THE UNIVERSITY of ADELAIDE

SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

The Compton Amplitude and Nucleon Structure Functions

K. Utku Can
CSSM, The University of Adelaide

partially based on:
PRD102, 114505(2020)
PoS LATTICE2021(2022)324

in collaboration with QCDSF-UKQCD-CSSM:
M. Batelaan, A. Hannaford-Gunn, R. D. Young, J. M. Zanotti (Adelaide), R. Horsley (Edinburgh), P. E. L. Rakow (Liverpool), H. Perlt (Leipzig), G. Schierholz (DESY, Hamburg), H. Stüben (Hamburg), Y. Nakamura (RIKEN, Kobe)

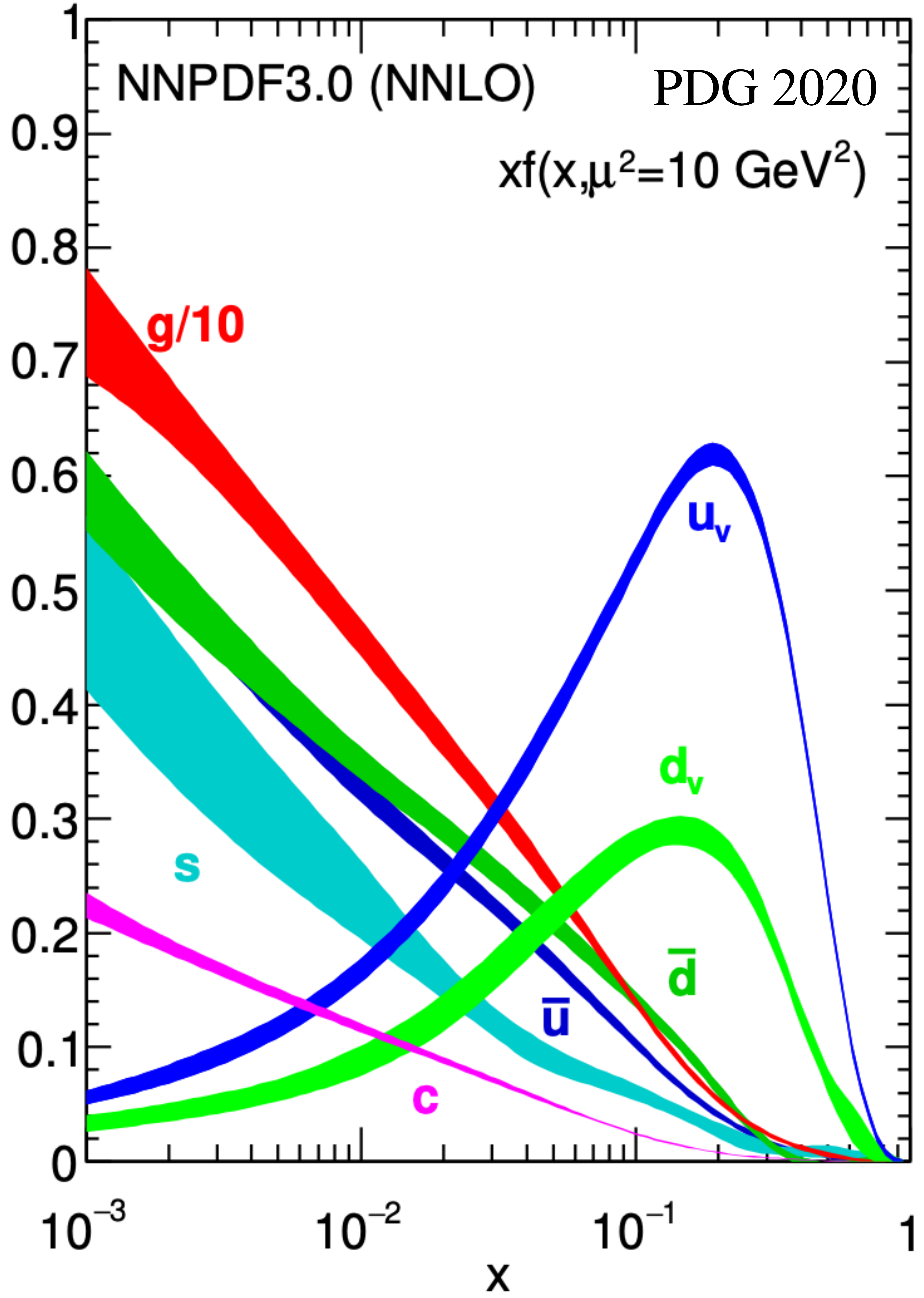
Lattice'22, Bonn, 8-13 Aug 2022

The numerical configuration generation (using the BQCD lattice QCD program)) and data analysis (using the Chroma software library) was carried on the DiRAC Blue Gene Q and Extreme Scaling (EPCC, Edinburgh, UK) and Data Intensive (Cambridge, UK) services, the GCS supercomputers JUQUEEN and JUWELS (NIC, Jülich, Germany) and resources provided by HLRN (The North-German Supercomputer Alliance), the NCI National Facility in Canberra, Australia (supported by the Australian Commonwealth Government) and the Phoenix HPC service (University of Adelaide). RH is supported by STFC through grant ST/P000630/1. PELR is supported in part by the STFC under contract ST/G00062X/1. KUC, RDY and JMZ are supported by the Australian Research Council grants DP190100297 and DP220103098

Motivation

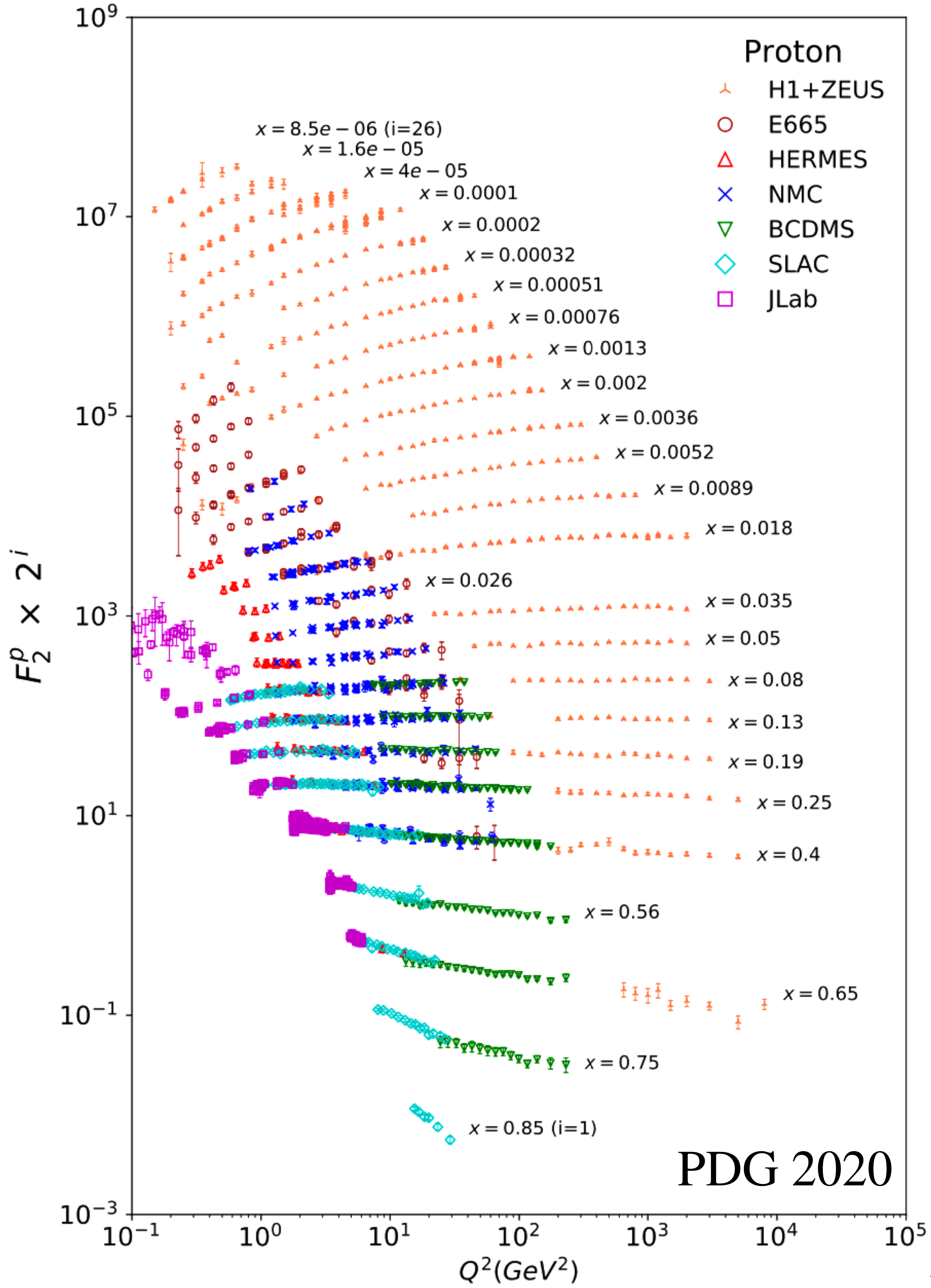
- Nucleon structure (leading twist)

  - Structure functions from first principles
  - Understanding the behaviour in the high- and low-x regions



Motivation

- Scaling
  - Q^2 cuts of global QCD analyses
- Power corrections / Higher twist effects
  - Target mass corrections
  - Twist-4 contributions



Motivation

- Technical issues:
  - Operator Product Expansion formalism to study DIS processes
  - Operator mixing/renormalisation issues in OPE approach in LQCD

physical observable

twist-2 twist-4

$$\mu(Q^2) = c_2(a^2Q^2)\, v_2(a) + \frac{c_4(a^2Q^2)}{Q^2}\, v_4(a) + \cdots$$

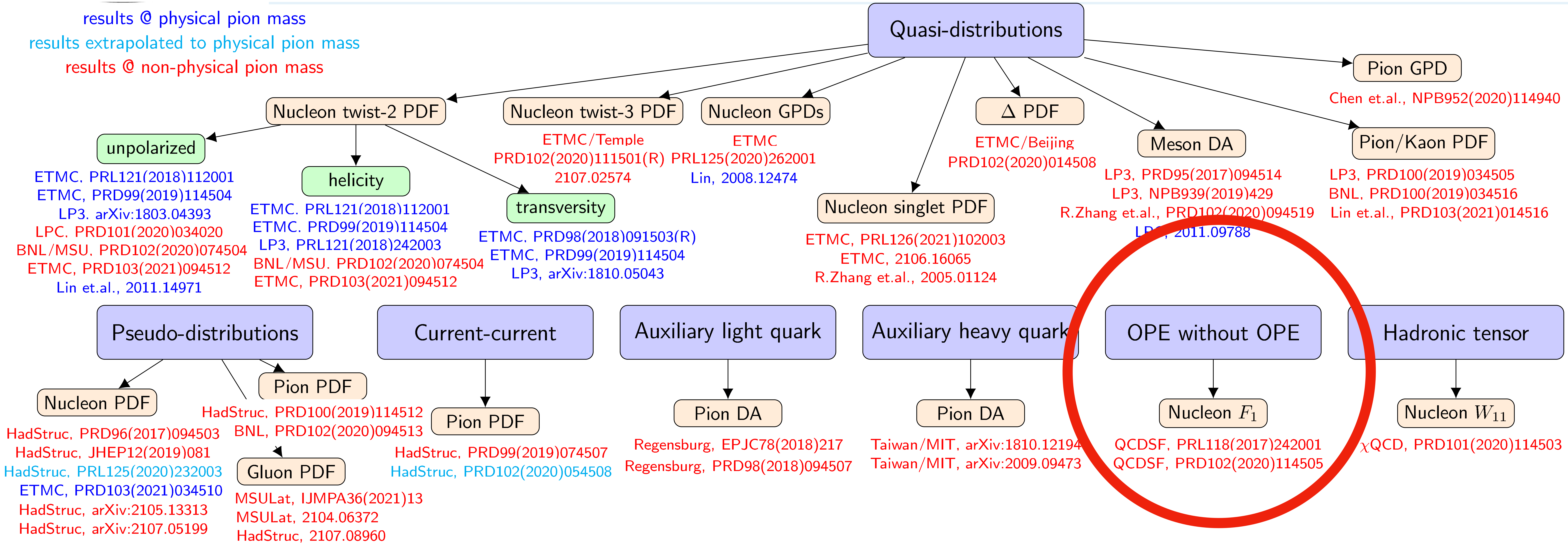
$1/a^2$ divergence

mixing

  - Why not calculate the observable directly?

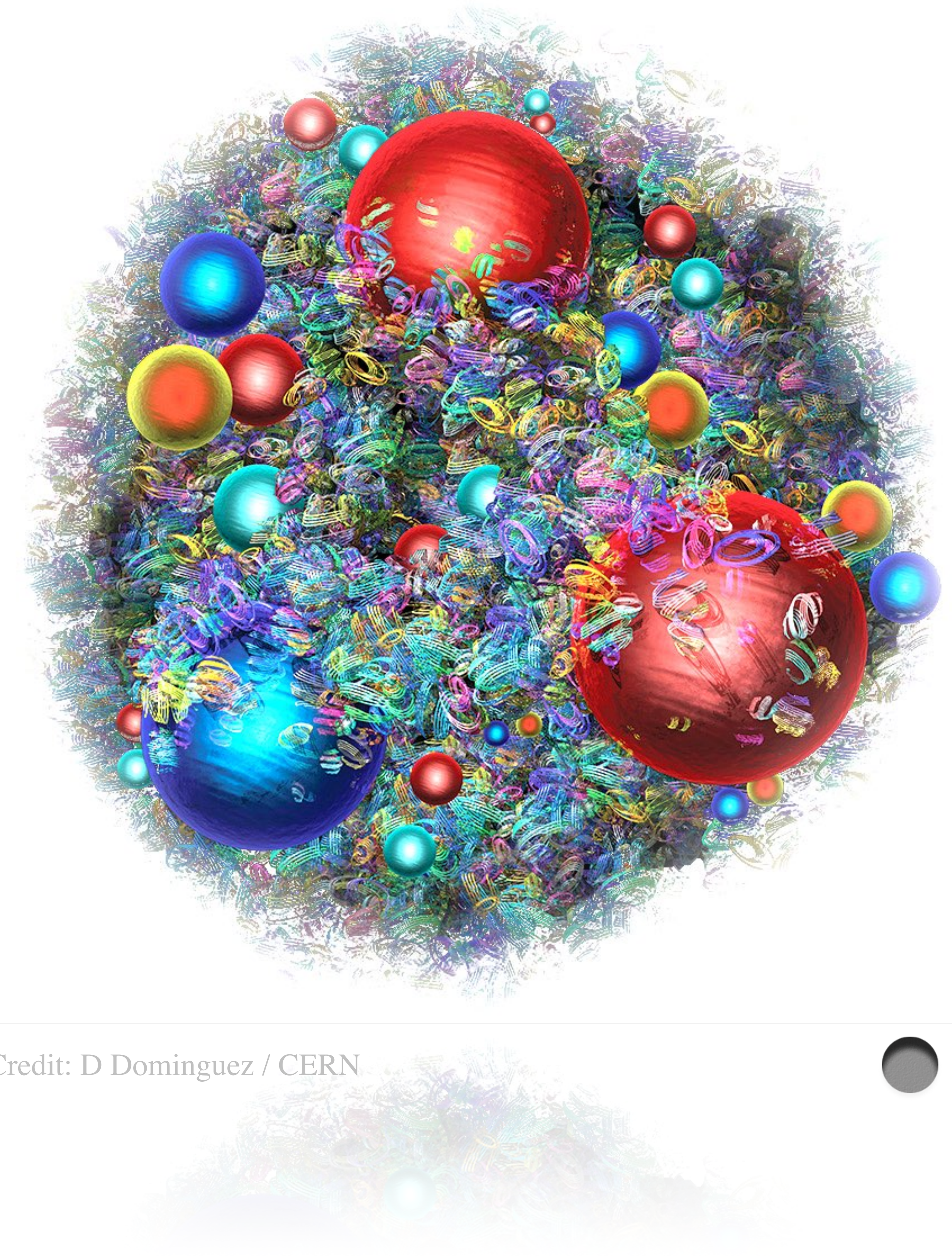
LQCD landscape

Krzysztof Cichy @ LATTICE'21 plenary
PoS(LATTICE2021)017



- **QCDSF-UKQCD-CSSM Collaboration**
- **Extended to nucleon F_2 and F_L**
- **Study of higher-twist effects**
- **also, a first look at g_1 and g_2**

Outline


Credit: D Dominguez / CERN

- Forward Compton Amplitude & the Nucleon Structure Functions
- Application of the Feynman-Hellmann Theorem
- Moments of the Nucleon Structure Functions
- Scaling and Power Corrections/Higher-twist effects

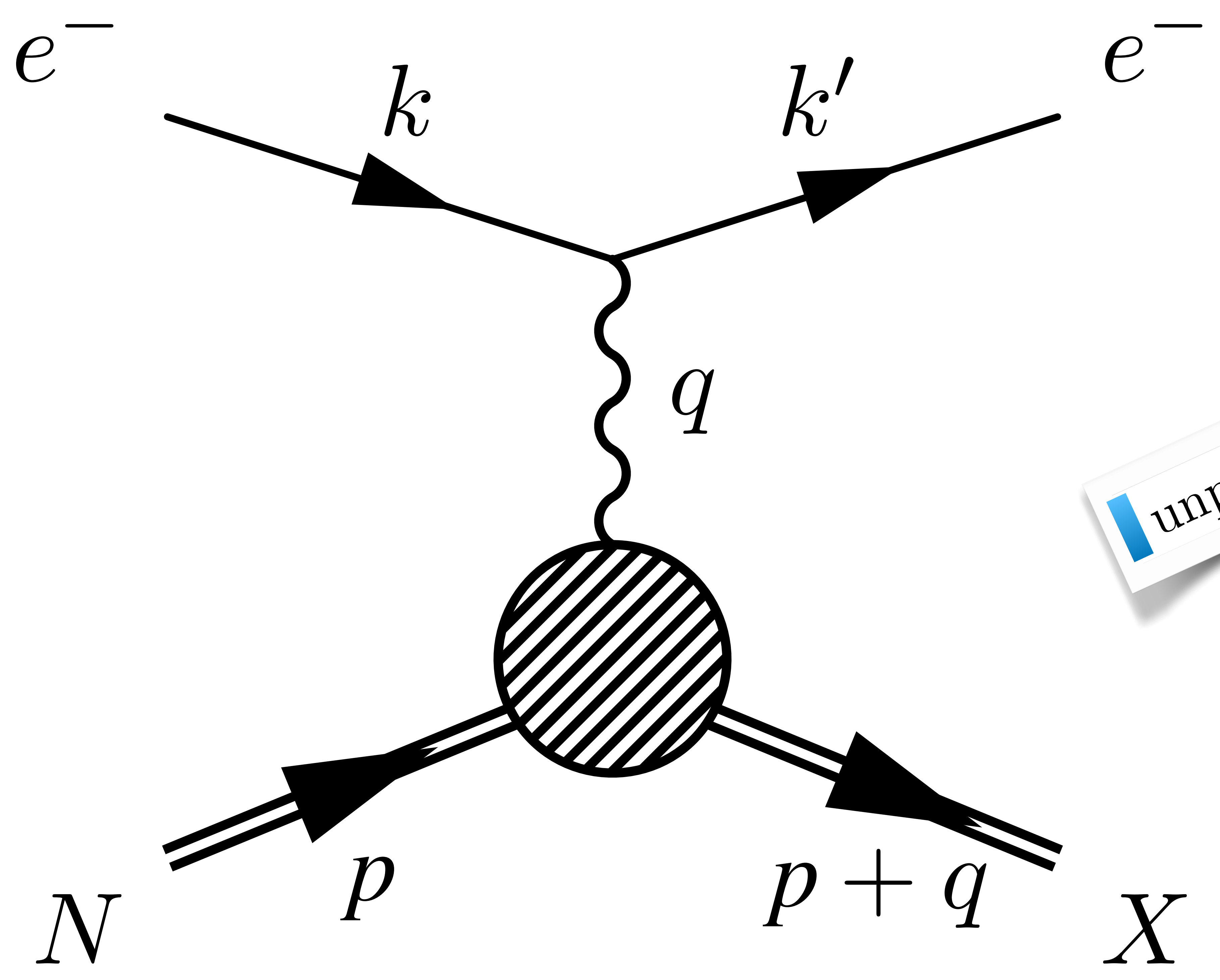
DIS and the Hadronic Tensor

Deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)

$$d\sigma \sim L_j^{\mu\nu} W_{\mu\nu}^j$$

$j = \gamma, Z,$ and γZ (neutral) or W (charged)

leptonic tensor ***hadronic tensor***



$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4 z e^{iq\cdot z} \rho_{ss'} \langle p, s' | [J_\mu(z), J_\nu(0)] | p, s \rangle$$

$$\rho_{ss'} = \tfrac{1}{2}\delta_{ss'}$$

unpolarised

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) \frac{F_2(x, Q^2)}{p\cdot q}$$

Structure Functions

Forward Compton Amplitude

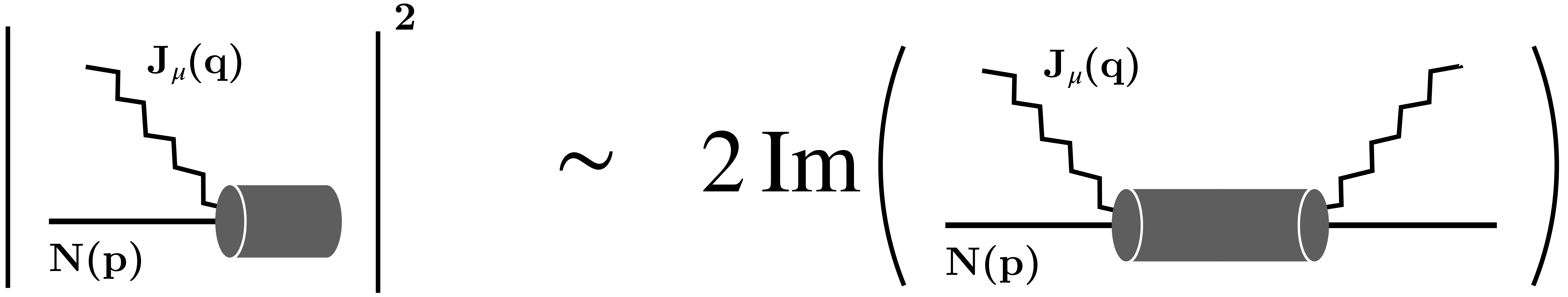
$$T_{\mu\nu}(p,q) = i\int d^4z\, e^{iq\cdot z}\rho_{ss'}\langle p,s'|\mathscr{T}\{J_\mu(z)J_\nu(0)\}|p,s\rangle \quad \text{, spin avg. } \rho_{ss'} = \frac{1}{2}\delta_{ss'}$$

$\omega = \dfrac{2p\cdot q}{Q^2}$

Same Lorentz decomposition as the Hadronic Tensor

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\mathscr{F}_1(\omega,Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{\mathscr{F}_2(\omega,Q^2)}{p\cdot q}$$

Compton Structure Functions (SF)



DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

Nucleon Structure Functions

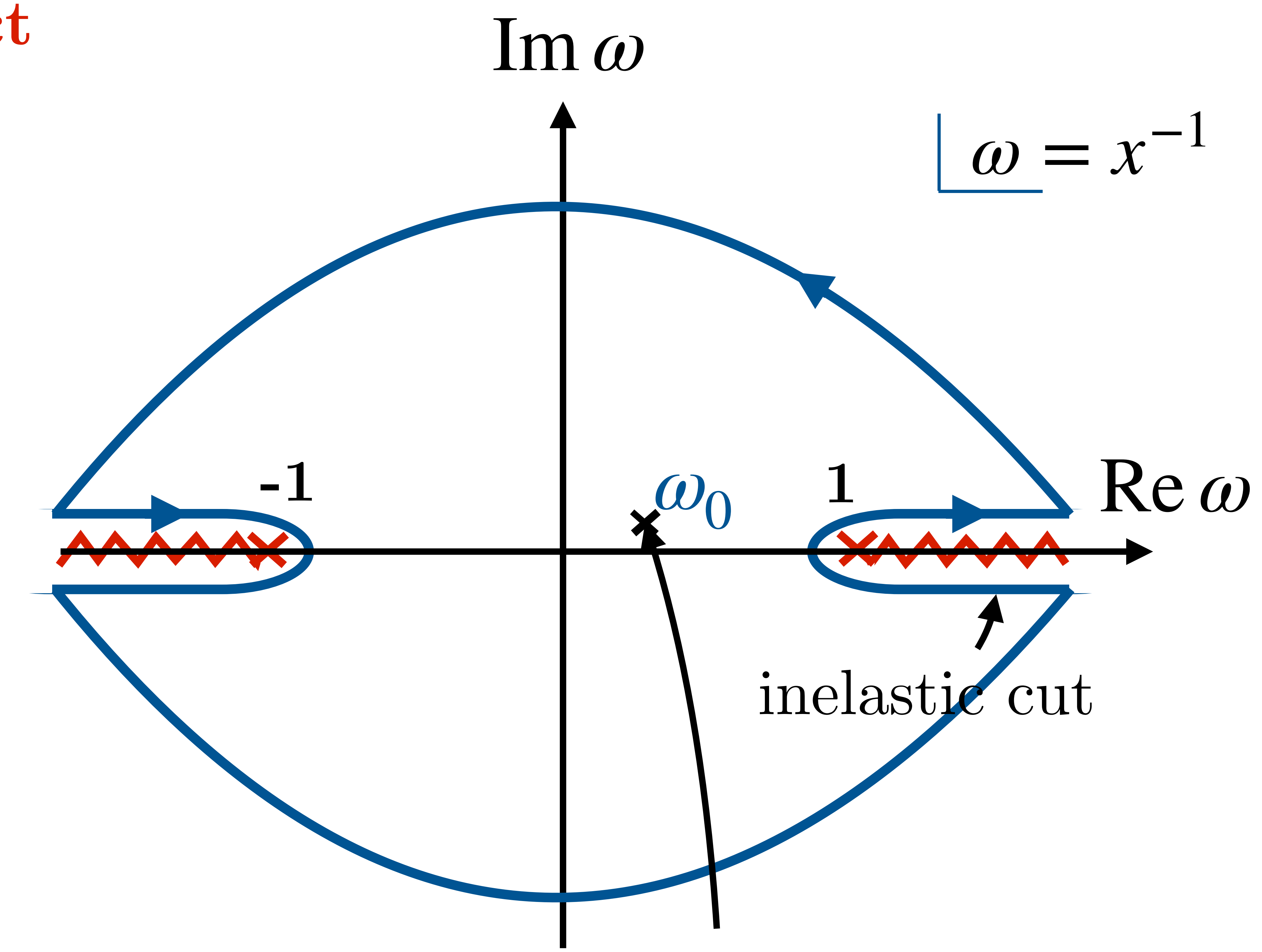
- **we can write down dispersion relations and connect Compton SFs to DIS SFs:**

$$\underbrace{\mathcal{F}_1(\omega,Q^2)-\mathcal{F}_1(0,Q^2)}_{\equiv\overline{\mathcal{F}}_1(\omega,Q^2)} = 2\omega^2\int_0^1 dx\frac{2x\,F_1(x,Q^2)}{1-x^2\omega^2-i\epsilon}$$

$$\mathcal{F}_2(\omega,Q^2) = 4\omega^2\int_0^1 dx\frac{F_2(x,Q^2)}{1-x^2\omega^2-i\epsilon}$$

$$\underbrace{\mathcal{F}_L(\omega,Q^2)+\mathcal{F}_1(0,Q^2)}_{\equiv\overline{\mathcal{F}}_L(\omega,Q^2)} = \frac{8M_N^2}{Q^2}\int_0^1 dxF_2(x,Q^2) +2\omega^2\int_0^1 dx\frac{F_L(x,Q^2)}{1-x^2\omega^2-i\epsilon}$$

$\omega = x^{-1}$



Compton Amplitude is an analytic function in the unphysical region $|\omega_0|<1$

Nucleon Structure Functions

- **using the Taylor expansion,** $\frac{1}{1-(x\omega)^2} = \sum_{n=1}^{\infty} (x\omega)^{2n-2}$

$$\omega = \frac{2p\cdot q}{Q^2} \equiv x^{-1}$$

$$\mathcal{F}_{1,L}(\omega,Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1,L)}(Q^2) \text{ , where } M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx\, x^{2n-1} F_1(x,Q^2) \text{ , and } \quad M_0^{(1)}(Q^2) = 0$$

$$\mathcal{F}_2(\omega,Q^2) = \sum_{n=1}^{\infty} 4\omega^{2n-1} M_{2n}^{(2)}(Q^2)\text{, where } M_{2n}^{(2,L)}(Q^2) = \int_0^1 dx\, x^{2n-2} F_{2,L}(x,Q^2) \text{ , and } \quad M_0^{(L)}(Q^2) = \frac{4M_N^2}{Q^2} M_2^{(2)}(Q^2)$$

- $\mu = \nu = 3$ and $p_3 = q_3 = 0 \implies \mathcal{F}_1(\omega,Q^2) = T_{33}(p,q)$

- $\mu = \nu = 0$ and $p_3 = q_3 = q_0 = 0 \implies \frac{\mathcal{F}_2(\omega,Q^2)}{\omega} = \left[T_{00}(p,q) + T_{33}(p,q)\right]\frac{Q^2}{2E_N^2}$

$$\mathcal{F}_L(\omega,Q^2) = -\mathcal{F}_1(\omega,Q^2) + \left(\frac{\omega}{2} + \frac{2M_N^2}{\omega Q^2}\right)\mathcal{F}_2(\omega,Q^2)$$

FH Theorem at 1st order

in Quantum Mechanics:

$$\frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle$$

H_λ: perturbed Hamiltonian of the system

E_λ: energy eigenvalue of the perturbed system

ϕ_λ: eigenfunction of the perturbed system

- **expectation value of the perturbed system is related to the shift in the energy eigenvalue**

in Lattice QCD: energy shifts in the presence of a weak external field

$$S \to S(\lambda) = S + \lambda \int d^4x\, \mathcal{O}(x) \xrightarrow{\text{e.g. local bilinear operator}} \bar{q}(x)\Gamma_\mu q(x) \qquad , \Gamma_\mu \in \{\mathbf{1}, \gamma_\mu, \gamma_5\gamma_\mu, \dots\}$$

real parameter

@ 1st order

$$\frac{\partial E_\lambda}{\partial \lambda} = \frac{1}{2E_\lambda} \langle 0 | \mathcal{O} | 0 \rangle$$

$\boxed{\mathbf{E_\lambda \to \text{spectroscopy, 2-pt function}}}$

$\boxed{\langle 0 | \mathcal{O} | 0 \rangle \to \textbf{determine 3-pt}}$

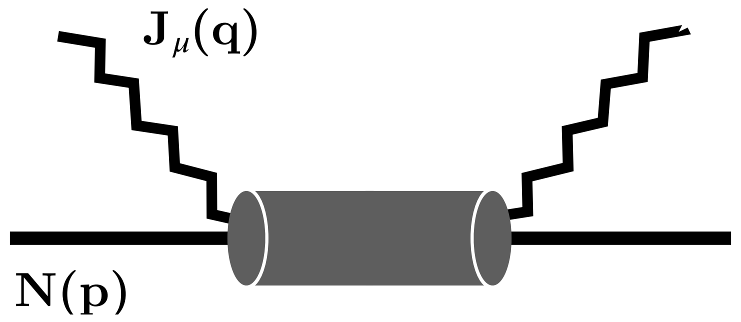
Applications:

- σ **- terms**
- **Form factors**

Compton amplitude via the FH relation at 2nd order

- **unpolarised Compton Amplitude**

$$T_{\mu\mu}(p,q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) \,|\, \mathscr{T}\{J_\mu(z) J_\mu(0)\} \,|\, N(p)\rangle$$



- **Action modification**

local EM current

$J_\mu(z) = \sum_q e_q \bar{q}(z)\gamma_\mu q(z)$

$$S \to S(\lambda) = S + \lambda \int d^4z \left(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}\right) J_\mu(z)$$

- **2nd order derivatives of the 2-pt correlator, $G_\lambda^{(2)}(\mathbf{p};t)$, in the presence of the external field**

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - tA(\mathbf{p})\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z \left(e^{iq\cdot z} + e^{-iq\cdot z}\right) \langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

from path integral

- **equate the time-enhanced terms:**

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z \left(e^{iq\cdot z} + e^{-iq\cdot z}\right) \langle N(\mathbf{p})|\mathcal{J}(z)\mathcal{J}(0)|N(\mathbf{p})\rangle}^{T_{\mu\mu}(p,q)} + (q \to -q)$$

<u>**Compton amplitude is related to the second-order energy shift**</u>

Compton amplitude via the FH relation at 2nd order

- relevant contribution comes from the ordering where the currents are sandwiched

$$\chi(t) \quad \mathcal{J}(z_4) \quad \mathcal{J}(y_4) \quad \bar{\chi}(0) \sim e^{-E_N(\mathbf{p})t} \int^t d\Delta\, e^{-\left(E_X(\mathbf{p}+\mathbf{q}) - E_N(\mathbf{p})\right)\Delta} (t-\Delta)$$

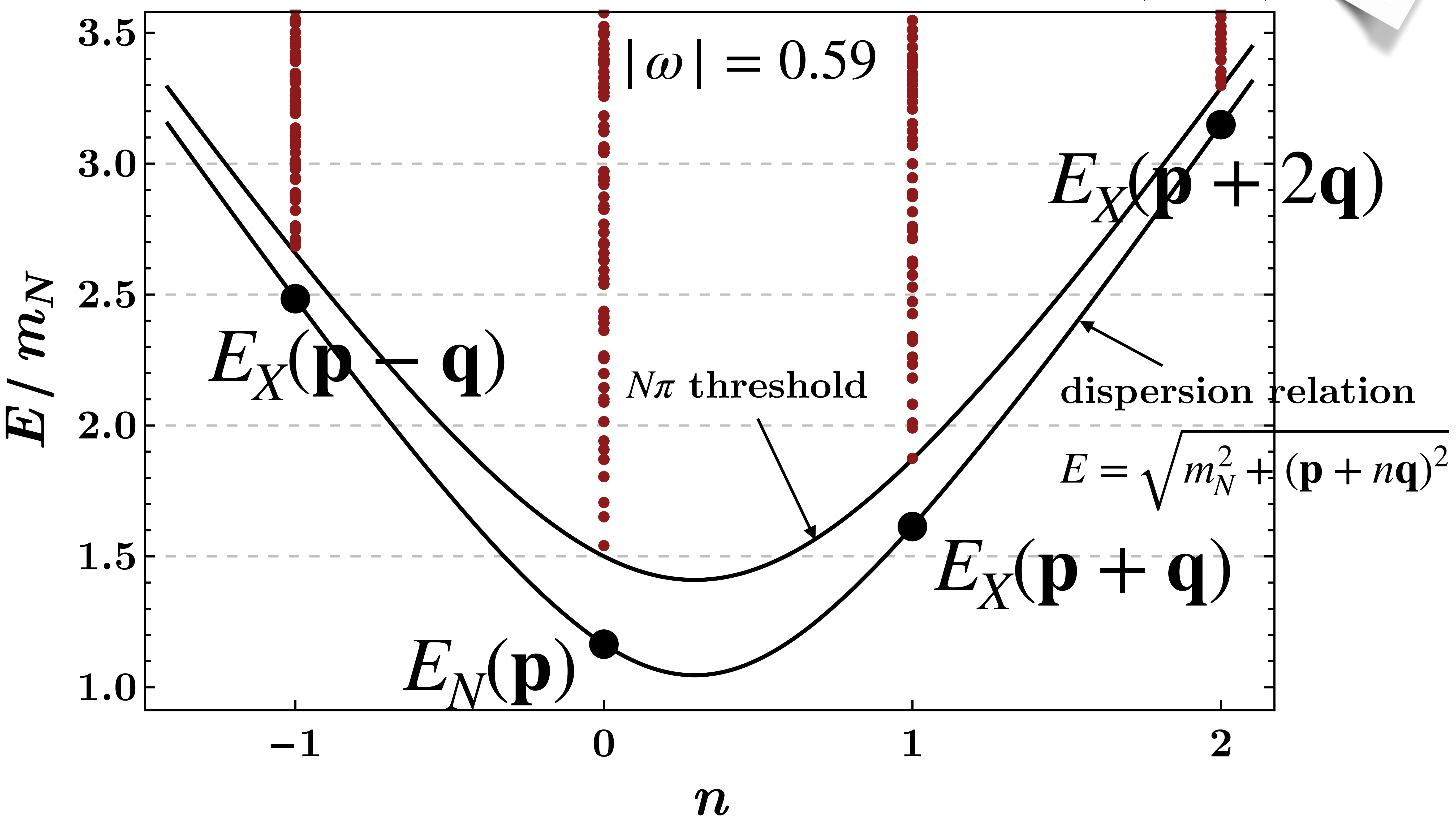
$\Delta = z_4 - y_4$

- under the condition $|\omega| < 1$, $E_X(\mathbf{p}+n\mathbf{q}) \gtrsim E_N(\mathbf{p})$, so the intermediate states <u>cannot go on-shell</u>

- ground state dominance is ensured in the large time limit

e.g.

$\mathbf{q} = 2\pi/L(4,1,0)$
$\mathbf{p} = 2\pi/L(-1,-1,0)$



Simulation Details

QCDSF/UKQCD configurations

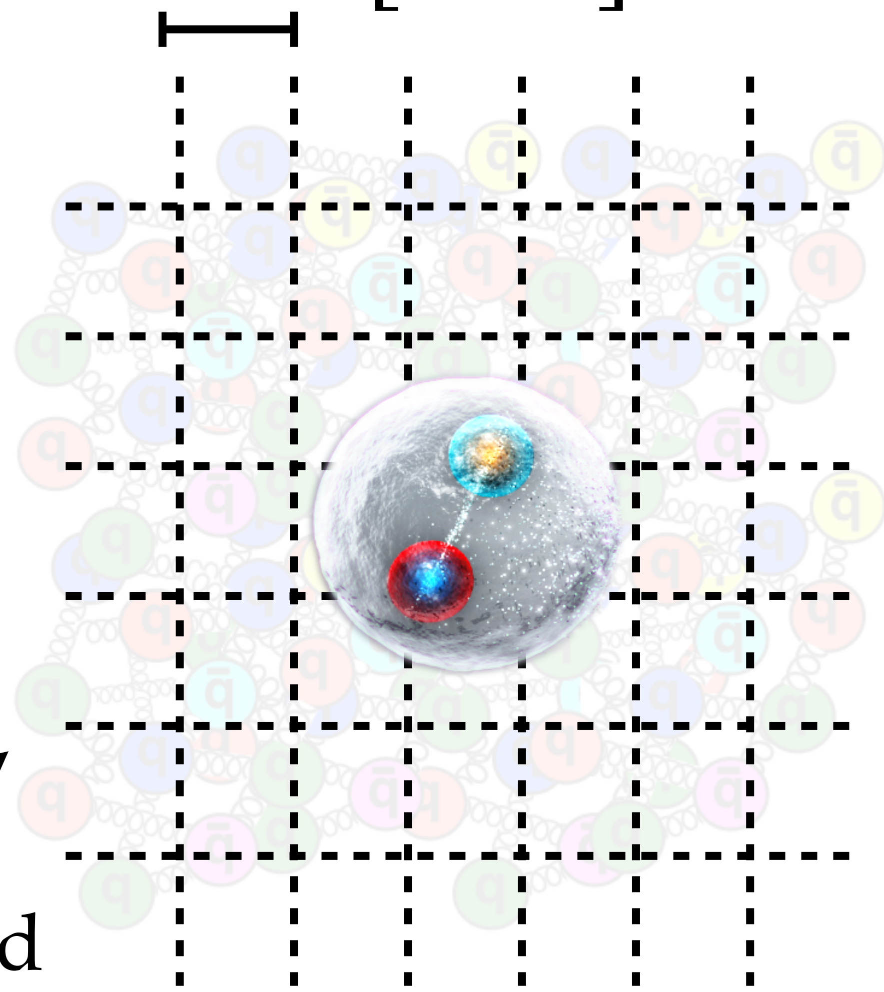
$\begin{pmatrix} 32^3 \times 64 \\ 48^3 \times 96 \end{pmatrix}$, 2+1 flavor (u/d+s)

$\beta = \begin{pmatrix} 5.50 \\ 5.65 \end{pmatrix}$, NP-improved Clover action

Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]

$m_\pi \sim \begin{bmatrix} 470 \\ 420 \end{bmatrix}$ MeV, ~SU(3) sym.

$m_\pi L \sim \begin{bmatrix} 5.6 \\ 6.9 \end{bmatrix}$ $\quad a = \begin{bmatrix} 0.074 \\ 0.068 \end{bmatrix}$ fm

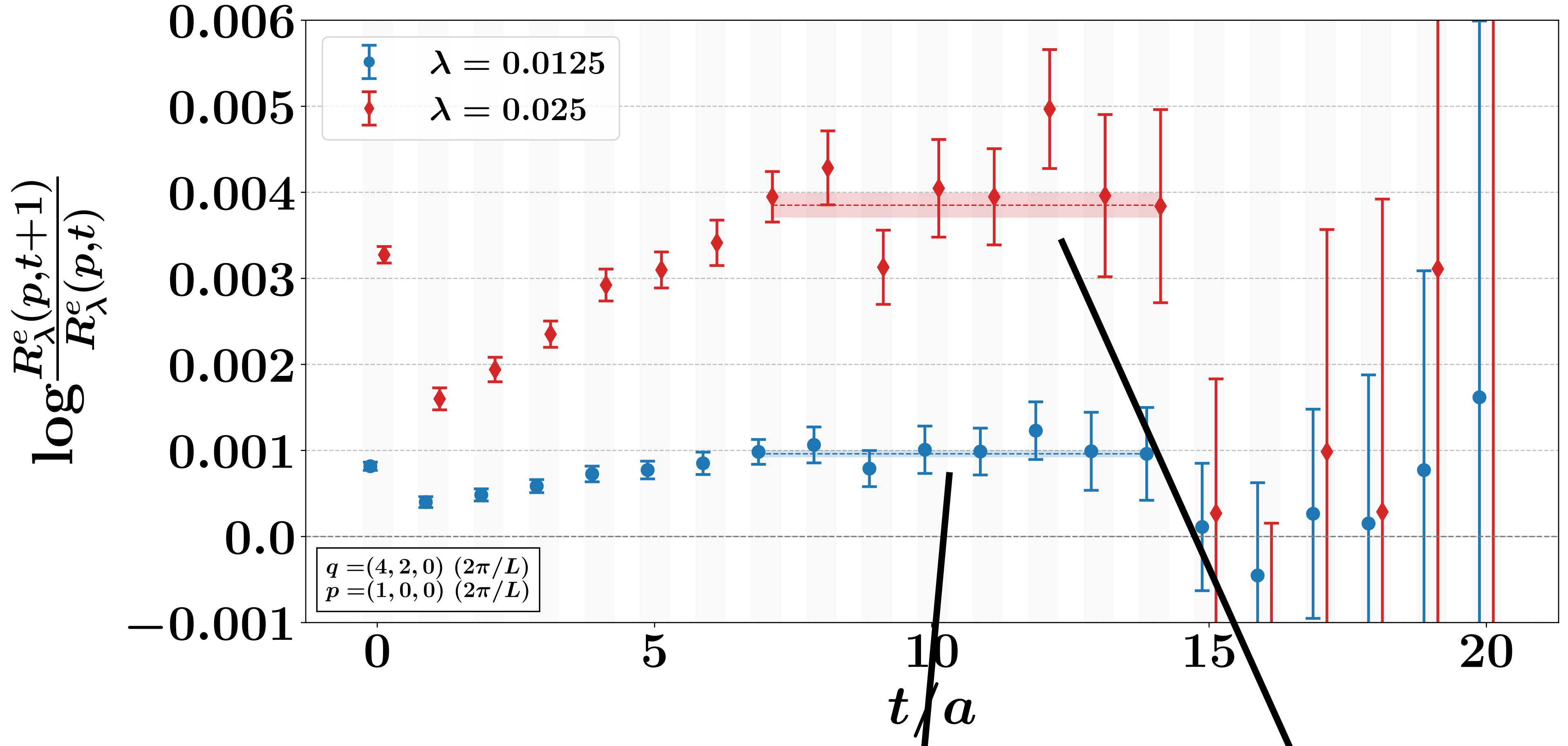


Unmodified QCD background

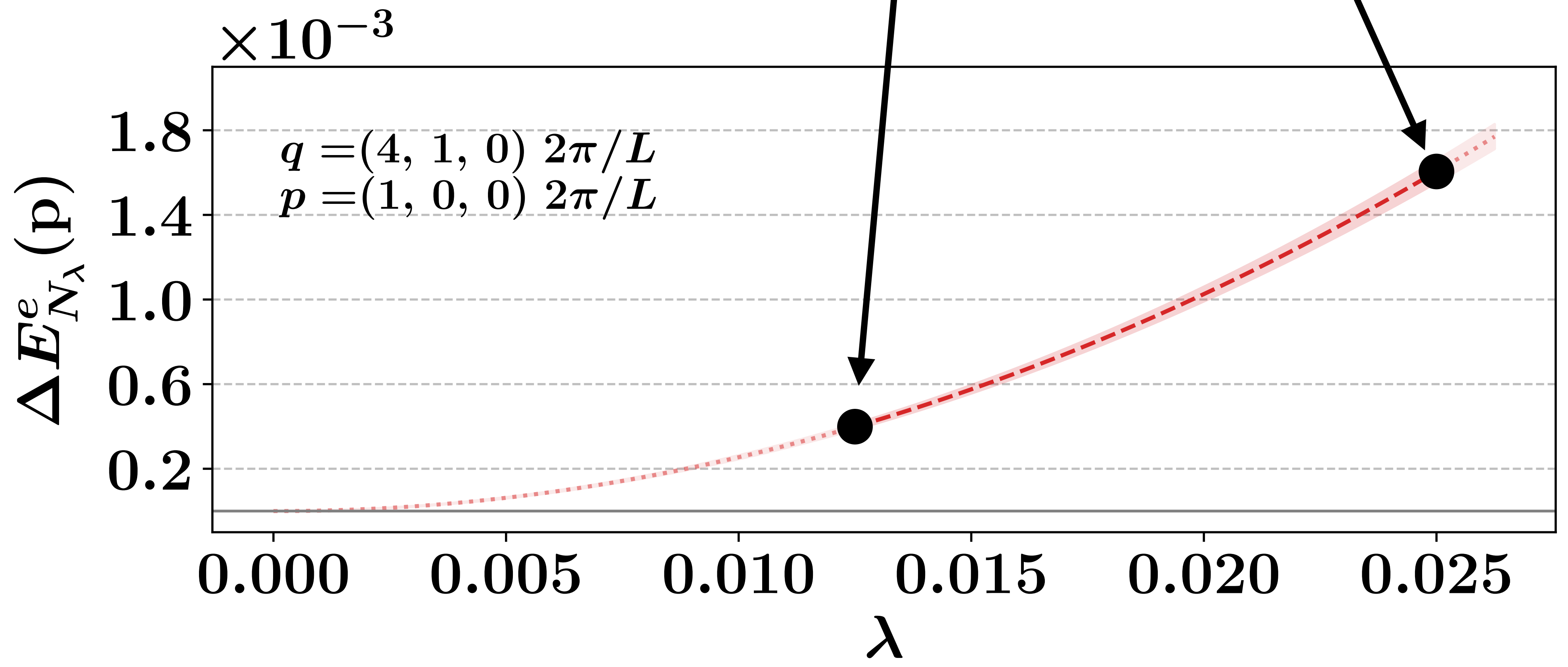
- FH implementation at the valence quark level
  - Valence u/d quark props with modified action, $S(\lambda)$
  - Local EM current insertion, $J_\mu(x) = Z_V \bar{q}(x)\gamma_\mu q(x)$
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- Several current momenta in the range, $1.5 \lesssim Q^2 \lesssim 7\, GeV^2$
- Up to $\mathcal{O}(10^4)$ measurements for each pair of Q^2 and λ
- Access to a range of $\omega = 2p\cdot q/Q^2$ values for several (p, q) pairs
  - An inversion for each q and λ, varying p is relatively cheap
- Connected 2-pt correlators calculated only, no disconnected

Strategy | Energy shifts

- **Extract energy shifts for each λ**



- **Get the 2nd order derivative**



Ratio of perturbed to unperturbed 2-pt functions

$$R^e_\lambda(\mathbf{p},t) \equiv \frac{G^{(2)}_{+\lambda}(\mathbf{p},t)\,G^{(2)}_{-\lambda}(\mathbf{p},t)}{\left(G^{(2)}(\mathbf{p},t)\right)^2}$$

$$\xrightarrow{t \gg 0} A_\lambda(\mathbf{p}) e^{-2\Delta E^e_{N_\lambda}(\mathbf{p})t}$$

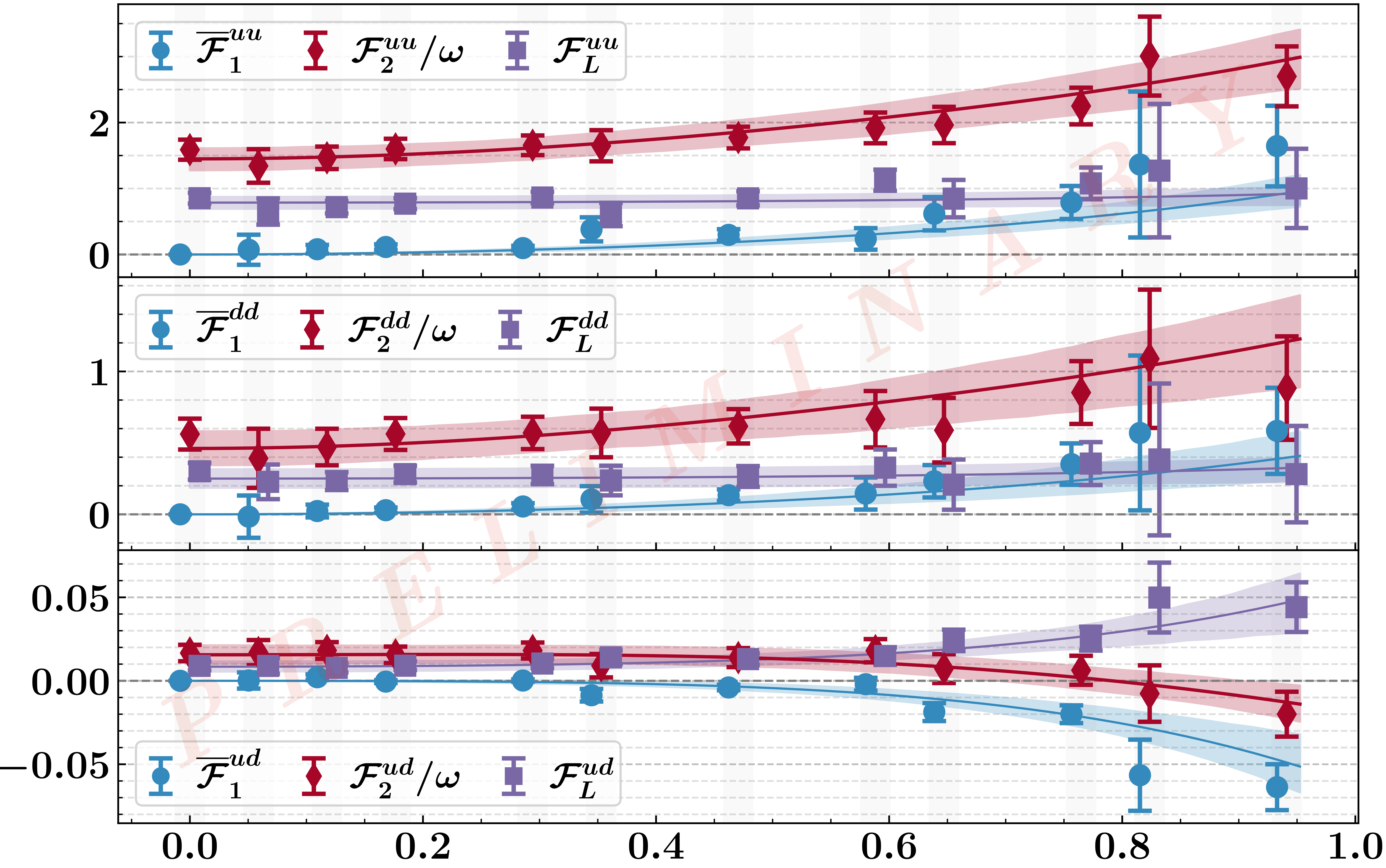
Isolates 2nd-order energy shift by construct considering,

$$G^{(2)}_\lambda(\mathbf{p};t) \sim A_\lambda(\mathbf{p}) e^{-E_{N_\lambda}(\mathbf{p})t}$$

$$E_{N_\lambda}(\mathbf{p}) = E_N(\mathbf{p}) + \lambda \left.\frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial \lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2!} \left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial^2 \lambda}\right|_{\lambda=0} + \mathcal{O}(\lambda^3)$$

$$= E_N(\mathbf{p}) + \Delta E^o_N(\mathbf{p}) + \Delta E^e_N(\mathbf{p})$$

Compton Structure Functions



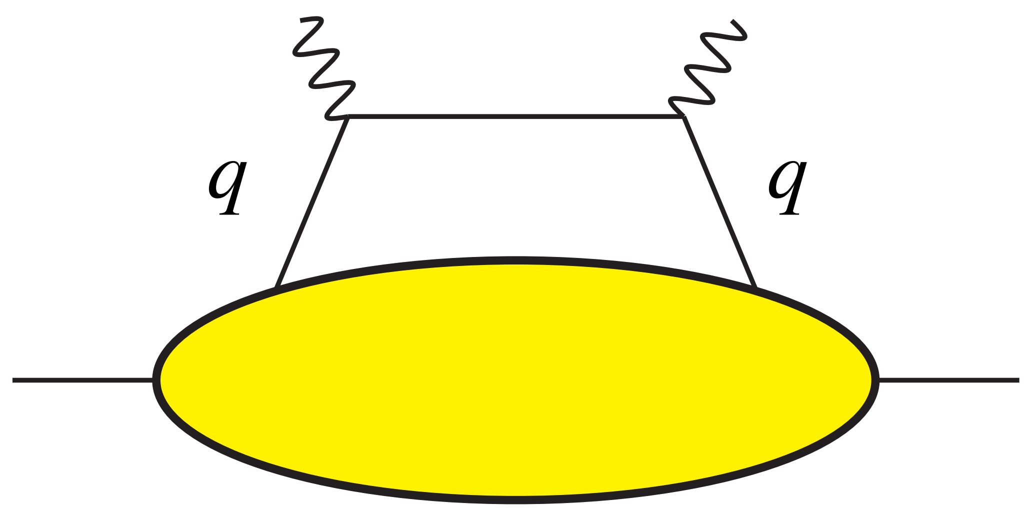
$\omega = 2p \cdot q/Q^2$

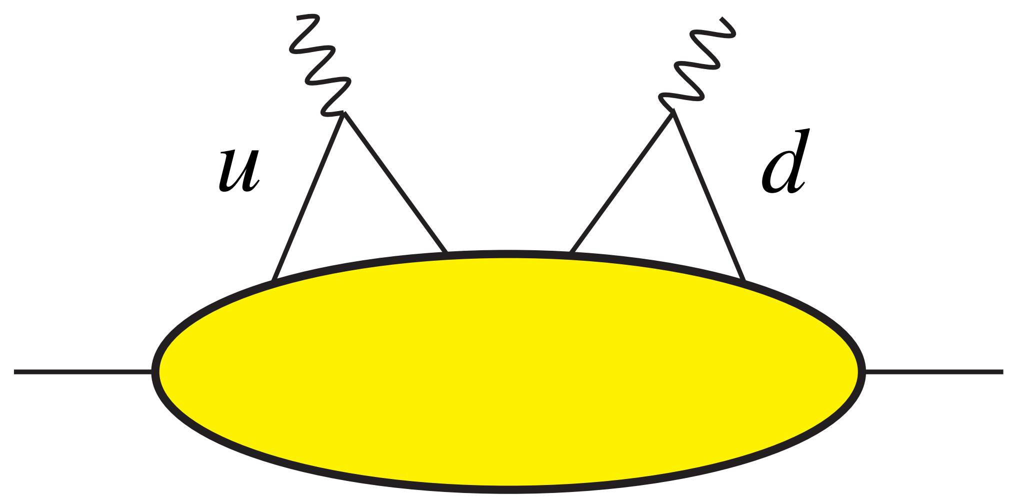
48^3x96, 2+1 flavour

$a = 0.068\,\mathrm{fm}$

$m_\pi \sim 420\,\mathrm{MeV}$

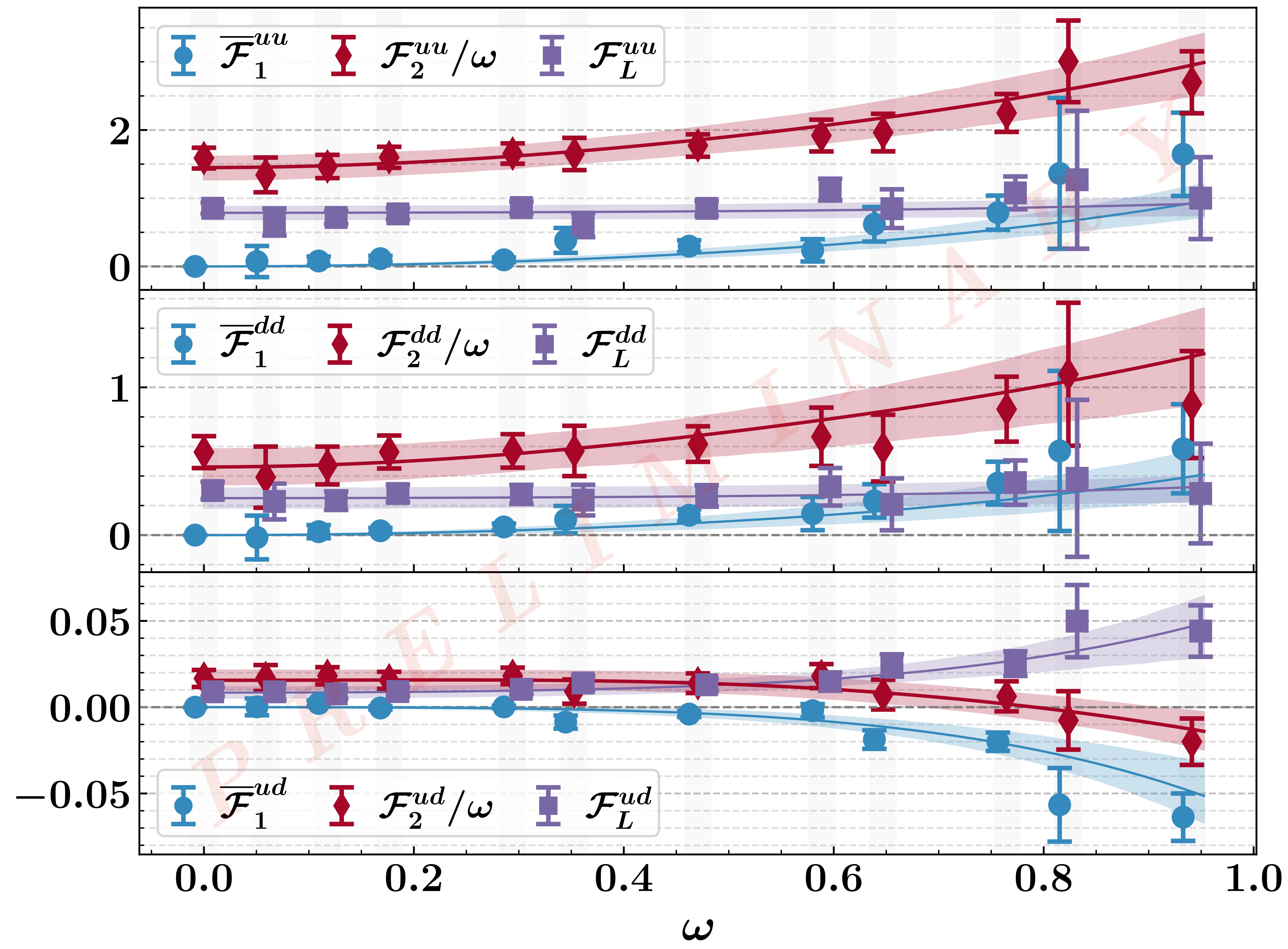
$Q^2 = 4.9\,GeV^2$





Moments | Fit details

48^3x96, 2+1 flavour
$a = 0.068$ fm
$m_\pi \sim 420$ MeV



$$\overline{\mathcal{F}}_1^{qq}(\omega, Q^2) = 2\sum_{n=1}^{\infty} M_{2n}^{(1)}(Q^2)\,\omega^{2n}$$

$$\frac{\mathcal{F}_2^{qq}(\omega, Q^2)}{\omega} = \frac{\tau}{1+\tau\omega^2}\sum_{n=0}^{\infty} 4\omega^{2n}\left[M_{2n}^{(1)} + M_{2n}^{(L)}\right](Q^2), \text{ where } \tau = \frac{Q^2}{4M_N^2}$$

- **Enforce monotonic decreasing of moments for uu and dd only, $|ud|^2 \leq 4uu * dd$**

$$M_2(Q^2) \geq M_4(Q^2) \geq \cdots \geq M_{2n}(Q^2) \geq \cdots \geq 0$$

We truncate at $n = 6$
No dependence to truncation order for $3 \leq n \leq 10$

- **Bayesian approach by MCMC method**

Sample the moments from Uniform priors
individually for u- and d-quark

$$M_2(Q^2) \sim \mathcal{U}(0, 1)$$
$$M_{2n}(Q^2) \sim \mathcal{U}\left(0, M_{2n-2}(Q^2)\right)$$

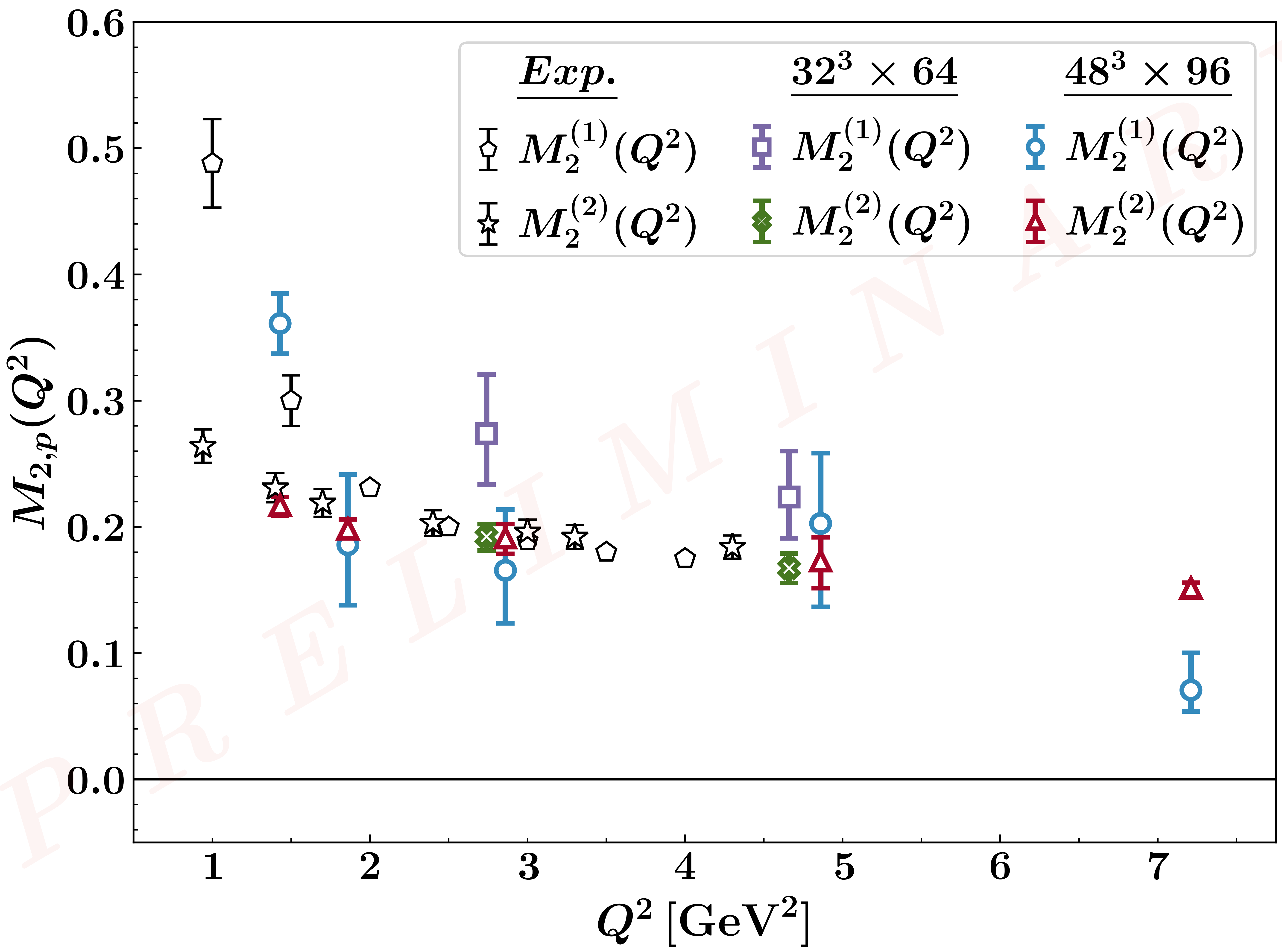
Normal Likelihood function, $exp(-\chi^2/2)$

$$\chi^2 = \sum_i \frac{\left(\overline{\mathcal{F}}_i - \overline{\mathcal{F}}^{obs}(\omega_i)\right)^2}{\sigma_i^2}$$

errors via bootstrap analysis

Moments of $F_{1,2}(x, Q^2)$

- **Unique ability to study the Q^2 dependence of the moments!**



- Lowest moments of $F_{1,2}(x, Q^2)$

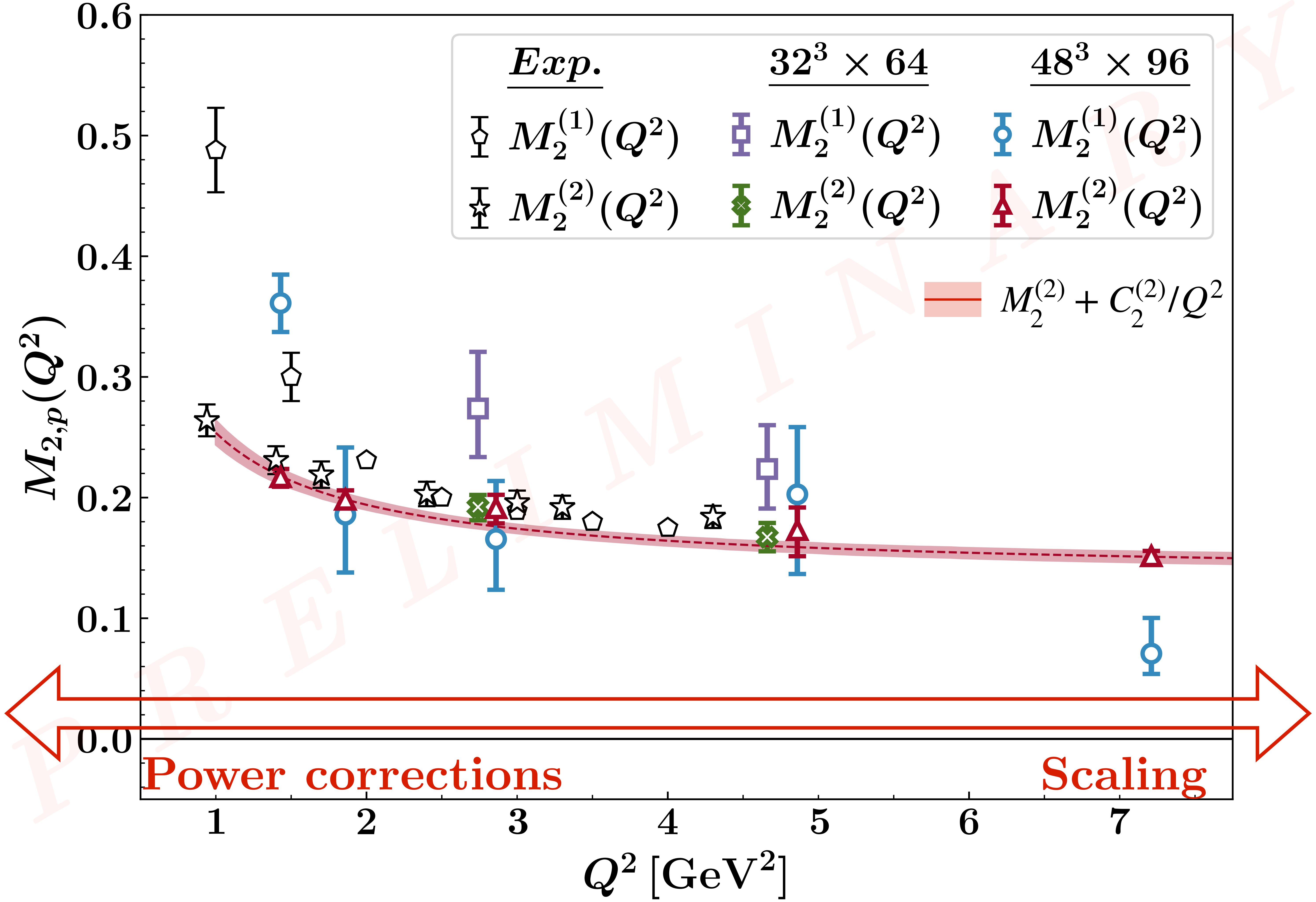
$$M_{2,p}^{(1,2)} = \frac{4}{9} M_{2,uu}^{(1,2)} + \frac{1}{9} M_{2,dd}^{(1,2)} - \frac{2}{9} M_{2,ud}^{(1,2)}$$

Exp $M_2^{(1)}$: W. Melnitchouk, R. Ent, and C. Keppel, Phys. Rept. 406, 127 (2005), arXiv:hep-ph/0501217.

Exp $M_2^{(2)}$: C. S. Armstrong, R. Ent, C. E. Keppel, S. Liuti, G. Niculescu, and I. Niculescu, Phys. Rev. D 63, 094008 (2001), arXiv:hep-ph/0104055.

Scaling and Power Corrections

- **Unique ability to study the Q^2 dependence of the moments!**



- Need $Q^2 \gtrsim 10\,GeV^2$ data to reliably constrain the partonic moments

- Power corrections below $\sim 3\,GeV^2$?
  - Naive modelling via
  - $M_2^{(2)}(Q^2) = M_2^{(2)} + C_2^{(2)}/Q^2$
  - $C_2^{(2)}$ contains:
    - TMC, elastic cont. ($x = 1$), $\ln Q^2$ scaling, and twist-4

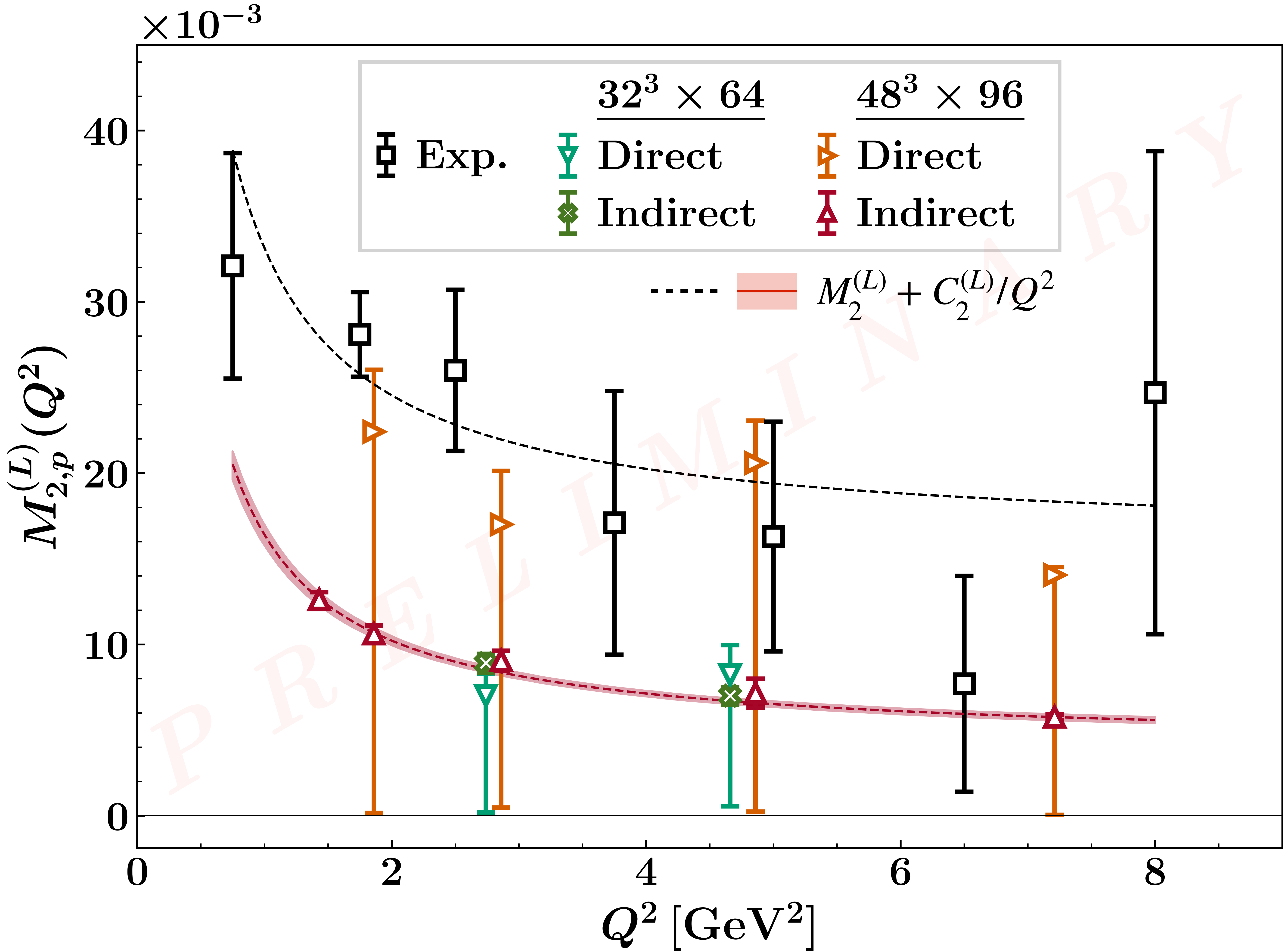
Exp $M_2^{(1)}$: W. Melnitchouk, R. Ent, and C. Keppel, Phys. Rept. 406, 127 (2005), arXiv:hep-ph/0501217.

Exp $M_2^{(2)}$: C. S. Armstrong, R. Ent, C. E. Keppel, S. Liuti, G. Niculescu, and I. Niculescu, Phys. Rev. D 63, 094008 (2001), arXiv:hep-ph/0104055.

Moments of $F_L(x, Q^2)$

Possible for the first time in a lattice QCD simulation!

- **Unique ability to study the moments of F_L!**



- $F_L(x, Q^2) \equiv \left(1 + \frac{4M_N^2 x^2}{Q^2}\right) F_2(x, Q^2) - 2xF_1(x, Q^2)$

$$\xrightarrow[Q^2 \to \infty]{} \mathcal{O}(\alpha_s(Q^2))$$

- **Direct:** Fit to data points
  - Determines upper bounds
- **Indirect:** Use the moments of F_2:
  - Leading twist contribution
  - $M_2^{(L),LT}(Q^2) = \frac{4}{9\pi}\alpha_s(Q^2)\, M_2^{(2)}(Q^2)$
  - Better precision, good agreement with exp. behaviour

Exp Nachtmann $M_2^{(L)}$: P. Monaghan, A. Accardi, M. E. Christy, C. E. Keppel, W. Melnitchouk, and L. Zhu, Phys. Rev. Lett. 110, 152002 (2013), arXiv:1209.4542 [nucl-ex].

Outlook

Polarised Structure Functions

$$T_{\mu\nu}(p,q,s) = i\varepsilon^{\mu\nu\alpha\beta}\frac{q_\alpha}{p\cdot q}\left[s_\beta\,\tilde{g}_1(\omega,Q^2) + \left(s_\beta - \frac{s\cdot q}{p\cdot q}p_\beta\right)\tilde{g}_2(\omega,Q^2)\right]$$

- Similar to the unpolarised case, we can extract $\tilde{g}_1$ and $\tilde{g}_2$
- Lowest moment of $g_1(x)$ is related to nucleon axial charge

$$\Gamma_1(Q^2) = \int_0^1 g_1^{(u-d)}(x,Q^2)\,dx = \underbrace{\left(\Delta u - \Delta d\right)}_{\equiv g_A} C_1(\alpha_s(Q^2))$$

where, $C_1(\alpha_s(Q^2)) = 1 - \dfrac{\alpha_s(Q^2)}{\pi} - \mathcal{O}(\alpha_s^2)$

- $g_2(x)$ is twist-3, holds information on quark-gluon correlations
- Wandzura-Wilczek decomposition

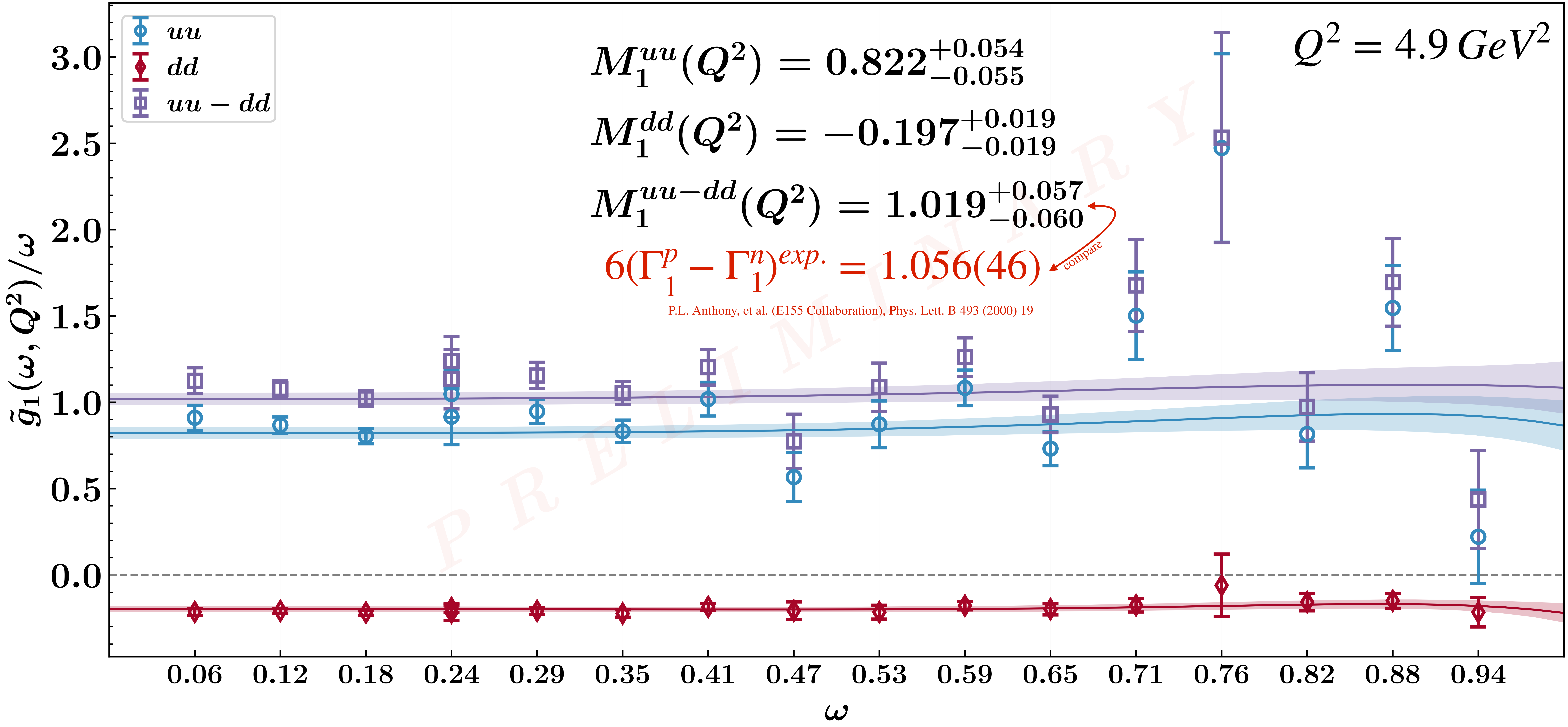
$$g_2(x,Q^2) = \underbrace{-g_1(x,Q^2) + \int_x^1 \frac{dy}{y}\,g_1(y,Q^2)}_{g_2^{WW}(x,Q^2)} + \bar{g}_2(x,Q^2)$$

- The Buckhardt — Cottingham sum rule

$$\int_0^1 g_2(x,Q^2)\,dx = 0$$

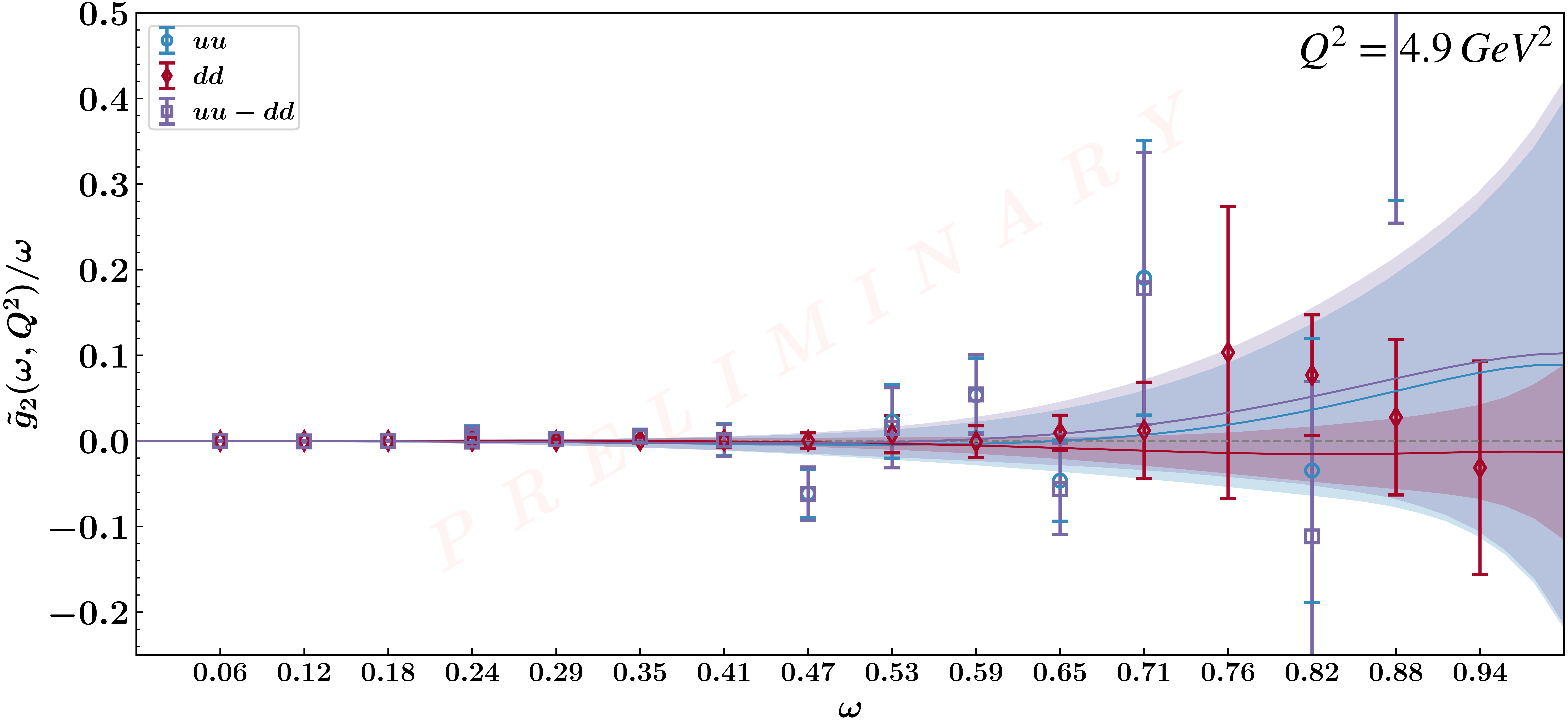
Polarised Structure Functions

48^3x96, 2+1 flavour
$a = 0.068$ fm
$m_\pi \sim 420$ MeV



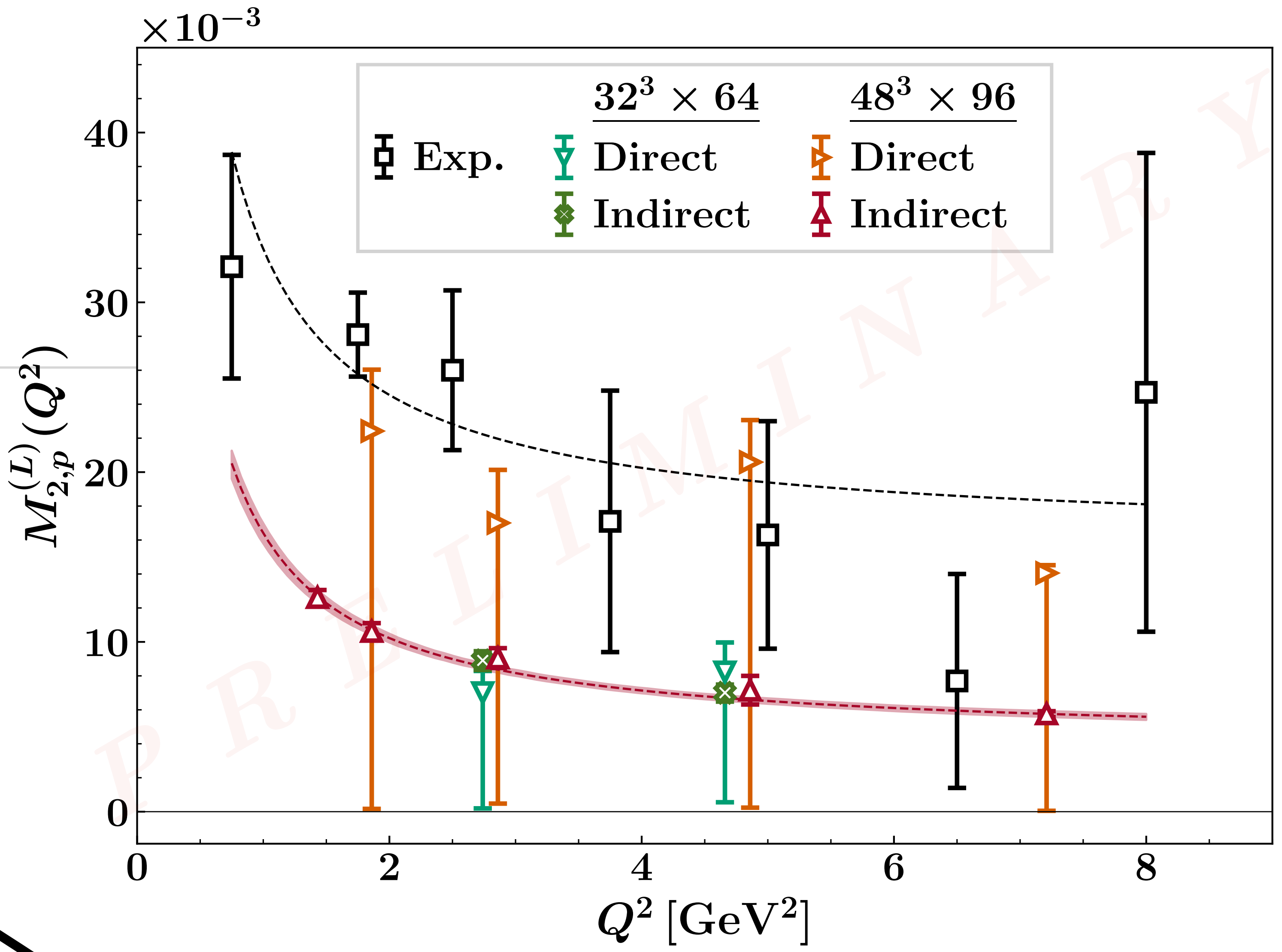
Polarised Structure Functions

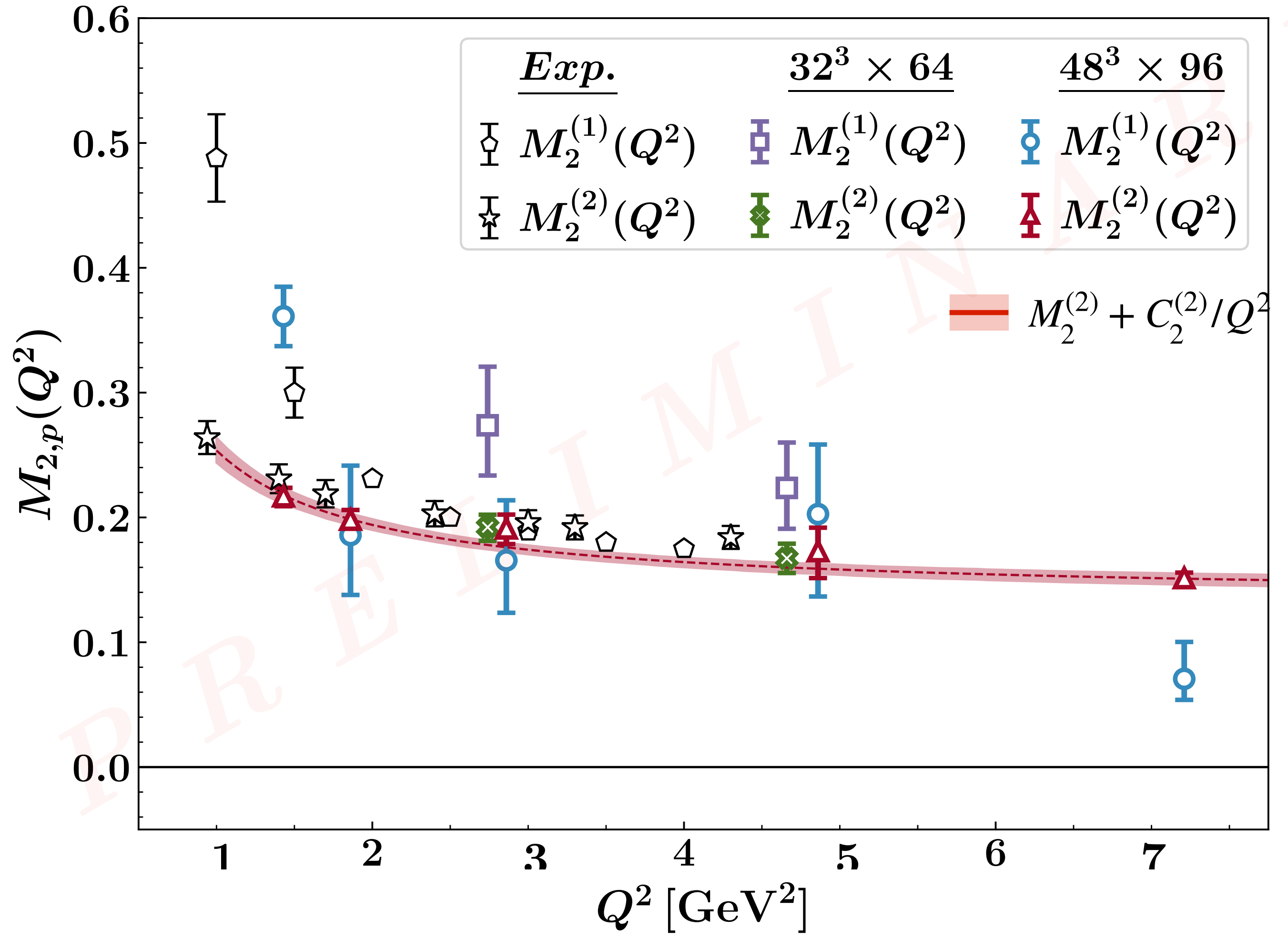
48^3x96, 2+1 flavour
$a = 0.068\,\mathrm{fm}$
$m_\pi \sim 420\,\mathrm{MeV}$

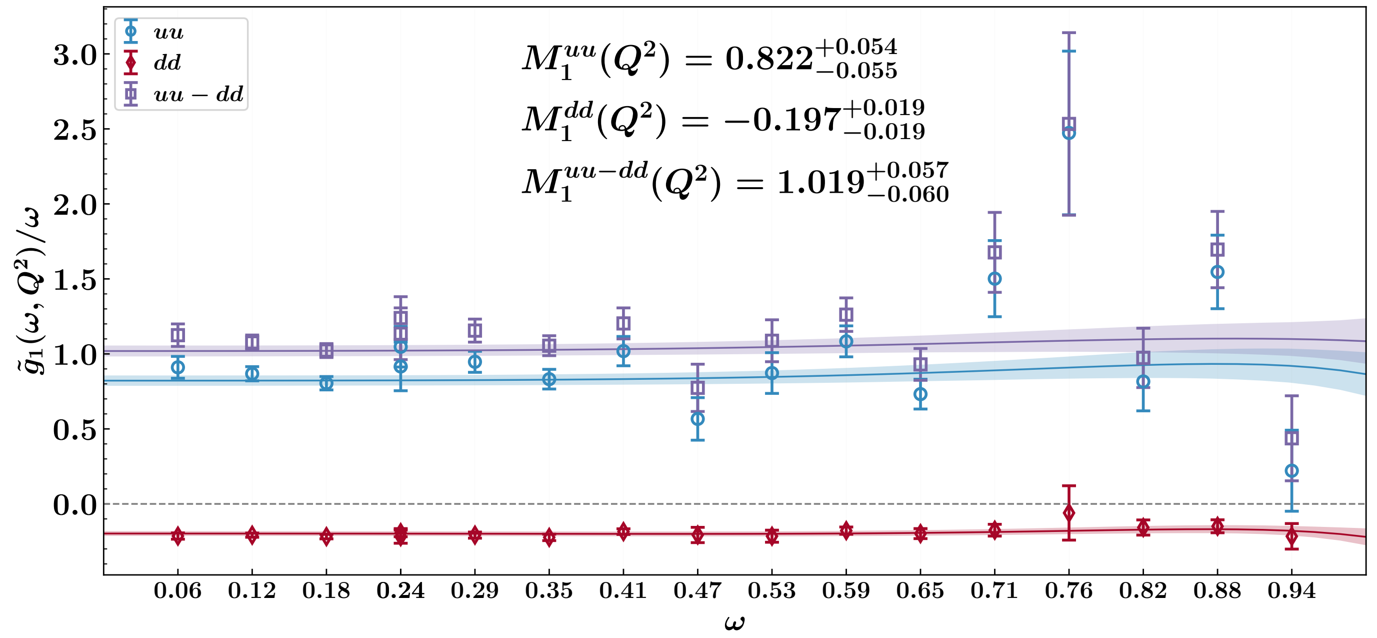


Summary

- A versatile approach! F_1, F_2, F_L and g_1, g_2
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Overcomes the operator mixing/renormalisation issues
- Can be extended to:
  - mixed currents, interference terms
  - spin-dependent structure functions (ongoing)
  - GPDs: A. Hannaford-Gunn et al. Phys. Rev. D **105**, 014502
    see Alec's talk on 10/08 (Wed) @ 18:10 Hadron Structure



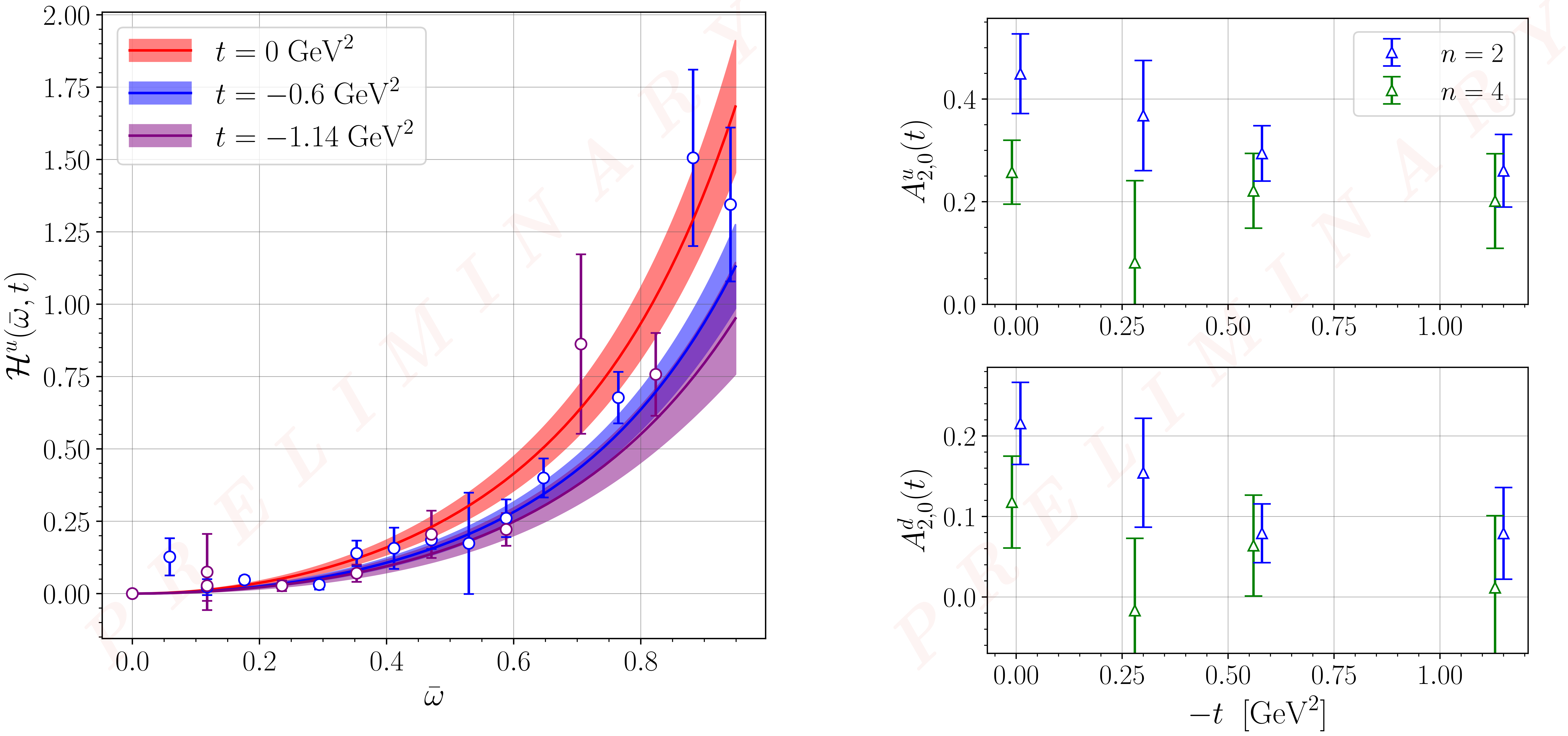




Teaser

off-forward kinematics:

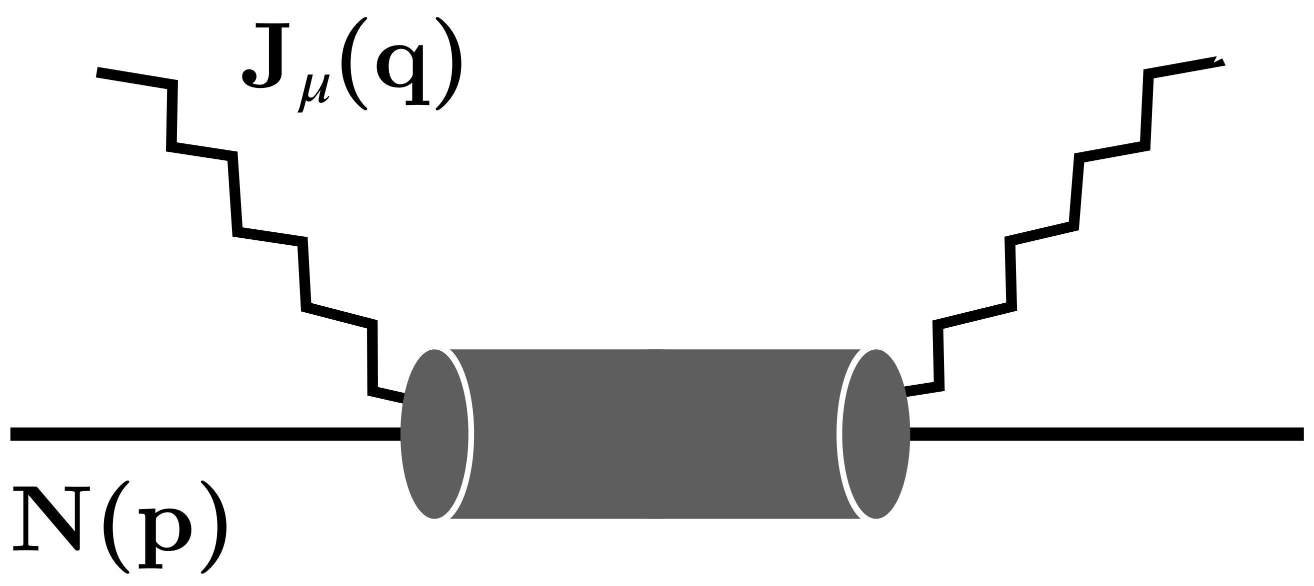
see Alec's talk on 10/08 (Wed) @ 18:10 Hadron Structure



Thank you

Backup

Compton amplitude via the FH relation at 2nd order



- **unpolarised Compton Amplitude**

$$T_{\mu\mu}(p,q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) | \mathscr{T}\{J_\mu(z) J_\mu(0)\} | N(p)\rangle$$

4-pt function

- **Action modification**

$$S \to S(\lambda) = S + \lambda \int d^4z \,(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})\, J_\mu(z)$$

local EM current

$$J_\mu(z) = \sum_q e_q \bar{q}(z)\gamma_\mu q(z)$$

@ 2nd order

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) | \mathscr{T}\{J_\mu(z) J_\mu(0)\} | N(p)\rangle}^{T_{\mu\mu}(p,q)} + (q \to -q)$$

Determine the Compton Amplitude from second order energy shifts!

Compton amplitude via the FH relation at 2nd order

- **Spectral decomposition of a 2-point nucleon correlator in an external field, Ω_λ,**

$$G_\lambda^{(2)}(\mathbf{p};t) \equiv \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \mathbf{\Gamma} \langle \Omega_\lambda | \chi(\mathbf{x},t) \bar{\chi}(0) | \Omega_\lambda \rangle \simeq A_\lambda(\mathbf{p}) e^{-E_{N_\lambda}(\mathbf{p})t}$$

- **Take the 2nd order derivative,**

Non-Breit frame, $|\mathbf{p}| \neq |\mathbf{p} \pm \mathbf{q}| \Rightarrow 0$

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial \lambda^2}\right|_{\lambda=0} = e^{-E_N(\mathbf{p})t} \left[\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial \lambda^2} - t \left(\cancel{2 \frac{\partial A_\lambda(\mathbf{p})}{\partial \lambda} \frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial \lambda}} + A(\mathbf{p}) \frac{\partial^2 E_{N_\lambda}}{\partial \lambda^2} \right) + \cancel{t^2 A(\mathbf{p}) \left(\frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial \lambda} \right)^2} \right]$$

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial \lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial \lambda^2} - tA(\mathbf{p}) \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right) e^{-E_N(\mathbf{p})t}$$

quadratic energy shift

temporal enhancement $\sim t\, e^{-E_N(\mathbf{p})t}$

Compton amplitude via the FH relation at 2nd order

- **2-point nucleon correlator in path integral formalism,**

$$ {}_{\lambda}\langle \chi(\mathbf{x},t)\bar{\chi}(0)\rangle_{\lambda} = \frac{1}{\mathcal{Z}(\lambda)}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}U \chi(\mathbf{x},t)\bar{\chi}(0)e^{-S(\lambda)} $$

, where

$$ S(\lambda) = S + \lambda\int d^4z(e^{iq\cdot z}+e^{-iq\cdot z})\mathcal{J}_{\mu}(z) $$

for simplicity define: $\mathcal{G} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\boldsymbol{\Gamma}\chi(\mathbf{x},t)\bar{\chi}(0)$

- **Take the 2nd order derivative,**

$$ \frac{\partial^2\langle\mathcal{G}\rangle_\lambda}{\partial\lambda^2} = \cancel{\langle\mathcal{G}\rangle_\lambda\left\langle\frac{\partial^2 S(\lambda)}{\partial\lambda^2}\right\rangle_\lambda} + \cancel{\left\langle\mathcal{G}\frac{\partial^2 S(\lambda)}{\partial\lambda^2}\right\rangle_\lambda} + \boxed{\langle\mathcal{G}\rangle_\lambda\left\langle\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle_\lambda} + \cancel{2\langle\mathcal{G}\rangle_\lambda\left\langle\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda\left\langle\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda} - \cancel{2\left\langle\mathcal{G}\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda\left\langle\frac{\partial S(\lambda)}{\partial\lambda}\right\rangle_\lambda} + \left\langle\mathcal{G}\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle_\lambda $$

no quadratic perturbation = 0

does not vanish in general, but only affects the free-field correlator

as $\lambda \to 0$, vacuum m.e. of ext. current $\langle\partial S(\lambda)/\partial\lambda\rangle = 0$, given that the operator does not carry vacuum quantum numbers. EM current satisfies this condition.

- **Thus (in the limit $\lambda \to 0$) the second order energy shift arises from,**

$$ \left.\frac{\partial^2\langle\mathcal{G}\rangle_\lambda}{\partial\lambda^2}\right|_{\lambda=0} = \left\langle\mathcal{G}\left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2\right\rangle + \ldots $$

terms that are not time enhanced

Compton amplitude via the FH relation at 2nd order

- **back to full form,**

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};y)}{\partial\lambda^2}\right|_{\lambda=0} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \mathbf{\Gamma} \left\langle \chi(\mathbf{x},t)\bar{\chi}(0) \left(\frac{\partial S(\lambda)}{\partial\lambda}\right)^2 \right\rangle, \text{ where } \frac{\partial S(\lambda)}{\partial\lambda} = \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z})\mathcal{J}_\mu(z)$$

note that $\langle\cdots\rangle$ is evaluated in the absence of the external field

- **writing the 2nd order derivative explicitly,**

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \mathbf{\Gamma} \int d^4y d^4z (e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}) \langle \chi(\mathbf{x},t)\mathcal{J}_\mu(z)\mathcal{J}_\mu(y)\bar{\chi}(0)\rangle$$

need to resolve the time ordering of the currents

Compton amplitude via the FH relation at 2<u>nd</u> order

- **possible time orderings and their contributions:**

$\mathcal{J}(z_4) \quad \mathcal{J}(y_4) \quad \chi(t) \quad \bar{\chi}(0)$

$$\sim e^{-E_X t}, \quad E_X \gtrsim E_N$$

no time enhancement

$\mathcal{J}(z_4) \quad \chi(t) \quad \bar{\chi}(0) \quad \mathcal{J}(y_4)$

$$\sim e^{-E_X t}, \quad E_X \gtrsim E_N$$

no time enhancement

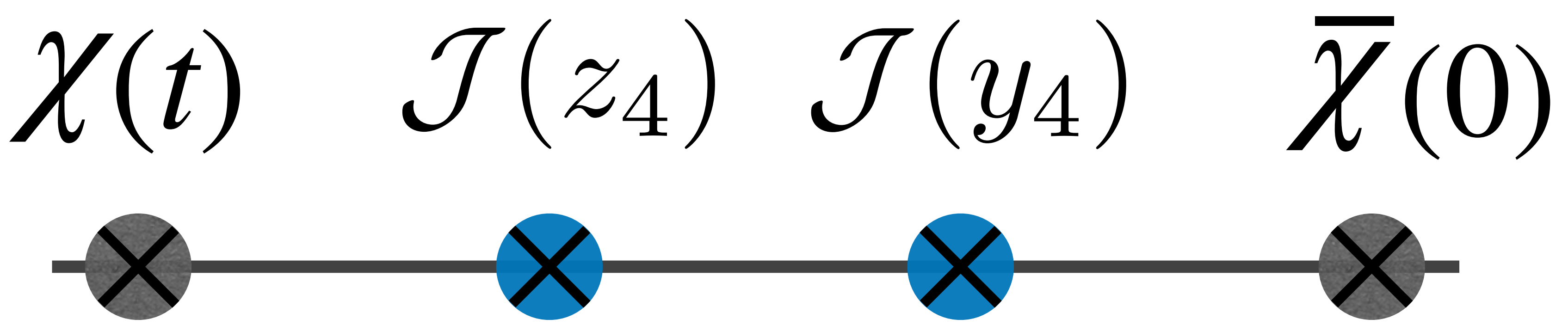
$\mathcal{J}(z_4) \quad \chi(t) \quad \mathcal{J}(y_4) \quad \bar{\chi}(0)$

$$\sim t\, e^{-E_N t} \frac{\partial E_N}{\partial \lambda} \to 0$$

there is time enhancement,
but due to non-Breit frame kinematics $\to 0$

Compton amplitude via the FH relation at 2$^{\underline{\text{nd}}}$ order

- **relevant contribution comes from the ordering where the currents are sandwiched**



$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = 2\int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \int d^3y d^3z \int_0^t d\tau' \int_0^{\tau'} d\tau (e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})\boldsymbol{\Gamma}\langle\chi(x)|\mathcal{J}_\mu(\mathbf{z},\tau')\mathcal{J}_\mu(\mathbf{y},\tau)|\bar{\chi}(0)\rangle$$

$\sum\!\!\!\!\!\!\int_X |X\rangle\langle X|$ $\sum\!\!\!\!\!\!\int_Y |Y\rangle\langle Y|$

insert sets of complete states, and use translational invariance,

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = 2\int d^3y d^3z \int_0^t d\tau' \int_0^{\tau'} d\tau \sum_{X,Y} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-E_X(\mathbf{p})t} e^{-(E_Y(\mathbf{k})-E_X(\mathbf{p}))\tau}}{4E_X(\mathbf{p})E_Y(\mathbf{k})} e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{y}} (e^{i\mathbf{q}\cdot\mathbf{y}} + e^{-i\mathbf{q}\cdot\mathbf{y}})(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})$$
$$\times\,\boldsymbol{\Gamma}\langle\Omega|\chi(0)|X(\mathbf{p})\rangle\langle X(\mathbf{p})|\mathcal{J}_\mu(\mathbf{z}-\mathbf{y},\tau'-\tau)\mathcal{J}_\mu(\mathbf{0},0)|Y(\mathbf{k})\rangle\langle Y(\mathbf{k})|\bar{\chi}(0)|\Omega\rangle.$$

carrying out the integrals and the remaining algebra,

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z})\langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

Compton amplitude via the FH relation at 2nd order

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - tA(\mathbf{p})\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

from path integral

- **equate the time-enhanced terms:**

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p})|\mathcal{J}(z)\mathcal{J}(0)|N(\mathbf{p})\rangle}^{T_{\mu\mu}(p,q)+T_{\mu\mu}(p,-q)}$$

Compton amplitude is related to the second-order energy shift

Recovering the x-dependence

- **determining the PDFs | x-coverage**

$$T_{33}(\omega,Q^2) = \overline{\mathcal{F}}_1(\omega,Q^2) = 4\omega^2\int_0^1 dx \frac{x\,F_1(x,Q^2)}{1-x^2\omega^2}$$ ← formalism in $\boldsymbol{\omega}$ space

$$\equiv \int_0^1 dx K(x,\omega) F_1(x,Q^2),$$ ← back to $\boldsymbol{x}$ space, **inverse problem!**

  - Fredholm integral eq. of the 1st kind: an ill-posed problem

- **starting from the phenom. ansatz**

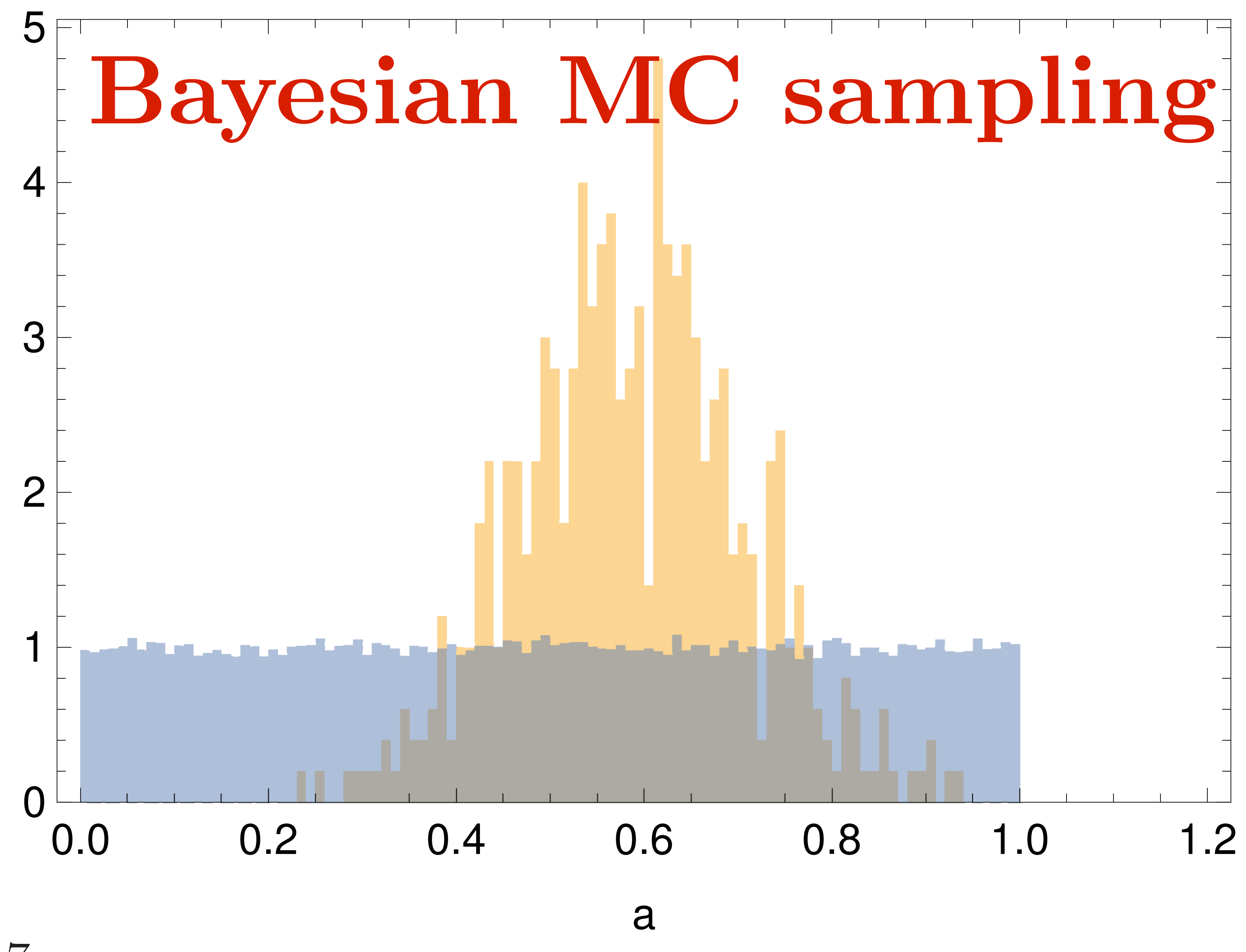
$$F_1(x,Q^2) \equiv p^{\mathrm{val}}(a,b,c) = \frac{a\,x^b\,(1-x)^c\Gamma(b+c+3)}{\Gamma(b+2)\Gamma(c+1)}$$

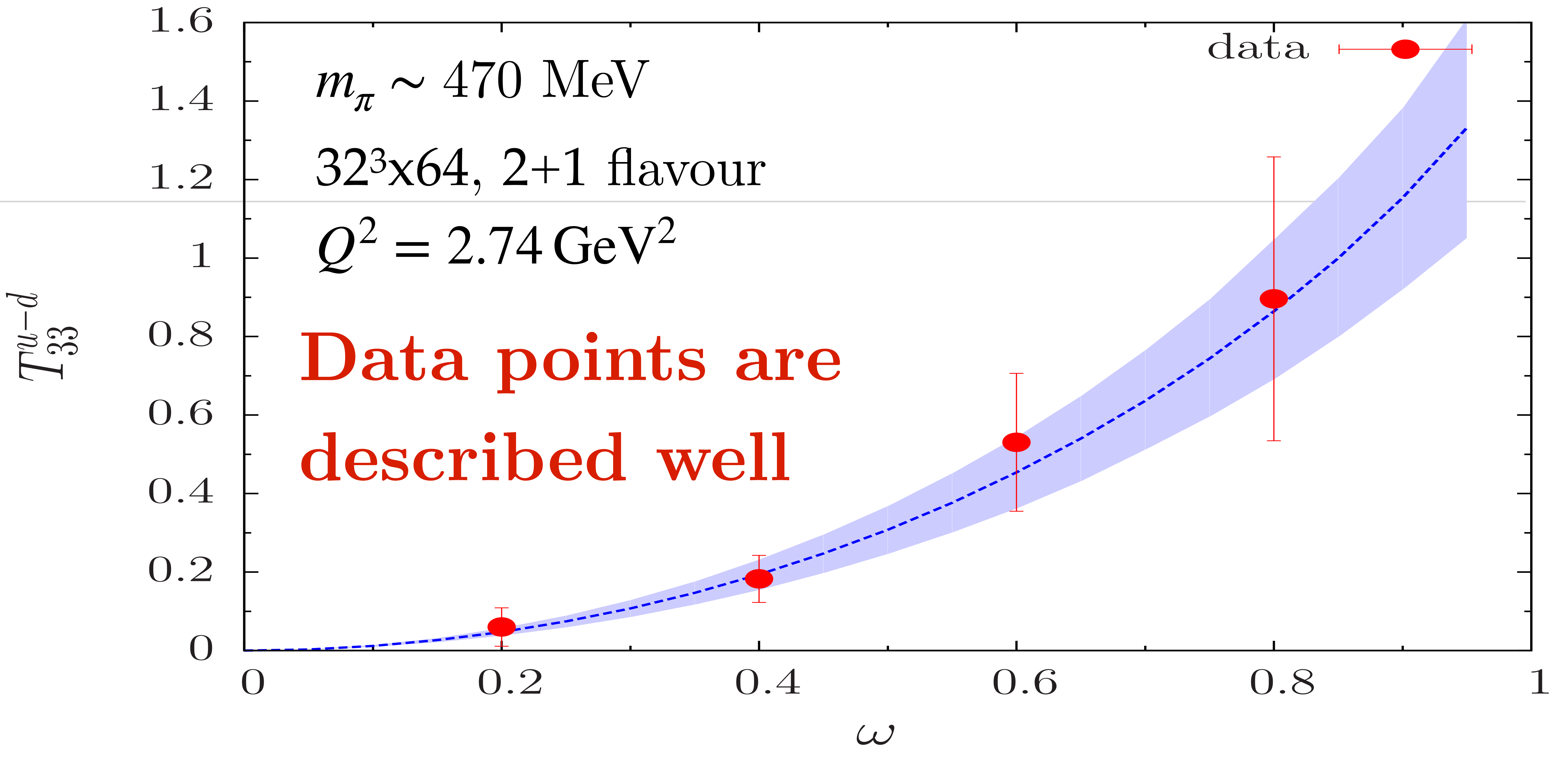
evaluate the dispersion integral

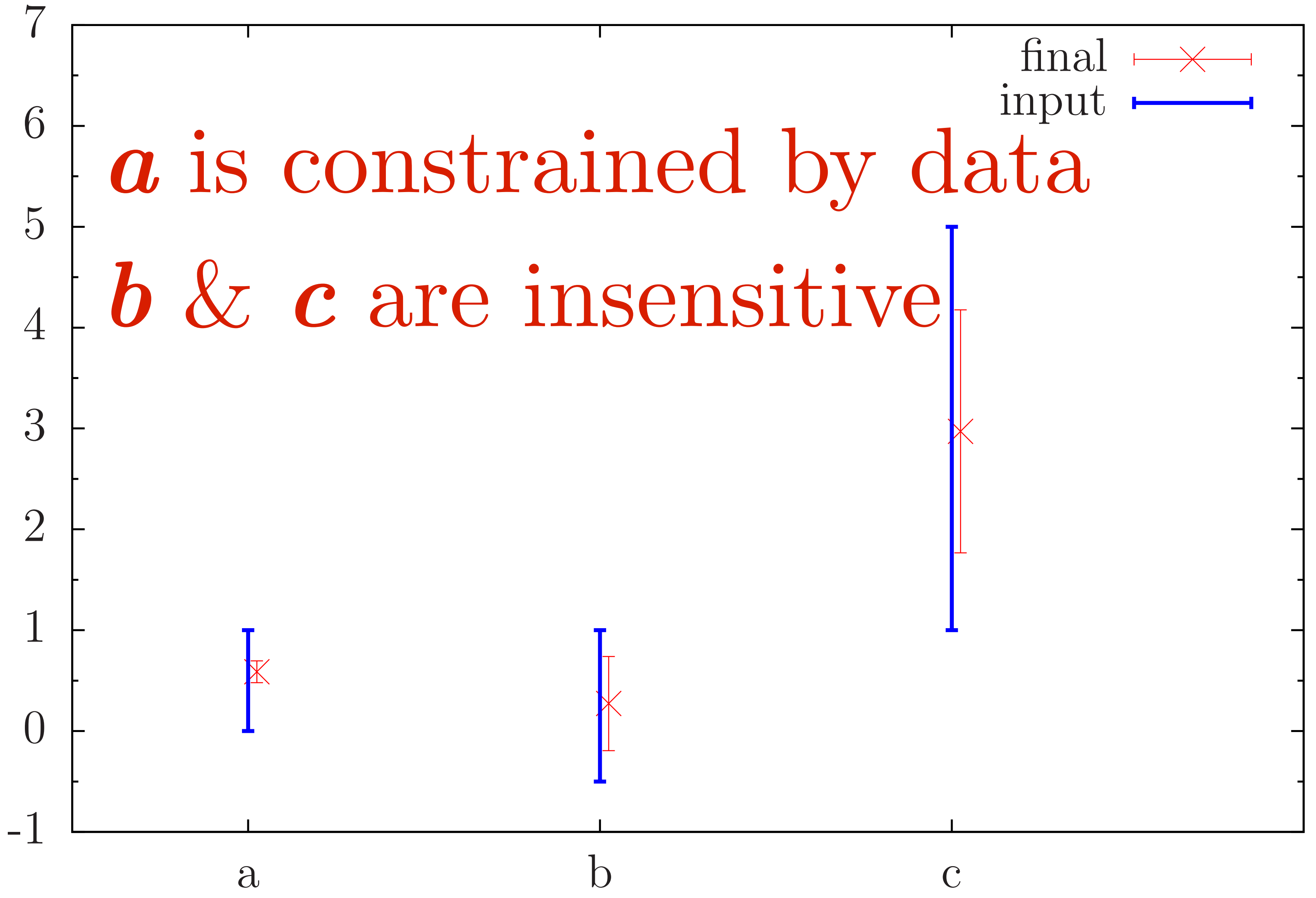
$$T_{33}^{\mathrm{val}}(\omega) = 4a\omega^2\,{}_3F_2\left[\begin{matrix}1,\ (b+2)/2,\ (b+3)/2 \\ (b+c+3)/2,\ (b+c+4)/2\end{matrix};\omega^2\right] = 4\,a\,\omega^2\big(\ c_0(a,b,c) + c_1(a,b,c)\,\omega^2 + c_2(a,b,c)\,\omega^4 + \cdots + c_n(a,b,c)\,\omega^{2n} + \cdots\big)$$

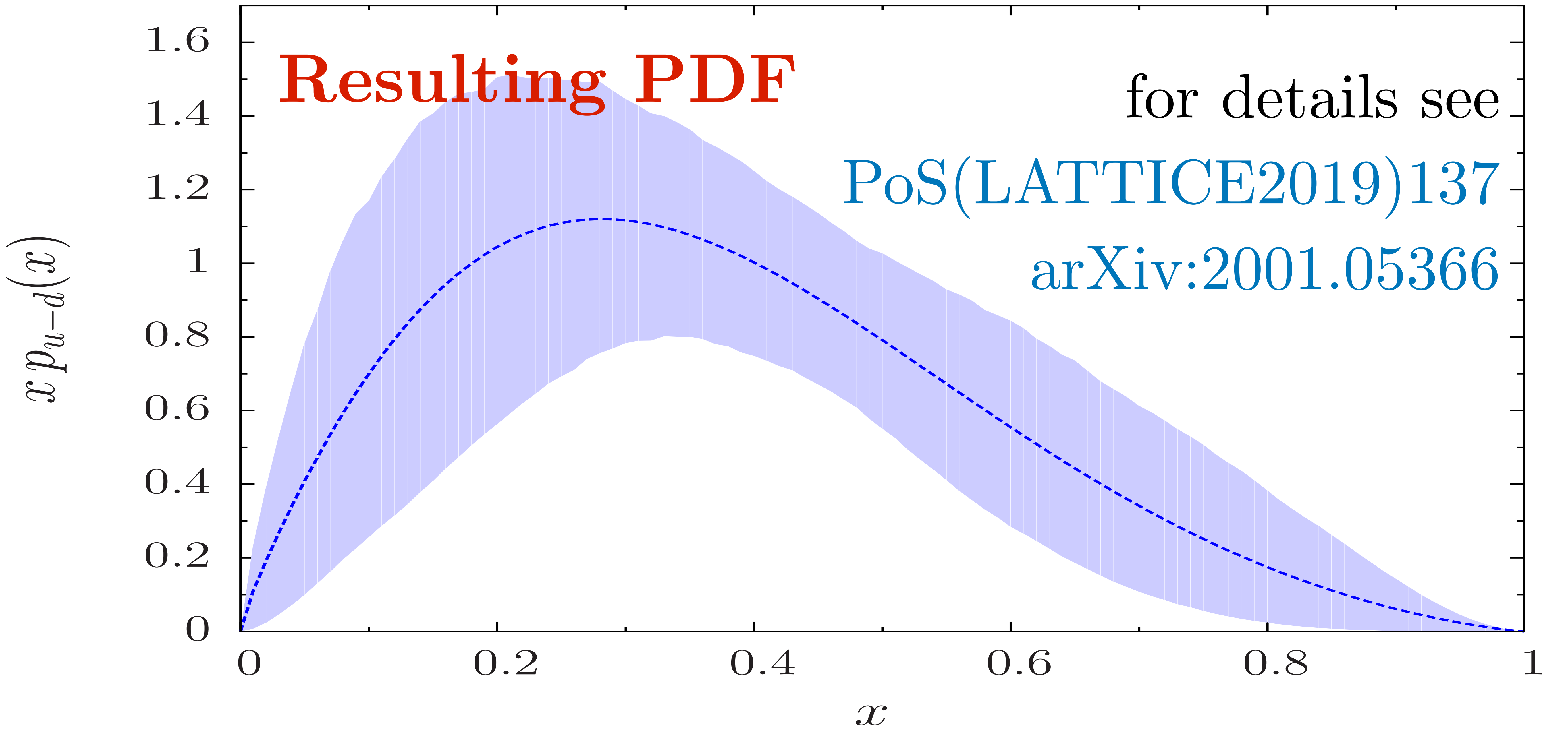
generalised hypergeometric function

PDFs


Bayesian MC sampling
a


data
$m_\pi \sim 470$ MeV
32^3x64, 2+1 flavour
$Q^2 = 2.74\,\mathrm{GeV}^2$
Data points are described well
T_{33}^{u-d}
ω


final
input
a is constrained by data
b & c are insensitive


Resulting PDF
for details see
PoS(LATTICE2019)137
arXiv:2001.05366
$x\,p_{u-d}(x)$
x