Defining Canonical Momenta for Discretised SU(2) Gauge Fields

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Motivation: Hamiltonian Formulation of Lattice Gauge Theories

- Hamiltonian for a non-Abelian Lattice Gauge Theory

$$H = \frac{1}{2}\sum_x \sum_a \left(L_a^2(x) + R_a^2(x)\right) + \sum_x \mathrm{Tr}_{\mathrm{colour}}\,\mathrm{Re}\,U_p(x)$$

  suppressing coefficients.

  [Kogut and Susskind, Phys.Rev.D 11 (1975)]

- requires discretisation in the group for implementation in practice
- wanted: most efficient formulation

  [see e.g. Davoudi et al. Phys.Rev.D 104 (2021) 7, 074505]

- we explore here a basis where the $\hat{U}$ are diagonal

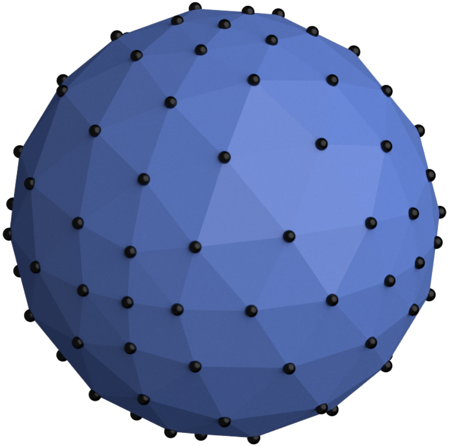
Linear Discretisation

- we have proposed a list of partitionings of SU(2)
  - generalisable to SU(3)
  - asymptotically isotropic in the group
  - freely adjustable number of elements

$\rightarrow$ see the talk of Timo Jakobs

[T. Hartung et al., Eur.Phys.J.C 82 (2022) 3, 237, arXiv:2201.09625]

- Here: so-called linear partitioning:
  - control parameter $M \in \mathbb{N}$, $M \rightarrow \infty$ continuous group
  - mean distance between elements $\propto 1/M$
  - number of elements grow roughly like M^3



Hilbert Space and Operators

- states $|U\rangle \in \mathcal{H}$ in Hilbert space $\mathcal{H}$
- SU(2) matrix parametrised by three real valued parameters x_0, x_1, x_2

$$\begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix} \in \mathrm{SU}(2)\,, \qquad x_3^2 = 1 - \sum_{i=0}^{2} x_i^2$$

- define operators $\hat{x}_i$ with

$$\hat{x}_i\,|U\rangle = x_i|U\rangle\,, \quad \text{and e.g.} \quad \hat{u}_{00} = \hat{x}_0 + \mathrm{i}\hat{x}_1$$

- this defines the action

$$\hat{U}\,|U\rangle = \begin{pmatrix} \hat{u}_{00} & \hat{u}_{01} \\ \hat{u}_{10} & \hat{u}_{11} \end{pmatrix}\,|U\rangle$$

- with a partitioning $\mathcal{H}$ is finite dimensional

Commutation Relations

- need to find operators L_a and R_a as follows

$$[\hat{L}_a, \hat{U}_{jl}] \;=\; (t_a)_{ji}\,\hat{U}_{il}\,, \qquad [\hat{R}_a, \hat{U}_{jl}] \;=\; \hat{U}_{ji}\,(t_a)_{il}$$

  with t_a the generators of SU(2), $a = 1, 2, 3$

- and

$$[\hat{L}_a, \hat{L}_b] = -2\,\mathrm{i}\,\epsilon_{abc}\,\hat{L}_c$$

- in the continuum this is fulfilled by the operators

$$L_a\, f(U) \;=\; -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\alpha}\, f\left(e^{\mathrm{i}\,\alpha t_a}\, U\right)|_{\alpha=0}\,, \qquad R_a\, f(U) \;=\; -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\alpha}\, f\left(U\, e^{\mathrm{i}\,\alpha t_a}\right)|_{\alpha=0}$$

- how to construct $\hat{L}_a$ and $\hat{R}_a$ for the discrete case?

Discretisation of $\hat{L}_a$

- in direction a use the finite difference

$$\begin{aligned}\frac{1}{\alpha}\left(e^{\mathrm{i}\alpha t_a}\,U - U\right) &= \frac{1}{\alpha}\left(U + \alpha\mathrm{i}\,t_a U - U + O(\alpha^2)\right)\\ &= \mathrm{i}\,t_a U \;+ O(\alpha)\,.\end{aligned}$$

- however, $e^{\mathrm{i}\alpha t_a}\,U$ not neccessarily in our set of elements!

- need to construct the directional derivative from existing neighbours

$\Rightarrow$ choose three neighbors and project onto the desired direction

Discrete Directional Derivative $\hat{L}_a$

1. find 3 neighbours V_i of element U_j, then there are 3 $W_i \in \mathrm{SU}(2)$

$$V_i(j) = W_i\, U \qquad \Leftrightarrow \qquad W_i = V_i U_j^{-1} = \exp(\mathrm{i}\alpha_b^i t_b)$$

2. solve

$$e_a = \gamma \cdot (\alpha^1 \alpha^2 \alpha^3)$$

   for vector γ with e_a unit vector in direction a

3. the elements of the discrete operator $\hat{L}_a$ are then given by

$$(L_a)_{j\,\#V_i(j)} = \gamma_i\,, \qquad (L_a)_{jj} = -\sum_i \gamma_i$$

   with $\#V_i(j)$ the index of neighbour $V_i(j)$

Proof

This leads to the desired result up to $O(\alpha)$

$$
\begin{aligned}
-\left(\sum_{i=1}^{n}\gamma_i\right)&U_j + \gamma_1 V_1 + \gamma_2 V_2 + \gamma_3 V_3 \\
&= -\left(\sum \gamma_i\right) U_j + \gamma_1 W_1 U_j + \gamma_2 W_2 U_j + \gamma_3 W_3 U_j \\
&\approx \left(-\sum_i \gamma_i + \gamma_1(1+\mathrm{i}\alpha_b^1 t_b) + \gamma_2(1+\mathrm{i}\alpha_b^2 t_b) + \gamma_3(1+\mathrm{i}\alpha_b^3 t_b)\right) U_j \\
&= \mathrm{i}\left(\gamma_1\alpha_b^1 t_b + \gamma_2\alpha_b^2 t_b + \gamma_3\alpha_b^3 t_b\right) U_j \\
&= \mathrm{i} t_a U_j\,.
\end{aligned}
$$

Numerical Test of Commutation Relations

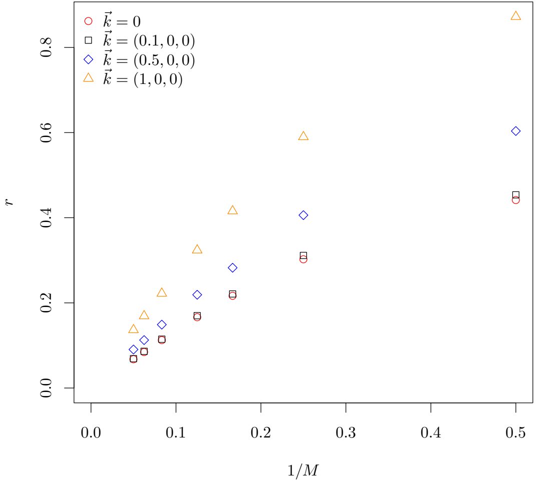
- we compute first

$$z \;=\; \Big([\hat{L}_a, \hat{U}_{jl}] - (t_a)_{ji}\hat{U}_{il}\Big)\cdot v(\vec{k})$$

  with $v(\vec{k})$ a Fourier mode in the algebra
- for each element expected convergence is $O(\alpha)$
- thus compute

$$r = \frac{1}{N}\sum_i |z_i|$$

  with N the number of elements



Numerical Test of Commutation Relations

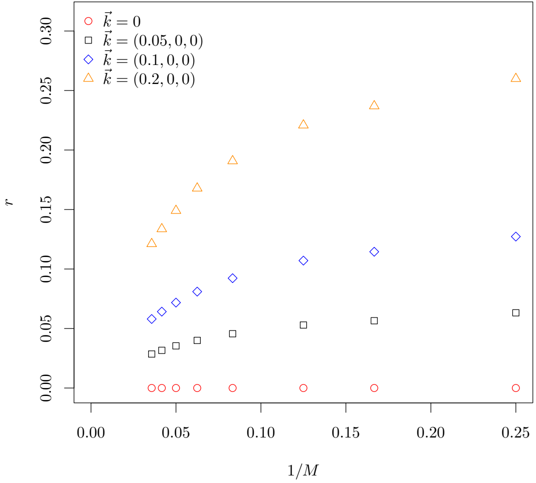
- similarly

$$z \;=\; ([L_a, L_b] \;+\; 2\mathrm{i}\,\epsilon_{abc}\,L_c) \cdot v(\vec{k})$$

- and again

$$r = \frac{1}{N}\sum_i |z_i|$$

- convergence slower in $1/M$



Spectrum of the Free Theory

Free Hamiltonian

$$H = \frac{1}{2}\sum_a \left(L_a^2 + R_a^2\right)$$

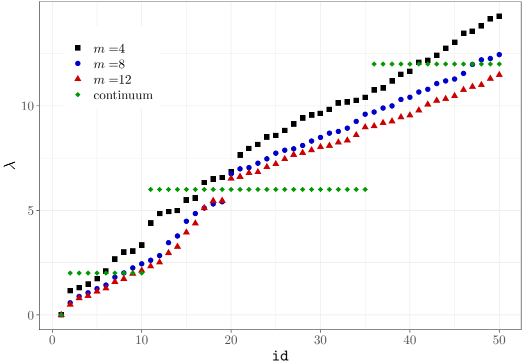
- the discrete L_a and R_a no longer hermitian
- we resort to $L^\dagger \cdot L$ instead
- we expect smallest deviations for small eigenvalues
- continuum spectrum

  $$\lambda_\ell = \ell(\ell+1)\,, \qquad (2\ell+1)^2 \text{ degenerate}$$

  with $\ell = 0, 1, \ldots$

Spectrum of the Free Theory

- numerically determine lowest lying spectrum
- agreement not breathtaking...
- possible reasons:
  - slow convergence
  - not consistently defined forward derivative



Summary

- work in a basis where $\hat{U}$ is diagonal
- defined discretised operators L_a, R_a based on SU(2) partitionings
- commutation relations converge in the SU(2) limit
- the free spectrum is not yet reproduced