Multi-hadron interactions from Lattice QCD

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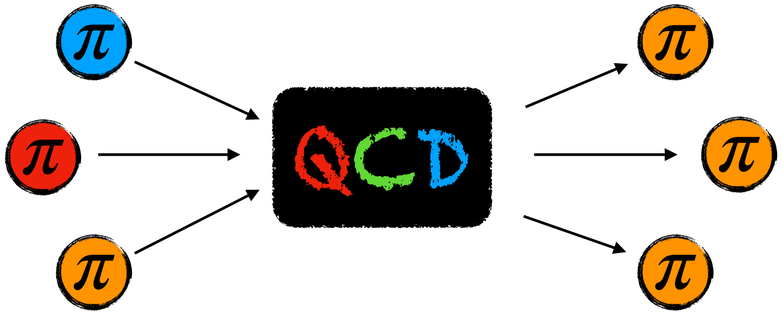
Bonn, 10th August 2022



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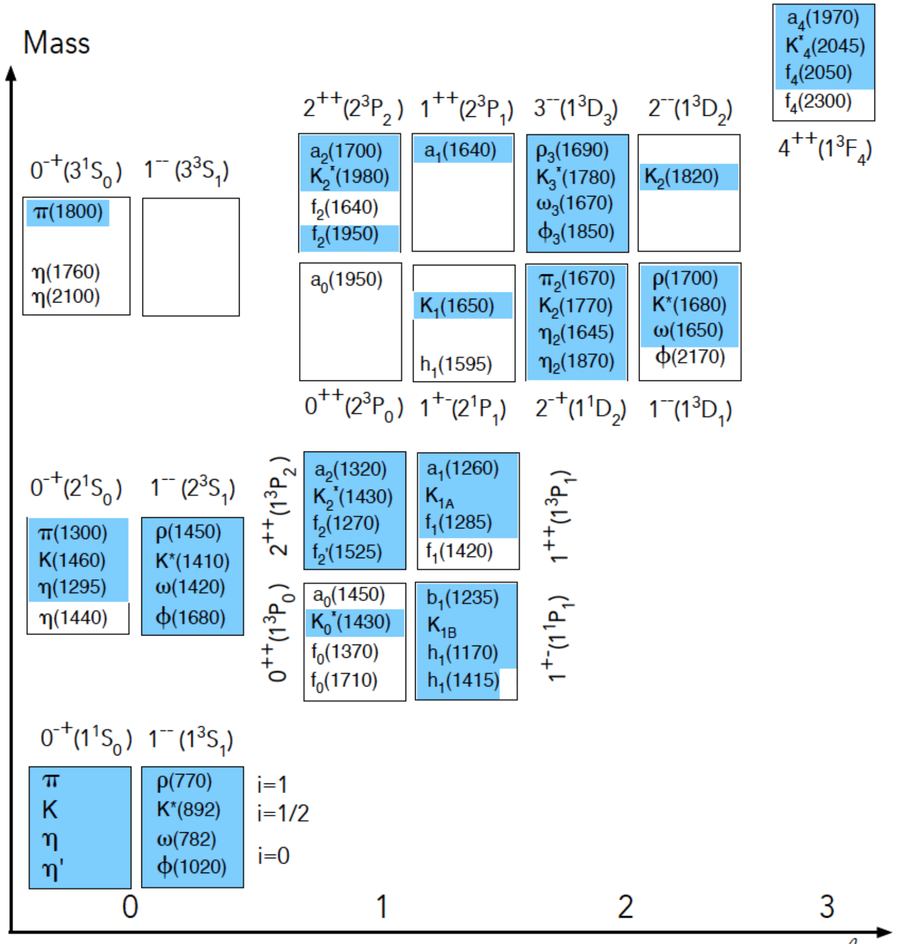

[PDG 2022]

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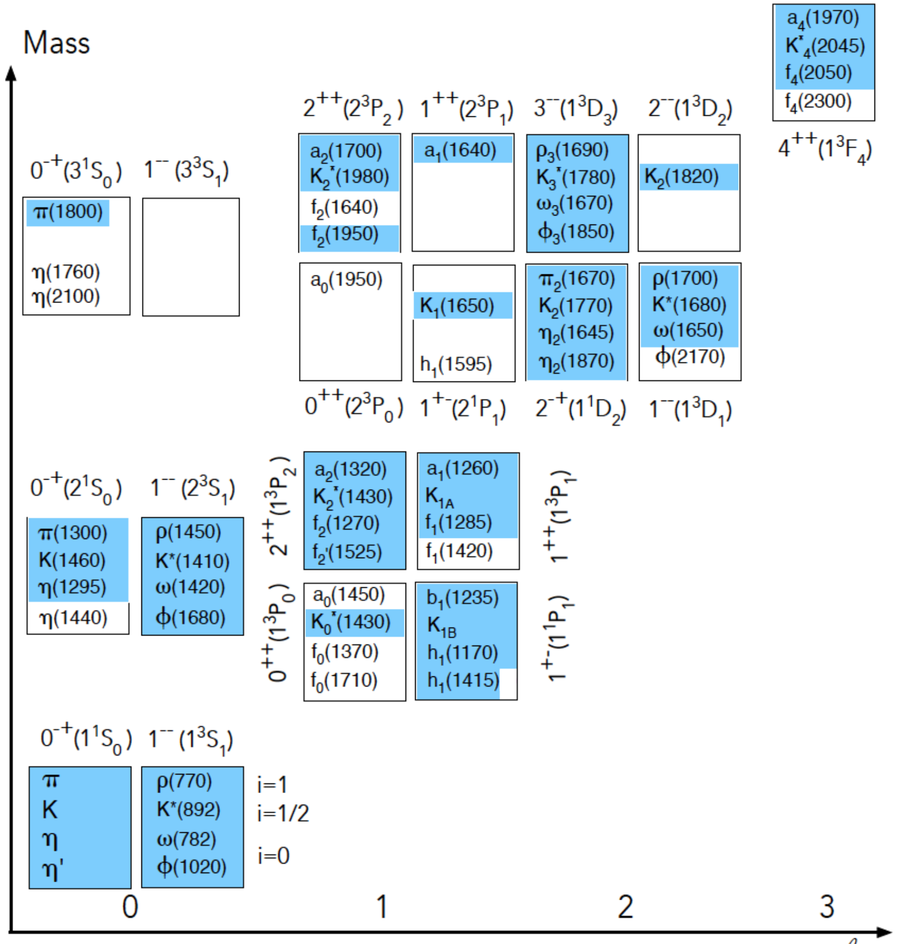
- **Tests of the Standard Model in meson weak decays**

CP violation in $D^0 \to K^+K^-/\pi^+\pi^-$ decays

$$\Delta a_{CP}^{\rm dir} = (-15.7 \pm 2.9) \times 10^{-4}$$ **[LHCb, 2019]**



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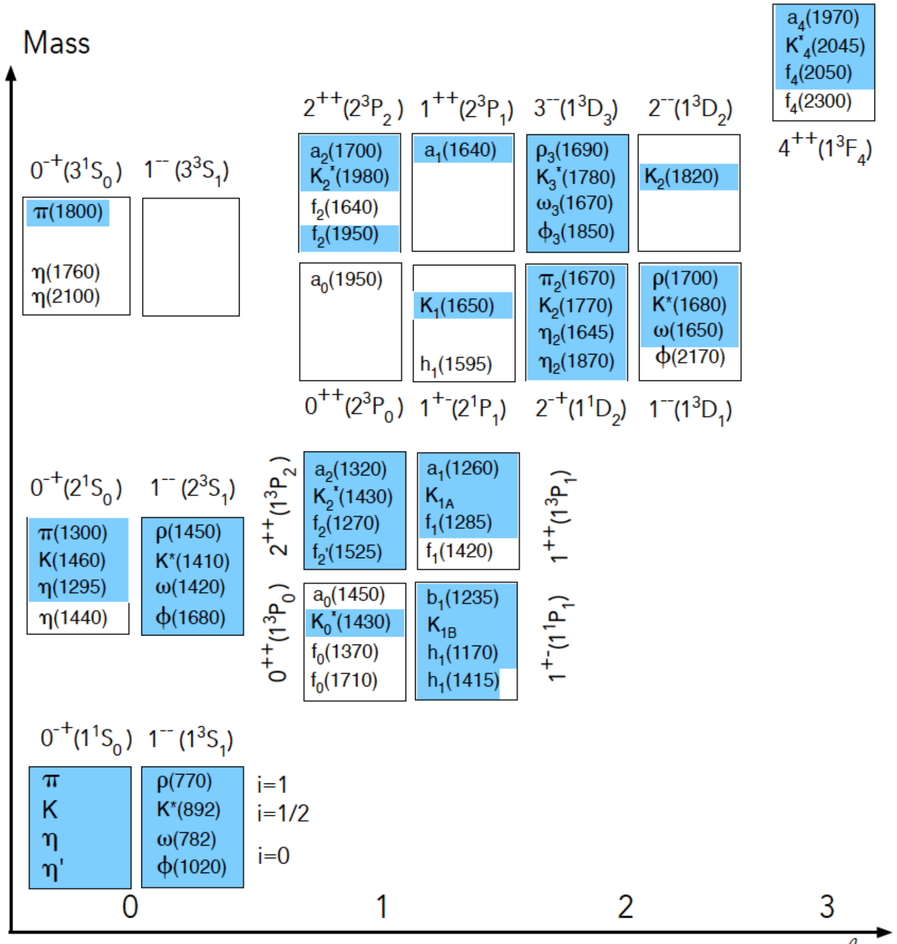
- **Nuclear interactions**
  - ➡ **Nucleon resonances:** $N(1440) \to \Delta\pi \to N\pi\pi$
  - ➡ **Binding energies and properties of nuclei**



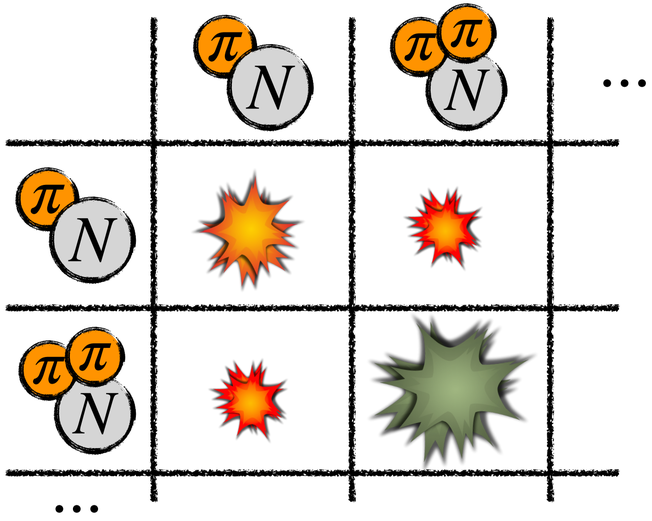
[PDG 2022]

Towards the QCD S-Matrix

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \text{out} | \mathcal{S} | \text{in} \rangle$$

Lattice QCD → QCD S-matrix



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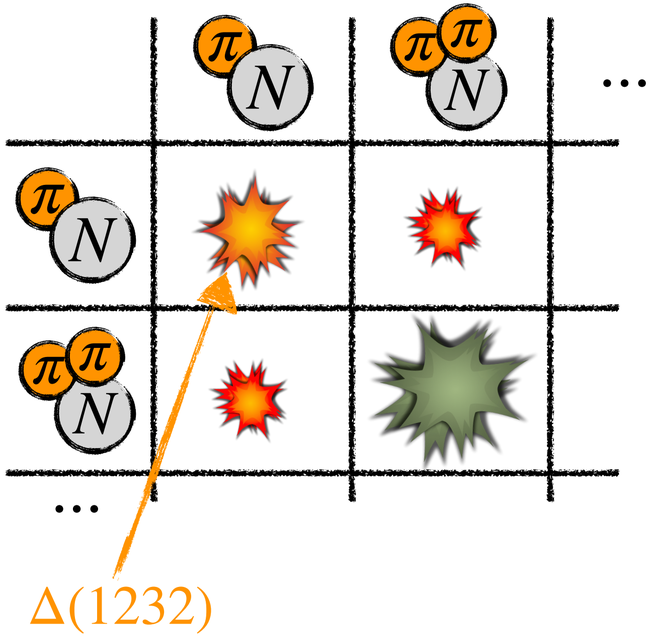

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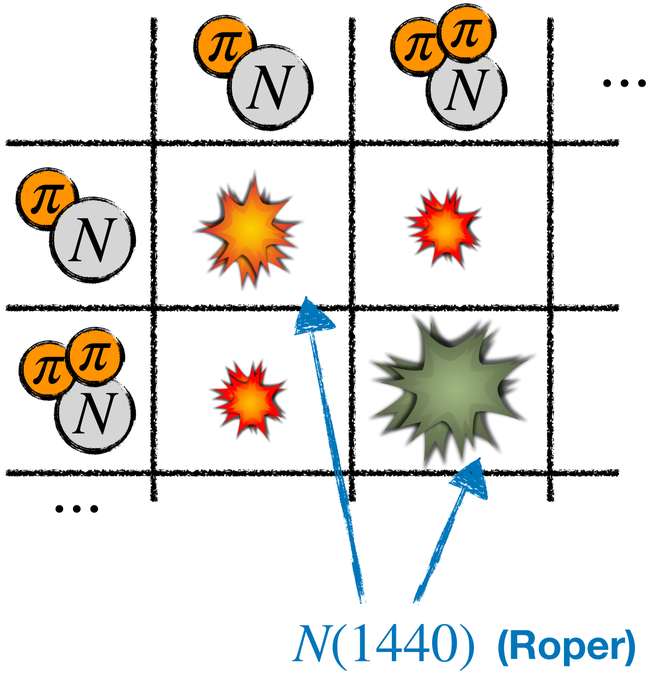

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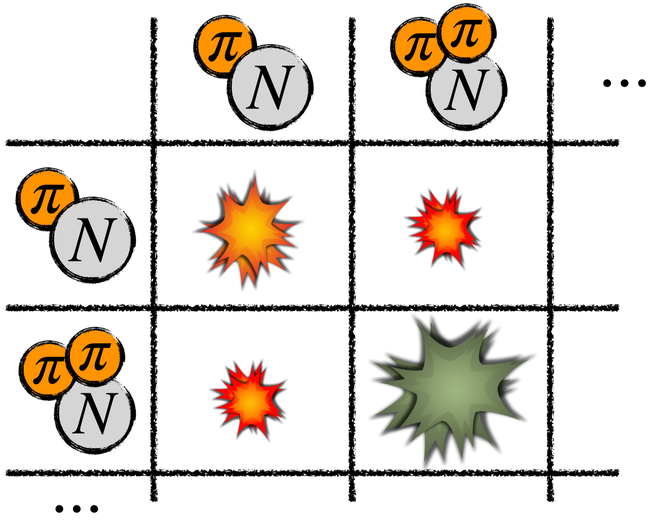
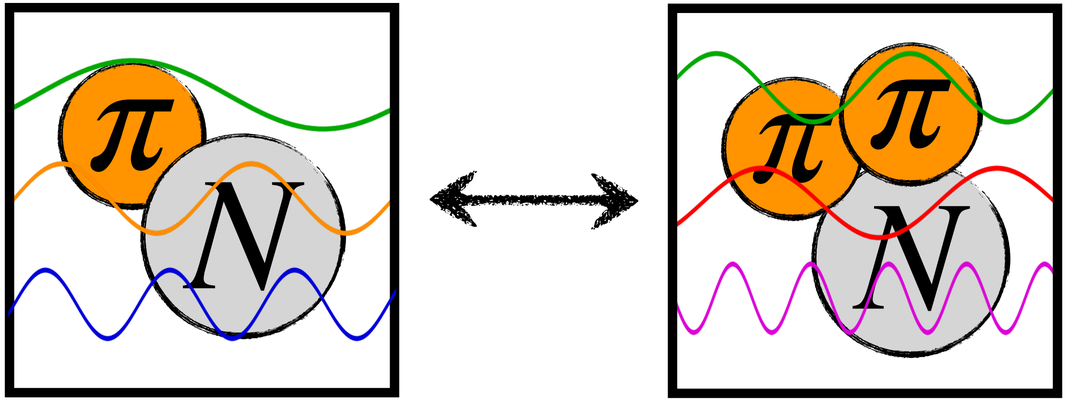


- However, the finite volume distorts multi-particle interactions



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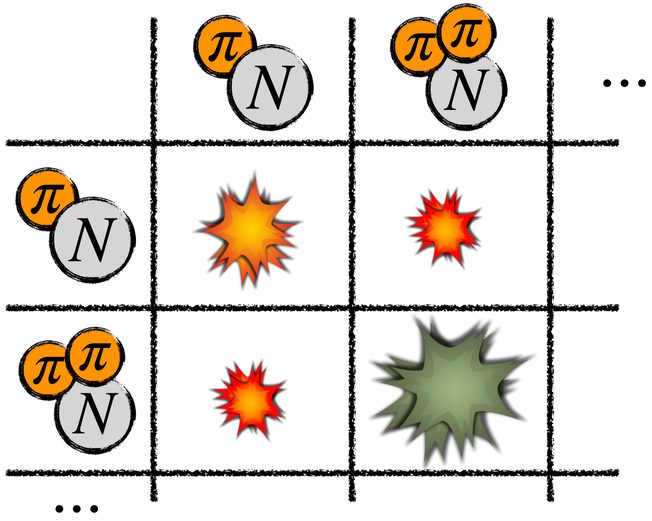

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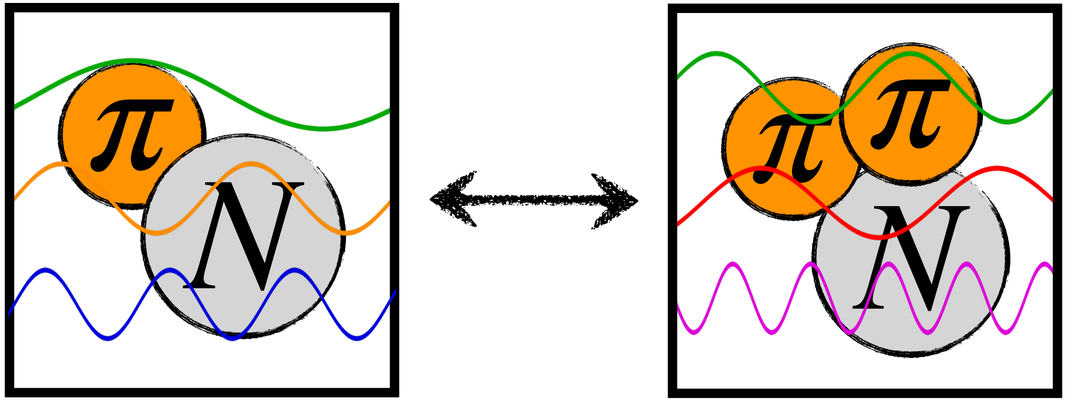
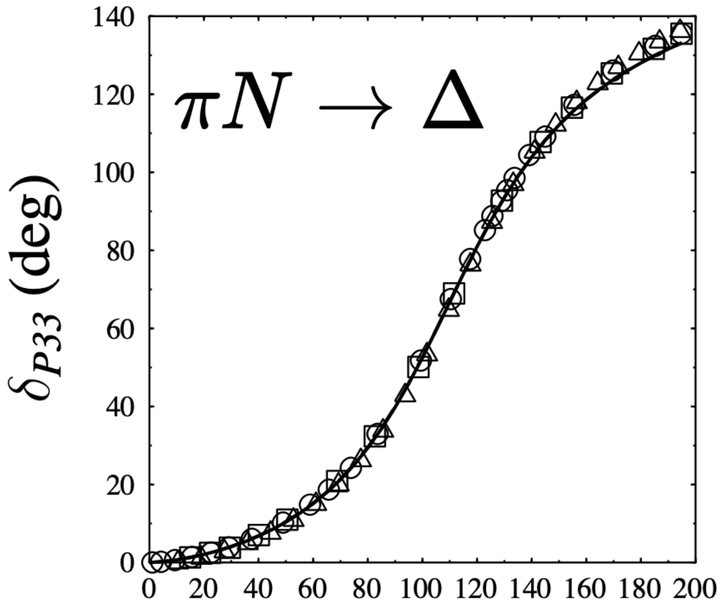


finite-volume formalism(s)

Infinite vs finite volume

Experiments

- Asymptotic states
- Direct access to scattering amplitudes



[Ellis, Tang (arXiv:9609459)]

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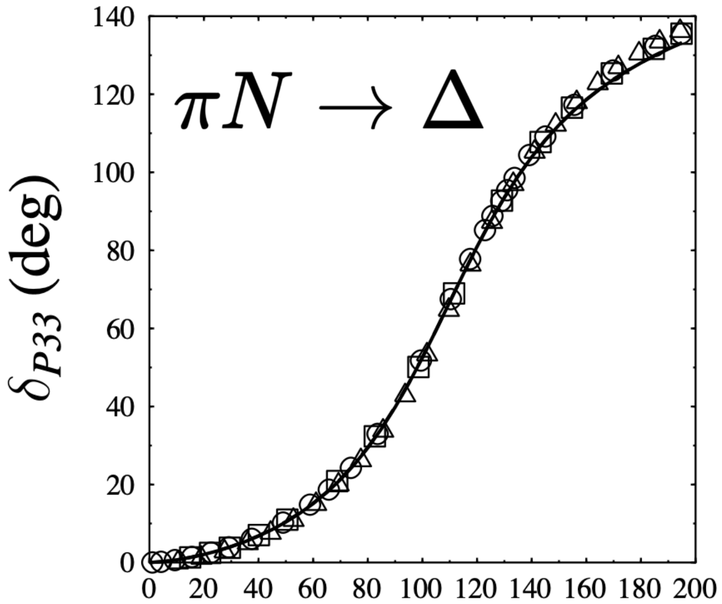

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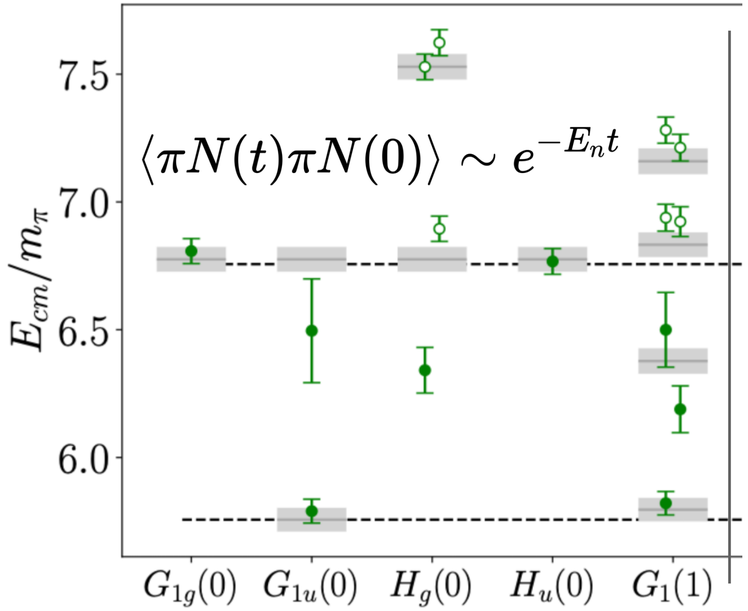
Lattice QCD

- Euclidean time
- Stationary states in a box



[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867]

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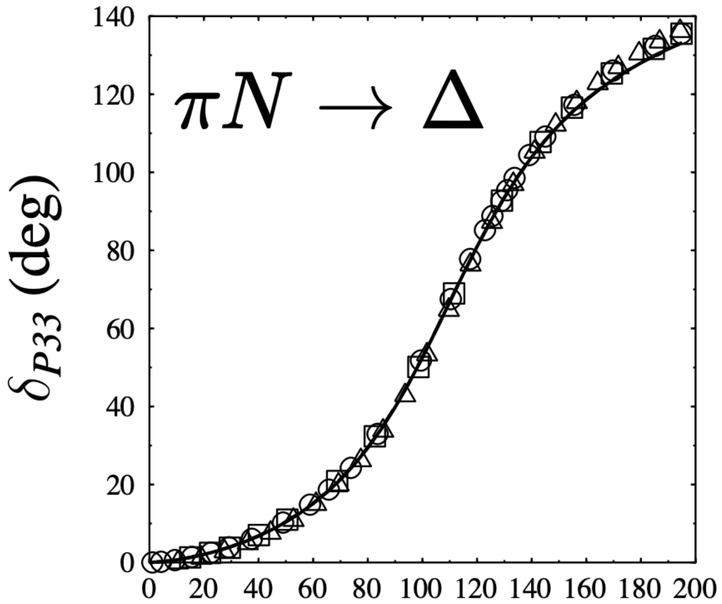

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Quantization condition

[Lüscher, 89']

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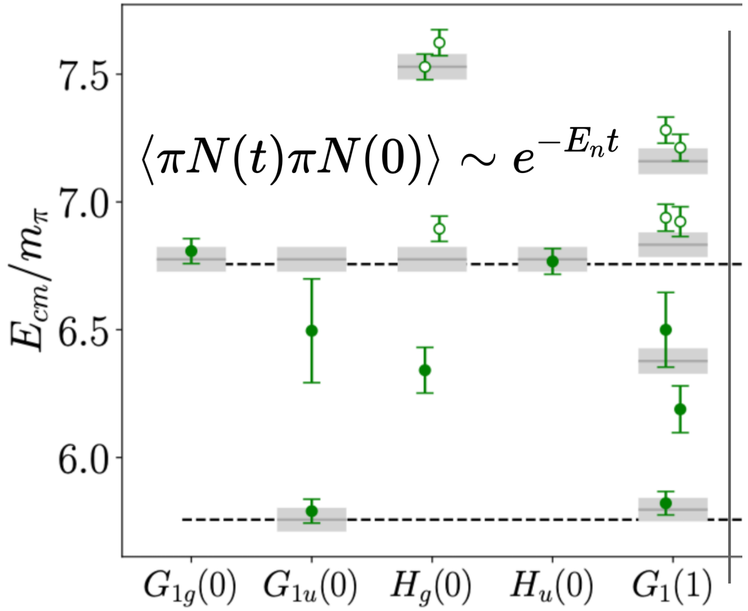

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Spectroscopy from Lattice QCD

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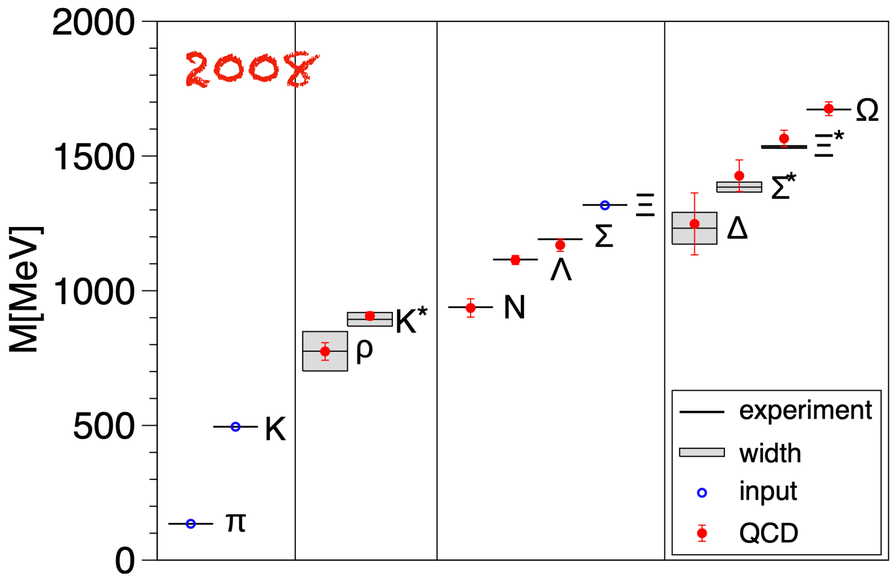

[BMW collaboration]

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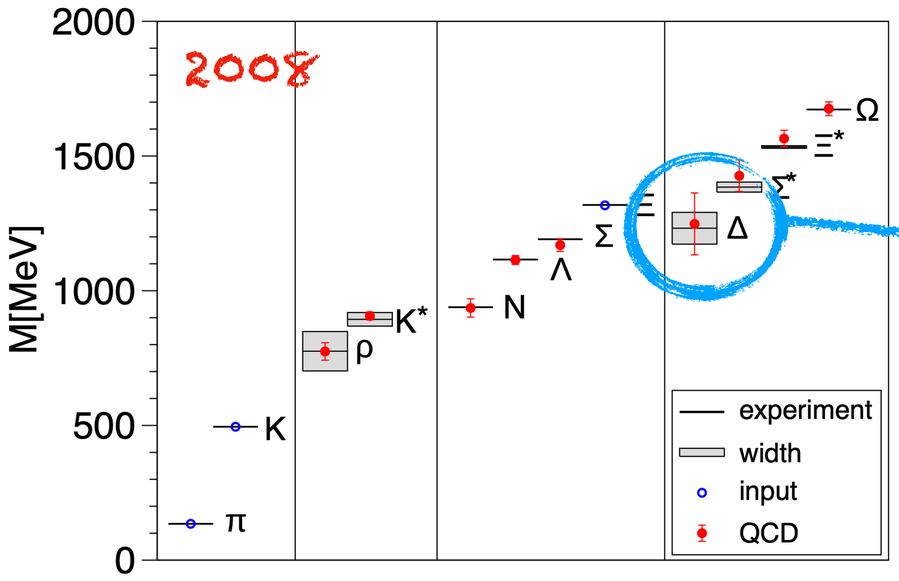

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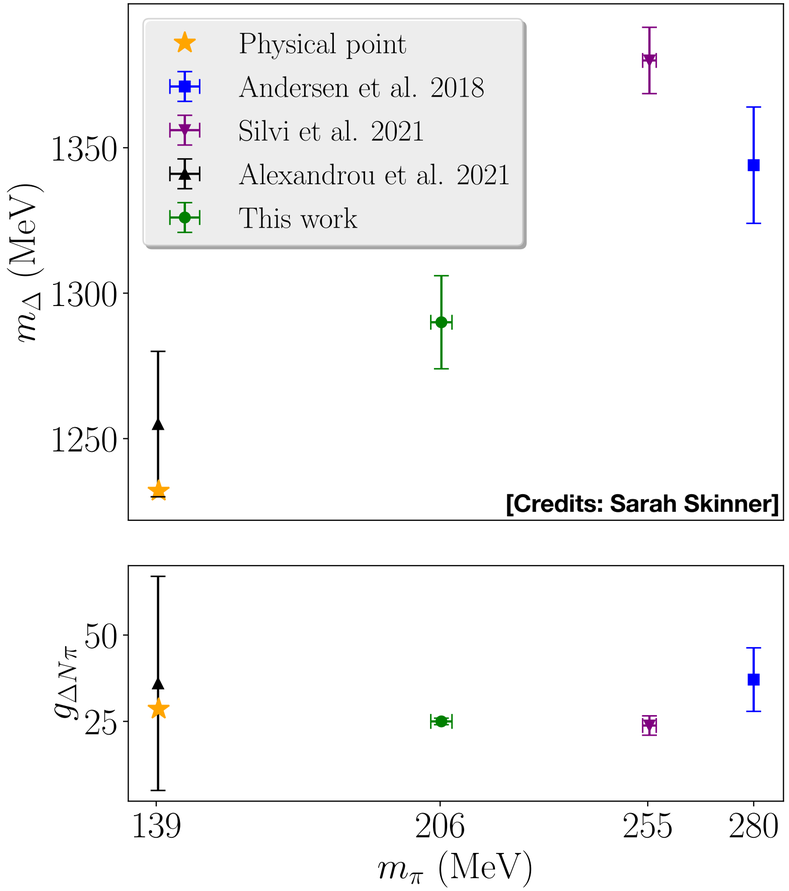

[Credits: Sarah Skinner]

[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867]

- ☑ State-of-the-art calculations can determine widths and masses accurately!

[see plenary by Liuming Liu]

$\mathcal{O}(40)$ talks at this conference!

Outline

1. Two-body interactions in finite-volume
2. Three-particle processes in finite volume
3. Outlook: towards many-hadron physics

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2. Three-particle processes in finite volume
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! Disclaimer: not a review talk. Apologies for missing references.

Two particles in finite-volume

The Lüscher method

- Indirect connection between the spectrum and the two-particle scattering amplitude [Lüscher 89']

Two-particle Quantization Condition

$$\det_{\ell m}\left[\underbrace{\mathcal{K}_2(E_n)}_{\text{Scattering K-Matrix}} + \underbrace{F^{-1}(E_n, \vec{P}, L)}_{\text{Known kinematic function}}\right]\Bigg|_{E=E_n} = 0$$

"QC2"

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K-matrix parametrized in terms of phase shift

$$\mathcal{K}_2^{\ell} = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

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Finite-volume information

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2 - q^2}$$

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- Fully general formalism (nondegenerate, spin, multi-channel):

[Rummukainen, Gottlieb, NPB 1995] [Kim, Sachrajda and Sharpe, NPB 2005]
[Bernard et al, JHEP 2008] [Davoudi, Savage, PRD 2011]
[Leskovec, Prevlosek, PRD 2012] [Göckeler et al, PRD 2012]
[Briceño, PRD 2014]

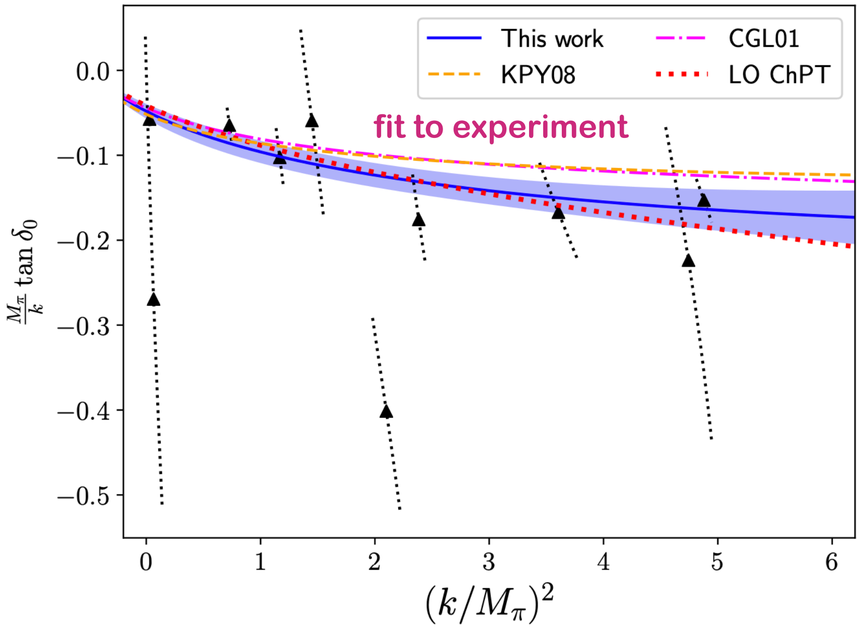
[see also talk by A. Baiao Raposo]

- See also the HALQCD method:
[Talk by S. Aoki]

Towards the physical point

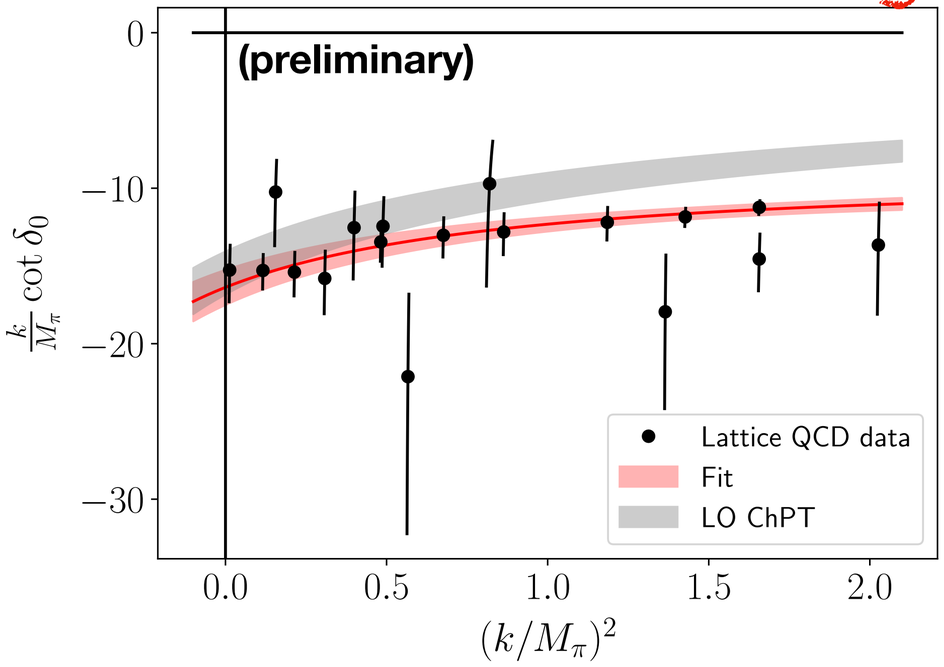
- Some systems already being studied at the physical point!

I=2 ππ scattering



[Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC), EPJC 2021]

I=3/2 πK scattering



[Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe (in preparation)]

- Several other physical point calculations:

[Fischer et al (ETMC), PLB 2021] $\rho(770)$

[Blum et al (RBC/UKQCD), PRD 2021] $(\pi\pi)_{I=2,0}$

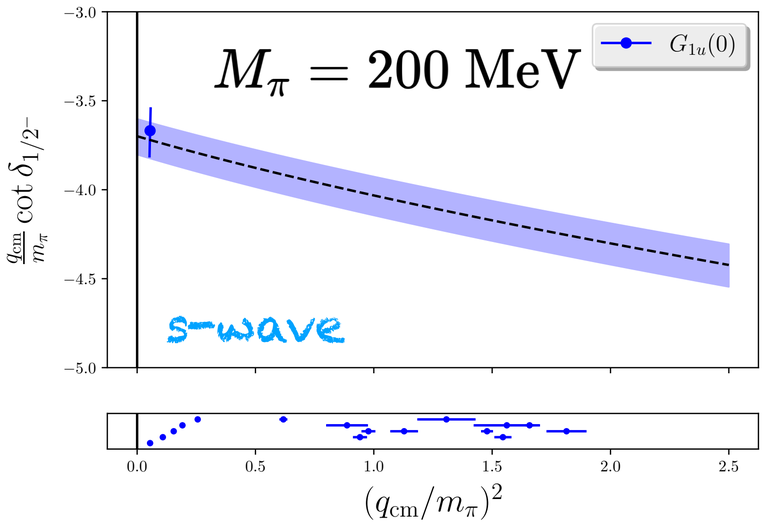
- Talks at this conference of scattering at the physical point:

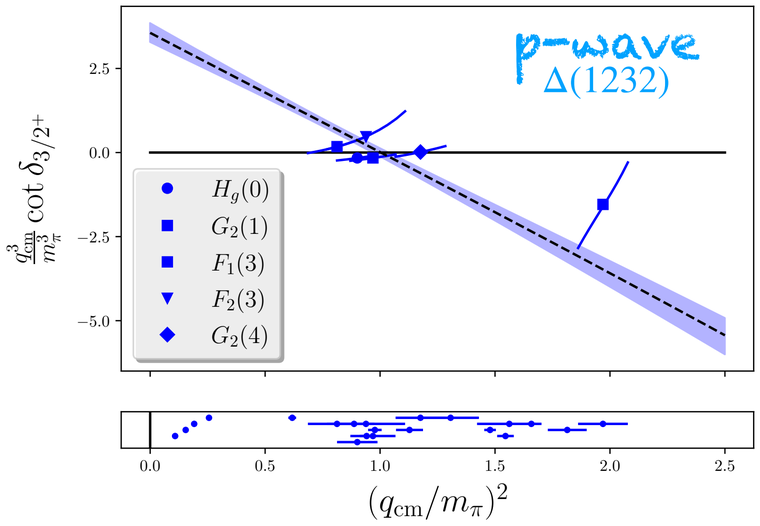
[R. Mukherjee], [N. Pitanga], [S. Paul]

The πN scattering amplitude

- The present frontier is scattering with nucleons ▶ See also talks by: [A. Nicholson], [J. Green], [F. Pittler], [M. Wagman]

Isospin 3/2





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$$\frac{m_\Delta}{m_\pi} = 6.257(35), \quad g_{\Delta N\pi} = 14.41(53)$$

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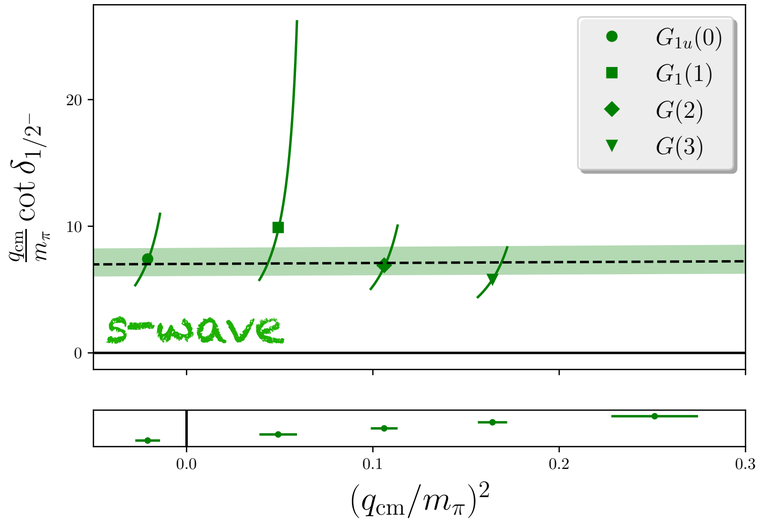

$$m_\pi a_0^{1/2} = 0.142(22)$$

[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867] [Talk by A. Walker-Loud]

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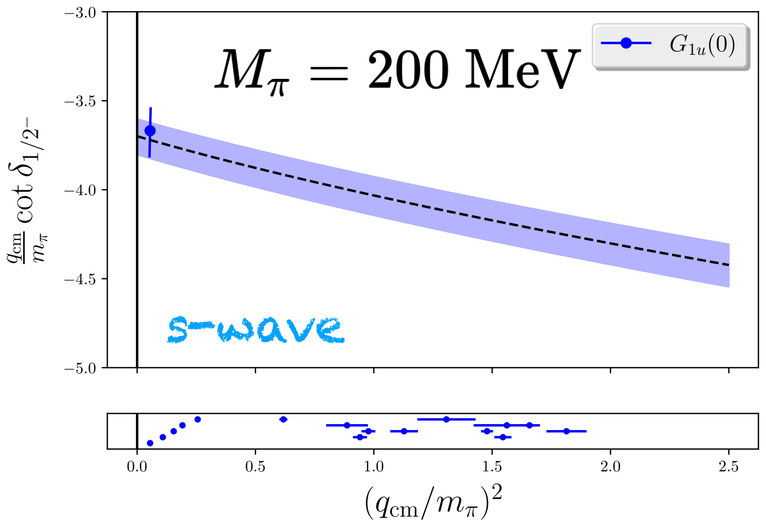

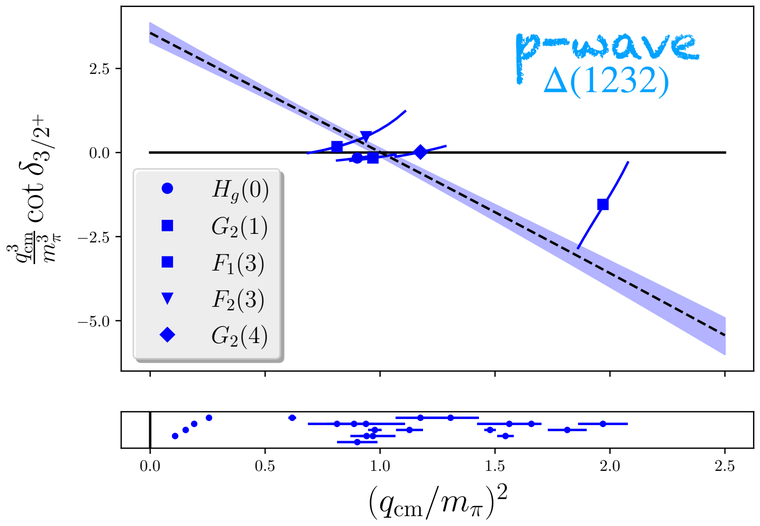

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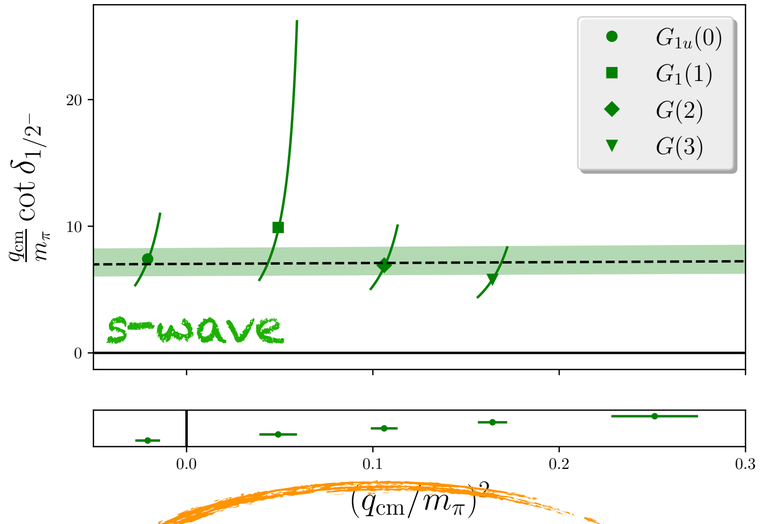

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First determination of scattering lengths in dynamical lattice QCD!

[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867] [Talk by A. Walker-Loud]

Matrix elements and form factors

- Finite-volume calculations also distort other matrix elements with two-body states

Decay processes

&

matrix elements

[See also talk by F. Ortega Gama]

$K \to \pi\pi$

[Lellouch, Luscher, Com. Math. Phys 2001]

$\pi + \gamma^* \to \pi\pi$

[Briceño, Hansen, Walker-Loud, PRD 2015]

$\rho + \gamma^* \to \rho$

[Baroni, Briceño, Hansen, Ortega-Gama, PRD 2019]

$\gamma^*\gamma^* \to \pi\pi$

[Sherman, Ortega-Gama, Briceño, Jackura, PRD 2022]

[Talk by R. Briceño]

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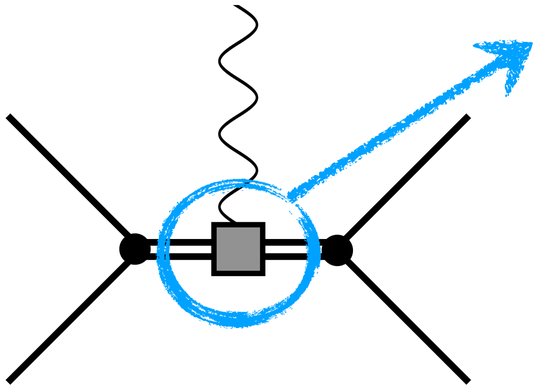
[Talk by R. Briceño]

- One approach is to use the Feynman-Hellmann theorem for resonance form factors:

[Lozano, Meißner, FRL, Rusetsky, Schierholz, arXiv:2205.11316]

[Talk by J. Lozano]



Form Factor

"$\langle\rho|V_0|\rho\rangle$"

$$F \propto \left.\frac{dP_R^0}{de}\right|_{e=0}$$

pole position in an external field

coupling to the external field

Three particles in finite volume

Why three particles?

- The two-body formalism is restricted to few interesting resonances
  - Even the ρ resonance couples (in principle) to four pions!

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Resonance	$I_{\pi\pi\pi}$	J^P
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$h_1(1170)$	0	1^+
$\omega_3(1670)$	0	3^-
$\pi(1300)$	1	0^-
$a_1(1260)$	1	1^+
$\pi_1(1400)$	1	1^-
$\pi_2(1670)$	1	2^-
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$a_4(1970)$	1	4^+

(with $\geq 3\pi$ decay modes)

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- Many-body nuclear physics: 3N force, tritium nucleus

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- Many-body nuclear physics: 3N force, tritium nucleus
- CP violation: $K \to 3\pi, \quad K^0 \leftrightarrow \overline{K}^0$ [Talk by A. Jackura / R. Briceño]

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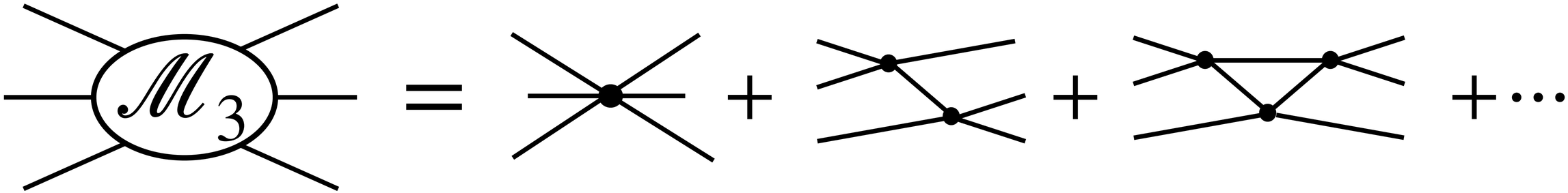
$a_4(1970)$	1	4^+

(with $\geq 3\pi$ decay modes)

Three-particle scattering

Qualitatively more complicated than the two-particle case!



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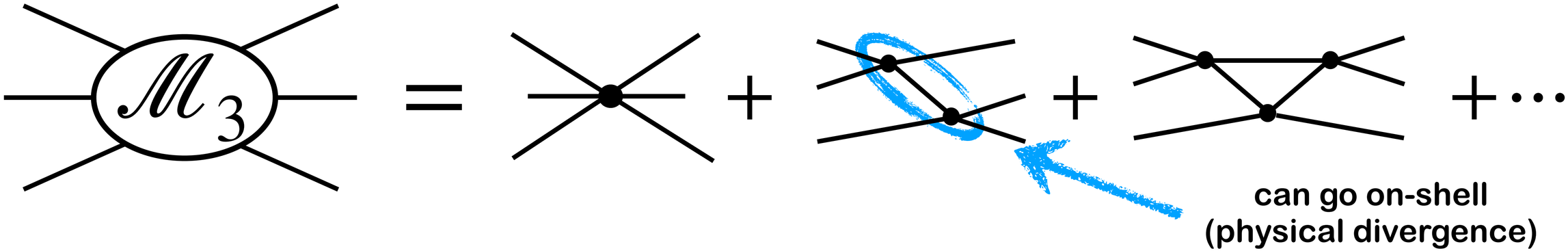

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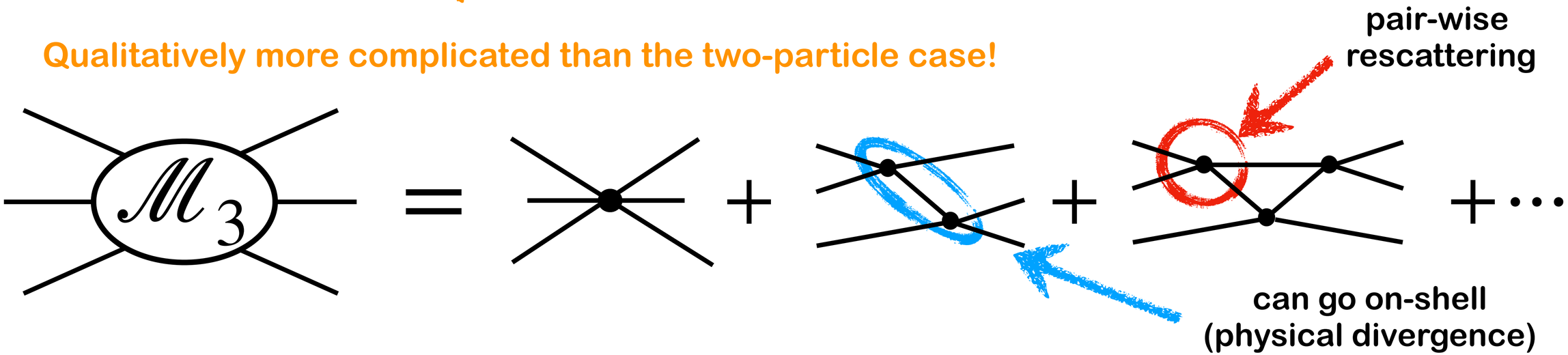

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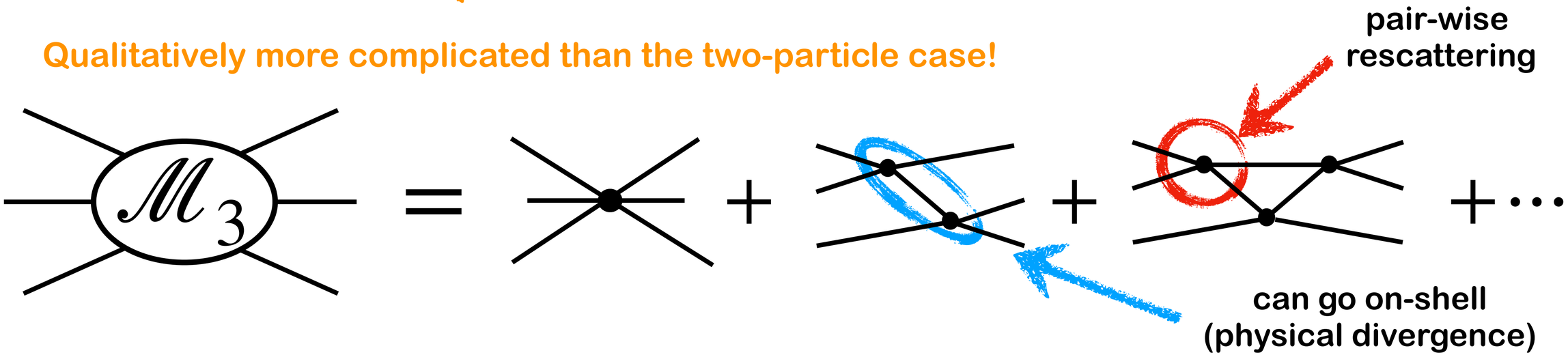
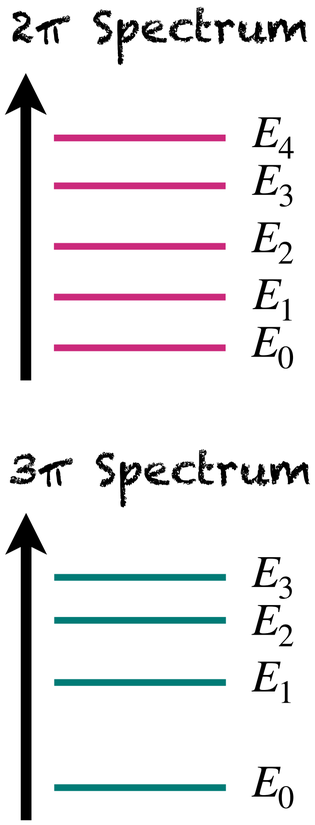


- Finite-volume formalism has been developed independently by three groups
  - Generic Relativistic Field Theory (RFT) [Hansen, Sharpe, PRD 2014 & 2015]
    [Talks by J. Baeza-Ballesteros, M. Hansen, S. Sharpe]
  - Non-Relativistic EFT (NREFT) [Hammer, Pang, Rusetsky, JHEP 2017] x 2
    [Talk by F. Müller]
  - Finite-Volume Unitarity (FVU) [Mai, Döring, EPJA 2017]
    [Talk by M. Mai]

Equivalence has been established
[Jackura et al. PRD 2019],[Blanton, Sharpe, PRD 2020],
[Jackura (to appear)]

Relativistic three-particle formalism for identical particles

[Hansen, Sharpe, PRD 2014 & 2015]



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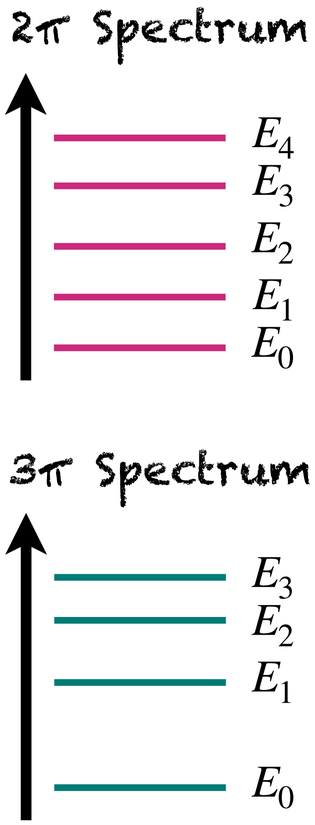

Quantization conditions

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

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Matrix indices describe three on-shell particles

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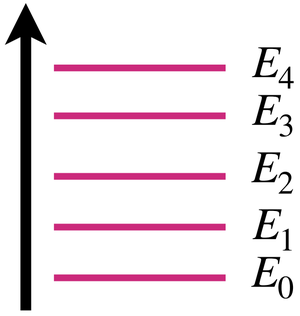

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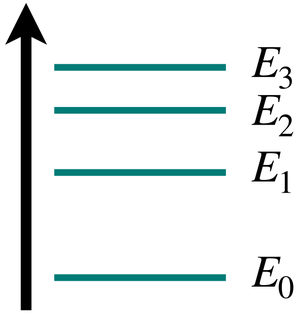

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Fit

K-matrices

$\mathcal{K}_2$

$\mathcal{K}_{df,3}$

Parametrize:

$$\mathcal{K}_2 = c_0 + c_1 k^2 + \ldots$$

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\left(\frac{s - 9m^2}{9m^2}\right) + \ldots$$

[Blanton, FRL, Sharpe, JHEP 2019]

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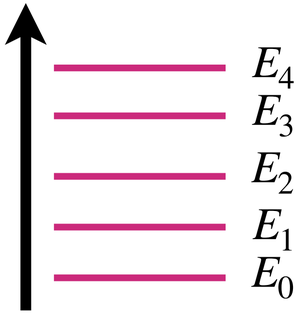

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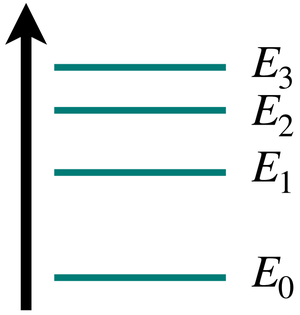

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Unitarity relations

Integral equations

[Briceño et al., PRD 2018]
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Scattering amplitudes

$\mathcal{M}_2$

$\mathcal{M}_3$

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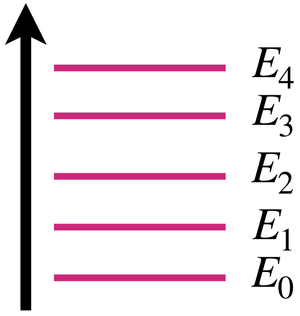

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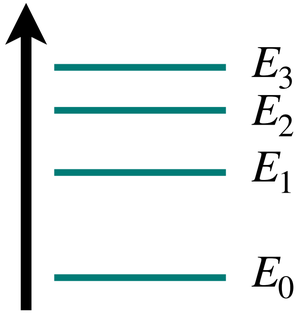

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Experiments

Applying the three-body formalism

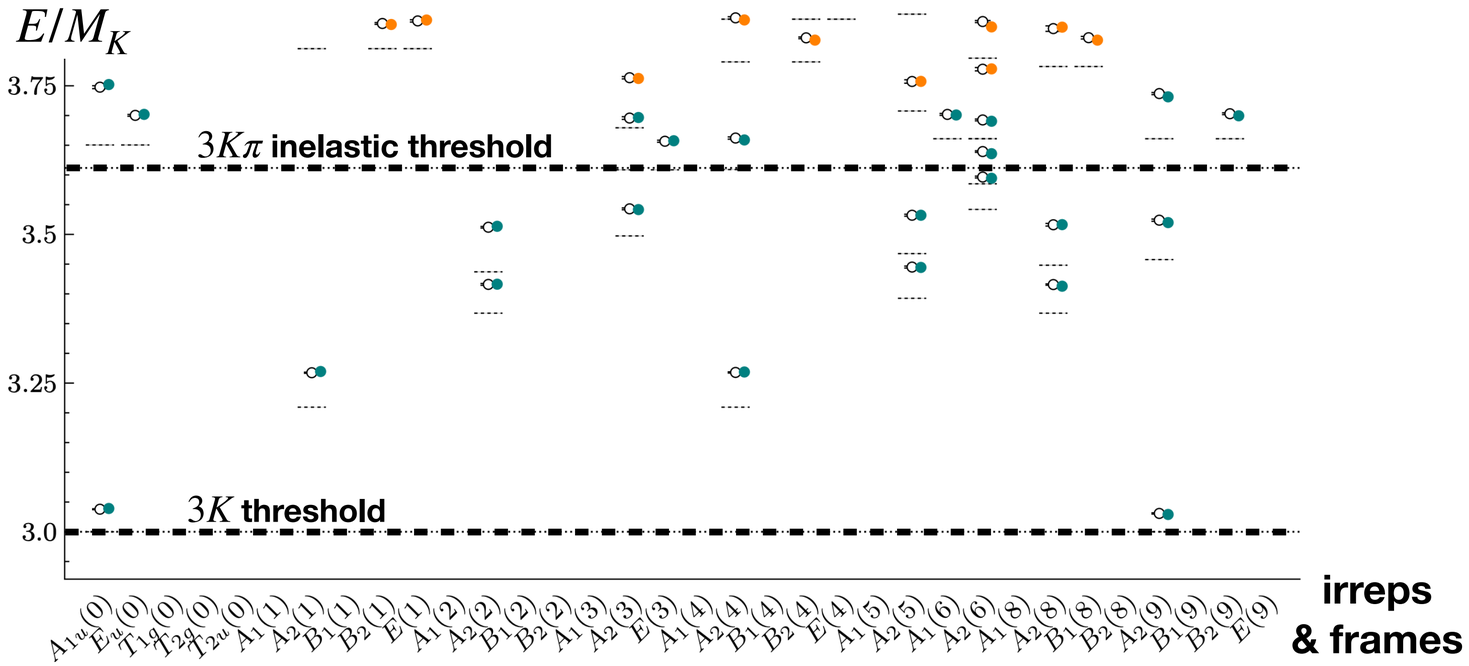
- **Three pions and three kaons at maximal isospin have been explored by different groups**

  [Alexandrou et al, PRD 2020], [Blanton et al., PRL 2020 & JHEP 2021], [Brett et al, PRD 2021], [Culver et al, PRD 2021], [Fischer et al, EPJC 2021], [Hansen et al, PRL 2021], [Mai et a PRL 2019 & 2021]

- **Requires large sets of energy levels obtained using variational techniques**

  Using stochastic LapH method [Morningstar et al, PRD 2011]

$3K^+$ **energy levels**



irreps & frames

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

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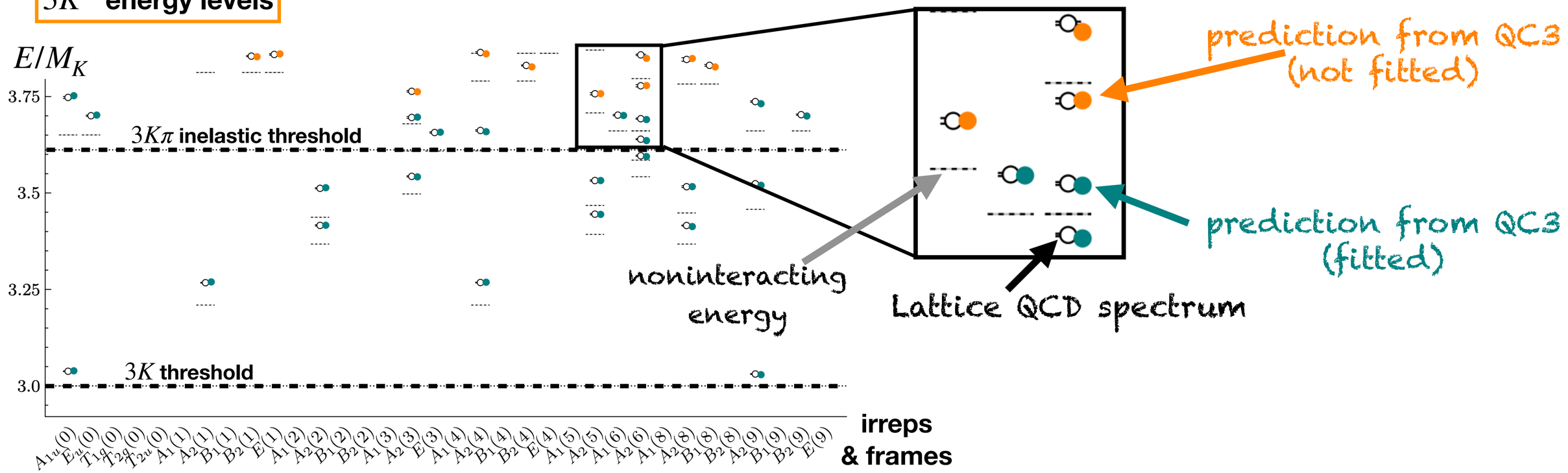

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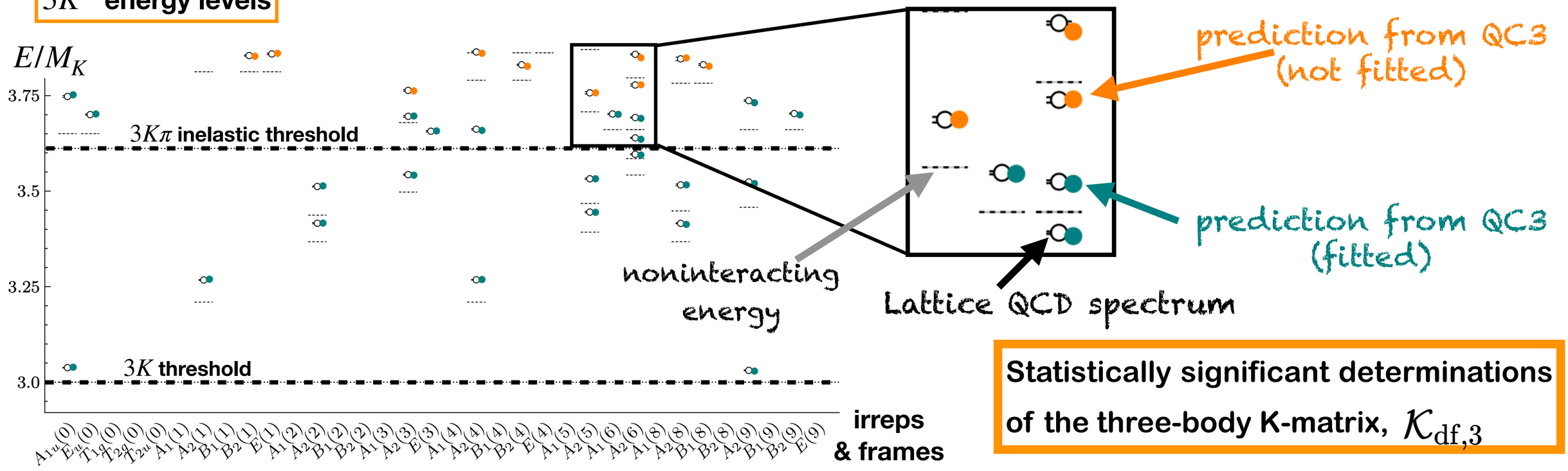

Statistically significant determinations of the three-body K-matrix, $\mathcal{K}_{\mathrm{df},3}$

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

$\mathcal{K}_{df,3}$ from Lattice QCD

Depend of CM energy

Angular dependence

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B,$$

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$$\Delta_B = \sum_{i,j=1}^{3} \tilde{t}_{ij}^2 - \Delta^2,$$

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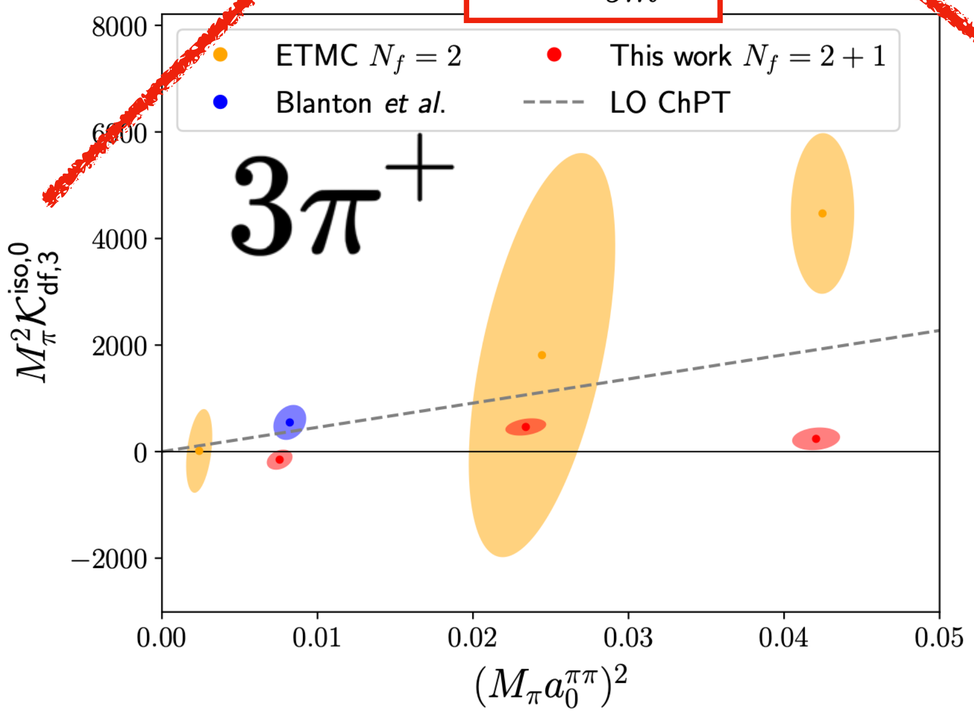

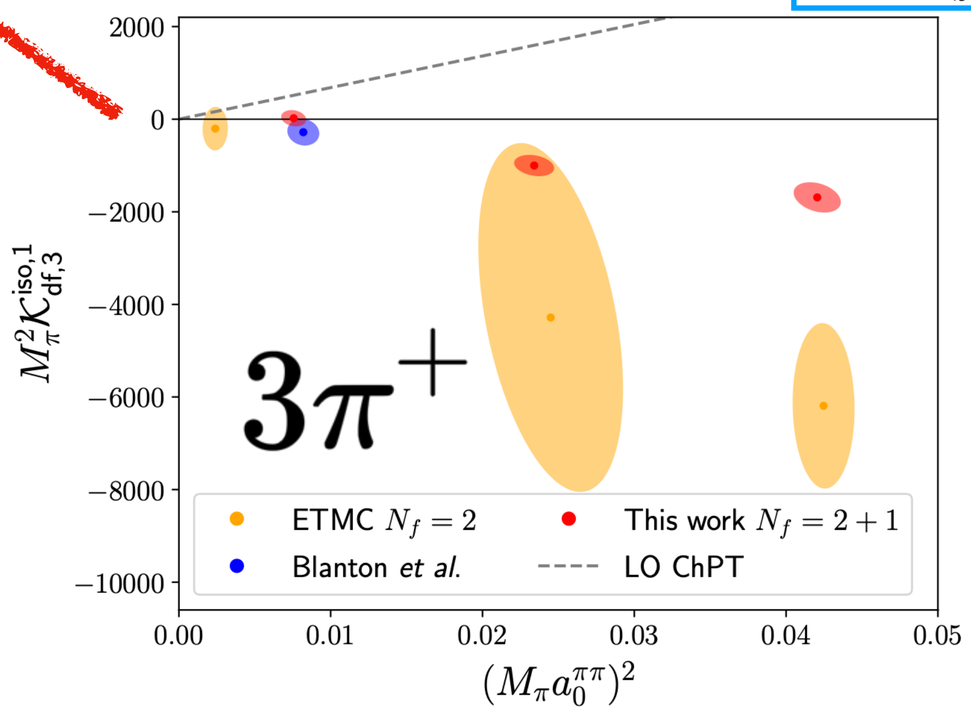

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

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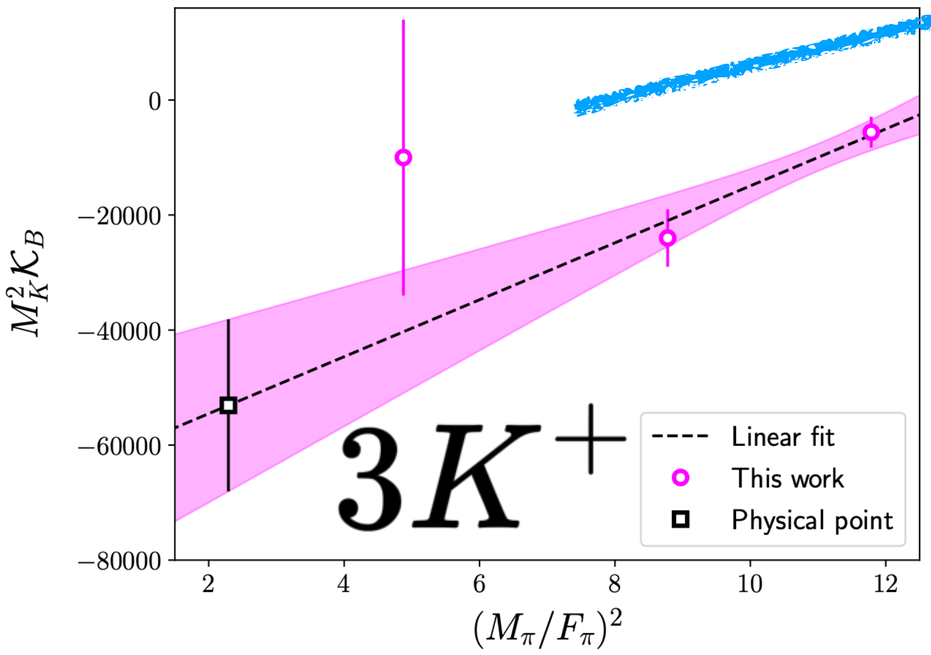
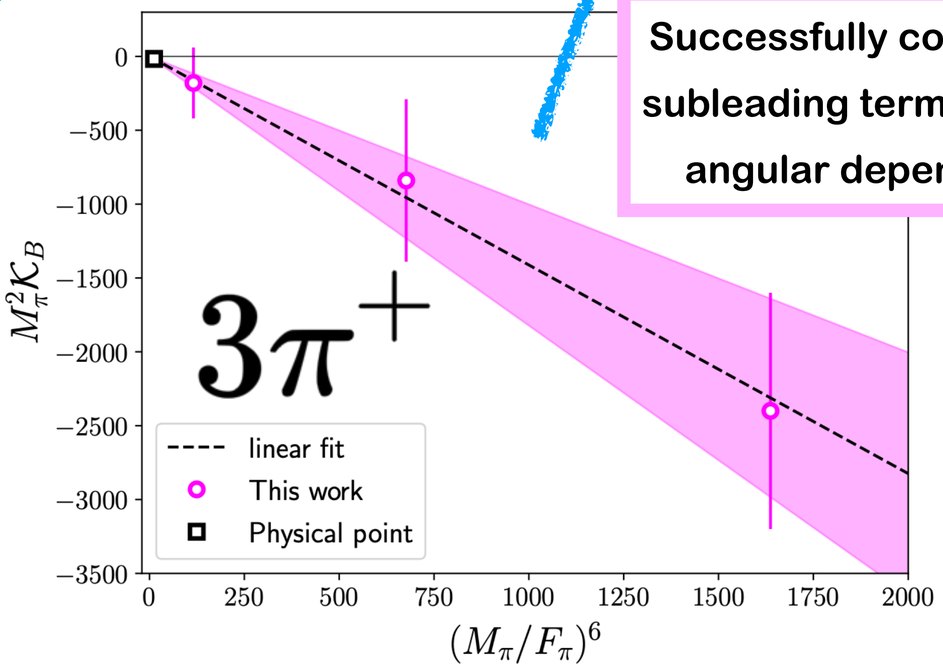
Successfully constrained subleading term including angular dependence!





[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

Nondegenerate systems

- Relevant three-body systems involve nonidentical particles ($\pi\pi N$)
- First step: formalism for three nonidentical scalars

e.g. $\pi^+\pi^0\pi^-$, $K^+K^+\pi^+$, $D_s^+D^0\pi^-$

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determinant runs over an additional "flavor" index

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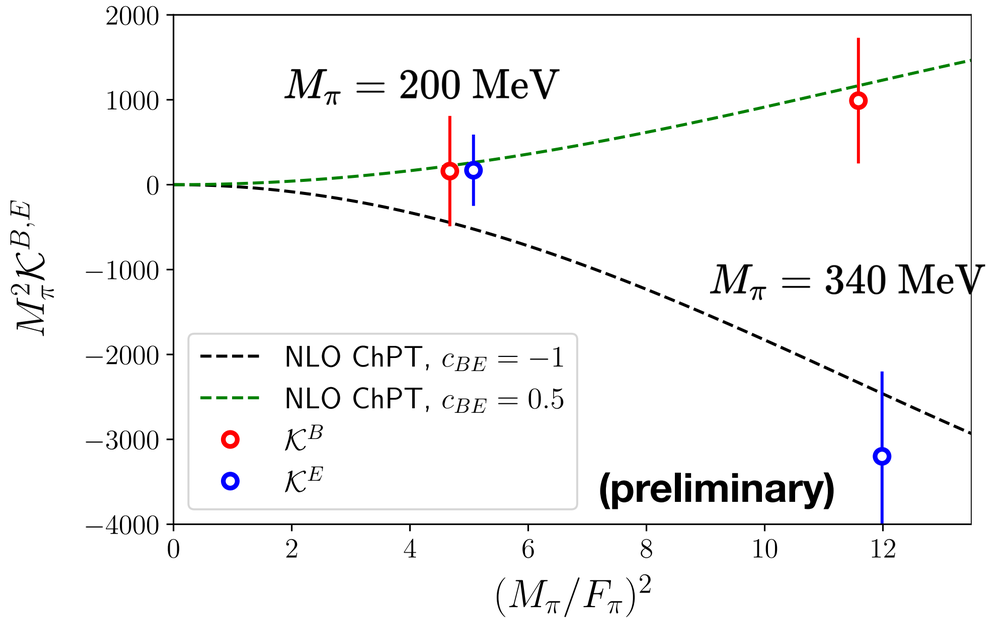
[Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe (in preparation)]

[Talk by S. Sharpe]



Three-body resonances

- Extract resonance properties on a toy model

$$\mathcal{L} = \sum_{i=0,1} \left(\partial_\mu \phi_i^\dagger \partial_\mu \phi_i + m_i \phi_i^\dagger \phi_i + \lambda_i \phi_i^4 \right) + \frac{g}{2} \phi_1^\dagger \phi_0^3 + h.c.$$

Induces transitions:

$$\phi_1 \to 3\phi_0$$

$$(M_1 > 3M_0)$$

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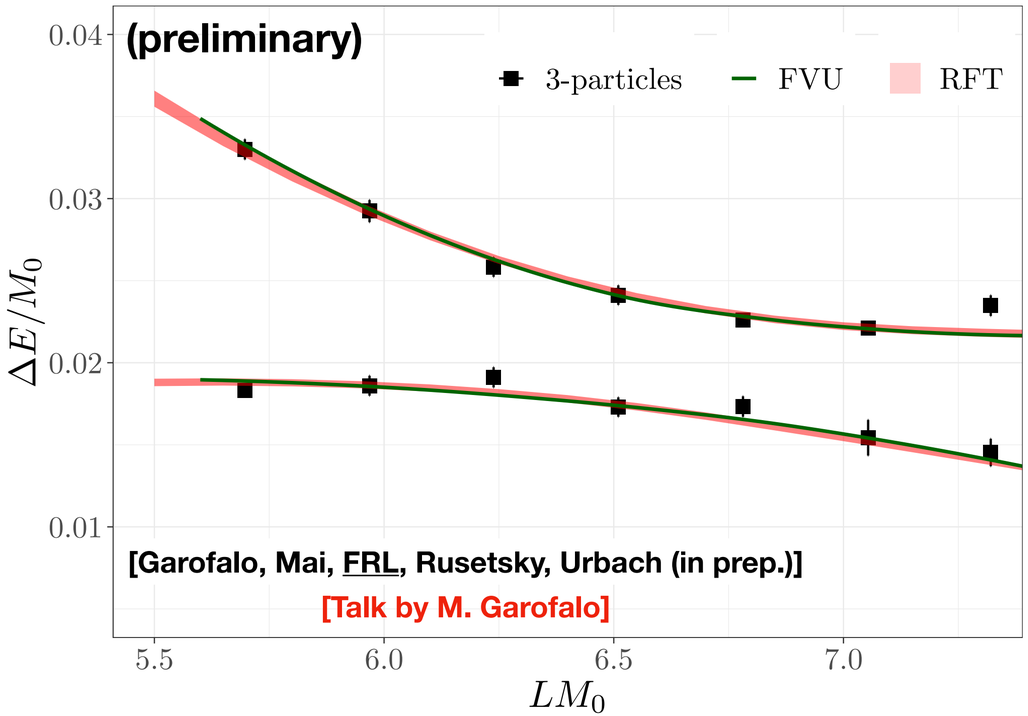

[Garofalo, Mai, FRL, Rusetsky, Urbach (in prep.)]

[Talk by M. Garofalo]

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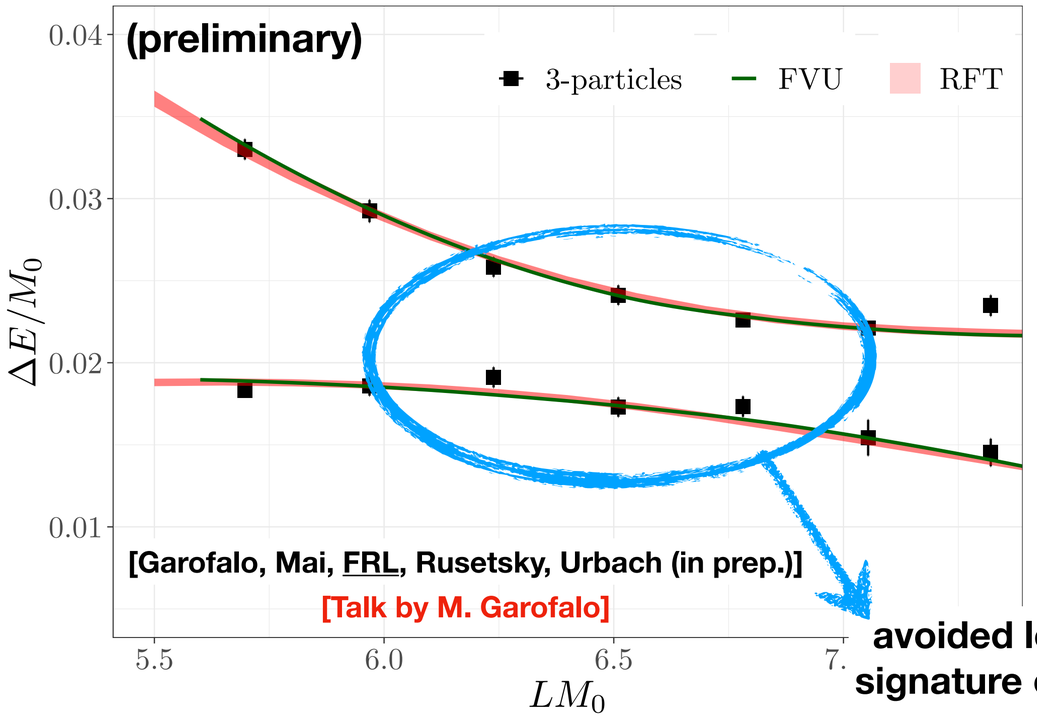

avoided level crossing:
signature of a resonance

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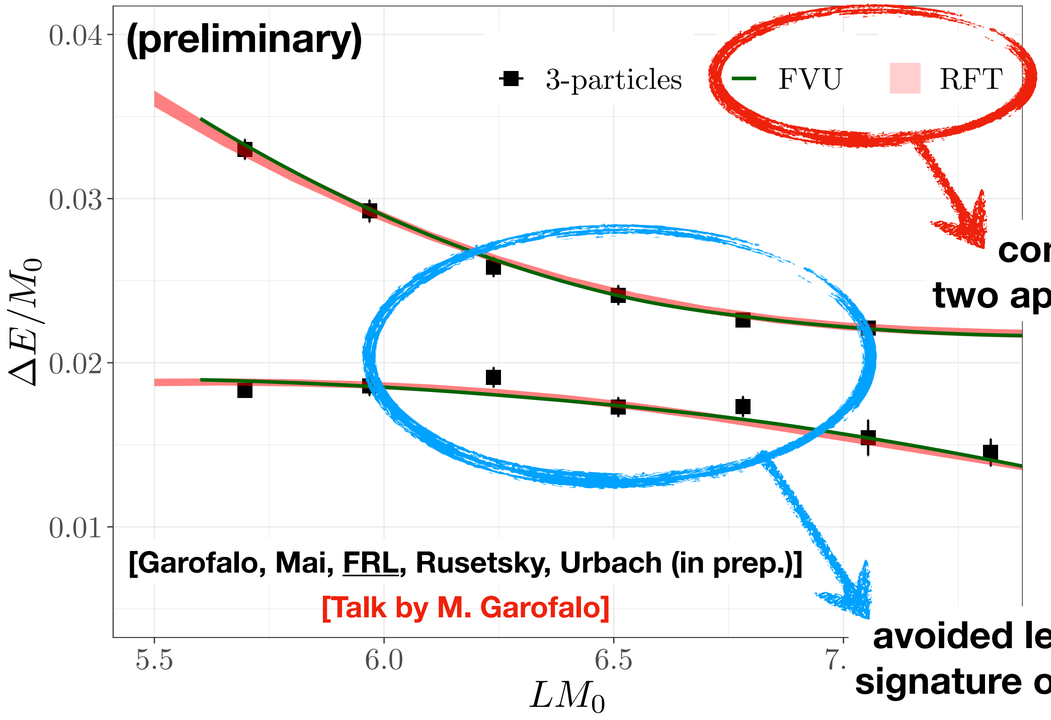

compare two approaches

avoided level crossing: signature of a resonance

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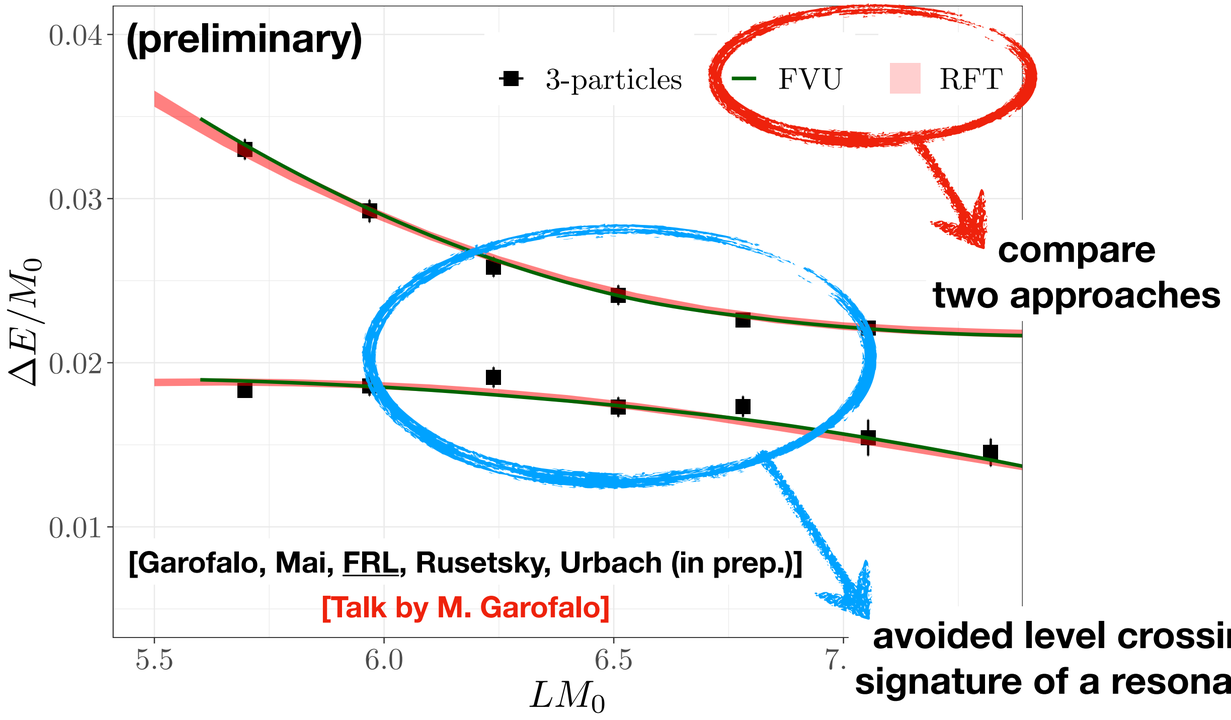
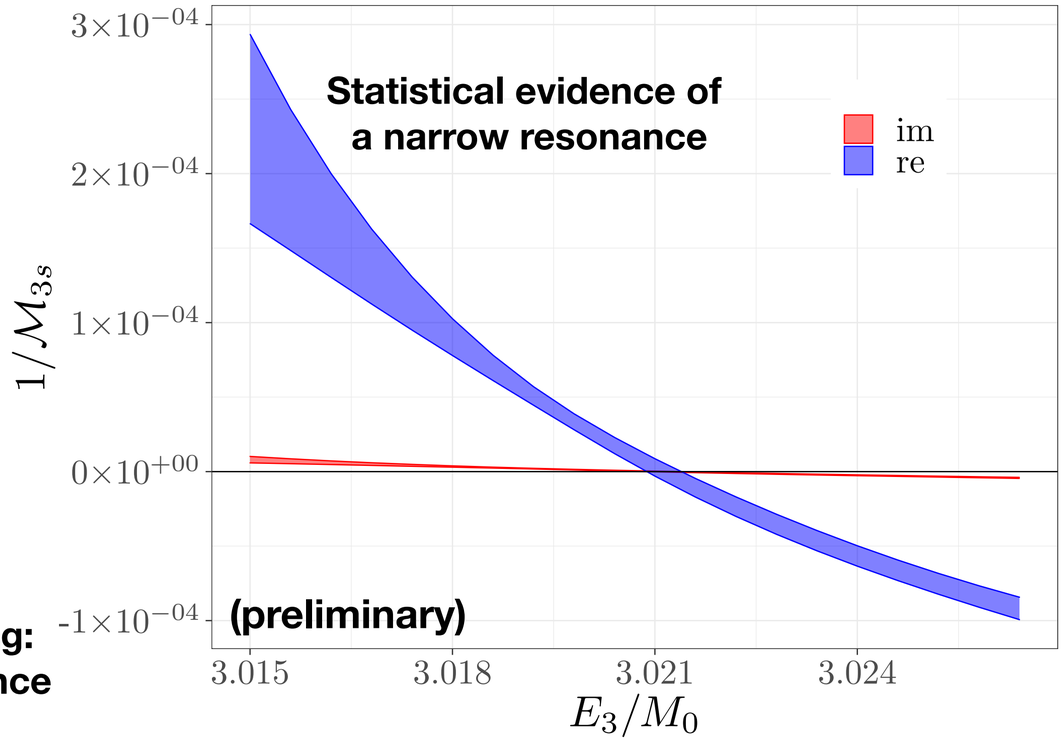




A three-particle era

- ☑ Many applications to simple system have been achieved or are under way
  - Three pions/kaons at maximal isospin and toy scalar theories

- Awaiting lattice QCD data to study resonances with three-particle decay channels, such as the ω(782) and the h_1(1170), as well as $K \to 3\pi$ decays.

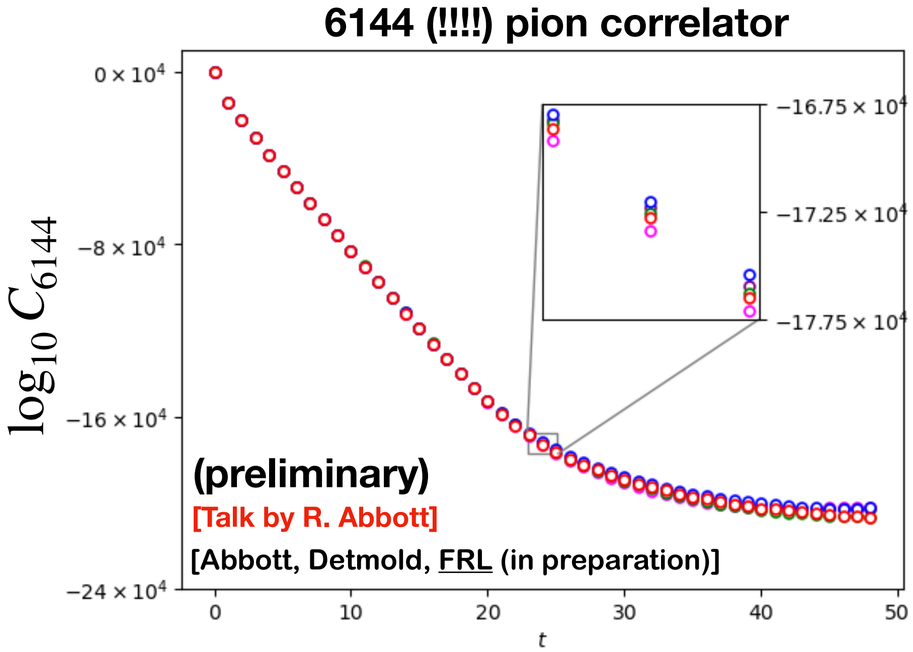
  [Hansen, Sharpe, FRL, JHEP 2020 & 2021] x 2, [Müller, Rusetsky, JHEP 2021]
  - First step in that direction with the $a_1(1260)$ [Mai et al (GWQCD), PRL 2021]

- However, some generalizations are still needed:
  - Three particles with spin [Draper, Hansen, Sharpe, FRL (in preparation)]
  - Particles with spin and 2-to-3 transitions $\longrightarrow$ $N(1440)$ "Roper resonance"
  - Matrix elements of currents with three-particle states [Talk by A. Jackura]

Towards many hadrons

Many-hadron interactions

- Need to compute many-hadron correlators.
  - Naive cost scales factorially with the number of particles
  - Smart algorithms can reduce the cost!



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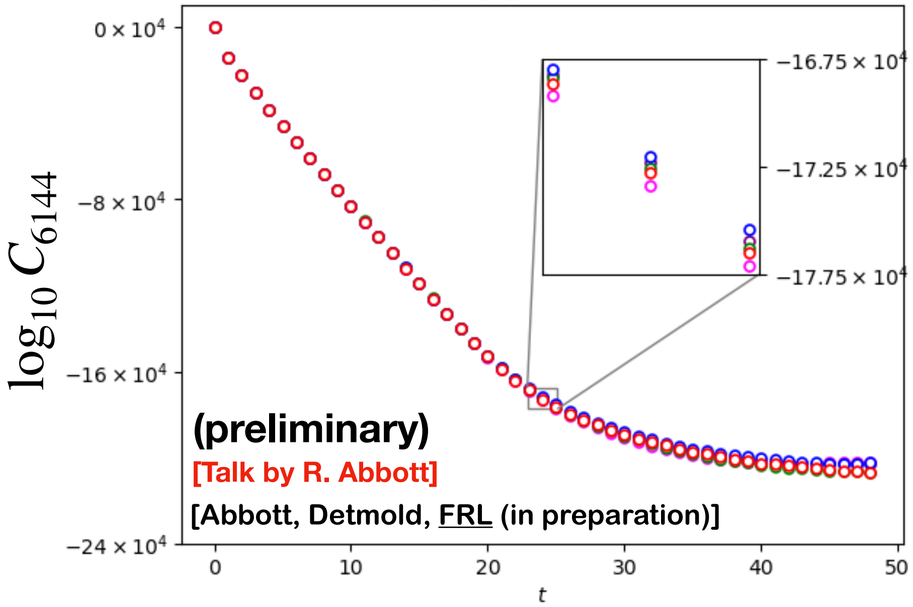

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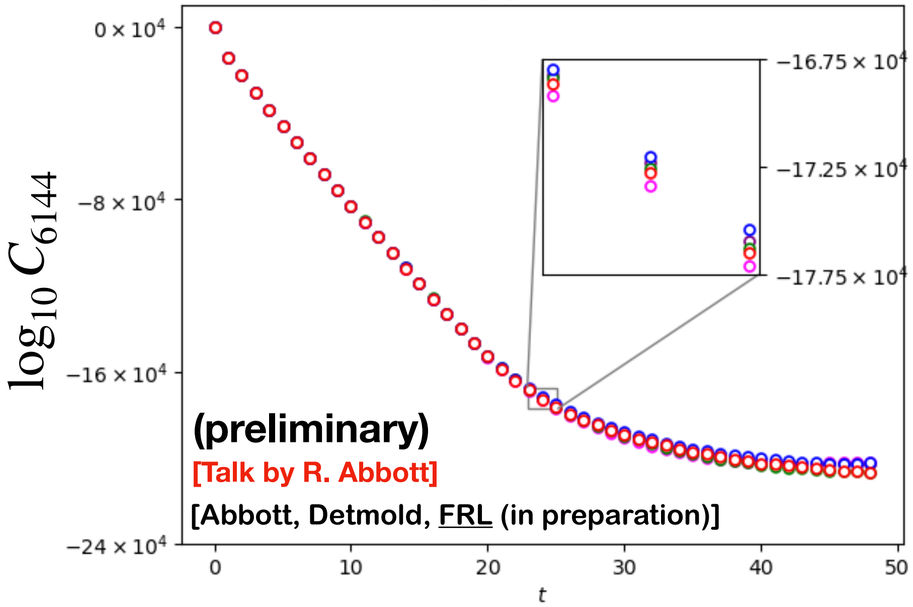

$$\text{finite-volume EFT}_{\not\pi}$$

$$\mathcal{L} = N^\dagger\left(iD_t + \frac{\mathbf{D}^2}{2M_N}\right)N - \frac{1}{2}\left[C_0\left(N^\dagger N\right)^2 + C_1\left(N^\dagger\vec{\sigma}N\right)^2\right] - \frac{D_0}{6}\left(N^\dagger N\right)^3 + \dots$$

Finite-Volume Pionless EFT

- Formulate EFTs for nucleons in a finite volume

$$H = -\frac{1}{2M_N}\sum_i \nabla_i^2 + \sum_{i<j} V_2(\mathbf{r}_{ij}) + \sum_{i<j<k} V_3(\mathbf{r}_{ij}, \mathbf{r}_{jk})$$

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Periodic Boundary Conditions

+ correlated gaussian Ansatz

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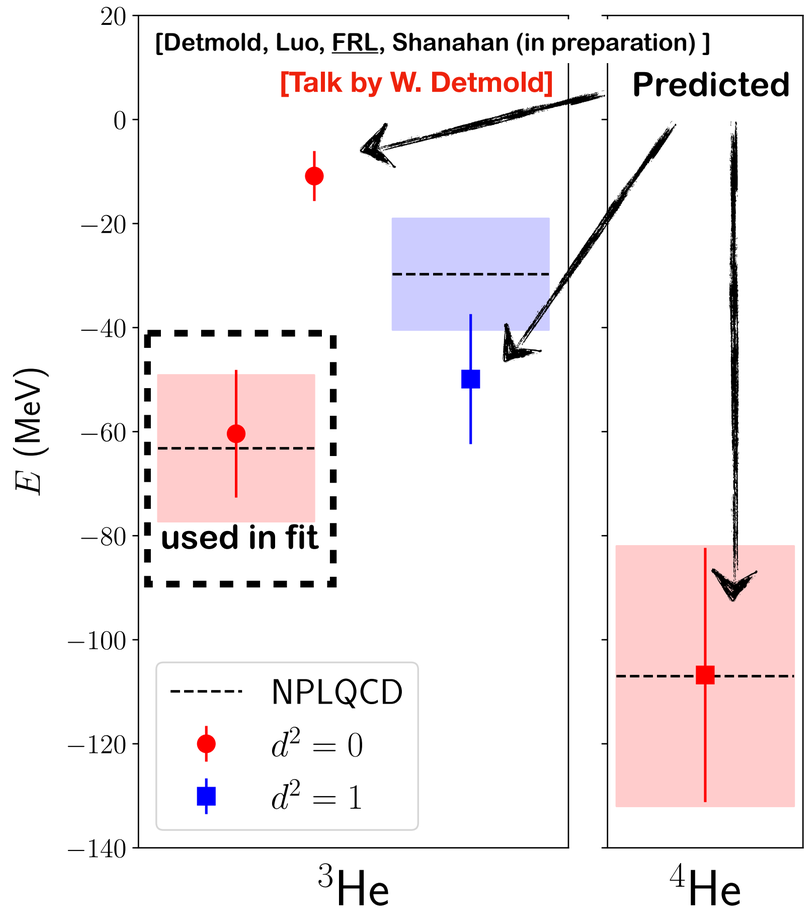


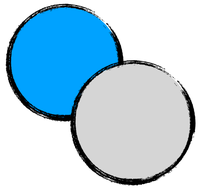
lattice QCD data vs. predicted FV spectrum
(including moving frames & excited states)

Summary and outlook

A roadmap for spectroscopy

- ☑ Well-understood formalism with many applications

A roadmap for spectroscopy



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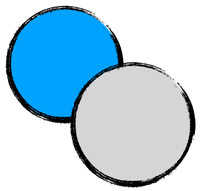

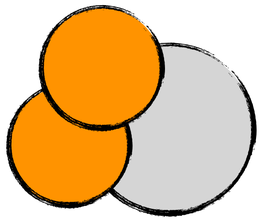

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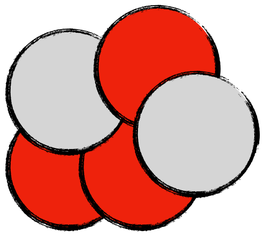

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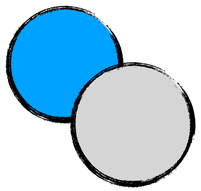

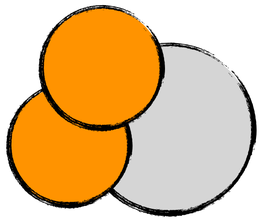

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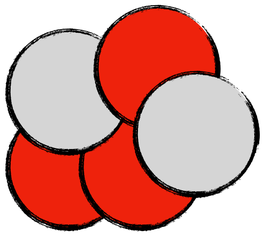

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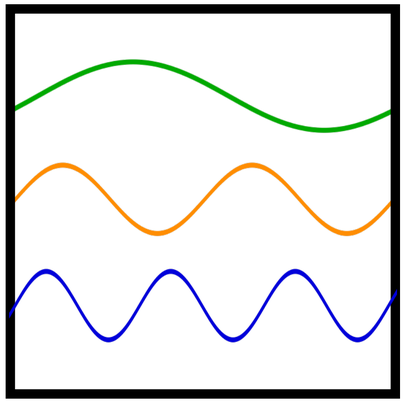
Thanks!

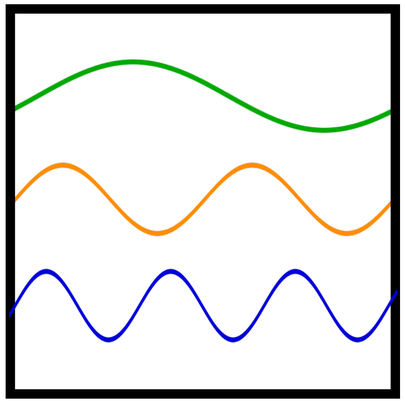
Back-up

Finite-Volume Spectrum



Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC



$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$

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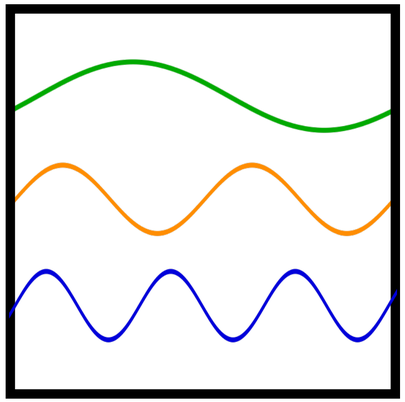
Free scalar particles in finite volume with periodic BC → Interactions change the spectrum!



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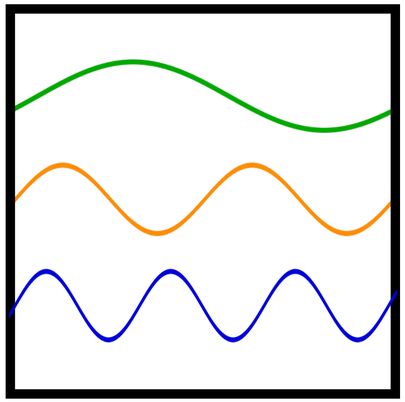

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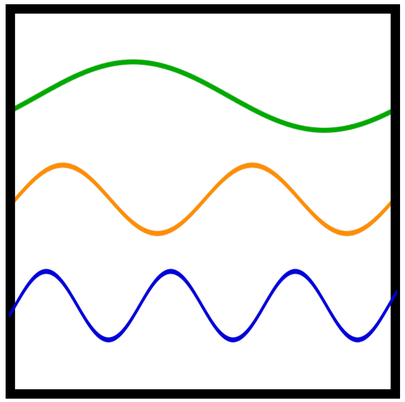

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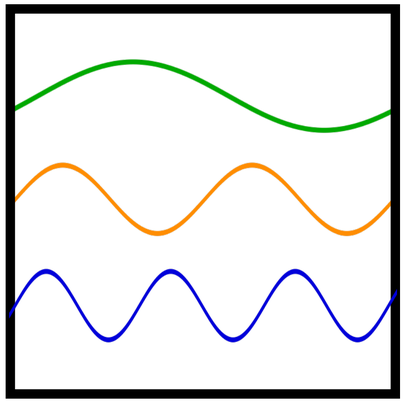

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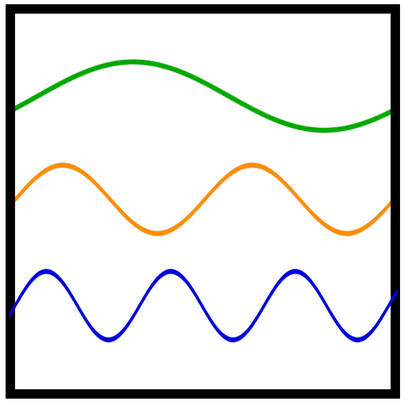

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In general a problem of Quantum Field Theory in finite volume

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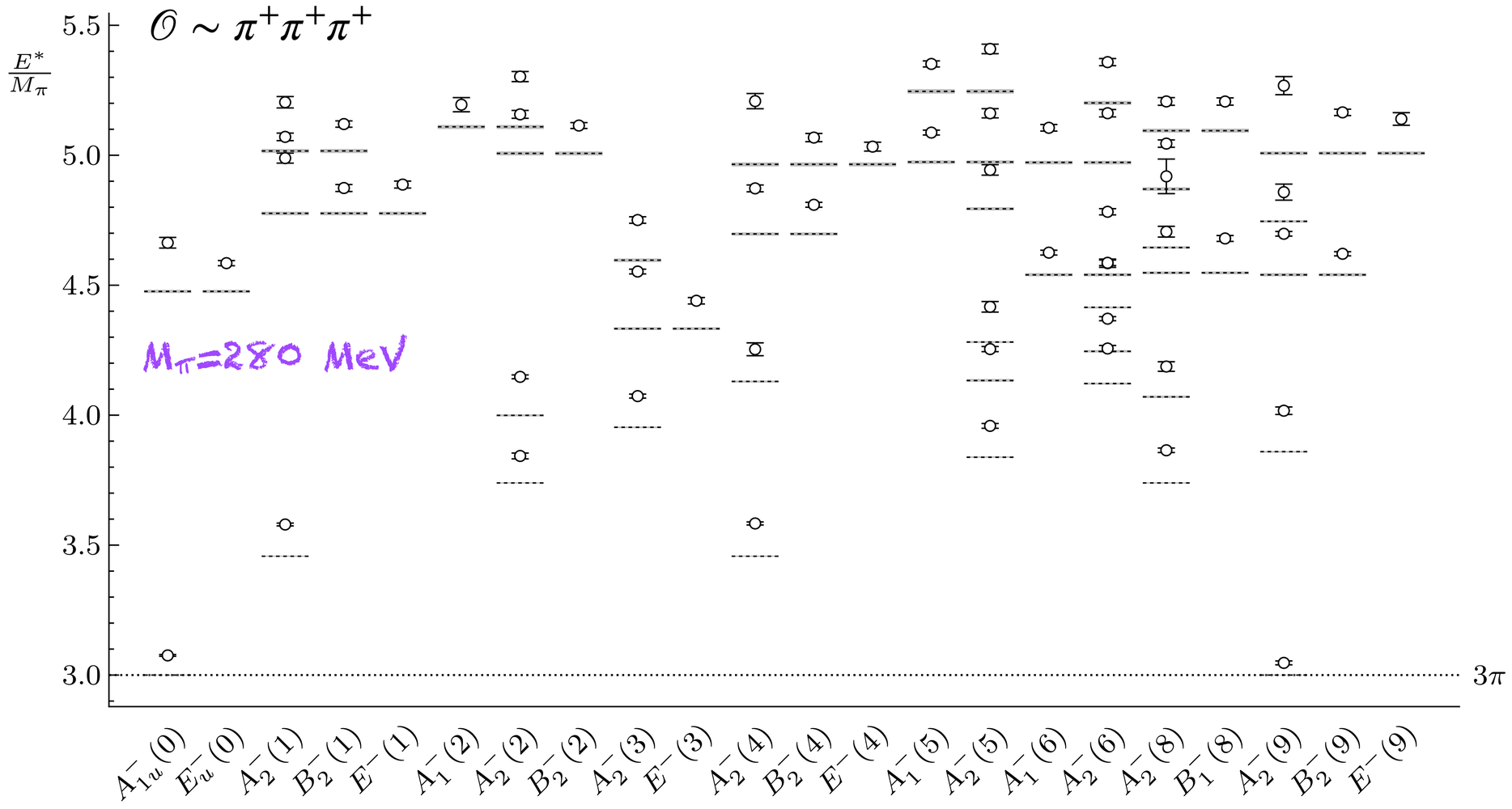

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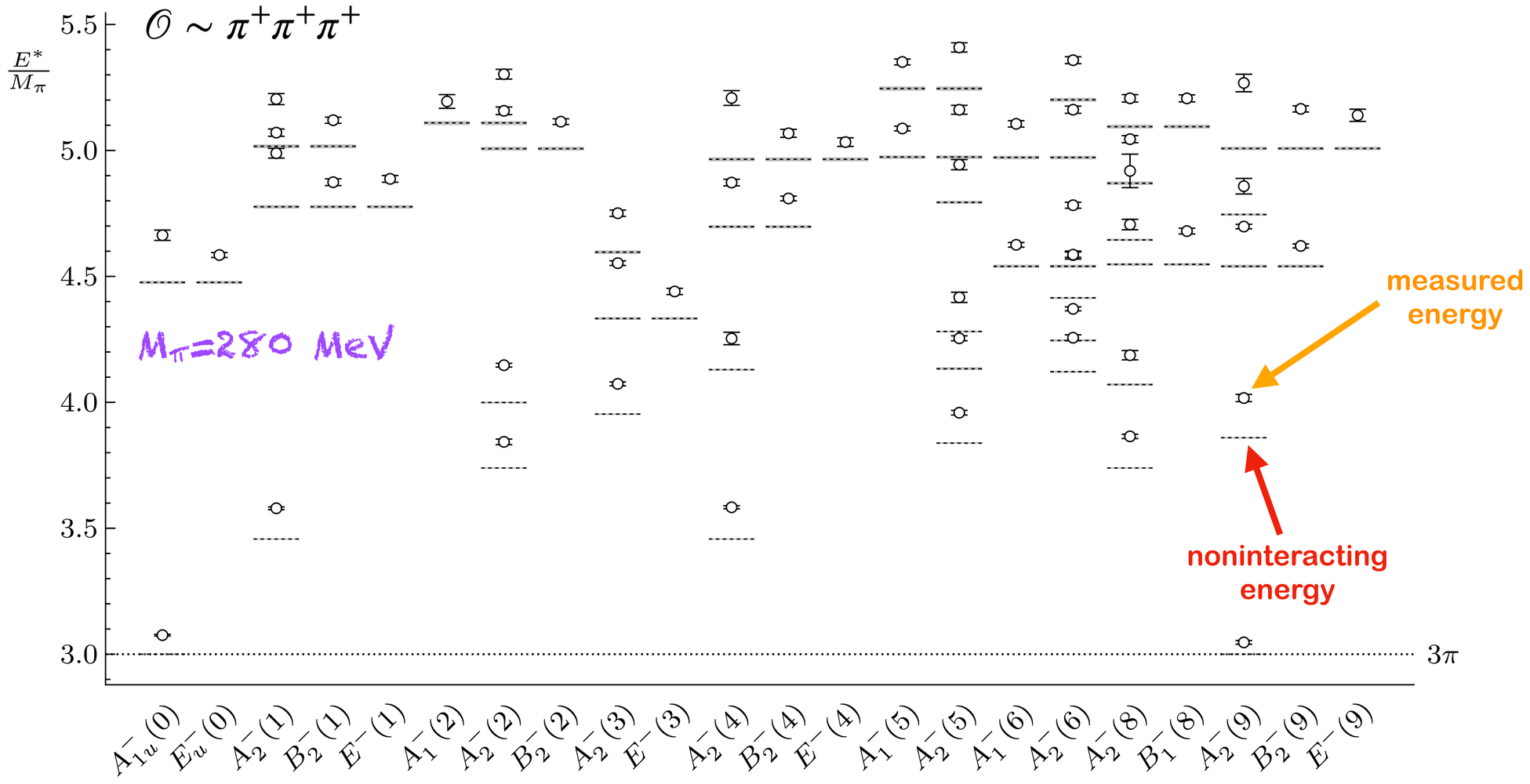

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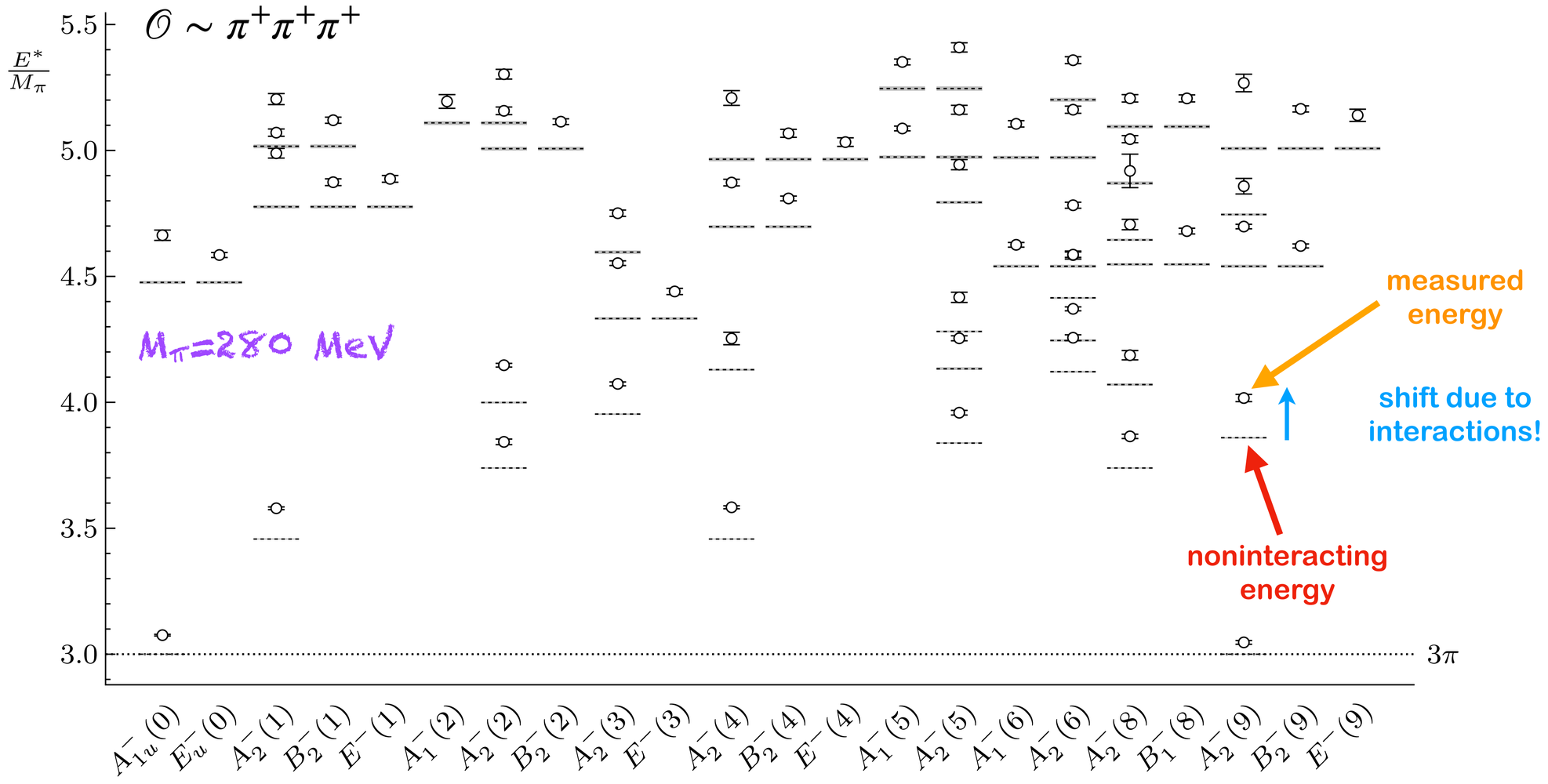

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Three-particle decays (K->3π)

- Decay processes get distorted in finite volume [Lellouch,Lüscher]

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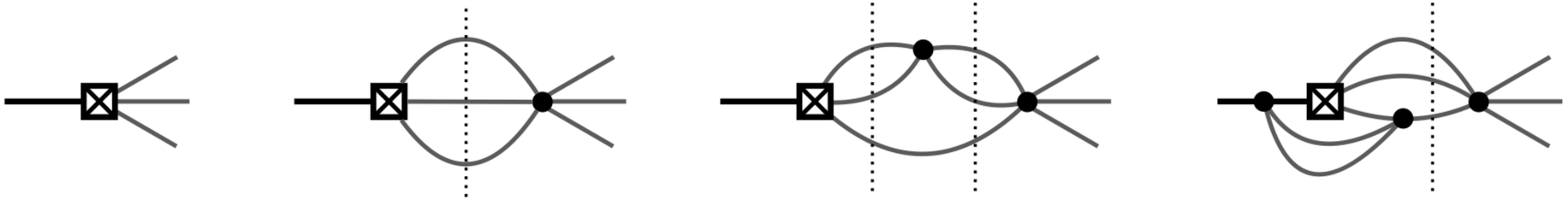

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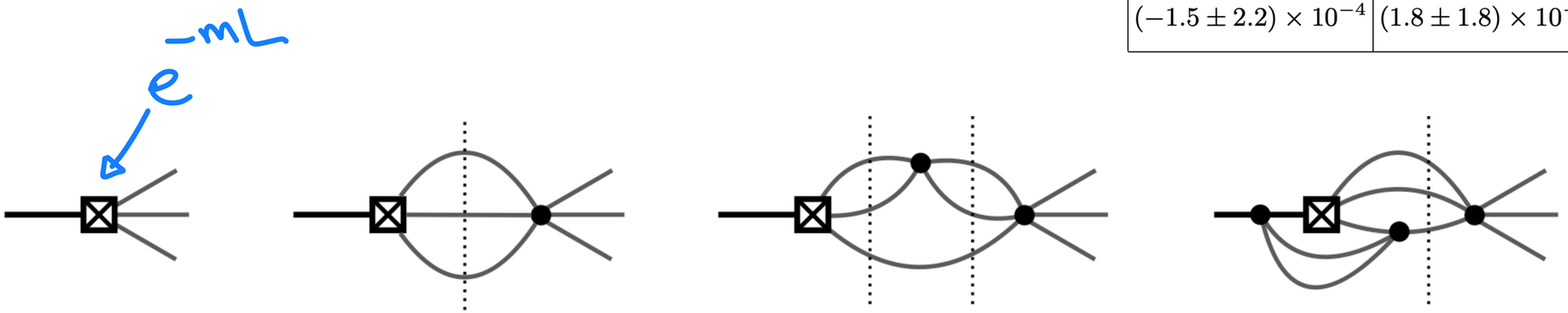

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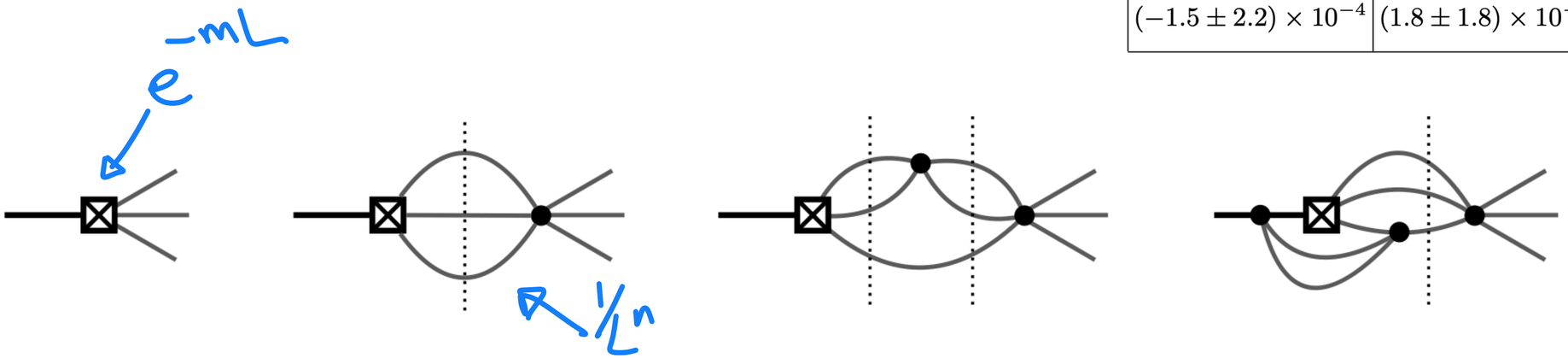

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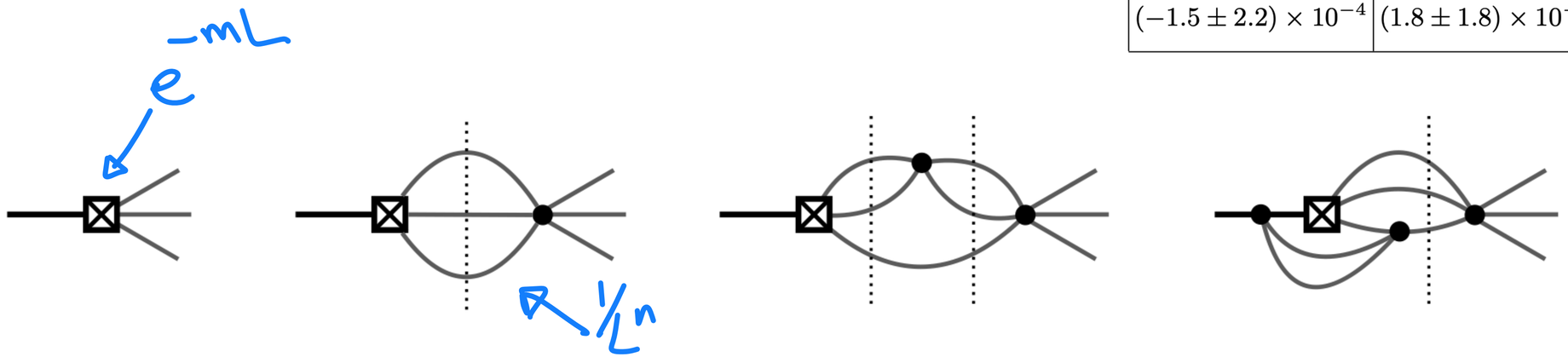

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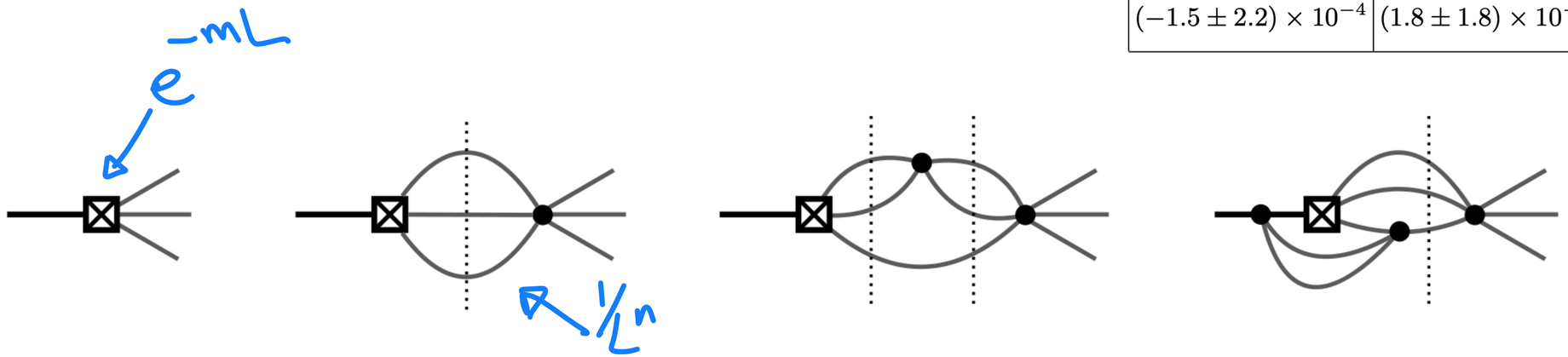

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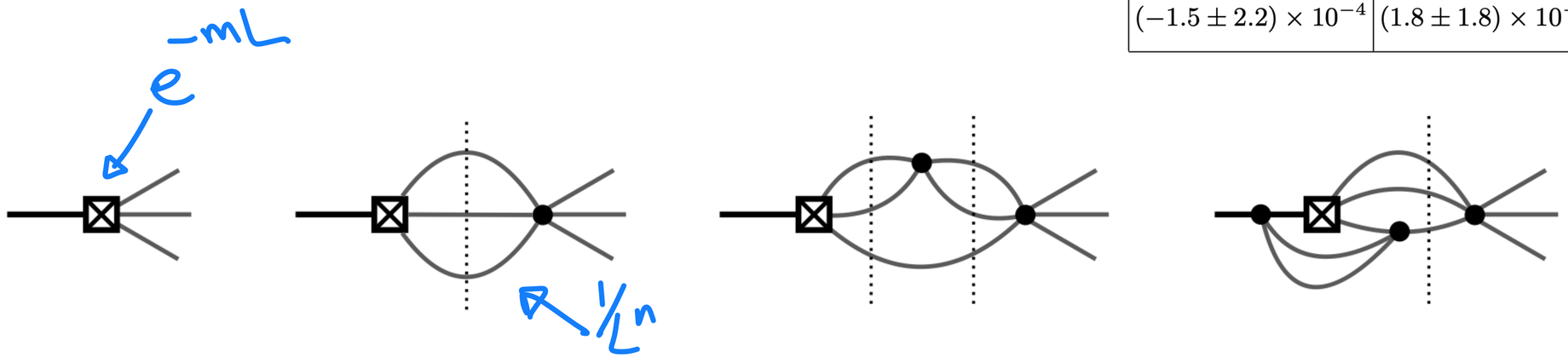

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Applying the RFT approach

- **Parametrize K-matrices with only s-wave interactions:**

$$\frac{q}{M}\cot\delta_0 = \frac{\sqrt{s}M}{s - z_2^2}\left(B_0 + B_1 q^2 + \cdots\right)$$

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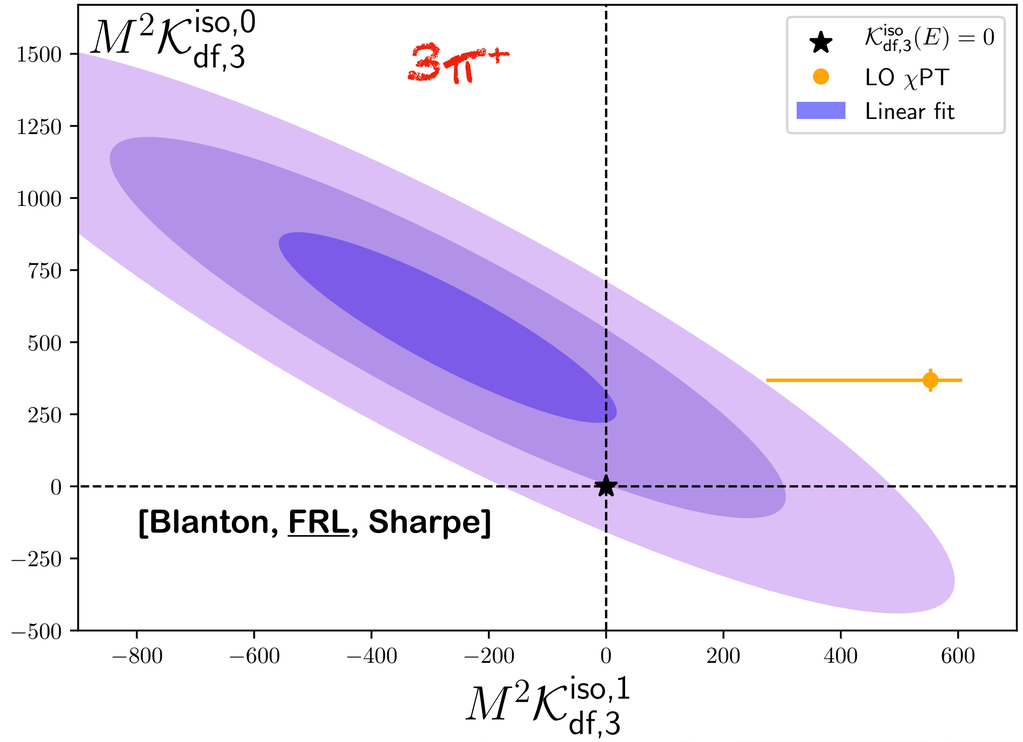
Fit	B_0	B_1	z_2^2/M^2	$M^2\mathcal{K}^{\mathrm{iso},0}_{\mathrm{df},3}$	$M^2\mathcal{K}^{\mathrm{iso},1}_{\mathrm{df},3}$	χ^2/dof	Ma_0	M^2ra_0
5	-11.1(7)	-2.4(3)	1 (fixed)	550(330)	-280(290)	26.04/(22-4)	0.090(5)	2.57(8)



Applying the RFT approach

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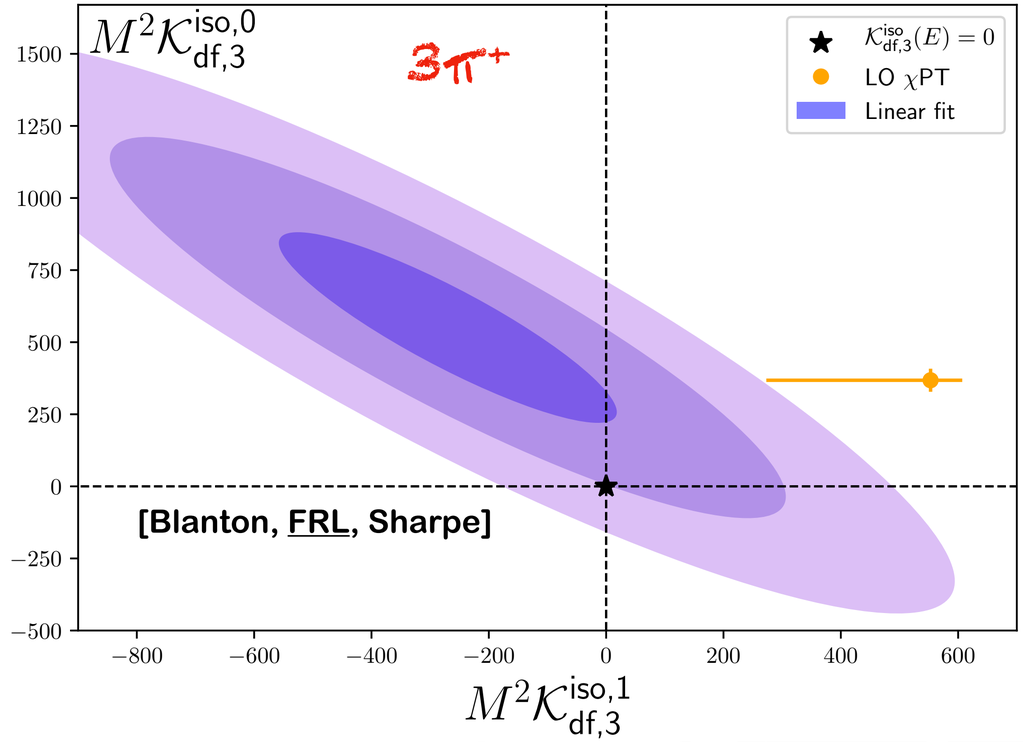

[Blanton, FRL, Sharpe]

1. 2σ evidence for $\mathcal{K}_{df,3} \neq 0$.

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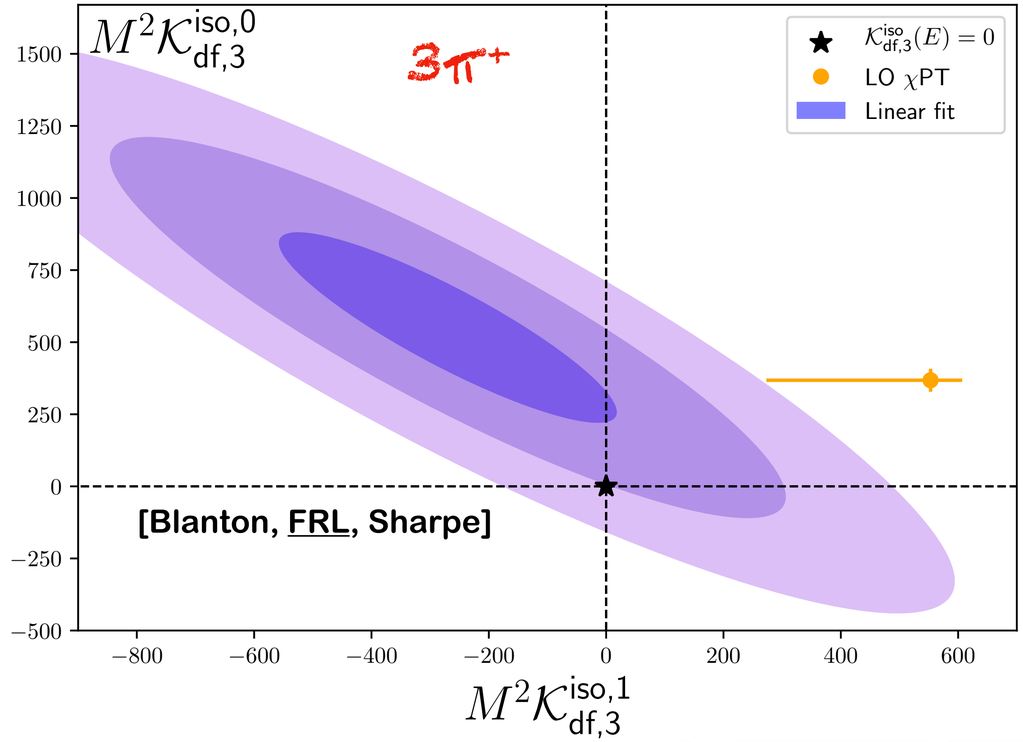

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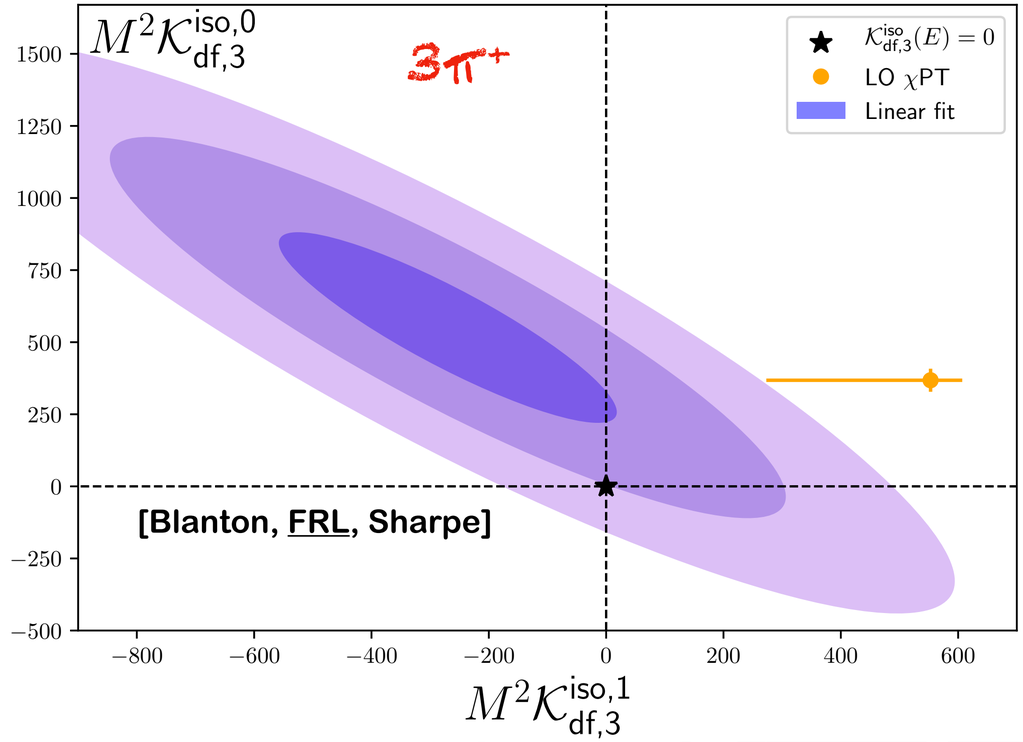

[Blanton, FRL, Sharpe]

1. 2σ evidence for $\mathcal{K}_{df,3} \neq 0$.
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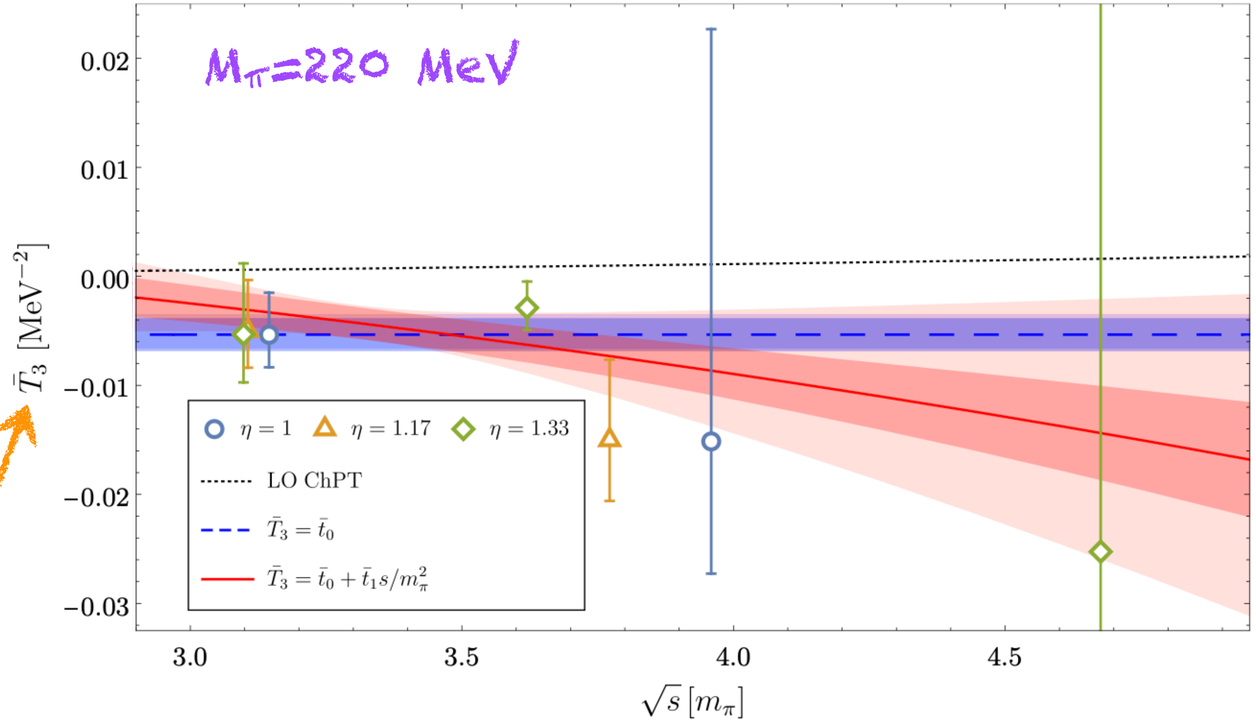
Same spectrum has been analyzed by [Mai, Döring, Culver, Alexandru]

Applying the FVU approach ($3\pi^+$)

- **The FVU formalism has also been applied to three-pion systems**
  [Brett et al.]



three-particle "contact" term

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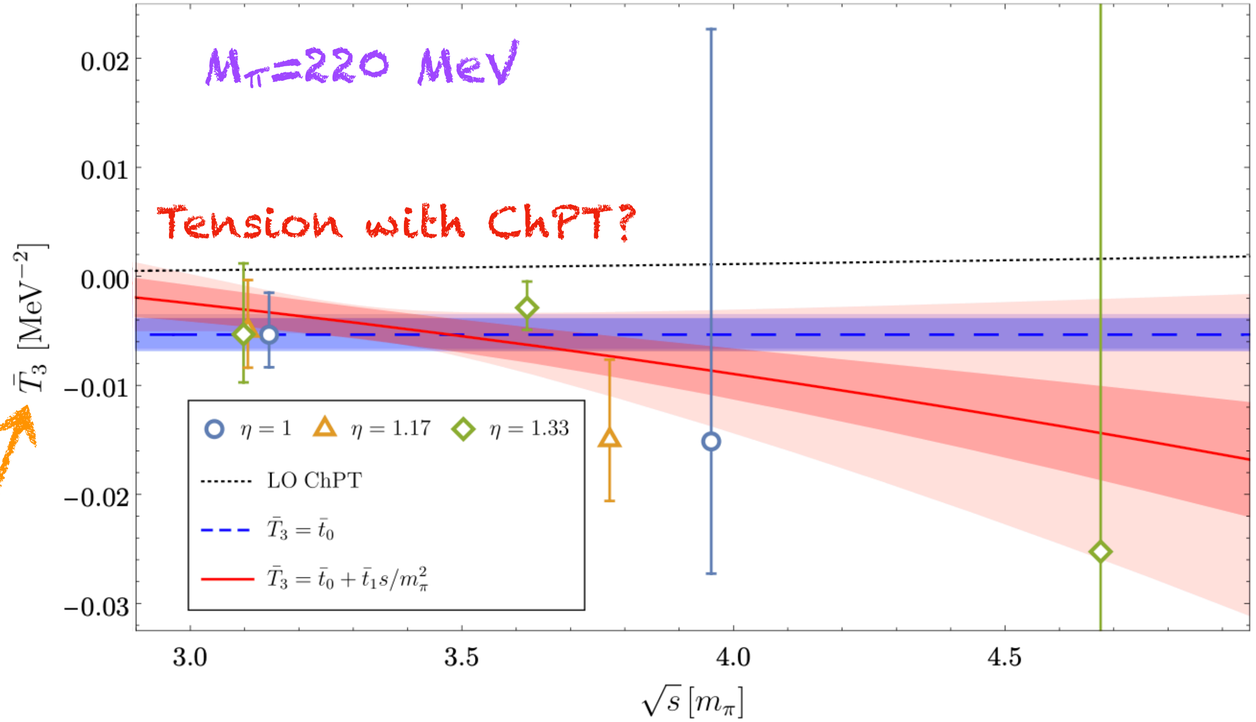

three-particle "contact" term

Integral equations (RFT)

Final step

Physical 3->3 amplitude

$$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow[\text{Integral equations}]{} \mathcal{M}_3$$

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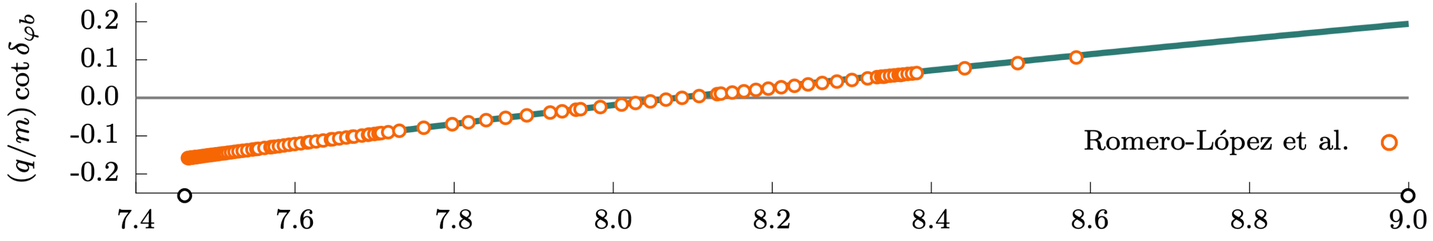
Particle-Dimer phase shift [Jackura et al.]



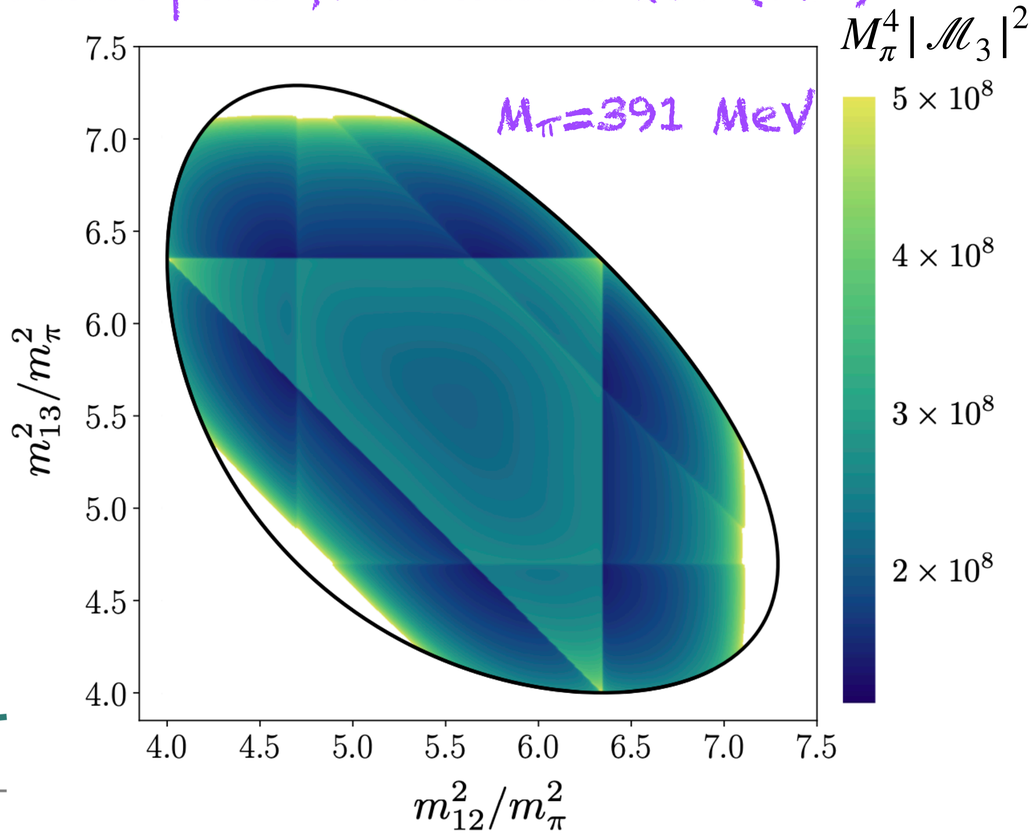
Integral equations (RFT)

Final step

$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow[\text{Integral equations}]{} \mathcal{M}_3$

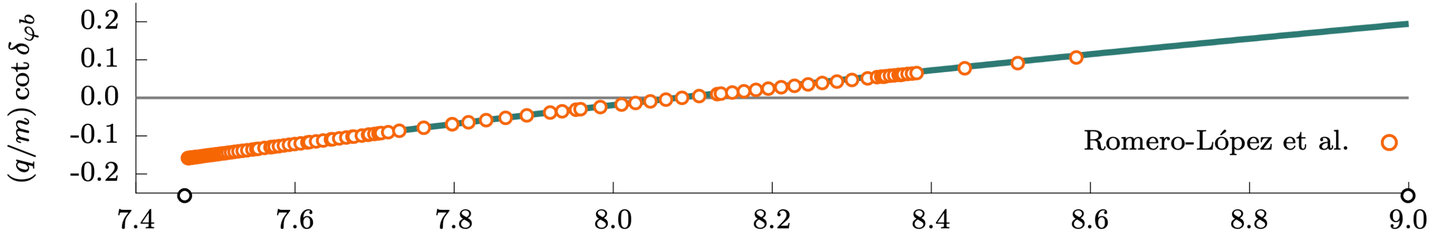
Physical 3->3 amplitude

Dalitz plots from lattice QCD ($3\pi^+$)



[Hansen et al. (HadSpec)]

Particle-Dimer phase shift [Jackura et al.]



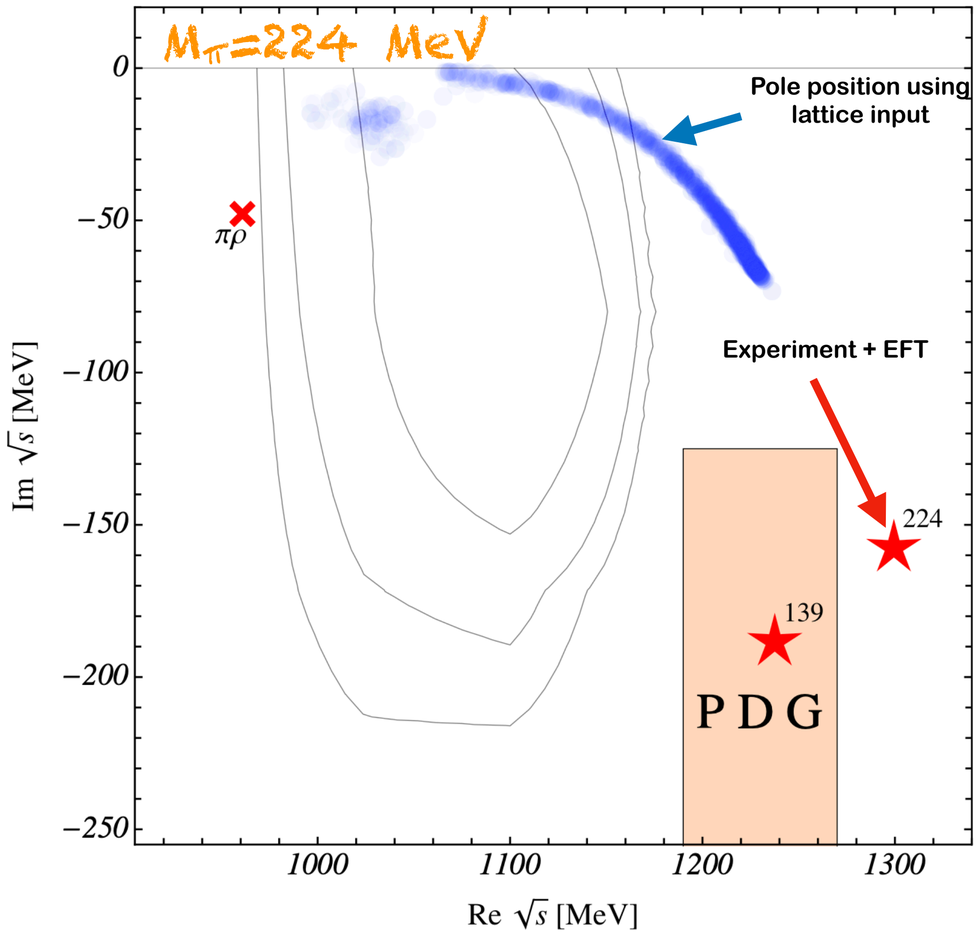
Integral equations (FVU)

Final step

$\mathcal{K}_2, C_{\ell\ell'} \xrightarrow[\text{Integral equations}]{} \mathcal{M}_3$

Physical 3->3 amplitude

Pole position of $a_1(1260)$



[Mai et al. (GWQCD)]

Quantization Condition

$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

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Finite-volume information & two-body interactions

$$F_3 = \frac{1}{L^3}\left[\frac{F}{3} - F\frac{1}{(\mathcal{K}_2)^{-1} + F + G}F\right]$$

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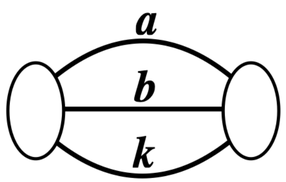
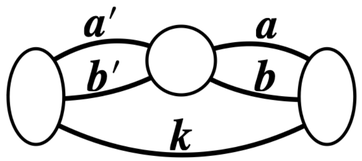
(a)



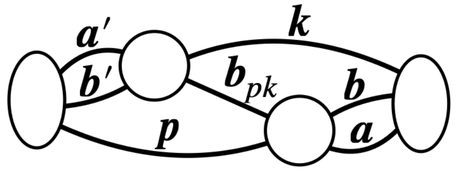
F

(b)



$\mathcal{K}_2$

(c)



G

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2 - q^2}$$

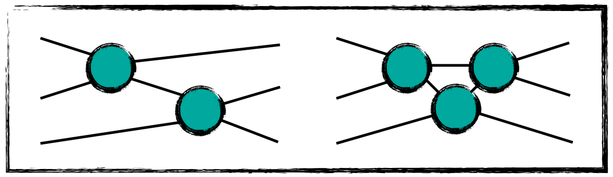
$$\mathcal{K}_2^\ell = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

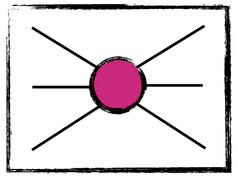
$$G_{p00;k00} \equiv \frac{1}{L^3}\frac{1}{2\omega_p}\frac{H(\vec{p})H(\vec{k})}{b_{pk}^2 - m^2}\frac{1}{2\omega_k}$$

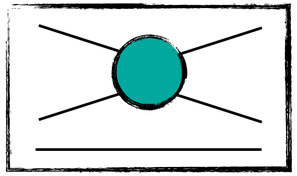
Integral equations

- Construct a "finite-volume amplitude" with spectator singled out

$$\mathcal{M}_{3,L}^{(u,u)}(P) \equiv \mathcal{D}^{(u,u)} + \mathcal{L}_L^{(u)} \frac{1}{1+\mathcal{K}_{\mathrm{df},3}F_3} \mathcal{K}_{\mathrm{df},3}\mathcal{R}_L^{(u)}$$

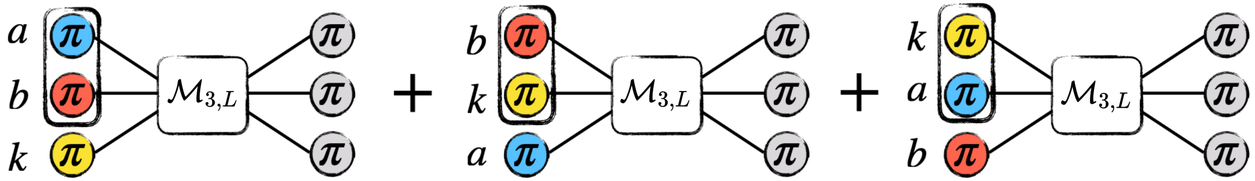






- Symmetrize (each particle gets a turn to be the spectator)

$$\mathcal{M}_{3,L}(P) \equiv \mathcal{S}\left[\mathcal{M}_{3,L}^{(u,u)}(P)\right] =$$



- Infinite-volume limit

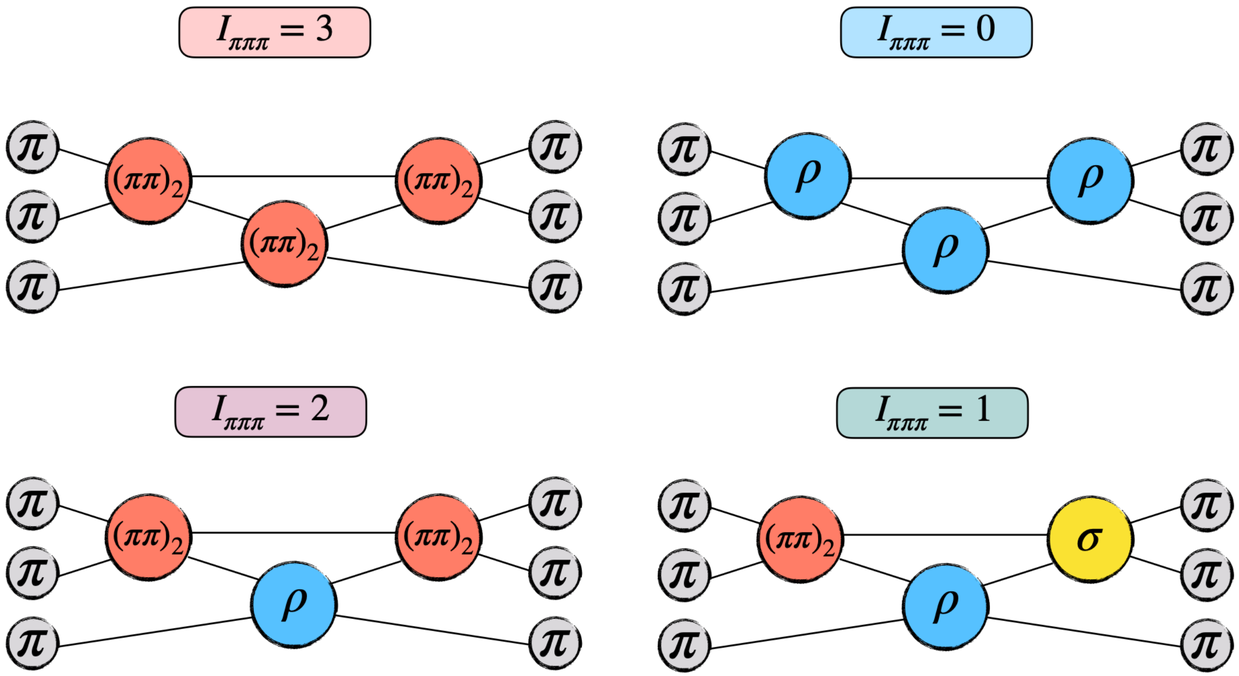
$$\mathcal{M}_3(E,\boldsymbol{P}) = \lim_{\epsilon\to 0^+} \lim_{L\to\infty} \mathcal{M}_{3,L}(E+i\epsilon,\boldsymbol{P})$$

Including isospin

- Relevant three-body systems involve nonidentical particles ($\pi\pi N$)
- Let us consider mass-degenerate pions with different flavor e.g. $\pi^+\pi^0\pi^-$

  [Hansen, FRL, Sharpe, JHEP 2020]

  - All pions have the same mass
  - Overall isospin is conserved
  - Presence of resonances
  - Example of multi-channel scattering



Including isospin

- **Three-pion formalism:**
  [Hansen, FRL, Sharpe, JHEP 2020]

$$\det_{k,\ell,m,\mathbf{f}}[1-\mathbf{K}_{\mathrm{df},3}(E^\star)\mathbf{F}_3(E,\boldsymbol{P},L)]=0$$

additional "flavor" index

$$\mathbf{F}_3 \equiv \frac{\mathbf{F}}{3}+\mathbf{F}\frac{1}{\mathbf{K}_2^{-1}-\mathbf{F}-\mathbf{G}}\mathbf{F}$$

- **Need to include all diagrams connecting all possible flavor combinations**
  - **7 possibilities in the zero electric charge sector**

$$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}=(\mathbf{0}\oplus\mathbf{1}\oplus\mathbf{2})\otimes\mathbf{1}=(\mathbf{1})\oplus(\mathbf{0}\oplus\mathbf{1}\oplus\mathbf{2})\oplus(\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3})$$

$$\begin{pmatrix}\tilde{\pi}_-(a)\ \tilde{\pi}_0(b)\ \tilde{\pi}_+(k)\\ \tilde{\pi}_0(a)\ \tilde{\pi}_-(b)\ \tilde{\pi}_+(k)\\ \tilde{\pi}_-(a)\ \tilde{\pi}_+(b)\ \tilde{\pi}_0(k)\\ \tilde{\pi}_0(a)\ \tilde{\pi}_0(b)\ \tilde{\pi}_0(k)\\ \tilde{\pi}_+(a)\ \tilde{\pi}_-(b)\ \tilde{\pi}_0(k)\\ \tilde{\pi}_0(a)\ \tilde{\pi}_+(b)\ \tilde{\pi}_-(k)\\ \tilde{\pi}_+(a)\ \tilde{\pi}_0(b)\ \tilde{\pi}_-(k)\end{pmatrix}$$

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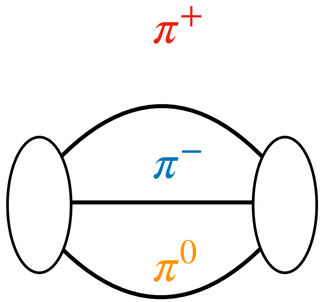
(a)



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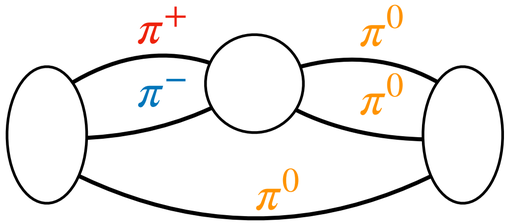

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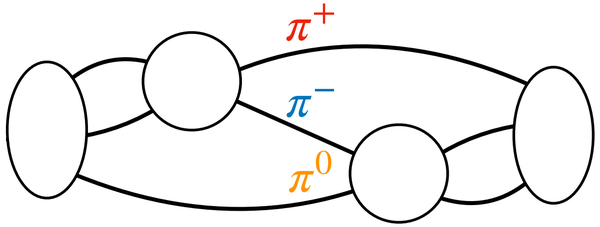

$$\left(\square = 0\,,\ \blacksquare = 1\right)$$

$$\mathbf{F} = F \times \begin{pmatrix} \blacksquare & \square & \square & \square & \square & \square & \square \\ \square & \blacksquare & \square & \square & \square & \square & \square \\ \square & \square & \blacksquare & \square & \square & \square & \square \\ \square & \square & \square & \blacksquare & \square & \square & \square \\ \square & \square & \square & \square & \blacksquare & \square & \square \\ \square & \square & \square & \square & \square & \blacksquare & \square \\ \square & \square & \square & \square & \square & \square & \blacksquare \end{pmatrix}$$

$$\mathbf{K}_2 \equiv \begin{pmatrix} \begin{pmatrix} [\pi_-\pi_0 \leftarrow \pi_-\pi_0] & [\pi_-\pi_0 \leftarrow \pi_0\,\pi_-] \\ [\pi_0\,\pi_- \leftarrow \pi_-\pi_0] & [\pi_0\,\pi_- \leftarrow \pi_0\,\pi_-] \end{pmatrix} & \\ & \begin{matrix} \mathcal{K}_0 & \\ & \mathcal{K}_- \end{matrix} \end{pmatrix}$$

$$\mathbf{G} = G \times \begin{pmatrix} \square & \square & \square & \square & \square & \square & \blacksquare \\ \square & \square & \square & \square & \blacksquare & \square & \square \\ \square & \square & \square & \square & \square & \blacksquare & \square \\ \square & \square & \square & \blacksquare & \square & \square & \square \\ \square & \blacksquare & \square & \square & \square & \square & \square \\ \square & \square & \blacksquare & \square & \square & \square & \square \\ \blacksquare & \square & \square & \square & \square & \square & \square \end{pmatrix}$$

- Diagonal in flavor
- Two-body interactions can change flavor
- Permute the spectator

Including isospin

- Project to definite total isospin

$$\det\left[1-\mathbf{K}_{\mathrm{df},3}^{[I]}(E^\star)\,\mathbf{F}_3^{[I]}(E,\boldsymbol{P},L)\right]=0$$

- Four independent QC3!

$$\mathbf{F}_3^{[I]}\equiv\frac{\mathbf{F}^{[I]}}{3}+\mathbf{F}^{[I]}\frac{1}{1-\mathbf{M}_{2,L}^{[I]}\mathbf{G}^{[I]}}\mathbf{M}_{2,L}^{[I]}\mathbf{F}^{[I]}\qquad\mathbf{M}_{2,L}^{[I]}\equiv\frac{1}{\mathbf{K}_2^{[I]-1}-\mathbf{F}^{[I]}}$$

I	$\mathbf{F}^{[I]}$	$\mathbf{K}_2^{[I]}$	$\mathbf{G}^{[I]}$
3	$\frac{iF}{2\omega L^3}$	$i[2\omega L^3]\mathcal{K}_{(\pi\pi)_2}$	$i\frac{1}{2\omega L^3}G$
2	$\frac{iF}{2\omega L^3}\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$i[2\omega L^3]\begin{pmatrix}\mathcal{K}_{(\pi\pi)_2}&0\\0&\mathcal{K}_\rho\end{pmatrix}$	$i\frac{1}{2\omega L^3}G\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$
1	$\frac{iF}{2\omega L^3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$i[2\omega L^3]\begin{pmatrix}\mathcal{K}_{(\pi\pi)_2}&0&0\\0&\mathcal{K}_\rho&0\\0&0&\mathcal{K}_\sigma\end{pmatrix}$	$i\frac{1}{2\omega L^3}G\begin{pmatrix}\frac{1}{6}&\frac{\sqrt{15}}{6}&\frac{\sqrt{5}}{3}\\\frac{\sqrt{15}}{6}&\frac{1}{2}&-\frac{1}{\sqrt{3}}\\\frac{\sqrt{5}}{3}&-\frac{1}{\sqrt{3}}&\frac{1}{3}\end{pmatrix}$
0	$\frac{iF}{2\omega L^3}$	$i[2\omega L^3]\mathcal{K}_\rho$	$-i\frac{1}{2\omega L^3}G$

Parametrizing $K_{df,3}$

- The three-particle K-matrix has the same symmetries as the scattering amplitude
- For identical particles:
  - Particle exchange symmetry: $\mathcal{K}_{\mathrm{df},3}(p_1, p_2, p_3; k_i) = \mathcal{K}_{\mathrm{df},3}(p_2, p_1, p_3; k_i) = \ldots$
  - Time reversal: $\mathcal{K}_{\mathrm{df},3}(p_i, k_i) = \mathcal{K}_{\mathrm{df},3}(k_i, p_i)$
- For there-pions with definite isospin, need to consider flavor indices. Example:
  - I=0 flavor wave function is fully antisymmetric
    $|3\pi, I=0\rangle \sim |+0-\rangle - |0+-\rangle + |0-+\rangle + |-0+\rangle - |+-0\rangle + |-+0\rangle$
  - Therefore, the K-matrix must also be antisymmetric

$$t_{ij} \equiv (p_i - k_j)^2$$

$$\mathbf{K}_{\mathrm{df},3}^{[I=0]} \supset \mathcal{K}_{\mathrm{df},3}^{\mathrm{AS}} \sum_{\substack{ijk \\ mnr}} \epsilon_{ijk}\epsilon_{mnr} t_{im} t_{jn}$$

  - Minimal parametrization for resonances:

$$\mathbf{K}_{\mathrm{df},3}^{[I=0,\omega(780)]} = \frac{c_X V_\alpha V'^\alpha}{s - M_\omega^2}$$

$$V^\alpha = P_\mu \sum_{ijk} \epsilon_{ijk}\, p_j^\mu p_k^\alpha$$