EXPLORING THE INFRARED STRUCTURE OF MASSLESS GAUGE THEORIES

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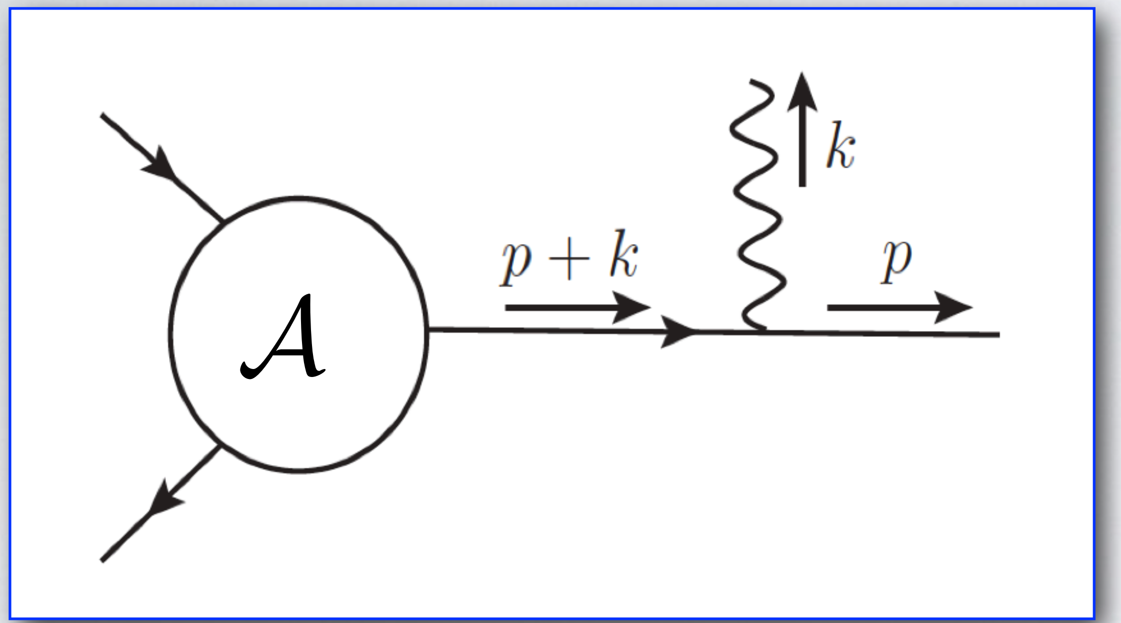
Outline

- Infrared factorisation of scattering amplitudes
- The subtraction problem
- A celestial viewpoint
- Outlook

INFRARED FACTORISATION

Textbook Infrared



Emission of a soft or collinear massless gauge boson

Singularities arise only when propagators go on shell

$$(p+k)^2 = 2p \cdot k = 2E_p w_k \left(1 - \cos\theta_{pk}\right) = 0$$
$$\Longrightarrow w_k = 0 \ \text{(soft)}; \quad \cos\theta_{pk} = 1 \ \text{(collinear)}$$

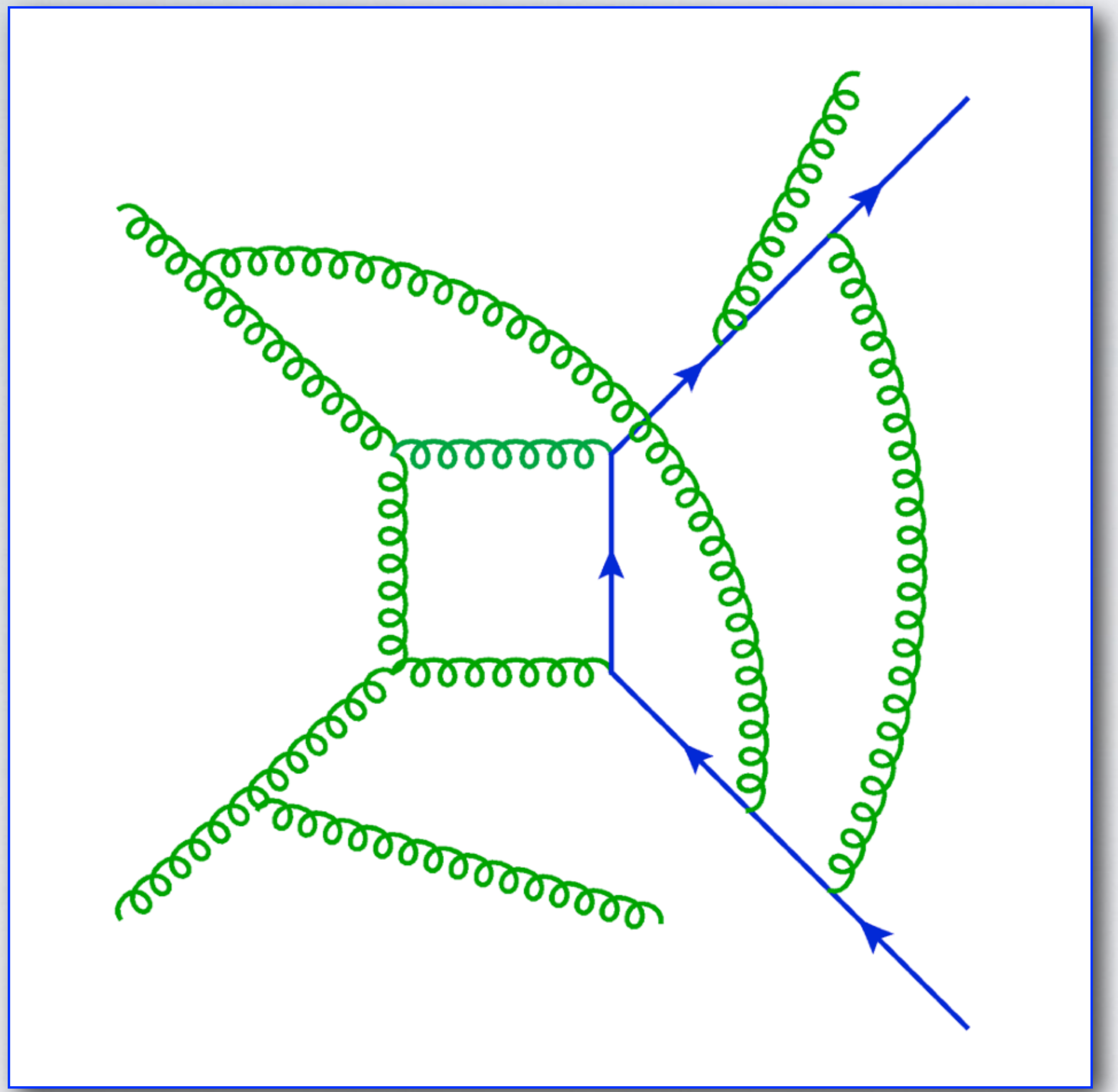
- Emission is not suppressed at long distances
- Isolated charged particles are not true asymptotic states of unbroken gauge theories

- A serious problem: the S matrix does not exist in the usual Fock space
- Possible solutions: construct finite transition probabilities (KLN theorem)
  construct better asymptotic states (coherent states)
- Long-distance singularities obey a pattern of exponentiation

$$\mathcal{A} = \mathcal{A}_0 \left[1 - \kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \ldots\right] \quad \Rightarrow \quad \mathcal{A} = \mathcal{A}_0 \exp\left[-\kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \ldots\right]$$

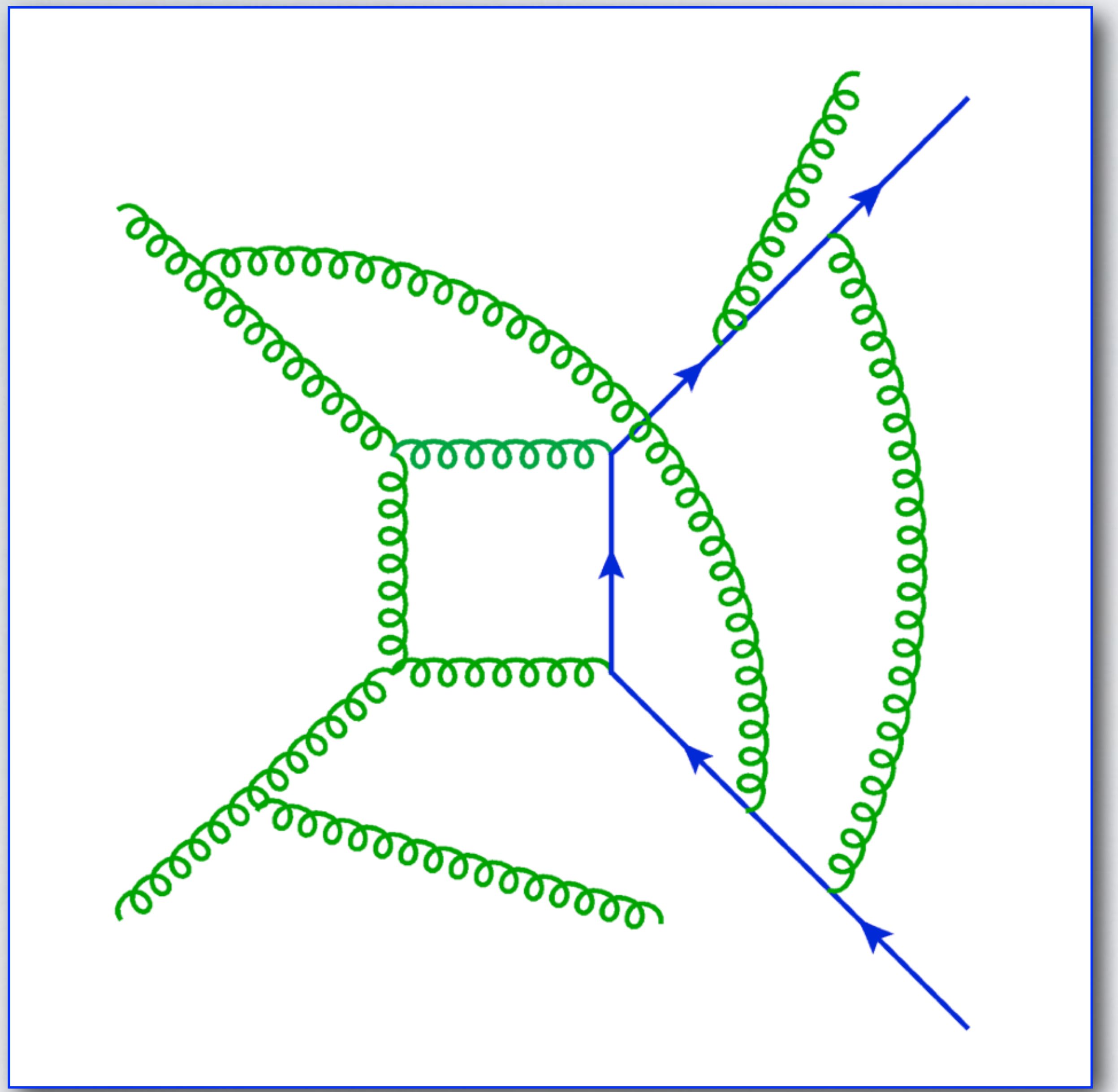
Soft-collinear factorisation

Soft-collinear factorisation



A gauge theory Feynman diagram with
soft and collinear enhancements

Soft-collinear factorisation



A gauge theory Feynman diagram with soft and collinear enhancements

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Potential singularities can be located using Landau equations.
- Actual singularities can be identified using power-counting techniques in the relevant regions.
- For renormalised massless theories only soft and collinear regions give divergences.
- Soft and collinear emissions have universal features, common to all hard processes.
- Ward identities can be used to prove decoupling of soft and collinear factors to all orders.
- A soft-collinear factorisation theorem for multi-particle matrix elements follows.
- Similar factorisation theorems hold for inclusive (soft and collinear safe) cross sections.

The factorised amplitude

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, LM, S. Catani, L. Dixon,
E. Gardi, M. Neubert, T. Becher, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

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The factorised amplitude

Infrared divergences in fixed-angle multi-particle scattering amplitudes factorise

$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

The infrared factor is a colour operator determined by a finite anomalous dimension matrix

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P} \exp\left[\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right)\right] ,$$

All infrared poles arise from the scale integration, through the d-dimensional running coupling

$$\lambda \frac{\partial \alpha_s}{\partial \lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s - \frac{\alpha_s^2}{2\pi} \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^k b_k \, .$$

For massless theories, the all-order structure of the anomalous dimension in known, up to corrections due to higher-order Casimir operators of the gauge algebra

$$\Gamma_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) + \Delta_n\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) ,$$

$$\rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_l \, p_j \cdot p_k} = \frac{s_{ij} s_{kl}}{s_{il} s_{jk}} \, .$$

N. Kidonakis, G. Oderda, G. Sterman; M. Sjödahl; Y. Dokshitzer, G. Marchesini.

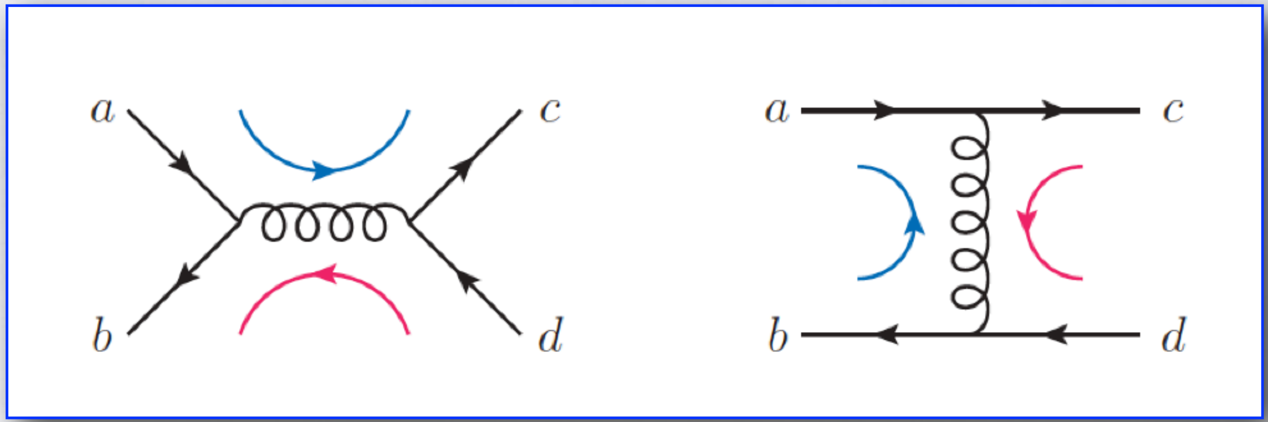
Color basis notation

The amplitude can be expressed in a process-dependent orthonormal basis of colour tensors

$$\mathcal{A}_n^{a_1 \ldots a_n}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \sum_L \mathcal{A}_n^L\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) c_L^{a_1 \ldots a_n} .$$

$$\sum_{\{a_i\}} c_L^{a_1 \ldots a_n} \left(c_M^{a_1 \ldots a_n}\right)^* = \delta_{LM} .$$

A simple example is quark-antiquark scattering, where colour space is two-dimensional



Tree-level diagrams and leading color flows for quark-antiquark scattering

The amplitude is a vector in colour space, to all perturbative orders

$$\mathcal{A}_{abcd} = \mathcal{A}_1 \, c^{(1)}_{abcd} + \mathcal{A}_2 \, c^{(2)}_{abcd} , \qquad c^{(1)}_{abcd} = \delta_{ac}\delta_{bd} , \quad c^{(2)}_{abcd} = \delta_{ab}\delta_{cd} .$$

The exchange of a virtual gluon will shuffle the colour components, even if the gluon is soft

$$\text{QED}: \quad \mathcal{A}_{\rm div} = \mathcal{Z} \, \mathcal{A}_{\rm Born} ; \qquad \text{QCD}: \quad \left[\mathcal{A}_{\rm div}\right]_J = \left[\mathcal{Z}\right]_{JK} \left[\mathcal{A}_{\rm Born}\right]_K .$$

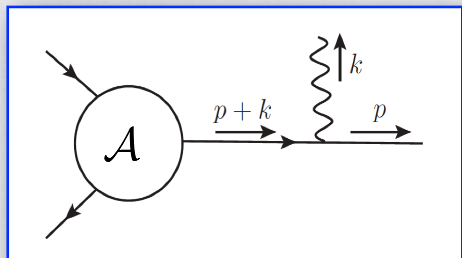
A. Bassetto, M. Ciafaloni, G. Marchesini; S. Catani, M. Seymour.

Color operator notation

A powerful basis-independent notation uses colour operators `inserting' soft gluons

$$\mathcal{A}_{n+1}^{a\, b_1 \ldots b_n}\Big|_{\rm soft} \propto \sum_{i=1}^{n} \left[\mathbf{T}_i^a\right]^{b_i}_{\;c_i} \mathcal{A}_n^{b_1 \ldots c_i \ldots b_n} \, ,$$

Soft gluon operators are generators of the algebra in the representation of the emitter



$$g \mu^\epsilon \, \overline{u}_{s_i}(p_i) \, \gamma_\alpha \, \frac{\not{p}_i + \not{k}}{2 p_i \cdot k} \left(T^c\right)_{c_i d_i} \widehat{\mathcal{A}}_{\,s_1 \ldots s_n}^{\,c_1 \ldots \, d_i \ldots \, c_n} \left(\{p_j\}, k\right) \epsilon_\lambda^{*\,\alpha}(k) \, ,$$

At leading power in k :

$$g \mu^\epsilon \, \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \left(T^c\right)_{c_i d_i} \left(\mathcal{A}_n\right)_{s_1 \ldots s_n}^{c_1 \ldots \, d_i \ldots \, c_n} \left(\{p_j\}\right) \equiv g \mu^\epsilon \, \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \, \mathbf{T}_i \, \mathcal{A}_n\left(\{p_j\}\right) .$$

For different emitters :

$$\mathbf{T}_i\Big|_{\rm q,\, out} \to T^a_{cd} \, , \qquad \mathbf{T}_i\Big|_{\rm \bar{q},\, out} \to - T^a_{dc} \, , \qquad \mathbf{T}_i\Big|_{\rm g,\, out} \to - {\rm i} f^a_{\;cd} \, ,$$

Colour operators obey identities inherited by the algebra and dictated by gauge invariance

$$\left[\mathbf{T}_i^a, \mathbf{T}_i^b\right] = {\rm i} f^{ab}_{\;\;c} \, \mathbf{T}_i^c \, , \qquad \mathbf{T}_i \cdot \mathbf{T}_i \equiv \mathbf{T}_i^a \mathbf{T}_i^b \, \delta_{ab} = C_i^{(2)} \, , \qquad \sum_{i=1}^{n} \mathbf{T}_i = 0 \, ,$$

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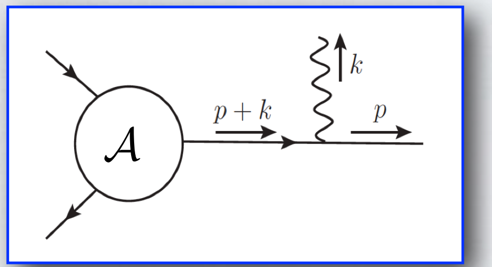

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when acting on the amplitude

The dipole formula

E. Gardi, LM; T. Becher, M. Neubert.

Let's take a closer look at the structure of the infrared anomalous dimension matrix.

The dipole term :

$$\Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2},\alpha_s(\mu^2)\right) = \frac{1}{2}\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\log\left(\frac{s_{ij}\,{\rm e}^{{\rm i}\pi\lambda_{ij}}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j+\sum_{i=1}^{n}\gamma_i\left(\alpha_s(\mu^2)\right),$$

The cusp anomalous dimension in the `Casimir scaling' limit:

$$\gamma_{K,r}(\alpha_s) = C_r^{(2)}\,\widehat{\gamma}_K(\alpha_s),$$

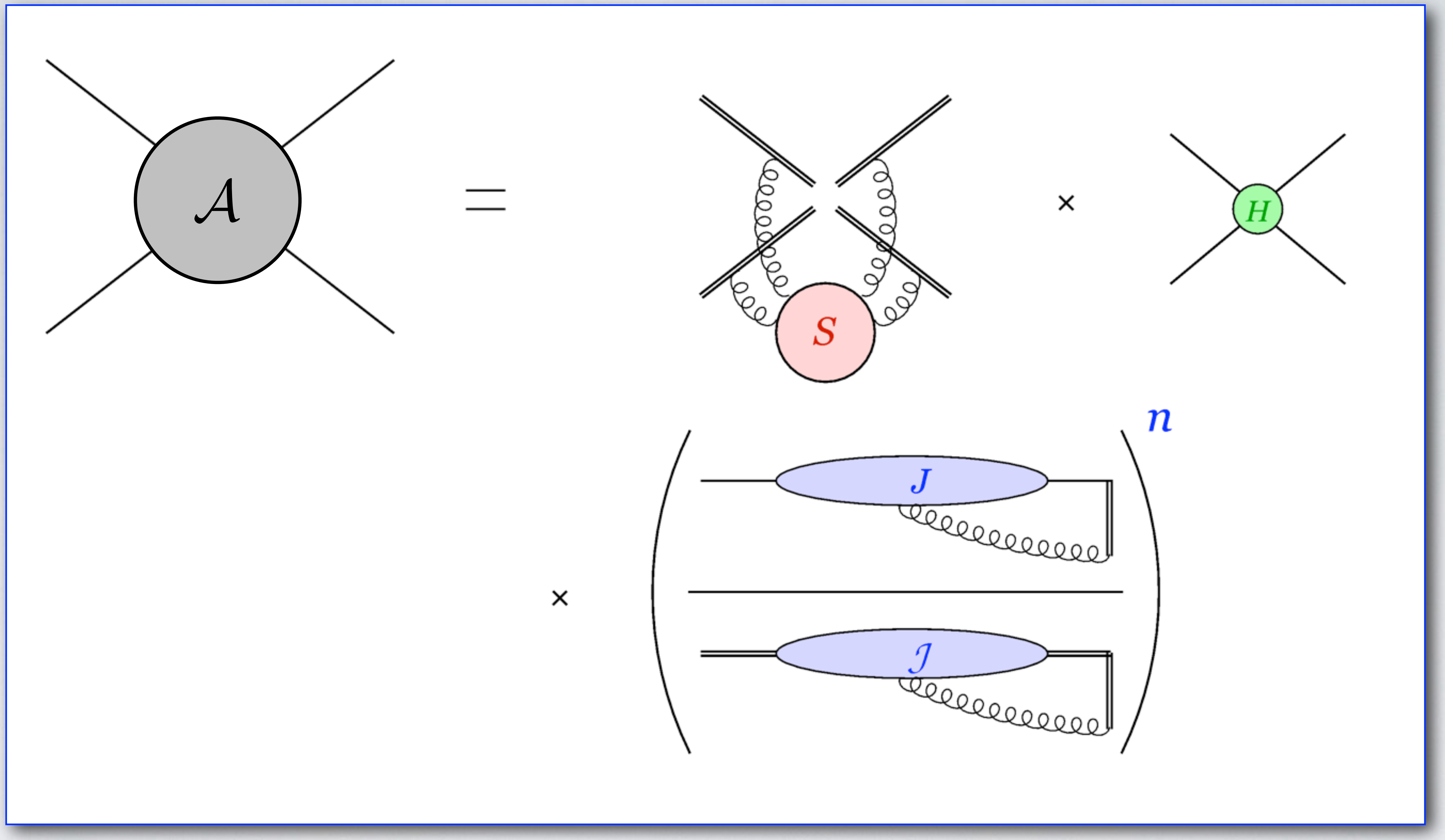
Corrections start at three loops, with quadrupoles:

$$F_{ijkl}(\{\rho\})\,f_{abe}f_{cd}{}^{e}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d,$$

Ø. Almelid, C. Duhr, E. Gardi; J. Henn, B. Mistlberger.

- The colour dipole is the natural structure arising at one loop from gluon exchange.
- The fact that it survives at two loops is a non-trivial consequence of symmetries.
- Field anomalous dimensions in color-uncorrelated terms govern collinear singularities.
- Unitarity phases contain crucial analytic information. For final-state pairs: $\lambda_{ij} = 1$.
- The cusp anomalous dimension plays a very special role: a universal infrared coupling.
- The structure emerges from the constraints of scale invariance in the soft limit.

Infrared factorisation: pictorial



A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

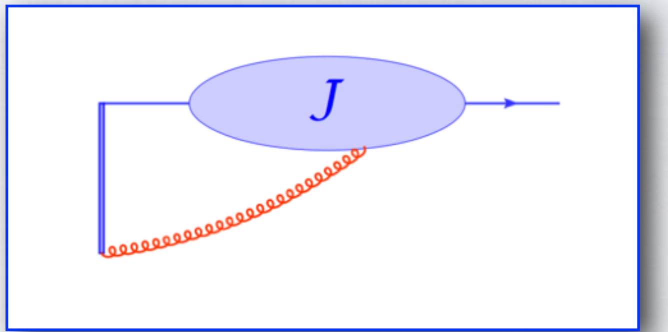
Operator Definitions

The precise functional form of this graphical factorisation is

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{E,\,i}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\;\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

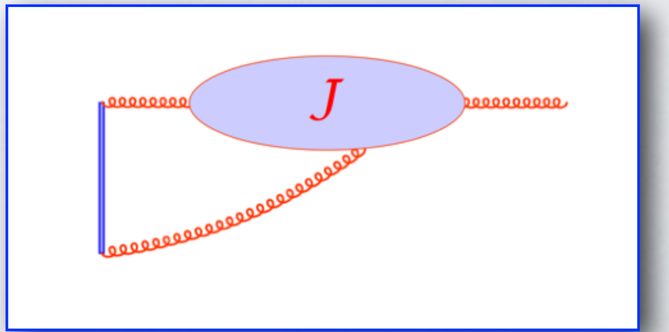
Here we introduced dimensionless four-velocities $\beta_i = p_i/Q$, and factorisation vectors n_i^μ , $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p, s\,|\,T\left[\overline{\psi}(0)\,\Phi_n(0,\infty)\right]\,|0\rangle\,,$$



where Φ_n is the Wilson line operator along the direction **n**. For outgoing gluons

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\;T\Big[\Phi_n(\infty,0)\,\mathrm{i}D^\nu\,\Phi_n(x,\infty)\Big]_{x=0}|0\rangle\;.$$

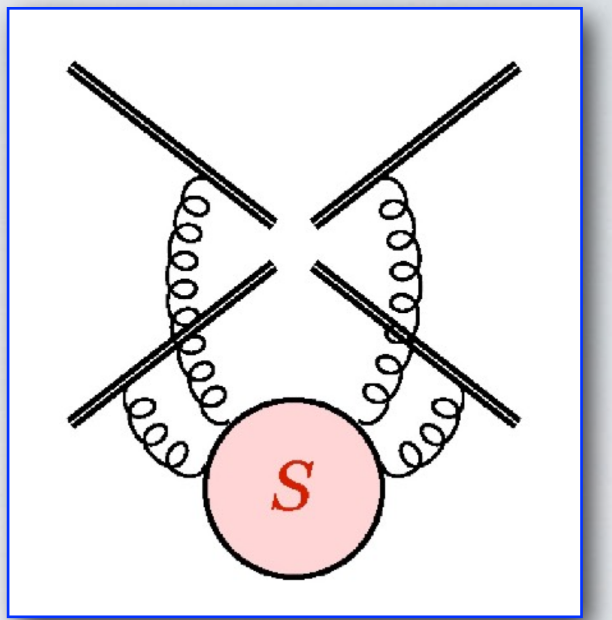


T. Becher, G. Bell.

Wilson line correlators

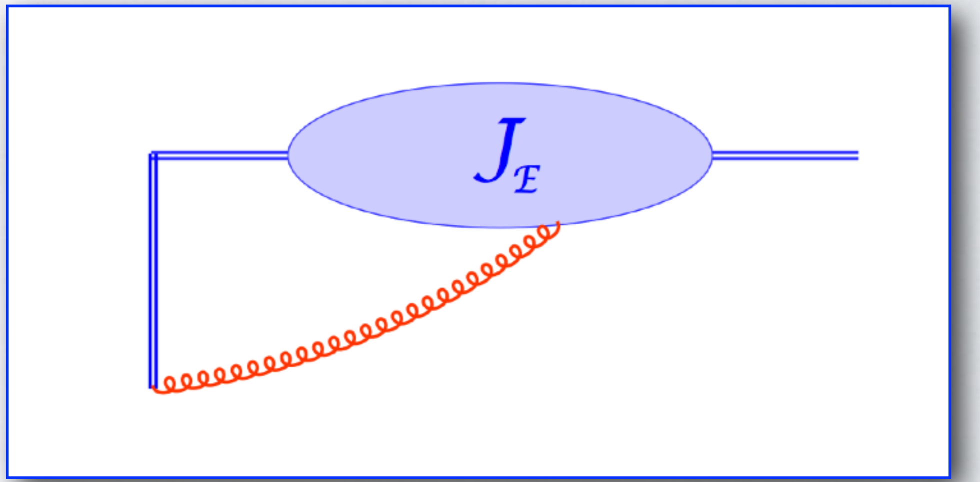
The soft function **S** is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0 | \, T \left[\prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) \right] |0\rangle \, ,$$



The eikonal jet function J_E contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

$$\mathcal{J}_{\rm E}\left(\frac{(\beta \cdot n)^2}{n^2}\right) = \langle 0 | \, T \big[\Phi_\beta(\infty, 0) \, \Phi_n(0, \infty) \big] \, |0\rangle \, .$$



Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

On functional dependences

Straight semi-infinite Wilson lines are scale-invariant

$$\Phi_\beta(\infty, 0) \equiv P \exp\left[ig \int_0^\infty d\lambda \beta \cdot A(\lambda\beta) \right] .$$

Correlators involving light-like Wilson lines break scale invariance due to collinear poles: a quantum `anomaly' proportional to the cusp anomalous dimension.

The anomaly must cancel in combination that are free from collinear poles

$$\widehat{\mathcal{S}}_{LK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{\mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^n \mathcal{J}_{E,i}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)} .$$

The reduced function depends only on scale-invariant combinations

$$\rho_{ij} \equiv \frac{(\beta_i \cdot \beta_j)^2 \, n_i^2 \, n_j^2}{(\beta_i \cdot n_i)^2 \, (\beta_j \cdot n_j)^2} .$$

At the level of anomalous dimensions the cancellation is particularly striking

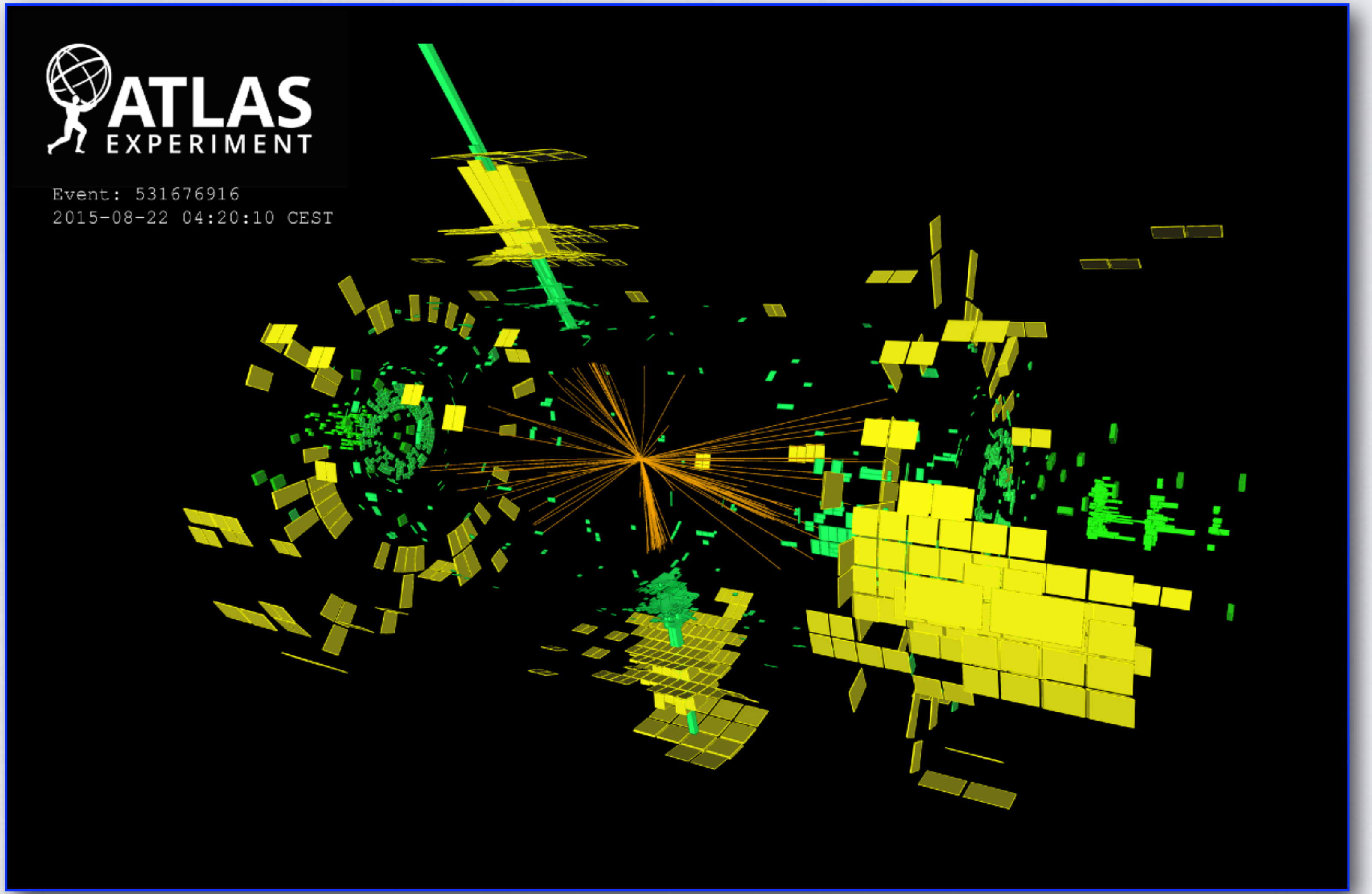
$$\Gamma_{KL}^{(\widehat{\mathcal{S}})}\left(\rho_{ij}, \alpha_s(\mu^2)\right) = \Gamma_{KL}^{(\mathcal{S})}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) - \delta_{KL} \sum_{i=1}^n \gamma_{\mathcal{J}_E}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Singular terms in Γ_S must be diagonal.
- Finite diagonal terms in Γ_S must form ρ_{ij}'s.
- Off-diagonal terms in Γ_S must be finite, and must depend only on cross-ratios ρ_{ijkl} .

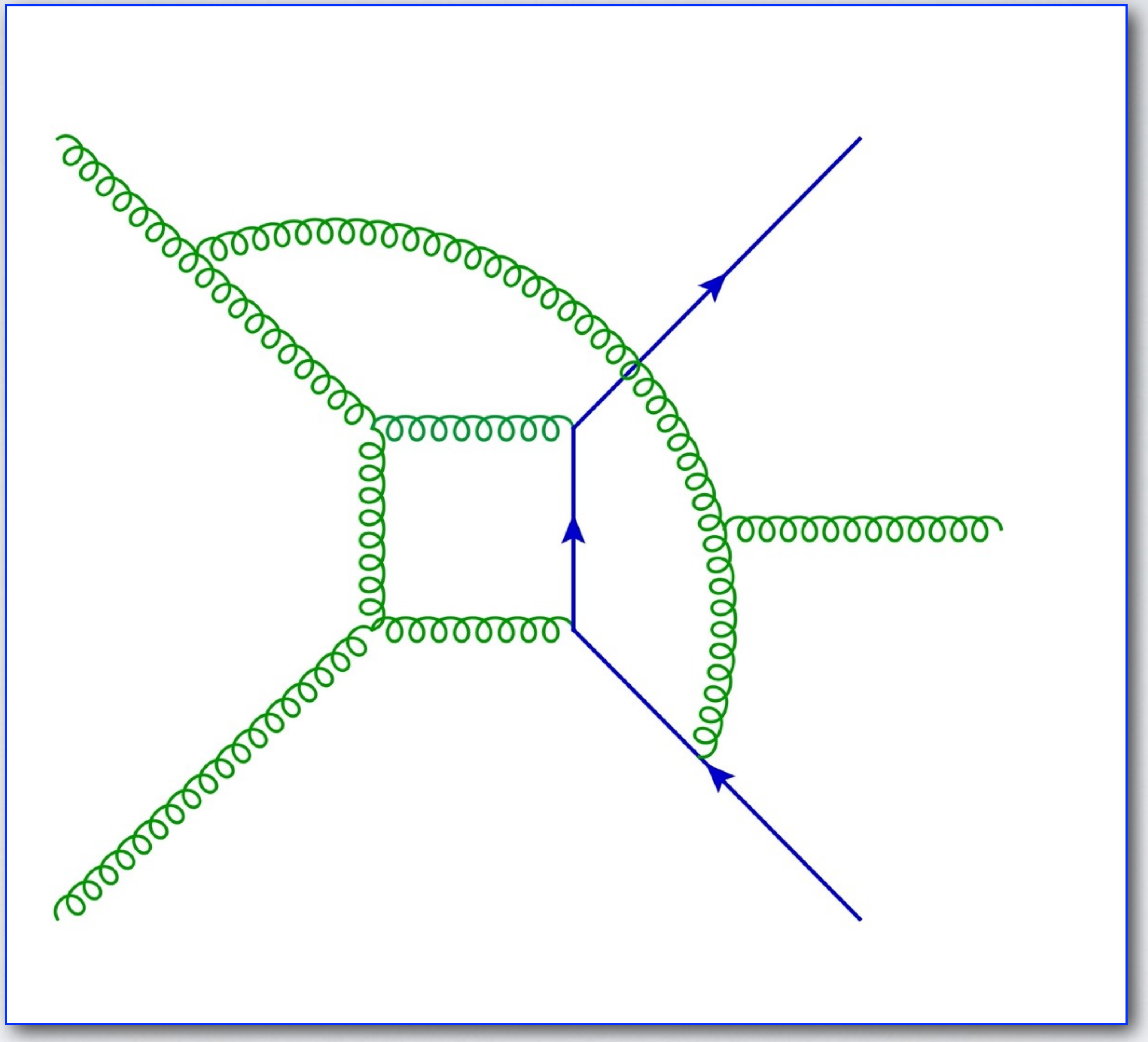
$$\sum_{j \neq i} \frac{\partial}{\partial \rho_{ij}} \Gamma_{LM}^{(\hat{S})}\left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \, \delta_{LM} .$$

An exact equation for the soft anomalous dimension

THE SUBTRACTION PROBLEM

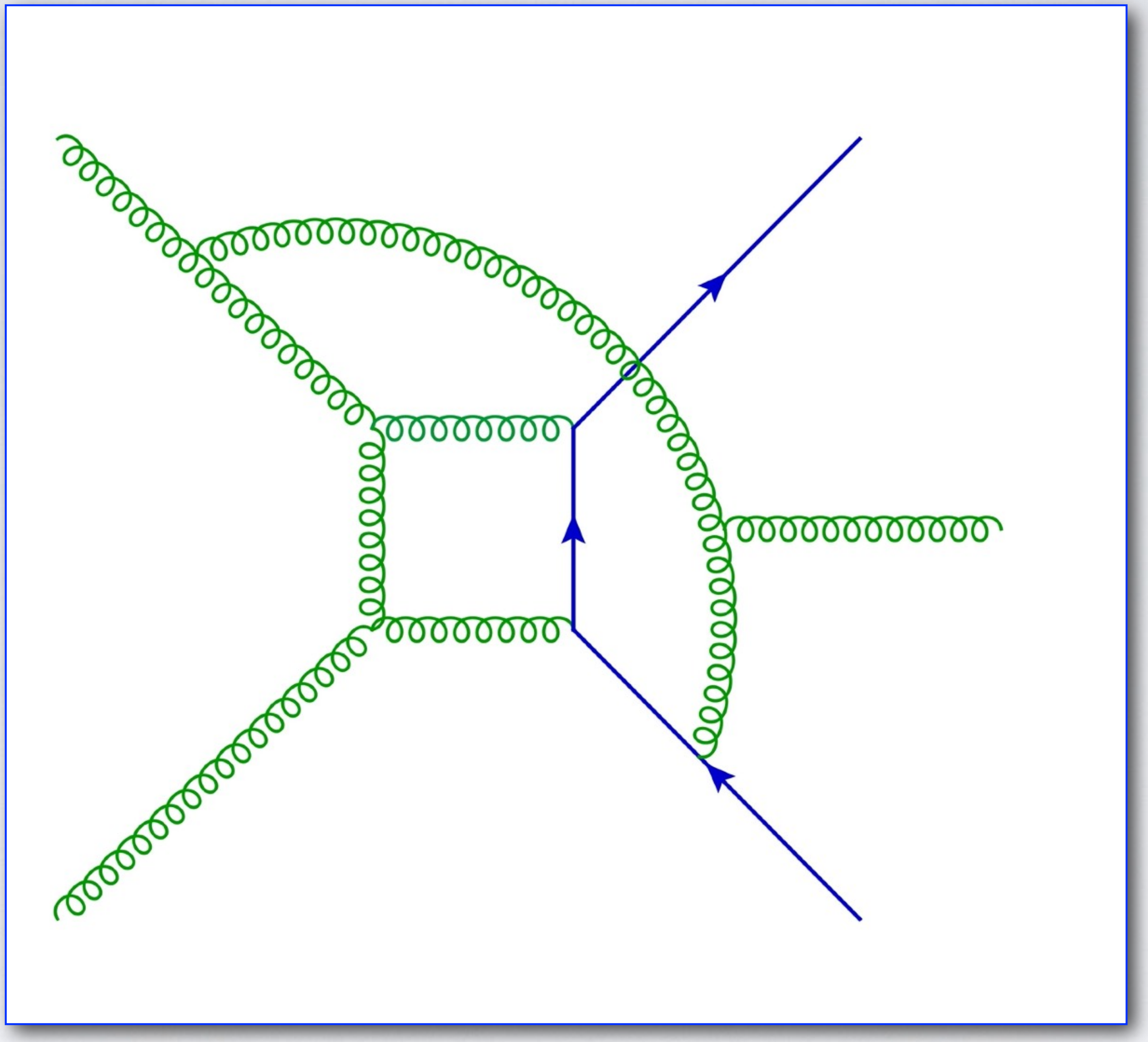


Pictorial infrared



A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared



$$\frac{1}{\epsilon^4}$$

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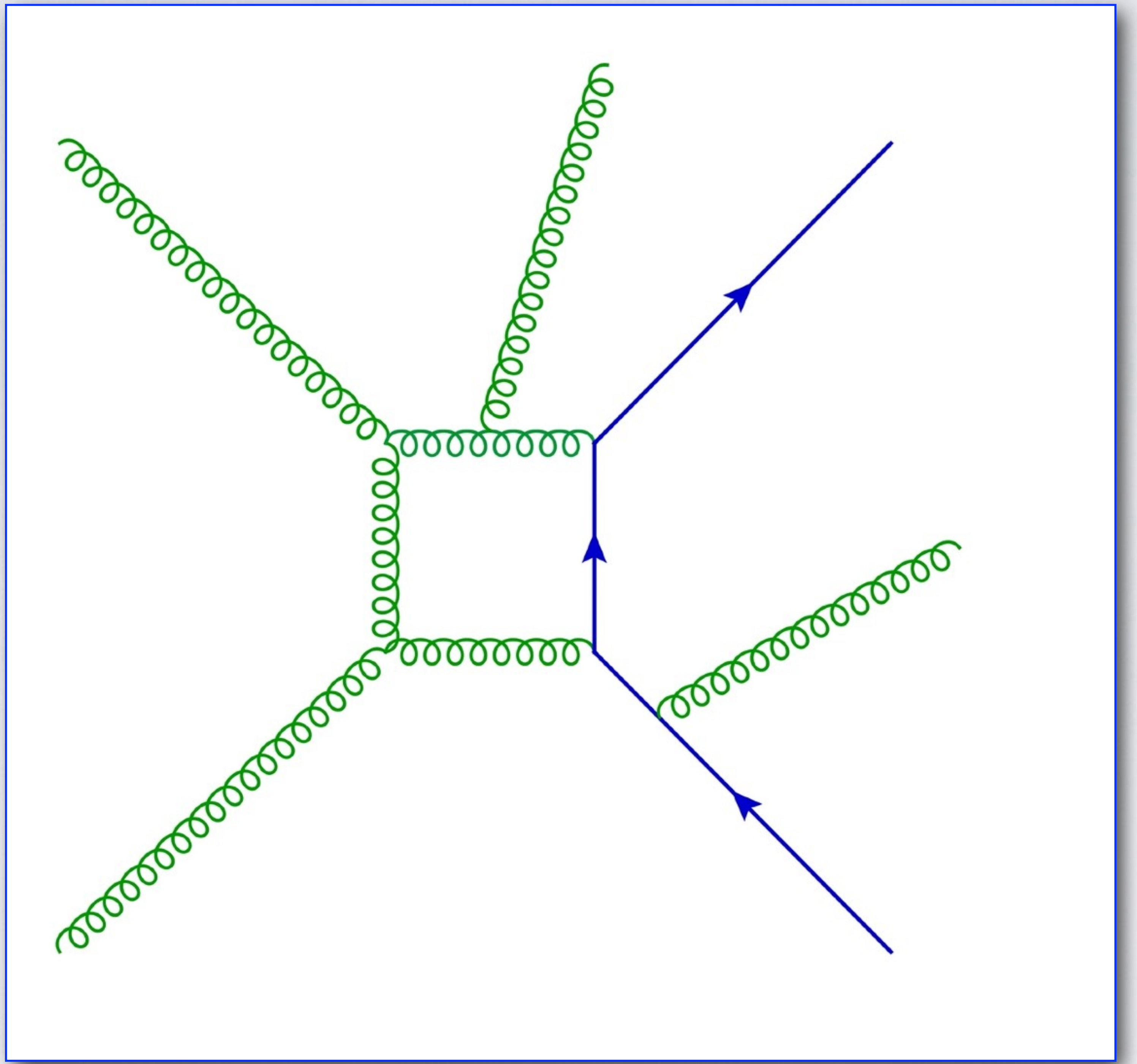

A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

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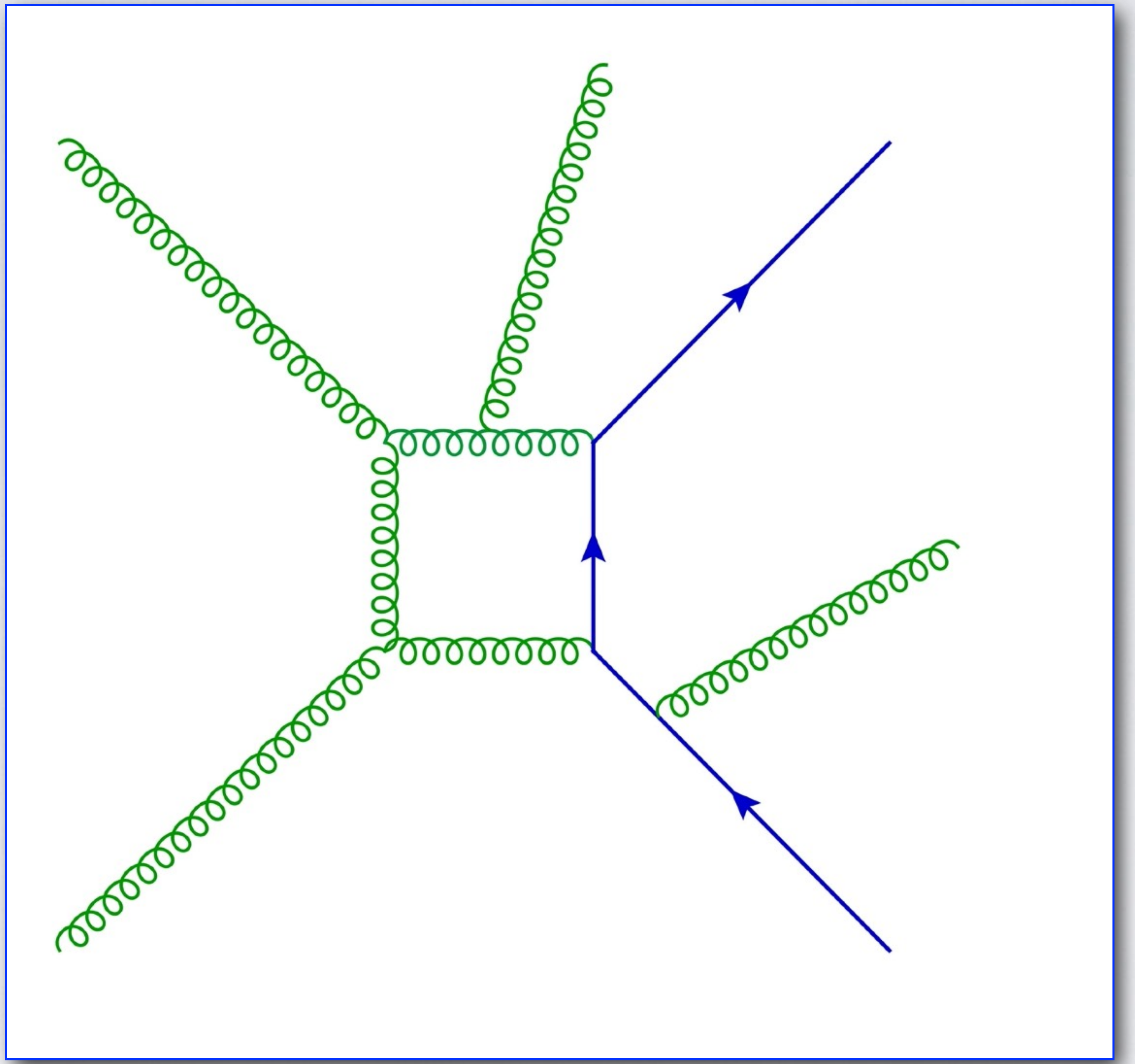

$$\frac{1}{\epsilon^2}$$

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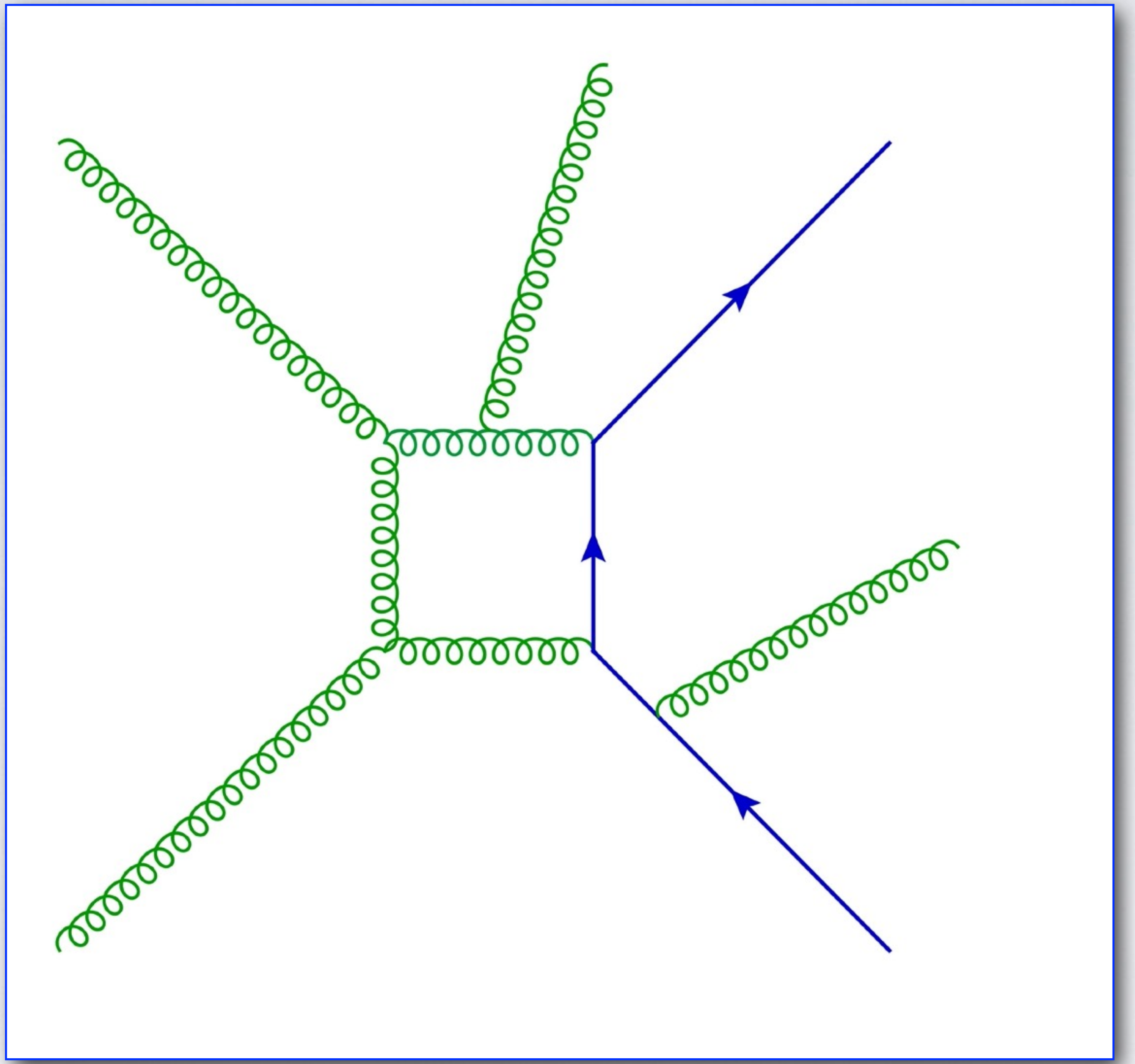

$$\frac{1}{\epsilon^2}$$

$$\frac{dE}{E}\ \frac{dk_\perp}{k_\perp}$$

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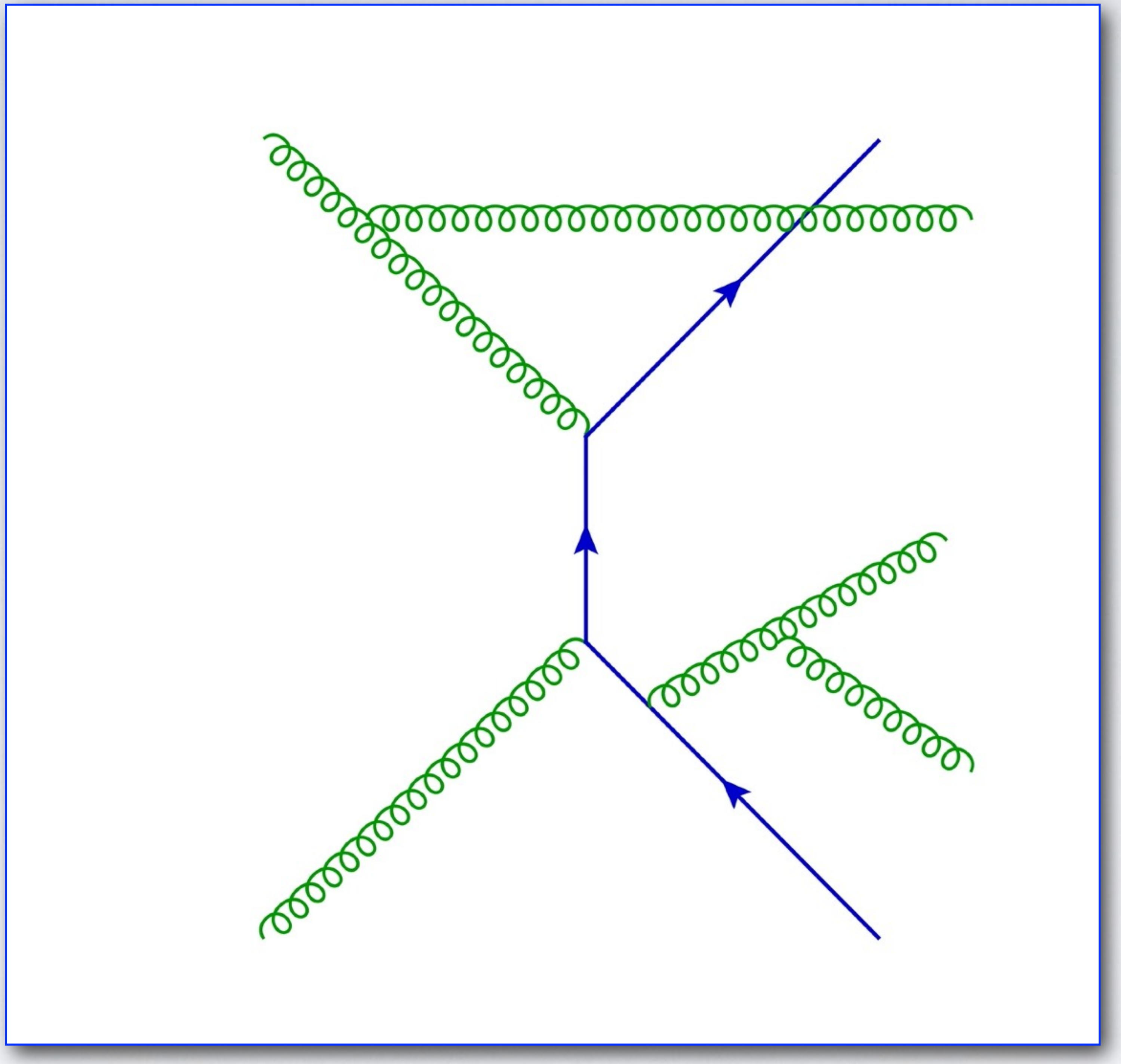

A diagram contributing a double-real NNLO correction to t-tbar-jet production

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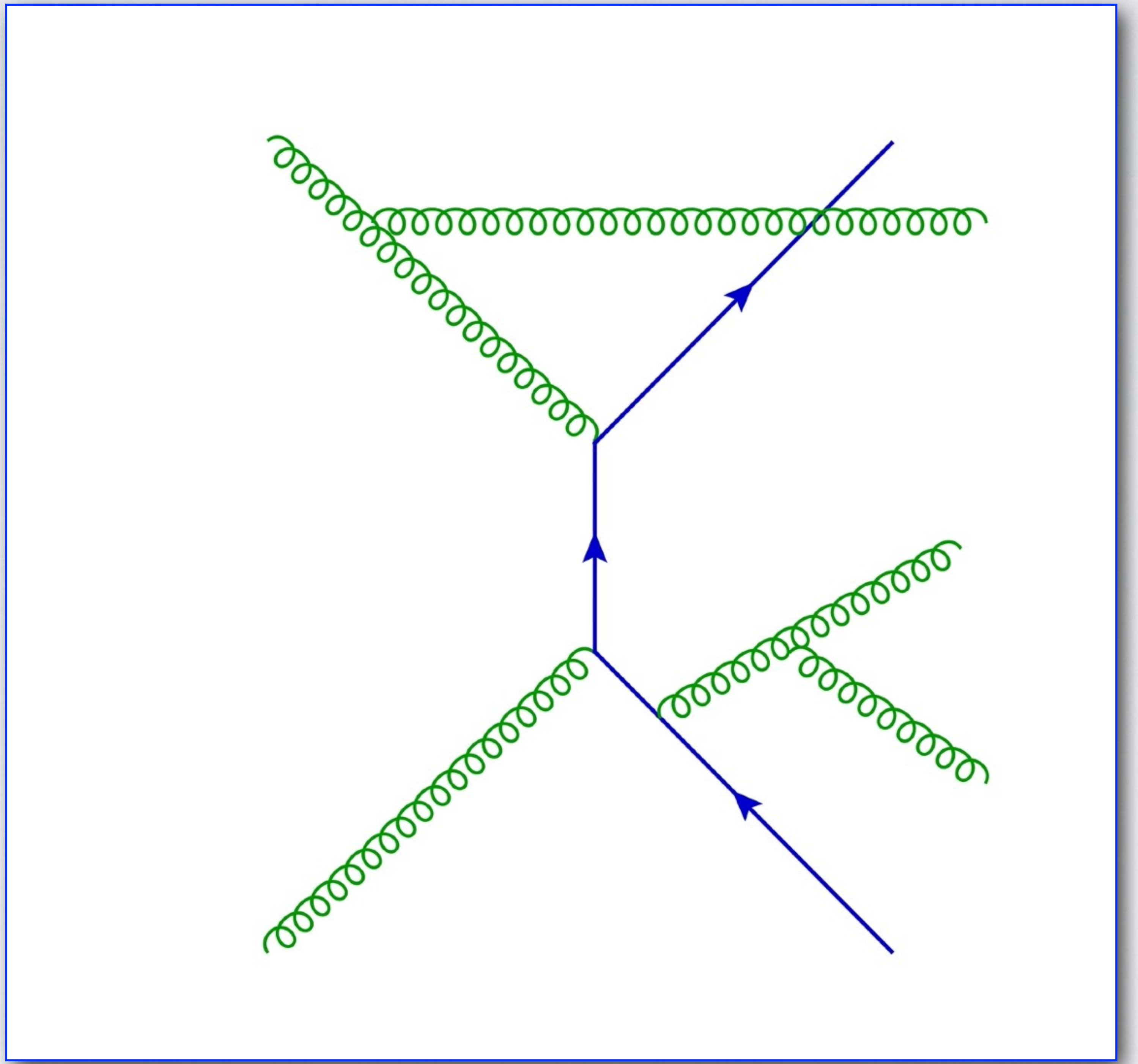
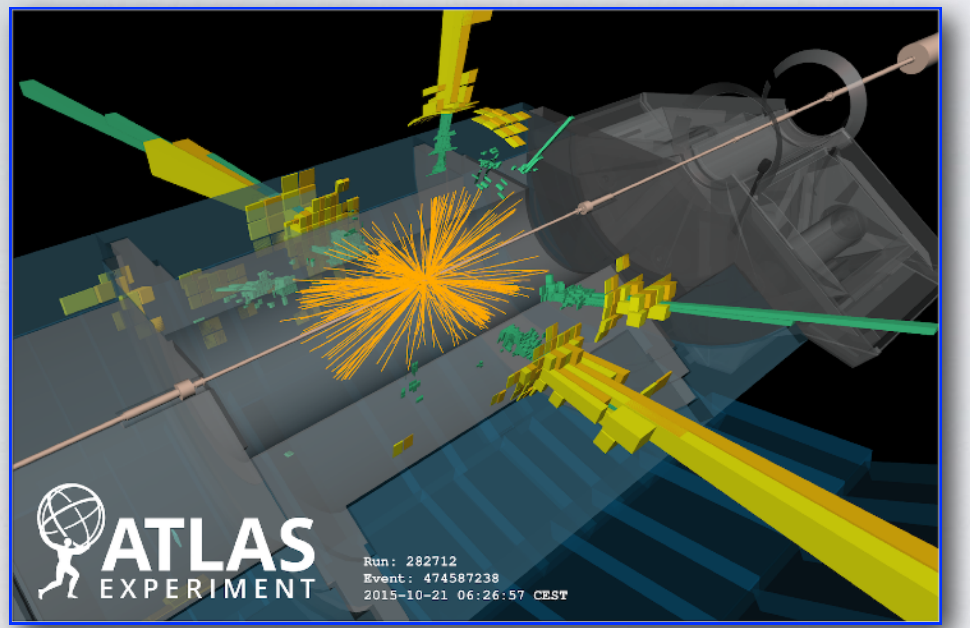
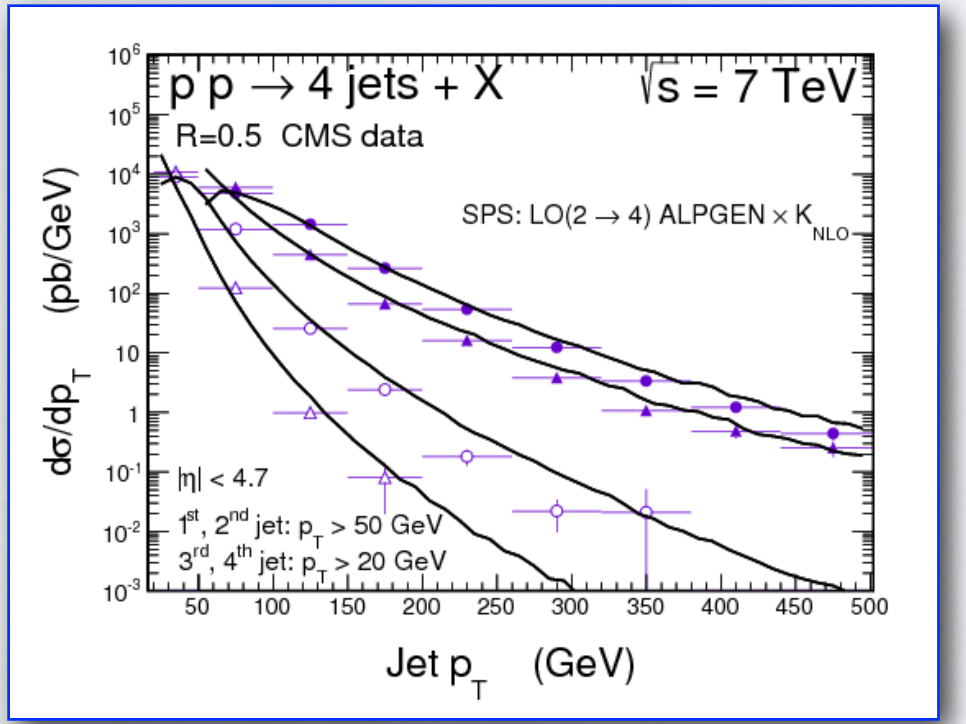


$$\left(\frac{dE}{E}\,\frac{dk_\perp}{k_\perp}\right)^2$$

A diagram contributing a double-real NNLO correction to t-tbar-jet production

The subtraction problem





- Infrared divergences (soft and collinear) cancel between configurations with different numbers of particles
- Collider observables are algorithmically complex and need elaborate phase-space constraints.
- Divergences must be canceled analytically before performing numerical integrations.
- Existing subtraction algorithms beyond NLO are computationally very intensive.
- LHC is now a precision machine: we are interested in subtraction for complicated process at very high orders.
- The factorisation of virtual corrections contains all-order information, not fully exploited.
- The structure of virtual singularities can be used as an organising principle for subtraction.

NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1} \, \delta_{n+1}(X) \right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K^{(\mathbf{1})}_{n+1} = \mathbf{L}^{(1)} \, R_{n+1} \, .$$

$$I^{(\mathbf{1})}_{n} \equiv \int d\Phi^{n+1}_{\rm r,\,1} \, K^{(\mathbf{1})}_{n+1} \, ,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(V_n + \underline{I^{(\mathbf{1})}_{n}} \Big) \, \delta_n(X) + \int d\Phi_{n+1} \Big(R_{n+1} \, \delta_{n+1}(X) - \underline{K^{(\mathbf{1})}_{n+1} \, \delta_n(X)} \Big) \, ,$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

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LM, E. Maina, G. Pellicciolì, C. Signorile-Signorile, P. Torrielli, S. Uccirati

Combinatorial complexity

Minimize complexity: split phase space in sectors with sector functions in order to have at most one soft (i) and one collinear (ij) singularity in each sector. S. Frixione, Z. Kunszt, A. Signer

- Sector functions must form a partition of unity.

- In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $\mathbf{S}_i$ the soft limit for parton i and $\mathbf{C}_{ij}$ the collinear limit for the ij pair,

$$\mathbf{S}_i \sum_{k \neq i} \mathcal{W}_{ik} = 1 \,, \qquad \mathbf{C}_{ij} \Big[\mathcal{W}_{ij} + \mathcal{W}_{ji} \Big] = 1 \,.$$

- Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$e_i \equiv \frac{s_{qi}}{s} \,, \qquad w_{ij} \equiv \frac{s \, s_{ij}}{s_{qi} s_{qj}} \,,$$

$$\sigma_{ij} \equiv \frac{1}{e_i w_{ij}} \,, \qquad \mathcal{W}_{ij} \equiv \frac{\sigma_{ij}}{\sum_{k \neq l} \sigma_{kl}} \,,$$

- With the help of sector functions, one can now define a candidate counterterm

$$\mathbf{L}^{(1)} R_{n+1} = \sum_i \sum_{j \neq i} \Big(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \Big) R_{n+1} \mathcal{W}_{ij} \,.$$

LM, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli, S. Uccirati

Kinematic complexity

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}}\, k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac}+s_{bc}}\, k_c\,,$$

S. Catani, M. Seymour

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\overline{\mathbf{S}}_i\, R(\{k\}) = -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}\, B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,,$$

$$\overline{\mathbf{C}}_{ij}\, R(k) = \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big]\,,$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R(\{k\}) = 2\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i,\, j \neq i} \overline{K}_{ij}\,, \qquad \overline{K}_{ij} \equiv \big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\big)\, R\, \mathcal{W}_{ij}\,,$$

NNLO Subtraction

The pattern of cancellations is more intricate at higher orders

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, VV_n\,\delta_n(X) + \int d\Phi_{n+1}\, RV_{n+1}\,\delta_{n+1}(X) + \int d\Phi_{n+2}\, RR_{n+2}\,\delta_{n+2}(X)\right\},$$

More counterterm functions need to be defined

$$K^{(\mathbf{1})}_{n+2} = \mathbf{L}^{(1)}\, RR_{n+2}\,, \qquad K^{(\mathbf{2})}_{n+2} = \mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{12})}_{n+2} = \mathbf{L}^{(1)}\,\mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{RV})}_{n+1} = \widetilde{\mathbf{L}}^{(1)}\, RV_{n+1}\,.$$

$$I^{(\mathbf{1})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(\mathbf{1})}_{n+2}\,, \quad I^{(\mathbf{12})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(\mathbf{12})}_{n+2}\,, \quad I^{(\mathbf{2})}_{n} = \int d\Phi^{n+2}_{\mathrm{r},2}\, K^{(\mathbf{2})}_{n+2}\,, \quad I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\mathrm{r},1}\, K^{(\mathbf{RV})}_{n+1}\,.$$

A finite expression for the observable in d=4 must combine several ingredients

$$\begin{aligned}\frac{d\sigma_{\text{NNLO}}}{dX} &= \int d\Phi_n\left[VV_n + \underline{I^{(\mathbf{2})}_n} + \underline{I^{(\mathbf{RV})}_n}\right]\delta_n(X)\\ &+ \int d\Phi_{n+1}\left[\left(RV_{n+1} + \underline{I^{(\mathbf{1})}_{n+1}}\right)\delta_{n+1}(X) - \left(\underline{K^{(\mathbf{RV})}_{n+1}} + \underline{I^{(\mathbf{12})}_{n+1}}\right)\delta_n(X)\right]\\ &+ \int d\Phi_{n+2}\left[RR_{n+2}\,\delta_{n+2}(X) - \underline{K^{(\mathbf{1})}_{n+2}}\,\delta_{n+1}(X) - \left(\underline{K^{(\mathbf{2})}_{n+2}} - \underline{K^{(\mathbf{12})}_{n+2}}\right)\delta_n(X)\right]\end{aligned}$$

A hard problem

A hard problem

A wishlist for an optimal subtraction algorithm at N^kLO

- Complete generality across all IR-safe observables with any number of particles.
- Exact locality of the IR and collinear counterterms.
- Exact independence on external slicing parameters.
- Complete analytical results for all integrated counterterms.
- Overall computational efficiency, including interfacing with MC codes.

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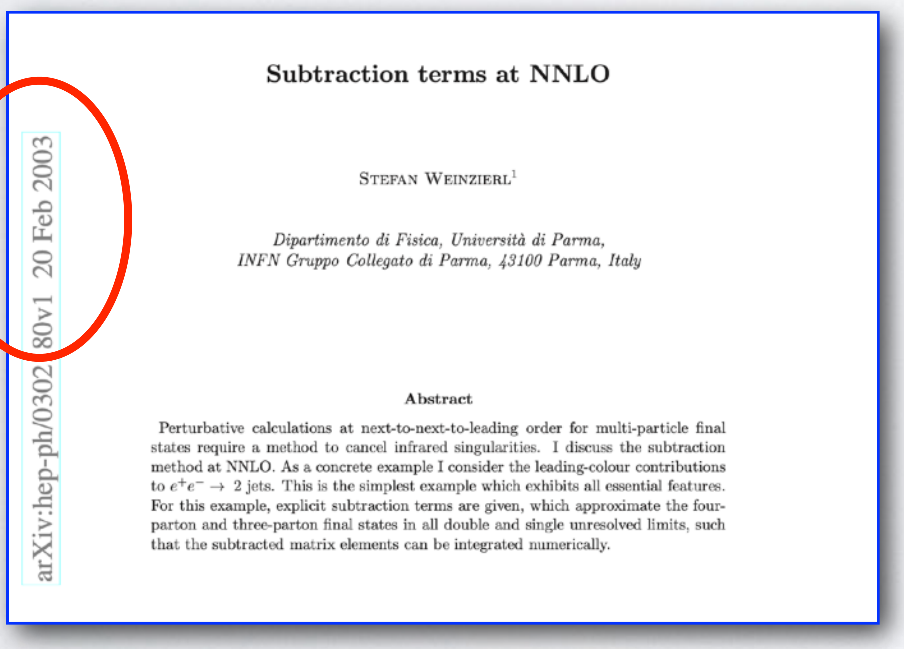

arXiv:hep-ph/0302180v1 20 Feb 2003

Subtraction terms at NNLO

STEFAN WEINZIERL[1]

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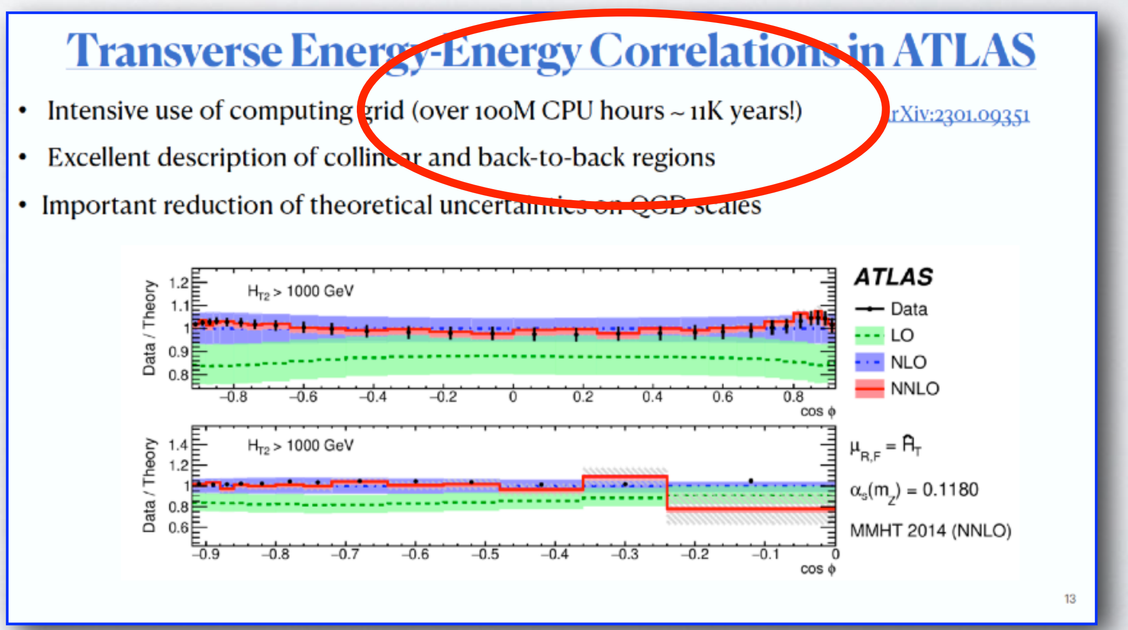
INFN Gruppo Collegato di Parma, 43100 Parma, Italy

Abstract

Perturbative calculations at next-to-next-to-leading order for multi-particle final states require a method to cancel infrared singularities. I discuss the subtraction method at NNLO. As a concrete example I consider the leading-colour contributions to $e^+e^- \to 2$ jets. This is the simplest example which exhibits all essential features. For this example, explicit subtraction terms are given, which approximate the four-parton and three-parton final states in all double and single unresolved limits, such that the subtracted matrix elements can be integrated numerically.



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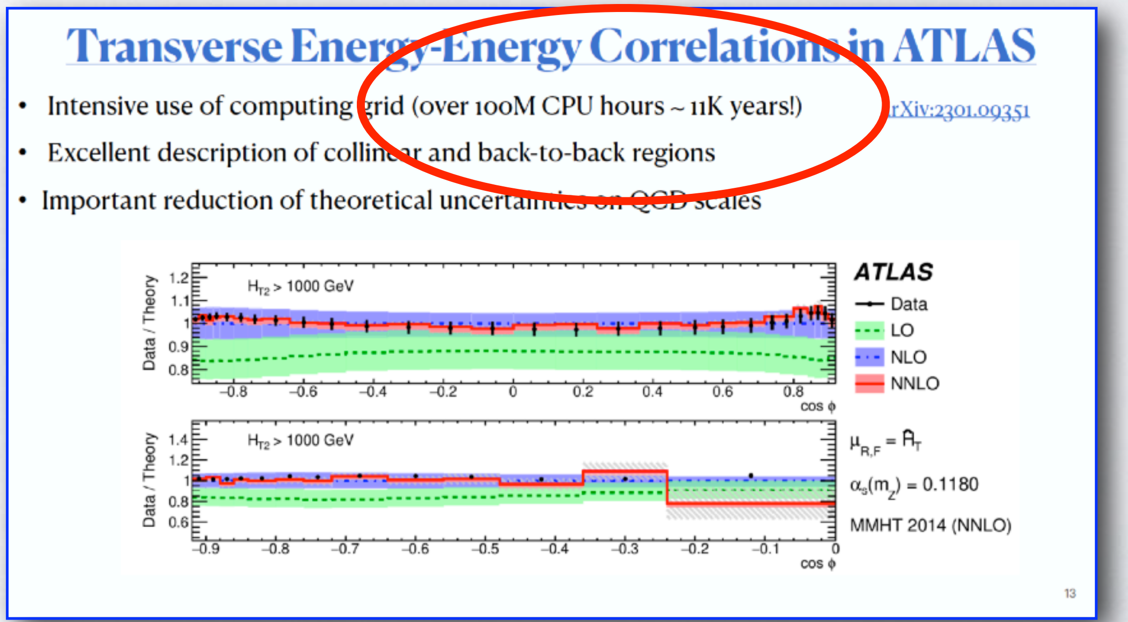
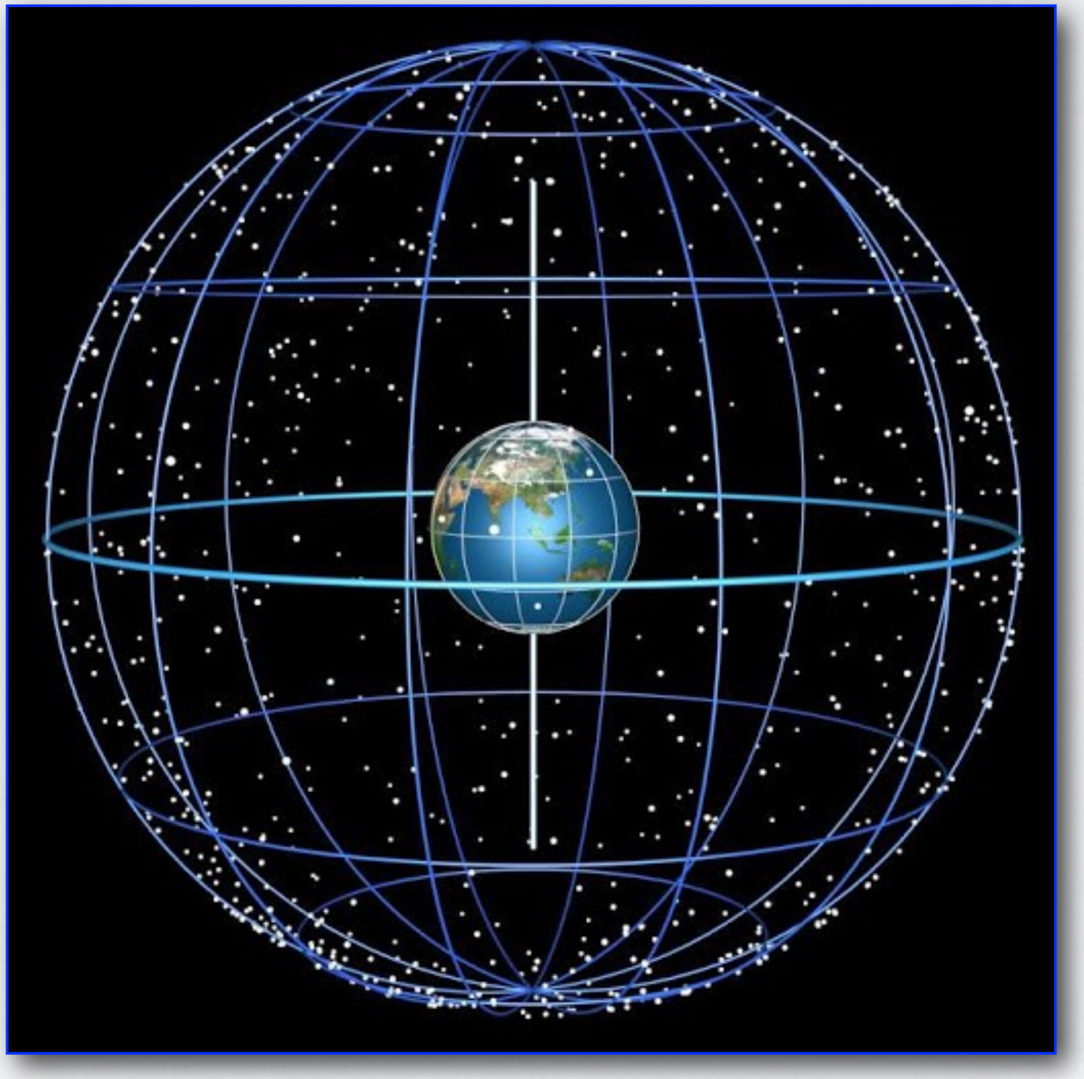
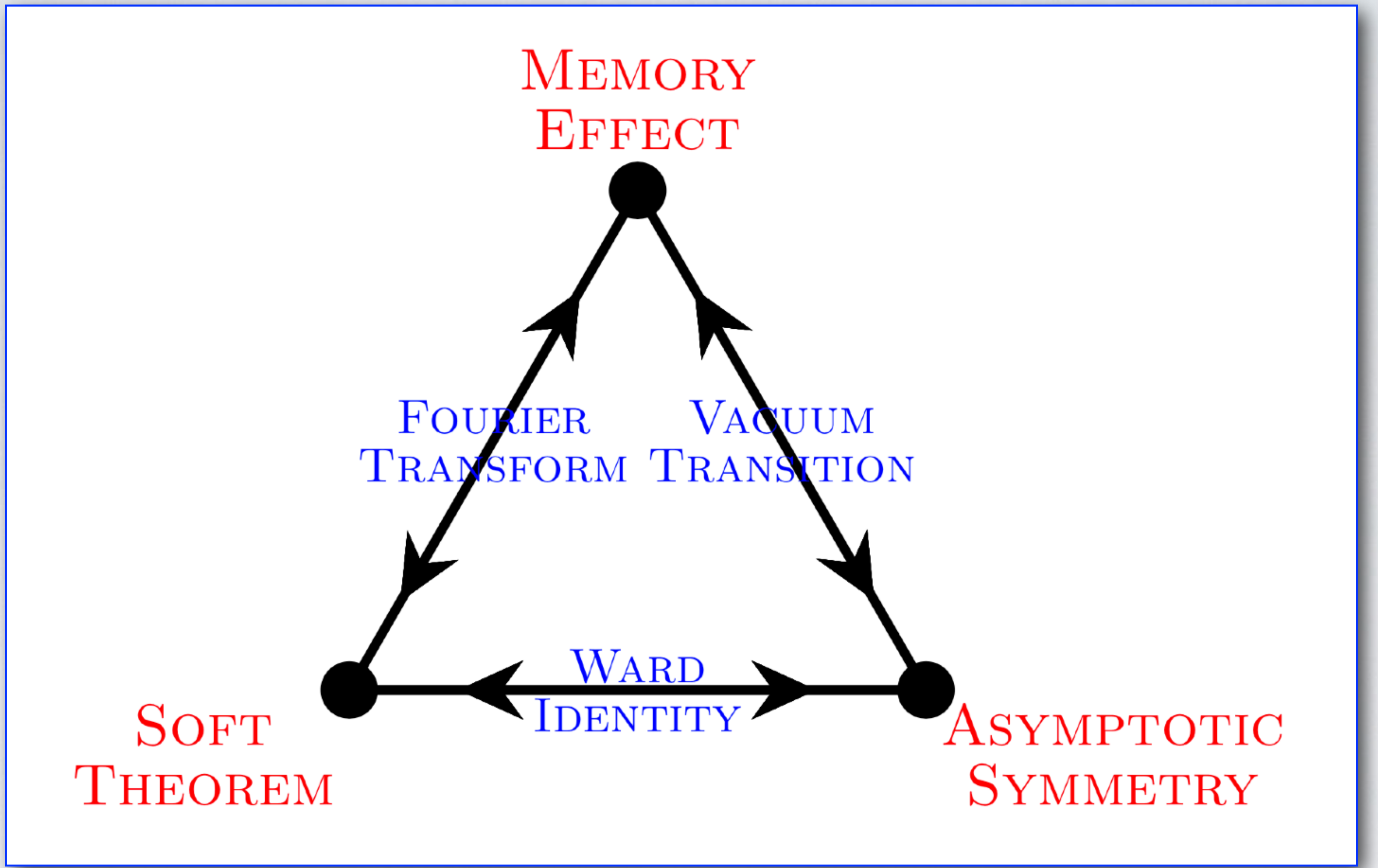


An extreme degree of optimisation will be necessary, and possibly completely new tools.

THE CELESTIAL SPHERE



The Strominger Triangle



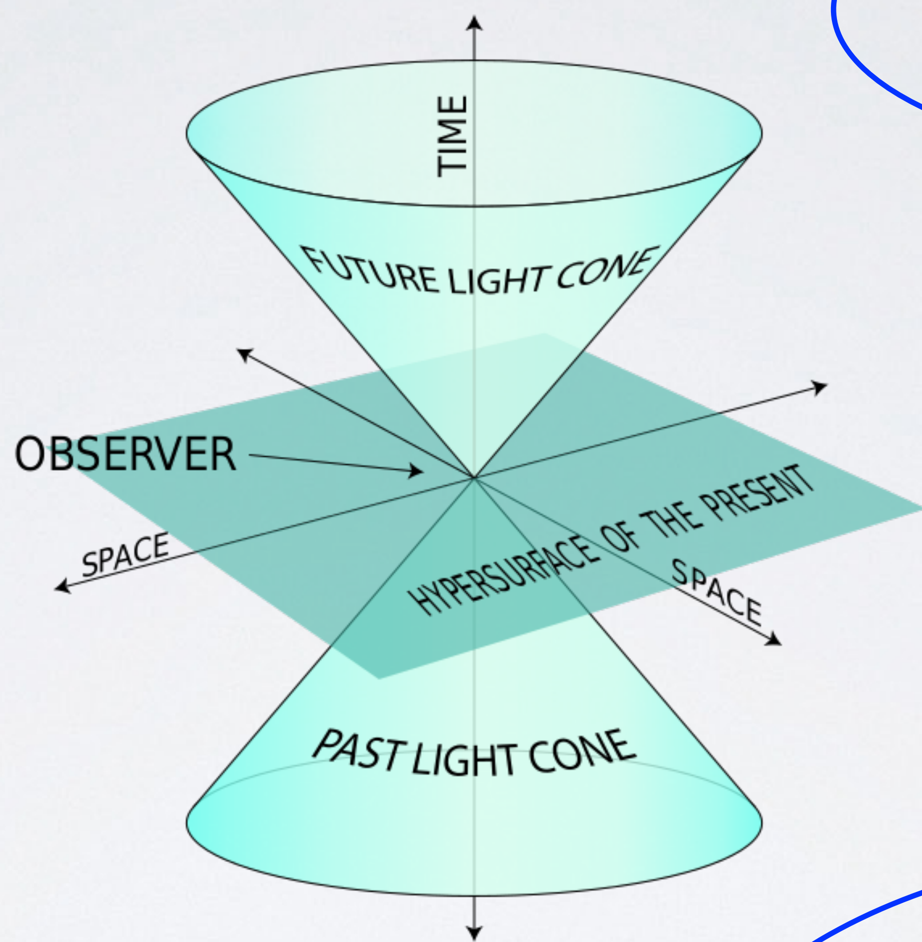
- A new viewpoint on infrared/long-distance phenomena in quantum field theory.
- A lesson from gravity: do not trivialise the behaviour and symmetries `at infinity'.
- Does this idea lead to new calculational techniques for non-abelian theories?

Many directions

Electromagnetic, colour and gravitational memory effects

Asymptotically flat spacetimes and holography

Full conformal symmetry on the celestial sphere



Black hole soft hair and the information paradox

Soft, next-to soft, next-to-next-to soft

$$\mathcal{A}(\Delta_j, z_j) = \left(\prod_{i=1}^{n} \int_0^\infty \frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\right) \mathbf{A}(\omega_j, z_j).$$

Celestial amplitudes

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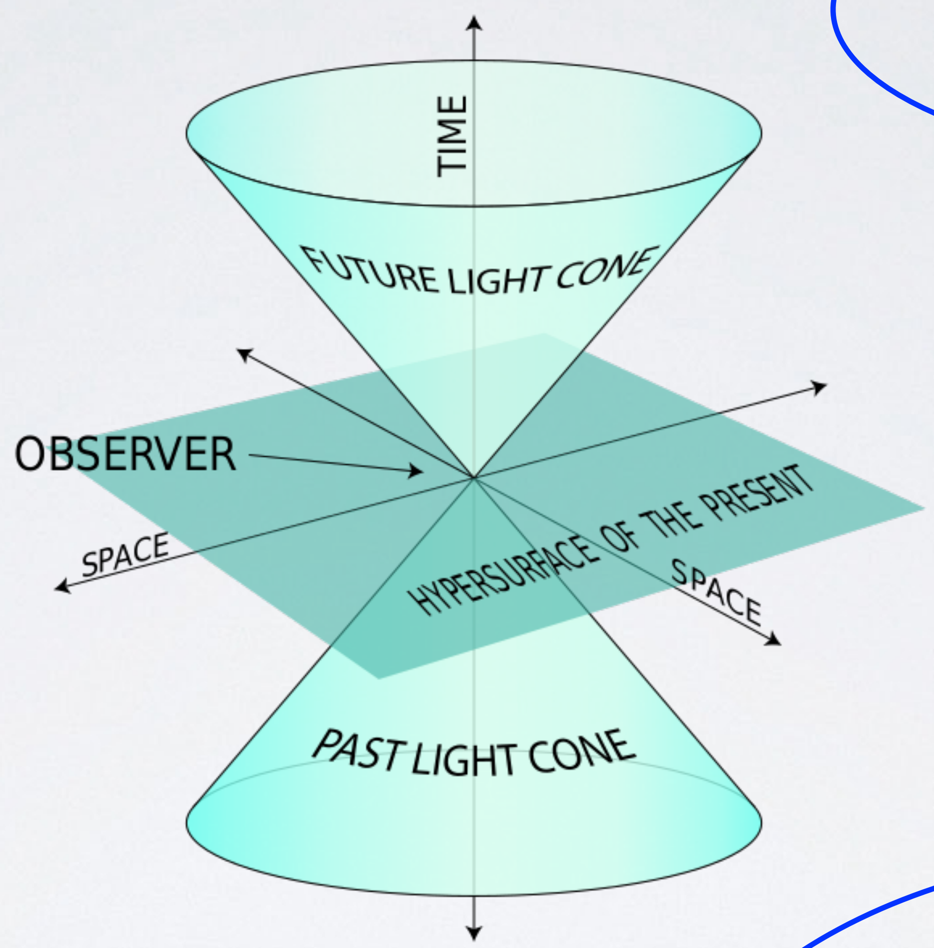

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Celestial amplitudes

See also: H.A. González, F. Rojas, 2104.12979

On dipole correlations

Let us begin by disentangling collinear poles (which are colour-singlets) from soft poles (which are colour-correlated). We replace the running scale λ with the fixed scale μ in the logarithmic term, and perform the colour sum using colour conservation.

$$\Gamma_n^{\rm dipole}\left(\frac{s_{ij}}{\lambda^2},\alpha_s(\lambda,\epsilon)\right) = \frac{1}{2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda,\epsilon)\right)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\left(\frac{-s_{ij}+{\rm i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i\left(\alpha_s(\lambda,\epsilon)\right) - \frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\lambda,\epsilon)\right)\ln\left(\frac{\mu^2}{\lambda^2}\right)\sum_{i=1}^{n}C_i^{(2)} \equiv \Gamma_n^{\rm corr.}\left(\frac{s_{ij}}{\mu^2},\alpha_s(\lambda,\epsilon)\right) + \Gamma_n^{\rm singl.}\left(\frac{\mu^2}{\lambda^2},\alpha_s(\lambda,\epsilon)\right),$$

At one loop, integrating the colour-correlated term yields single soft poles, while the singlet term yields single collinear and double soft-collinear poles

$$\alpha_s\left(\lambda,\epsilon\right) = \alpha_s\left(\mu\right)\left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon},$$

$$\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\alpha_s(\lambda,\epsilon) = -\frac{1}{\epsilon}\,\alpha_s(\mu)\,, \quad \int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\ln\left(\frac{\lambda^2}{\mu^2}\right)\alpha_s(\lambda,\epsilon) = -\frac{1}{\epsilon^2}\,\alpha_s(\mu)\,, \quad (\epsilon<0)\,.$$

At h loops, multiple poles (up to order h+1) are generated by the β function. For conformal gauge theories the logarithm of the infrared factor has only single and double poles.

Celestial dipoles

Crucially, we now parametrise the light-cone momenta in celestial coordinates

$$p_i^\mu = \omega_i \left\{1 + z_i \bar{z}_i,\, z_i + \bar{z}_i,\, -\mathrm{i}\big(z_i - \bar{z}_i\big),\, 1 - z_i \bar{z}_i\right\},$$

where the energy ω_i and the sphere coordinates z_i have simple transformation properties under the Lorentz group acting as SL(2, C):

$$\omega' = |cz + d|^2 \omega, \qquad z' = \frac{az + b}{cz + d},$$

Mandelstam invariants are distances on the sphere

$$s_{ij} = 2 p_i \cdot p_j = 4 \omega_i \omega_j \big|z_i - z_j\big|^2,$$

which unpacks the logarithms

$$\log\left(-s_{ij} + \mathrm{i}\eta\right) = \log\left(\big|z_i - z_j\big|^2\right) + \log \omega_i + \log \omega_j + 2 \log 2 + \mathrm{i}\pi,$$

Energies give new singlet terms

$$\Gamma_n^{\mathrm{dipole}}\left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon)\right) \equiv \widehat{\Gamma}_n^{\mathrm{corr.}}\Big(z_{ij}, \alpha_s(\lambda, \epsilon)\Big) + \widehat{\Gamma}_n^{\mathrm{singl.}}\left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon)\right),$$

which take the form

$$\widehat{\Gamma}_n^{\mathrm{singl.}}\left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon)\right) = -\sum_{i=1}^{n} \gamma_i\big(\alpha_s(\lambda, \epsilon)\big) - \frac{1}{4}\, \widehat{\gamma}_K\big(\alpha_s(\lambda, \epsilon)\big) \sum_{i=1}^{n} \ln\left(\frac{-4\omega_i^2 + \mathrm{i}\eta}{\lambda^2}\right) C_i^{(2)},$$

Celestial dipoles

The colour-correlated term, responsible for all soft poles, is remarkably simple

$$\widehat{\Gamma}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\lambda,\epsilon)\Big) = \frac{1}{2}\,\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\,.$$

Scale and coupling dependence are completely factored from colour and kinematics, and equal for all dipoles. The scale integral can this be performed in full generality, yielding

$$\mathcal{Z}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\mu),\epsilon\Big) \equiv \exp\left[\int_0^\mu\frac{d\lambda}{\lambda}\,\widehat{\Gamma}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\lambda,\epsilon)\Big)\right] = \exp\left[-K\big(\alpha_s(\mu),\epsilon\big)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\right],$$

The scale factor K is well-known in QCD from form-factor calculations, and gives the perturbative Regge trajectory in the high-energy limit of four-point amplitudes. It is

J. Collins, D. Soper; G. Korchemsky, I.A. Korchemskaya;
V. Del Duca, C. Duhr, E. Gardi, LM, C. White;
G. Falcioni, L. Vernazza, …

$$K\big(\alpha_s(\mu),\epsilon\big) = -\frac{1}{2}\int_0^\mu\frac{d\lambda}{\lambda}\,\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)\,.$$

The function K can be computed order by order in terms of the cusp and the β function

$$K(\alpha_s,\epsilon) = \frac{\alpha_s}{\pi}\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon} + \frac{b_0\widehat{\gamma}_K^{(1)}}{32\epsilon^2}\right) + \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\widehat{\gamma}_K^{(3)}}{12\epsilon} + \frac{b_0\widehat{\gamma}_K^{(2)} + b_1\widehat{\gamma}_K^{(1)}}{48\epsilon^2} + \frac{b_0^2\,\widehat{\gamma}_K^{(1)}}{192\epsilon^3}\right) + \mathcal{O}\big(\alpha_s^4\big),$$

$\beta \to 0$

$$K(\alpha_s,\epsilon) = \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n\frac{\widehat{\gamma}_K^{(n)}}{4n\epsilon},$$

A celestial conformal theory

It is natural to mimic the bosonic string, considering free bosons spanning the gauge algebra.

$$S(\phi) \,=\, \frac{1}{2\pi} \int d^2 z \, \partial_z \phi^a(z,\bar{z}) \, \partial_{\bar{z}} \phi_a(z,\bar{z}) \,,$$

The free bosons could be organised in a matrix field :
gauge generators at different points must then be taken to commute

$$\Phi_r(z,\bar{z}) \,\equiv\, \phi_a(z,\bar{z}) \, T^a_{r,\,z} \,,$$

The well-known results for free bosons in d=2 can be directly transcribed.

The equations of motions are:

$$\partial_z \, \partial_{\bar{z}} \, \phi^a(z,\bar{z}) \,=\, 0 \,,$$

implying that the derivatives of the fields are (anti)holomorphic

A normal-ordered product can be defined, obeying the classical equation of motion

$$:\phi^a(z,\bar{z}) \, \phi^b(w,\overline{w}): \,=\, \phi^a(z,\bar{z}) \, \phi^b(w,\overline{w}) + \frac{1}{2} \, \delta^{ab} \log |z-w|^2 \,,$$

There is a traceless conserved energy-momentum tensor, and a conserved Noether current

$$T(z) \,=\, - \, :\partial_z \phi^a(z,\bar{z}) \, \partial_z \phi_a(z,\bar{z}): \,,$$

$$j^a(z) \,=\, \partial_z \phi^a(z,\bar{z}) \,,$$

A. Nande, M. Pate, A. Strominger; N. Kalyanapuram (QED). LM.

Matrix vertex operators

Guided by the QED example, we can tentatively define a matrix-valued vertex operator

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} \,: \,=\, :\, \mathrm{e}^{\mathrm{i}\kappa \Phi(z,\bar{z})} \,: \,,$$

Colour-kinematic dual of the string vertex operator

In colour space, this is a matrix in the representation of $\mathbf{T}_z$, defined on the boundary sphere and acting on the bulk colour degrees of freedom. But is it a conformal primary field?

For conventional vertex operators (as for example for bosonic strings)

$$V_{\mathrm{c.s.}}(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}k^\mu X_\mu(z,\bar{z})} \,: \quad \longrightarrow \quad h = \frac{1}{4}\, k^\mu k^\nu \eta_{\mu\nu} = \frac{k^2}{4}\,,$$

The same calculation yields

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} \,: \quad \longrightarrow \quad h = \frac{\kappa^2}{4}\, \mathbf{T}_z \cdot \mathbf{T}_z = \frac{\kappa^2}{4}\, C_r^{(2)}\,,$$

Crucially, this is a positive real number and not a matrix. For consistency, two-point functions must evaluate to a power of the distance given by the conformal weight $\Delta = h + \bar{h}$. Indeed

$$\big\langle V(z_1,\bar{z}_1) V(z_2,\bar{z}_2) \big\rangle \sim |z_{12}|^{-2\Delta}\,,$$

by colour conservation $\quad \mathbf{T}_1 + \mathbf{T}_2 = 0$

Note analogies with other constructions.

Vertex operator construction of Kac-Moody algebras:

$$U^\alpha(z) = z^{\alpha^2/2} :e^{i\alpha\cdot Q(z)}:.$$

Reggeon fields for high-energy scattering:

$$U(z) = e^{i g_s T^a W^a(z)}\,.$$

(Caron-Huot 2013)

A conformal correlator

Our construction from the beginning targeted the n-point correlator

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) \equiv \Big\langle \prod_{i=1}^{n} V(z_i,\bar{z}_i)\Big\rangle .$$

The calculation is a textbook exercise: it can be done with oscillators, after expanding the free fields in modes on the sphere, or computing the path integral (Polchinski). The result is

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) = C(N_c)\,\exp\left[\frac{\kappa^2}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\right],$$

$$\sum_{i=1}^{n}\mathbf{T}_i = 0\,,$$

reproducing the structure of the gauge theory infrared operator. Note that

- The correlator has support only on colour conserving configurations
- The field normalisation κ maps to the integral K, carrying scale and regulator dependence.
- In a path integral evaluation on a curved surface (say, a finite sphere with radius R) the correlator acquires a scale-dependent `Weyl' factor, which in this setting maps to an (undetermined) colour-singlet collinear contribution.

$$\mathcal{W}_n\Big(\{z_i\},\kappa\Big) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n}C_i^{(2)}\,g(z_i,\bar{z}_i)\right],$$

MANY QUESTIONS

Many Questions

- The choice of the gauge coupling.

  Our construction lends support to the idea the the cusp anomalous dimension should be taken as the definition of the strong coupling in the infrared.
  How far can one take this definition?

  S. Catani, B. Webber, G. Marchesini; A. Grozin et al.; A. Banfi et al.; O. Erdogan, G. Sterman; S. Catani, D. DeFlorian, M. Grazzini.

- Scale and regulator dependence.

  It is remarkable, and necessary, that infrared singularities be hidden in the matching condition between the gauge theory and the conformal theory.
  How can one make this correspondence more precise?

- Beyond the free theory.

  The celestial conformal theory certainly has corrections involving structure constants (as confirmed by the structure of Δ). The deformed theory is still scale invariant.
  What drives the deformation?

- Constraints from vast field theory data.

  Soft and collinear factorisation kernels are known to three loops, and in the massive case to two loops. In most cases their remarkable simplicity is only partly explained.
  How can we harness these data to constrain the celestial theory?

The exploration has just begun

OUTLOOK

Outlook

- The infrared structure of gauge theory scattering amplitudes is theoretically interesting and phenomenologically relevant.

- Factorisation of physics at different length scales is the key to progress: it leads to universality, evolution equations, and predictive exponentiation.

- The problem of subtraction of IR-singular configurations beyond NLO is intricate both theoretically and computationally.

- Infrared factorisation provides general tools to understand subtraction to all orders in perturbation theory. Much technical work however remains to be done.

- A new theoretical viewpoint on infrared dynamics emerges from asymptotic symmetries of the S-matrix and expresses infrared properties of d=4 amplitudes in terms of a d=2 conformal field theory, to all orders. Powerful new calculation tools may be at hand.

THANK YOU