + \overbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)} \longrightarrow \text{(schematically)}\; H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \cos \sum_{p=1}^{3} \tilde{B}_p + \cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)$$

$B_p \to \mathcal{W}_{pp'} \tilde{B}_{p'}$



$\longrightarrow$



$\text{Gates}(n_q = 2) \sim \mathcal{O}(2^{n_q N_p}) \sim \mathcal{O}(2^{32}) \sim \mathcal{O}(10^9)$

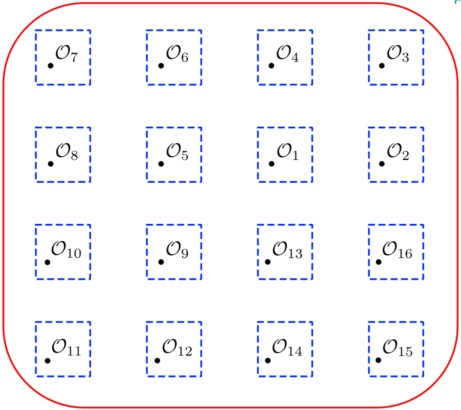
Basis change example: $N_p = 16$

$$H_B \sim \sum_{p=1}^{N_p} \overbrace{\cos B_p} + \overbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)}$$

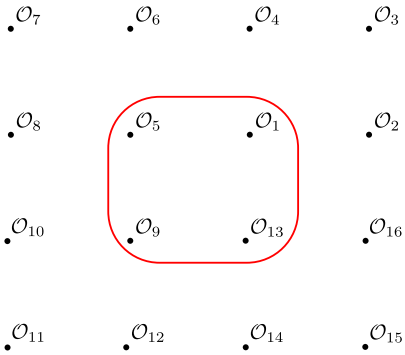
$\longrightarrow$ $B_p \to \mathcal{W}_{pp'}\tilde{B}_{p'}$

(schematically)

$$H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \cos \sum_{p=1}^{3} \tilde{B}_p + \overbrace{\cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)}$$



$\longrightarrow$



$\text{Gates}(n_q = 2) \sim \mathcal{O}(2^{n_q N_p}) \sim \mathcal{O}(2^{32}) \sim \mathcal{O}(10^9)$

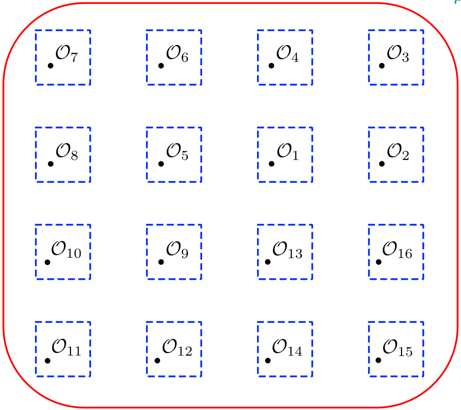
Basis change example: $N_p = 16$

$$H_B \sim \overbrace{\sum_{p=1}^{N_p} \cos B_p}^{} + \overbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)}^{}$$

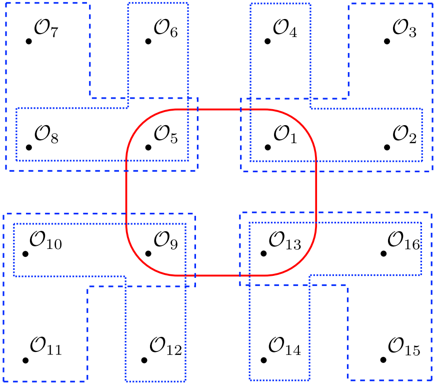
$$\xrightarrow{B_p \to \mathcal{W}_{pp'}\tilde{B}_{p'}}$$

(schematically)

$$H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \overbrace{\cos \sum_{p=1}^{3} \tilde{B}_p}^{} + \overbrace{\cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)}^{}$$



$\longrightarrow$



$\text{Gates}(n_q = 2) \sim \mathcal{O}(2^{n_q N_p}) \sim \mathcal{O}(2^{32}) \sim \mathcal{O}(10^9)$

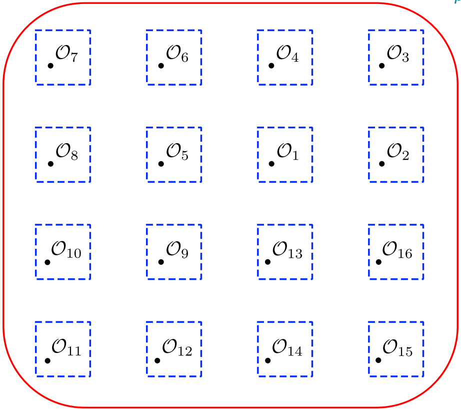
Basis change example: $N_p = 16$

$$H_B \sim \sum_{p=1}^{N_p} \overbrace{\cos B_p} + \overbrace{\cos\left(\sum_{p=1}^{N_p} B_p\right)} \longrightarrow$$

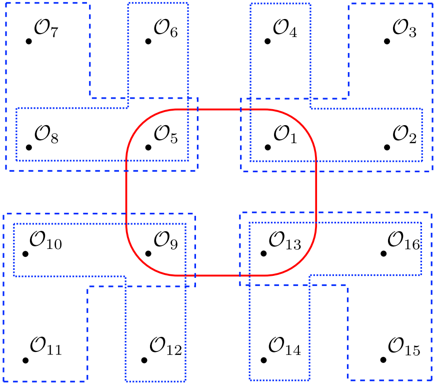
$B_p \to \mathcal{W}_{pp'}\tilde{B}_{p'}$

(schematically)

$$H_B^{\text{weaved}} \sim \sum_{p=1}^{N_p} \overbrace{\cos \sum_{p=1}^{3} \tilde{B}_p} + \overbrace{\cos\left(\sum_{p=1}^{4} \tilde{B}_{p'}\right)}$$



$\longrightarrow$



Gates$(n_q = 2) \sim \mathcal{O}(2^{n_q N_p}) \sim \mathcal{O}(2^{32}) \sim \mathcal{O}(10^9)$

Gates$(n_q = 2) \sim \mathcal{O}(N_p^{n_q}) \sim \mathcal{O}(16^2) \sim \mathcal{O}(10^2)$

Conslusions

Quantum computers have a fundamentally different computational strategy and provide novel probes of fundamental questions in particle and nuclear physics

It is important to carefully study the scaling of quantum computing resources for simulating gauge theories on quantum computers

Main Take-Away Point 1: Naive implementation using only physical states has exponential volume scaling in gate count

Main Take-Away Point 2: Scaling can be made polynomial with carefully applied change of operator basis

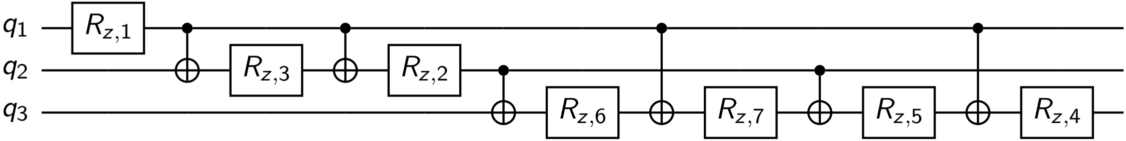
(for more details: D. Grabowska, **C. Kane**, B. Nachman, C. Bauer, arXiv:[2208.03333])

(Implementation of this method is in progress)

Backup slides

Diagonal operators on a quantum comptuer

Implementing n qubit diagonal operators without ancillary qubits $\rightarrow 2^{n+1} - 3$ gates



Certain class of simple operators require less than $2^{n+1} - 3$ gates [J. Welch, et. al., arXiv:1306.3991]