CP and other Outer Automorphisms of Modular Flavor Symmetries

Andreas Trautner

contact: trautner AT mpi-hd.mpg.de

based on:

NPB 883 (2014) 267-305	1402.0507	w/ M.-C. Chen, M. Fallbacher, K.T. Mahanthappa and M. Ratz
JHEP 1702 (2017) 103	1612.08984	w/ M. Ratz
PLB 786 (2018) 283-287	1808.07060	w/ H.P. Nilles, M. Ratz, P. Vaudrevange
PLB 795 (2019) 7-14	1901.03251	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 947 (2019) 114737	1908.00805	w/ A. Baur, H.P. Nilles, P. Vaudrevange
NPB 971 (2021) 115534	2105.08078	w/ H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange
PRD 105 (2022) 5 055018	2112.06940	w/ A.Baur, H.P. Nilles, S. Ramos-Sánchez, P. Vaudrevange
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CP and other Outer Automorphisms

- Outer automorphisms and CP in Standard Model
- General vs. genera**lized** CP
- Two types of groups
- $\Delta(54)$ example with CP-like symmetry
- CP properties of modular group
- Relevance of Outer automorphisms for derivation of the eclectic flavor symmetry
- Summary

Outer automorphisms 101

Example: $\mathbb{Z}_3$ symmetry, generated by $\mathsf{a}^3 = \mathsf{id}$.

$\mathbb{Z}_3$	id	a	a^2
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	ω	ω^2
$\mathbf{1}''$	1	ω^2	ω

$(\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3})$

- All elements of $\mathbb{Z}_3$: $\{\mathsf{id}, \mathsf{a}, \mathsf{a}^2\}$.
- Outer automorphism group ("Out") of $\mathbb{Z}_3$: generated by

$$u(\mathsf{a}) : \mathsf{a} \mapsto \mathsf{a}^2. \quad \left(\text{think: } \mathsf{u}\,\mathsf{a}\,\mathsf{u}^{-1} \;=\; \mathsf{a}^2\right)$$

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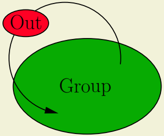

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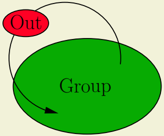

Abstract: Out is a reshuffling of symmetry elements.

In words: Out is a **"symmetry of the symmetry"**.

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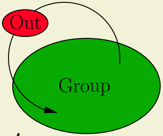

Concrete: Out is a 1:1 mapping of representations $\boldsymbol{r} \mapsto \boldsymbol{r}'$.
Comes with a transformation matrix U, which is given by

$$U\rho_{\boldsymbol{r}'}(\mathrm{g})U^{-1} = \rho_{\boldsymbol{r}}(u(\mathrm{g}))\,, \qquad \forall \mathrm{g} \in G\,.$$

(consistency condition)

[Holthausen, Lindner, Schmidt, '13]
[Chen, Fallbacher, Mahanthappa, Ratz, AT '14]
[Fallbacher, AT, '15]

- $\rho_{\boldsymbol{r}}(g)$: representation matrix for group element $g \in G$
- $u : g \mapsto u(g)$: **outer** automorphism
- U unique only up to phase + central element

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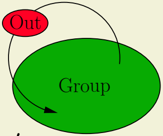

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Physical CP trafo
$\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{r}^*$
is a special case of this!

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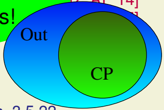

CP transformation in the Standard Model

In the Standard Model

$$\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1) \qquad \text{and} \qquad \mathrm{SO}(3,1)\;,$$

physical CP is described by a *simultaneous* outer automorphism transformation of all symmetries which maps

$$\boldsymbol{r}_i \;\longleftrightarrow\; {\boldsymbol{r}_i}^*\;,$$

$$\Big(\text{e.g. } (\mathbf{3},\mathbf{2})^{\mathrm{L}}_{1/6} \;\;\longleftrightarrow\;\; (\overline{\mathbf{3}},\overline{\mathbf{2}})^{\mathrm{R}}_{-1/6}\Big)\;,$$

for *all* representations of *all* symmetries.

[Grimus, Rebelo '95]
[Buchbinder et al. '01]
[AT '16]

Conservation of such a transformation warrants $\overline{\theta},\,\delta_{\cancel{\mathrm{CP}}} = 0$.

Violation of such a transformation is implied by experiment, and necessary requirement for baryogenesis. [Sakharov '67]

However: Why $\delta_{\mathrm{CKM}} \sim \mathcal{O}(1)$ while $\overline{\theta}_{\mathrm{exp}} \;<\; 10^{-10}$?

General vs. generalized CP

Schematically, QFT with symmetry

$$G_1 \otimes G_2 \otimes \cdots \ ,$$

and quantum fields

$$\psi \in \boldsymbol{r}_{G_1} \otimes \boldsymbol{r}_{G_2} \otimes \cdots \ .$$

CP trafo based on complex conjugation outer automorphism.

$$\psi(x) \xmapsto{\mathcal{CP}} \left(U_{\boldsymbol{r}_{G_1}} \otimes U_{\boldsymbol{r}_{G_2}} \otimes \cdots\right) \psi^*(\mathcal{P}\,x) \ .$$

- Each U ***has to*** fulfill its own **consistency condition**. There is **no choice**: No **"generalization"** necessary or possible.
- Only in specific cases a basis maybe *chosen* such that $U = \mathbb{1}$.
  see e.g. [Ecker, Grimus, Neufeld '87]
- Often such a basis is actually an inconvenient choice.
  [Chen, Fallbacher, Mahanthappa, Ratz, AT '14]
- This is different for **unconstrained** spaces $\mathcal{H}$. For example flavorspace of the SM! *Here generalization is possible.*

Notion of general**ized** CP *for symmetry constrained spaces* should be abandonned. There is only **general CP!**

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The most **general** CP transformation

One generation of (chiral) fermion fields with gauge symmetry $[T_a, T_b] = \mathrm{i} f_{abc}\, T_c$

$$\mathscr{L} \;=\; \mathrm{i}\,\overline{\Psi}\,\gamma^\mu \left(\partial_\mu - \mathrm{i}\,g\,T_a\,W_\mu^a\right)\Psi - \frac{1}{4}\,G_{\mu\nu}^a\,G^{\mu\nu,a}\;.$$

The most **general** possible **CP** transformation:

$$\begin{aligned} W_\mu^a(x) \;&\mapsto\; R^{ab}\,\mathcal{P}_\mu^{\;\nu}\,W_\nu^b(\mathcal{P}\,x)\;,\\ \Psi_\alpha^i(x) \;&\mapsto\; \eta_{\mathsf{CP}}\,U^{ij}\,\mathcal{C}_{\alpha\beta}\,\Psi^{*\,j}_{\;\beta}(\mathcal{P}\,x)\;. \end{aligned}$$

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This is (can be) a conserved symmetry of the *action* iff,

$\curvearrowright$ Three **consistency conditions!**

$$\begin{aligned} &\text{(i)} & R_{aa'}\, R_{bb'}\, f_{a'b'c} \;&=\; f_{abc'}\, R_{c'c}\;, \\ &\text{(ii)} & U\,(-T_a^{\mathrm{T}})\, U^{-1} \;&=\; R_{ab}\, T_b\;, \\ &\text{(iii)} & \mathcal{C}\,(-\gamma^{\mu\,\mathrm{T}})\,\mathcal{C}^{-1} \;&=\; \gamma^\mu\;. \end{aligned}$$

This implies:

- (i) CP is an automorphism of the gauge group.
- (ii) CP maps representations to their complex conjugate representations. $\left(T_a \mapsto -T_a^{\mathrm{T}}\right)$
- (iii) CP is an automorphism of the Lorentz group which maps representations to their complex conjugate representation. $\left(\chi_{\mathrm{L}} \mapsto (\chi_{\mathrm{L}})^\dagger\right)$

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$\Rightarrow \mathcal{C} = \mathrm{e}^{\mathrm{i}\eta}\,\gamma_2\gamma_0$

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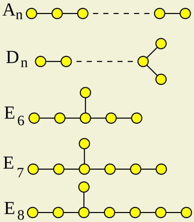
Outer automorphisms of groups

Outer automorphisms exist for continuous & discrete groups.

There are easy ways to depict this:

Continuous groups:

Outer automorphisms of a simple Lie algebra are the symmetries of the corresponding Dynkin diagram.



	Lie Group	Out	Action on reps
$A_{n>1}$	$\mathrm{SU}(N)$	$\mathbb{Z}_2$	$\boldsymbol{r} \rightarrow \boldsymbol{r}^*$
$D_{n>4}$	$\mathrm{SO}(2N)$	$\mathbb{Z}_2$	$\boldsymbol{r} \rightarrow \boldsymbol{r}^*$
E_6	E_6	$\mathbb{Z}_2$	$\boldsymbol{r} \rightarrow \boldsymbol{r}^*$
$D_{n=4}$	$\mathrm{SO}(8)$	S_3	$\boldsymbol{r}_i \rightarrow \boldsymbol{r}_j$
all others		/	/

Outer automorphisms of groups

Discrete groups:

Outer automorphisms of a discrete group are symmetries of the character table (not 1:1).

$\mathbb{Z}_2$

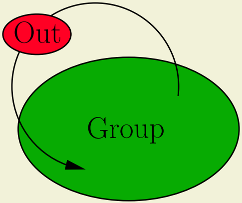
T_7	C_{1a}	C_{3a}	C_{3b}	C_{7a}	C_{7b}
$\mathbf{1}_0$	1	1	1	1	1
$\mathbf{1}_1$	1	ω	ω^2	1	1
$\overline{\mathbf{1}}_1$	1	ω^2	ω	1	1
$\mathbf{3}_1$	3	0	0	η	η^*
$\overline{\mathbf{3}}_1$	3	0	0	η^*	η

$\Delta(54)$	C_{1a}	C_{3a}	C_{3b}	C_{3c}	C_{3d}	C_{2a}	C_{6a}	C_{6b}	C_{3e}	C_{3f}
$\mathbf{1}_0$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	1	1	1	-1	-1	-1	1	1
$\mathbf{2}_1$	2	2	-1	-1	-1	0	0	0	2	2
$\mathbf{2}_2$	2	-1	2	-1	-1	0	0	0	2	2
$\mathbf{2}_3$	2	-1	-1	2	-1	0	0	0	2	2
$\mathbf{2}_4$	2	-1	-1	-1	2	0	0	0	2	2
$\mathbf{3}_1$	3	0	0	0	0	1	ω^2	ω	3ω	$3\omega^2$
$\overline{\mathbf{3}}_1$	3	0	0	0	0	1	ω	ω^2	$3\omega^2$	3ω
$\mathbf{3}_2$	3	0	0	0	0	-1	$-\omega^2$	$-\omega$	3ω	$3\omega^2$
$\overline{\mathbf{3}}_2$	3	0	0	0	0	-1	$-\omega$	$-\omega^2$	$3\omega^2$	3ω

The outer automorphisms group of any ("small") discrete group can easily be found with `GAP` [GAP] .

Group	Out	Action on reps
$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\boldsymbol{r} \to \boldsymbol{r}^*$
$\mathrm{A}_{n\neq 6}$	$\mathbb{Z}_2$	$\boldsymbol{r} \to \boldsymbol{r}^*$
$\mathrm{S}_{n\neq 6}$	/	/
$\Delta(27)$	$\mathrm{GL}(2,3)$	$\boldsymbol{r}_i \to \boldsymbol{r}_j$
$\Delta(54)$	S_4	$\boldsymbol{r}_i \to \boldsymbol{r}_j$
$\ldots$		

Two types of groups (without mathematical rigor)



List of representations: $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_k, {\boldsymbol{r}_k}^*, \ldots$

$$\mathrm{Out}\text{ in general }: \qquad \boldsymbol{r}_i \;\mapsto\; \boldsymbol{r}_j \quad \forall \text{ irreps } i,j \; (1:1)$$

Criterion:

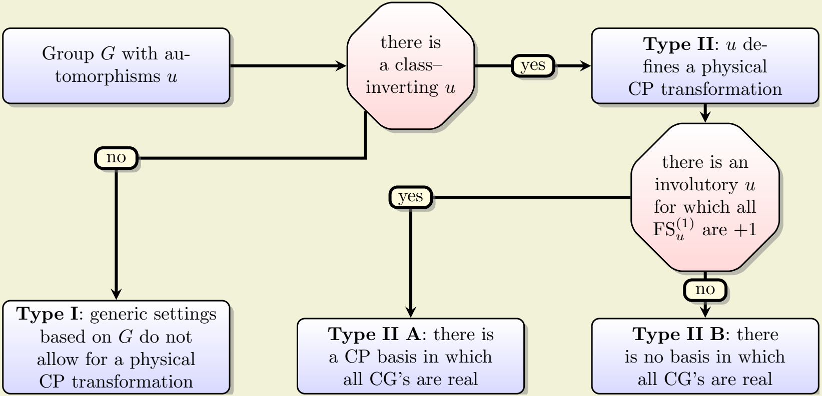
Is there an (outer) automorphism transformation that maps

$$\boldsymbol{r}_i \;\mapsto\; {\boldsymbol{r}_i}^* \qquad \text{for all irreps } i\,?$$

No	**Yes**
$\Rightarrow$ Group of “**type I**”	$\Rightarrow$ Group of “**type II**”

This tells us whether a CP transformation is possible, or not!

Systematic classification of **finite Groups** G



(For details see [Chen, Fallbacher, Mahanthappa, Ratz, AT, '14])

Mathematical tool to decide: Twisted Frobenius-Schur indicator FS_u(Backup slides)

Do CP transformations exist for all symmetries?

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General answer: **No.**

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For example: Discrete groups of **type I**:

G	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	T_7	$\Delta(27)$	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	$\ldots$
SG id	$(20,3)$	$(21,1)$	$(27,3)$	$(27,4)$	

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- These are **inconsistent** with the trafo $\boldsymbol{r}_i \;\mapsto\; \boldsymbol{r}_i^* \;\forall\, i$.

$\Rightarrow$ CP transformation is inconsistent with a type I symmetry.
(assuming sufficient # of irreps are in the model)

There are models in which CP is violated
as a consequence of unbroken type I symmetry.

[Chen, Fallbacher, Mahanthappa, Ratz, AT '14]

The corresponding CPV phases are calculable and quantized (e.g. $\delta_{\cancel{CP}} = 2\pi/3, ...$) stemming from the necessarily complex Clebsch-Gordan coefficients of the "type I" group. This has been termed "explicit geometrical" CP violation.

[Chen, Fallbacher, Mahanthappa, Ratz, AT '14]
[Branco, '15], [de Medeiros Varzielas, '15]

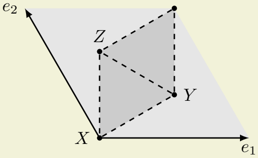
Example with $\Delta(54)$

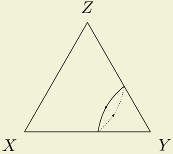
"CP Violation from String Theory"

[Nilles, Ratz, Trautner, Vaudrevange '18]

CP violation from string theory

- Heterotic orbifold theory compactified on $\mathbb{T}^2/\mathbb{Z}_3$. [Ibáñez, Kim, Nilles, Quevedo '87]
- This theory has $\Delta(54)$ flavor symmetry. [Kobayashi, Nilles, Plöger, Raby, Ratz '07]
- These models are "semi-realistic" (MSSM from heterotic orbifolds) SM families + RHν's are $\Delta(54)$-triplets. [Carballo-Perez, Peinado, Ramos-Sanchez '16]
- Light spectrum consist *only* of $\Delta(54)$ singlets and triplets.





- $\Delta(54)$ is a group of type I, can lead to "geometrical CP violation".
- Identification of source of CP violation: Type I flavor symmetry & presence of heavy winding strings.

CP violation from string theory

$\Delta(54)$ is group of type I $\iff$ $\mathrm{Out}\,[\Delta(54)] \cong \mathrm{S}_4$ does not contain simultaneous CP trafo for *all* states.

- **However**, there exist trafos in $\mathrm{Out}\,[\Delta(54)]$ which correspond to CP trafos for the singlets and triplets (the light spectrum!).
- **Crucial**: these are **no** physical CP transformations *IF* there are *more than* two doublet states $\mathbf{2}_{1,2,3,4}$!

This is what we call a **"CP-like"** transformation.

$\Delta(54)$	C_{1a}	C_{3a}	C_{3b}	C_{3c}	C_{3d}	C_{2a}	C_{6a}	C_{6b}	C_{3e}	C_{3f}
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$\mathbf{1}_1$	1	1	1	1	1	-1	-1	-1	1	1
$\mathbf{2}_1$	2	2	-1	-1	-1	0	0	0	2	2
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$\mathbf{3}_2$	3	0	0	0	0	-1	$-\omega^2$	$-\omega$	3ω	$3\omega^2$
$\overline{\mathbf{3}}_2$	3	0	0	0	0	-1	$-\omega$	$-\omega^2$	$3\omega^2$	3ω

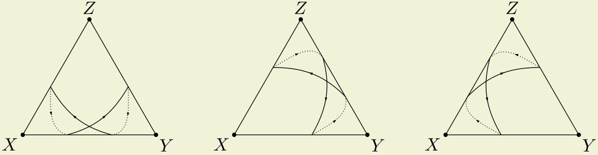
- Are there doublets in the string model?

Doublets in the string model

Easy trick to see (three of) the doublets: Technical details see [Lauer, Mas, Nilles '89,'91]

$$\boldsymbol{3}_i \otimes \overline{\boldsymbol{3}}_i = \boldsymbol{1}_0 \oplus \boldsymbol{2}_1 \oplus \boldsymbol{2}_2 \oplus \boldsymbol{2}_3 \oplus \boldsymbol{2}_4 .$$

(Heavy) string winding modes transform as doublets.



Interactions between light (triplets) and heavy (doublet) modes:

EFT superpotential: $\mathscr{W} \supset \sum_k (c_k)^{mab}\, \phi_m^{(\boldsymbol{2}_k)} \chi_a^{(\boldsymbol{3}_1)} \psi_b^{(\overline{\boldsymbol{3}}_1)}$.

Explicit identification of CPV

Convenient explicit proof for presence of CP Violation:

Construct CP-odd basis invariants (like Jarlskog Inv.)

see e.g. [Bernabeau, Branco, Gronau '86], [Lavoura, Silva '94]
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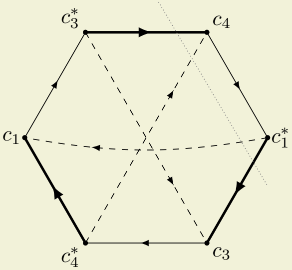
Lowest order invariant here is at four loop, and contains **three** doublets.



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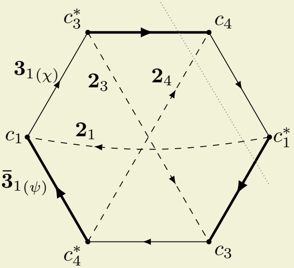

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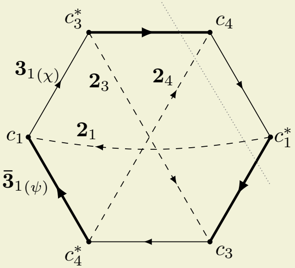

Explicit expression

$$\mathcal{I}_{\mathrm{CP-odd}} = \frac{\mathbf{1+3\,e^{4\pi i/3}}}{\mathbf{36}} |c_1|^2 \, |c_3|^2 \, |c_4|^2 \ .$$

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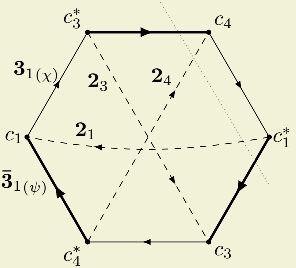

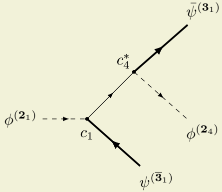

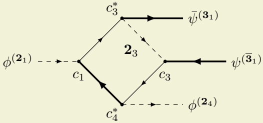

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Comments on this example

- This is a proof-of-principle that type I groups and (thereby caused) geometrical CP violation exists in potentially realistic string theory models.
- There exist many more semi-realistic string theory examples with type I groups.

  [Olguin-Trejo, Perez-Martinez, Ramos-Sanchez '18]
- Many (very model dependent) details remain to be worked out:
  - Decay of heavy modes is CP violating: B/L violation? Baryogenesis?
  - Does integrating out the heavy modes give rise to CP violation among the light modes? (no)
  - Yukawa couplings and low energy CP violation (CKM and θ)?

CP transformation of Modular Symmetry

CP transformation of modular symmetry

[Baur, Nilles, AT, Vaudrevange '19], [Novichkov, Penedo, Petcov, Titov '19]

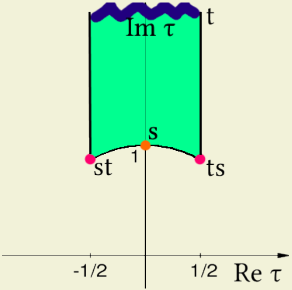
$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle\, \mathsf{s}, \mathsf{t} \;\middle|\; \mathsf{s}^4 = 1, \mathsf{s}^2 = (\mathsf{st})^3 \,\right\rangle$$

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\;, \quad \tau \xmapsto{\gamma} \frac{a\,\tau + b}{c\,\tau + d}\;, \quad \Phi \xmapsto{\gamma} (c\,\tau + d)^n \rho(\gamma)\Phi\;,$$

$$\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) / \{\mathbb{1}, -\mathbb{1}\}$$

$$\mathsf{s} : \tau \mapsto -\frac{1}{\tau}\;, \quad \mathsf{t} : \tau \mapsto \tau + 1\;,$$

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\; \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$



Class inverting outer automorphism?

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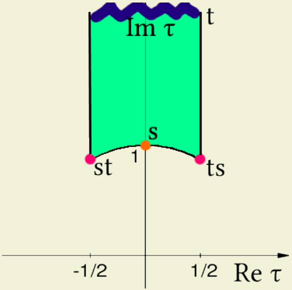

Class inverting outer automorphism? ✓

$$\mathsf{u}(\mathsf{s}) = \mathsf{s}^{-1},\ \mathsf{u}(\mathsf{t}) = \mathsf{t}^{-1}\ .$$

Coresponds to $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$ transformation

$$\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^{\mathcal{CP}} = \mathrm{GL}(2,\mathbb{Z})$$

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$$\det\left[\,\overline{\gamma} \in \mathrm{GL}(2,\mathbb{Z})\,\right] = -1\ , \quad \tau \xmapsto{\overline{\gamma}} \frac{a\overline{\tau}+b}{c\overline{\tau}+d}\ , \quad \Phi \xmapsto{\overline{\gamma}} (c\overline{\tau}+d)^n \rho(\overline{\gamma})\overline{\Phi}\ ,$$

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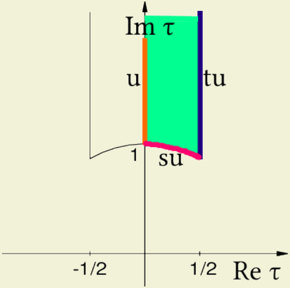

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Relevance of Outs for derivation of the Eclectic Flavor Symmetry

Origin of eclectic flavor symmetry in heterotic orbifolds

Narain lattice formulation of heterotic string theory: [Narain '86]

[Narain, Samardi, Witten '87],[Narain, M. H. Sarmadi, and C. Vafa,'87],[Groot Nibbelink & Vaudrevange '17]

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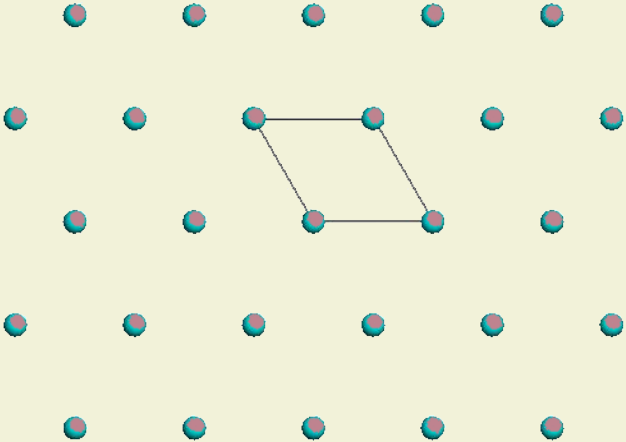

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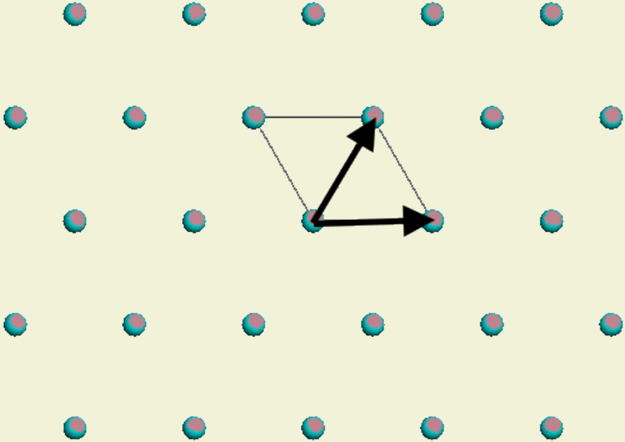

discrete translations

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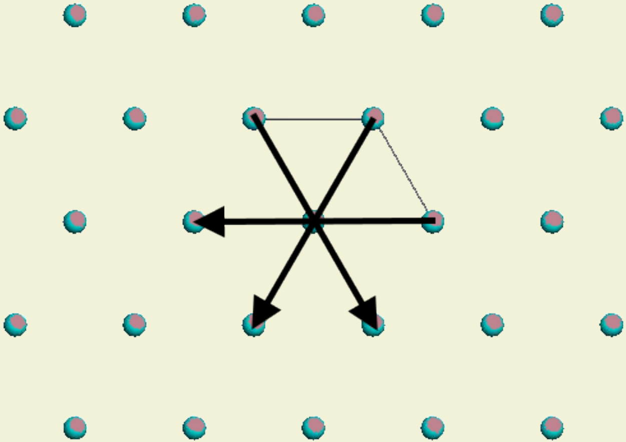

reflections / inversions

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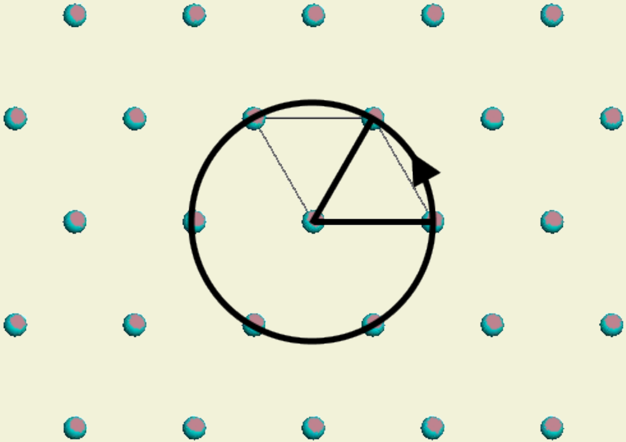

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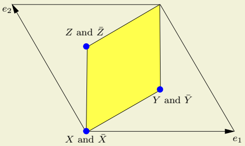
e.g. $\mathbb{T}^2/\mathbb{Z}_3$ (with $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$)



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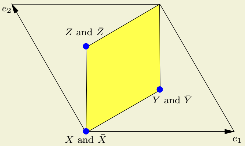

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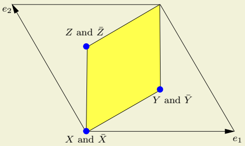

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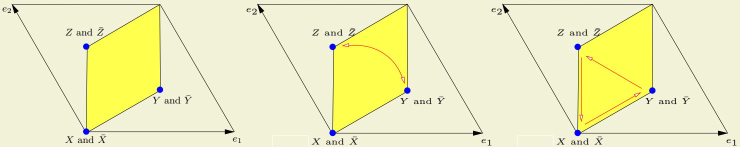

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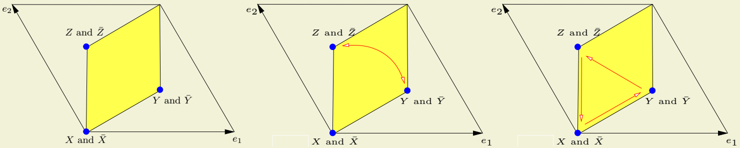

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New insight: Flavor symmetries are given by outer automorphisms of the Narain lattice space group! [Baur, Nilles, AT, Vaudrevange '19]

In this way we can unambiguously compute them in the top-down approach.

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- Bosonic string coordinates, D right- and D left-moving, $y_{\mathrm{R,L}}$, *compactified* on $2D$ torus:

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- $\Theta^K = \mathbb{1}$, is an "orbifold twist" with $\theta_{\mathrm{R,L}} \in \mathrm{SO}(D)$.
- "Narain lattice":

$$\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$$

(Γ is even, self-dual lattice with metric $\eta = \mathrm{diag}(-\mathbb{1}_D, \mathbb{1}_D)$.)

- $\hat{N} = (n, m) \in \mathbb{Z}^{2D}$, n: winding number, m: Kaluza-Klein number of string boundary condition.
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$$\mathcal{H}(T,U) = \frac{1}{\mathrm{Im}\,T\,\mathrm{Im}\,U} \begin{pmatrix} |T|^2 & |T|^2\,\mathrm{Re}\,U & \mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,T \\ |T|^2\,\mathrm{Re}\,U & |T\,U|^2 & |U|^2\,\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U \\ \mathrm{Re}\,T\,\mathrm{Re}\,U & |U|^2\,\mathrm{Re}\,T & |U|^2 & -\mathrm{Re}\,U \\ -\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,U & 1 \end{pmatrix}.$$

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Narain space group $g = (\Theta^k, E\,\hat{N}) \in S_{\mathrm{Narain}}$ is given by multiplicative closure of all twist and shifts

$$S_{\mathrm{Narain}} \;:=\; \langle\; (\Theta, 0)\;\;,\;\; (\mathbb{1}, E_i)\;\; \text{for}\;\; i \in \{1, \ldots, 2D\}\;\rangle\;.$$

Outs of the Narain lattice

Maps beween Narain lattice Γ to an equivalent lattice Γ' are given by outer automorphisms of the **Narain lattice**

$$\mathrm{O}_{\hat{\eta}}(D,D,\mathbb{Z}) \;:=\; \left\langle\, \hat{\Sigma} \;\middle|\; \hat{\Sigma} \,\in\, \mathrm{GL}(2D,\mathbb{Z}) \quad \mathrm{with} \quad \hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma} \,=\, \hat{\eta} \,\right\rangle .$$

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For example, specializing to $D=2$, $\curvearrowright$ d.o.f. in E are Kähler (T) and complex strucutre moduli (U). Outs of Narain lattice:

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With $\mathrm{SL}(2,\mathbb{Z})$ and action on the moduli $M=\{T,U\}$ given by

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$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle\, \mathsf{s}, \mathsf{t} \;\middle|\; \mathsf{s}^4 = 1, \mathsf{s}^2 = \mathsf{s}\mathsf{t}^3 \,\right\rangle .$$

$$\mathsf{s} : M \mapsto -\frac{1}{M} \qquad \text{and} \qquad \mathsf{t} : M \mapsto M+1\ ,$$

Outer automorphisms of Γ contain the **modular transformations**, including T-duality transformations, $T \leftrightarrow U$ mirror symmetry and a $\mathcal{CP}$-like transformation $M \mapsto -\overline{M}$.

[Baur, Nilles, AT, Vaudrevange '19]

Outs of the Narain space group

For the full **Narain space group**, the outer automorphisms are given by transformations $h := (\hat{\Sigma}, \hat{T}) \notin S_{\text{Narain}}$ such that

$$g \overset{h}{\mapsto} h\,g\,h^{-1} \overset{!}{\in} S_{\text{Narain}}\;.$$

Outs are given by the solutions to the **consistency conditions**

$$\begin{aligned} \hat{\Sigma}\,\Theta^k\,\hat{\Sigma}^{-1} &\overset{!}{=} \Theta^{k'}\;, \\ \left(\mathbb{1} - \hat{\Sigma}\,\Theta^k\,\hat{\Sigma}^{-1}\right)\hat{T} &\overset{!}{=} \hat{N}'\;. \end{aligned}$$

Solutions yield a set of generators of the Out group as

$$\left\{(\hat{\Sigma}_1, 0),\ (\hat{\Sigma}_2, 0),\ \ldots,\ (\mathbb{1}, \hat{T}_1),\ (\mathbb{1}, \hat{T}_2),\ \ldots\right\}\;.$$

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Note: These Outs also act on the moduli. $M \equiv T, U$

$M \stackrel{h}{\longmapsto} M' = M$ $\rightarrow$ “traditional flavor trafo”

$M \stackrel{h}{\longmapsto} M' \neq M$ $\rightarrow$ “modular flavor trafo”

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Outer automorphisms of Narain space group unify flavor symmetries with **modular transformations**, including $\mathcal{CP}$-like transformations.

The eclectic flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

<table>
<tr><th colspan="2">nature of symmetry</th><th>outer automorphism of Narain space group</th><th colspan="5">flavor groups</th></tr>
<tr><td rowspan="6">eclectic</td><td rowspan="2">modular</td><td>rotation $\mathrm{S} \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="3" rowspan="2">T'</td><td rowspan="6">$\Omega(2)$</td></tr>
<tr><td>rotation $\mathrm{T} \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="4">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation $\mathrm{R} \in \mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

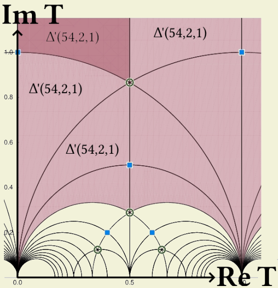
table from [Nilles, Ramos-Sánchez, Vaudrevange '20]

Action on the T modulus as

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{K}_*^{\mathcal{CP}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{R}$: trivial!



The eclectic flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

(For this specific orbifold, $\langle U \rangle = \exp({}^{2\pi\mathrm{i}}\!/\!{}_3)$.)

The outer automorphisms of the corresponding Narain space group yield the following symmetries:

[Baur, Nilles, AT, Vaudrevange '19; Nilles, Ramos-Sánchez, Vaudrevange '20]

- a $\Delta(54)$ traditional flavor symmetry,
- an $\mathrm{SL}(2,\mathbb{Z})_T$ modular symmetry which acts as a $\Gamma'_3 \cong T'$ finite modular symmetry on matter fields and their couplings,
- a $\mathbb{Z}_9^R$ discrete R-symmetry as remnant of $\mathrm{SL}(2,\mathbb{Z})_U$, and
- a $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation.

$$\boxed{G_{\textbf{eclectic}} \;=\; G_{\text{traditional}} \;\cup\; G_{\text{modular}} \;\cup\; G_{\mathrm{R}} \;\cup\; \mathcal{CP}\,,}$$

Together, the full eclectic group of this setting is of order 3888 given by

$$G_{\text{eclectic}} \;=\; \Omega(2) \rtimes \mathbb{Z}_2^{\mathcal{CP}}\,, \qquad \text{with} \quad \Omega(2) \cong [1944, 3448]\,.$$

Summary

- CP is a special outer automorphism, corresponding to complex conjugation outer automorphism of **every** group.
- Groups which don't have such an automorphism (type I) violate CP in generic settings.
- Example: $\Delta(54)$, arising in semi-realistic string theory models.
- CP doesn't need to be "generalized", just applied correctly.
- Modular symmetry is of type II (has class-inverting Out).
- Outer automorphisms beyond CP: The complete eclectic flavor symmetry in top-down approach (modular+traditional+R+CP) can unambiguously be derived by the outer automorphisms of the Narain space group:

$$\boxed{G_{\textbf{eclectic}} \;=\; G_{\text{traditional}} \,\cup\, G_{\text{modular}} \,\cup\, G_{\text{R}} \,\cup\, \mathcal{CP}\,.}$$

Thank You

Backup slides

Physical CP transformations

Physical observable: Asymmetry $\Leftrightarrow$ Basis-invariants, e.g. J.

$$\varepsilon_{i\to f} \;=\; \frac{\left|\Gamma(i\to f)\right|^2-\left|\Gamma(\bar{\imath}\to\overline{f})\right|^2}{\left|\Gamma(i\to f)\right|^2+\left|\Gamma(\bar{\imath}\to\overline{f})\right|^2} \;\Leftrightarrow\; J \;=\; \det\left[M_u M_u^\dagger, M_d M_d^\dagger\right]$$

CP conservation: $\varepsilon, J \stackrel{!}{=} 0$.

see also [Bernabéu, Branco, Gronau '86], [Botella, Silva '94]

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To warrant this: **need** a map $M_{u/d} \to M_{u/d}^*$.
Equivalently:

$$\mathscr{L} \;\supset\; c\,\boxed{\mathcal{O}\,(x)} + c^*\,\boxed{\mathcal{O}^\dagger(x)} \qquad \Rightarrow \qquad \text{Fields} \xmapsto{\mathcal{CP}} (\text{Fields})^*$$

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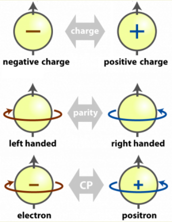
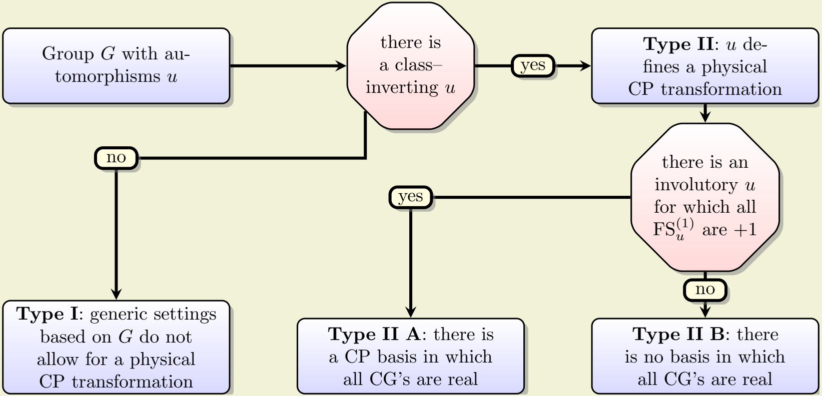


$$\Psi_{\text{Dirac}} = \begin{pmatrix} \chi_{\mathrm{L}} \\ \xi_{\mathrm{R}}^\dagger \end{pmatrix}$$

$$\begin{tikzcd}
\underset{(\mathrm{L},-)}{\chi} \arrow[rr, leftrightarrow, "\mathrm{P}"] \arrow[d, leftrightarrow, "\mathrm{CP}"] \arrow[drr, leftrightarrow, "\mathrm{C}"] & & \underset{(\mathrm{R},-)}{\xi^\dagger} \arrow[d, leftrightarrow, "\mathrm{CP}"] \\
\underset{(\mathrm{R},+)}{\chi^\dagger} \arrow[rr, leftrightarrow, "\mathrm{P}"] \arrow[urr, leftrightarrow, "\mathrm{C}"] & & \underset{(\mathrm{L},+)}{\xi}
\end{tikzcd}$$

CP symmetries in settings with discrete G



(For details see [Chen, Fallbacher, Mahanthappa, Ratz, AT, '14])

Mathematical tool to decide: Twisted Frobenius-Schur indicator FS_u(Backup slides)

Twisted Frobenius–Schur indicator

Criterion to decide: existence of a CP outer automorphism.
$\curvearrowright$ can be probed by computing the

"twisted Frobenius–Schur indicator" $\mathbf{FS}_{\boldsymbol{u}}$

$$\mathrm{FS}_u(\boldsymbol{r}_i) \ := \ \frac{1}{|G|}\sum_{g\in G}\chi_{\boldsymbol{r}_i}(g\,u(g))$$

($\chi_{\boldsymbol{r}_i(g)}$: Character)
[Chen, Fallbacher, Mahanthappa, Ratz, AT, 2014]

$$\mathrm{FS}_u(\boldsymbol{r}_i) \ = \ \begin{cases} +1 \text{ or } -1 \quad \forall\, i, & \Rightarrow\ u \text{ is good for CP,} \\ \text{different from } \pm 1, & \Rightarrow\ u \text{ is no good for CP.} \end{cases}$$

In analogy to the Frobenius–Schur indicator

$\mathrm{FS}_{\cancel{u}}(\boldsymbol{r}_i) = +1, -1, 0$ for real / pseudo–real / complex irrep.

Do type I groups occur in Nature?

- Discrete groups? $\rightarrow$ Crystals?

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  - ✗ no type I point groups in $2\mathrm{D}$ $(\mathrm{SO}(2))$, $3\mathrm{D}$ $(\mathrm{SO}(3))$.
  - ✗ no type I subgroups of $\mathrm{SU}(2)$.
  - ✗ no type I subgroups of the Lorentzgroup. (Open question: Type I "spacetime crystals"? [Wilczek '12]).
  - ✓ In $\geq 4\mathrm{D}$: crystals with type I point groups [Fischer, Ratz, Torrado and Vaudrevange '12]

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- Discrete flavor symmetries?
  - Many models with type I groups:
    $$\mathrm{T}_7, \Delta(27), \Delta(54), \mathcal{PSL}_2(7), ...$$
    e.g. [Björkeroth, Branco, Ding, de Anda, Ishimori, King, Medeiros Varzielas, Neder, Stuart et al. '15-'18]
    [Chen, Pérez, Ramond '14], [Krishnan, Harrison, Scott '18]
  - These can originate from extra dimensions, e.g. in string theory. [Kobayashi et al. '06], [Nilles, Ratz, Vaudrevange '12]

Types of (discrete) flavor symmetries

Schematically for the example of $\mathcal{N}=1$ SUSY.

x: spacetime, θ: superspace, Φ : (Super-)fields, T: modulus.
$K(T,\Phi)$: Kähler potential, $W(T,\Phi)$: Superpotential

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta} K(T,\overline{T},\Phi,\overline{\Phi}) + \int d^4x\, d^2\theta\, W(T,\Phi) + \int d^4x\, d^2\overline{\theta}\, \overline{W}(\overline{T},\overline{\Phi})\ .$$

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- **“traditional” Flavor symmetries** $\quad \Phi \mapsto \rho(\mathrm{g})\Phi\,, \quad \mathrm{g}\in G$

  for a review, see e.g. [King & Luhn ’13]

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- "traditional" Flavor symmetries $G_{\mathrm{traditional}}$
- **modular Flavor symmetries** **[Feruglio '17]**

$$\Phi\xmapsto{\gamma}(c\,T+d)^n\rho(\gamma)\Phi\;,\quad T\xmapsto{\gamma}\frac{a\,T+b}{c\,T+d}\;,\quad \gamma:=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z})\;.$$

Couplings are modular forms: $Y=Y(T)$, $Y(\gamma T)=(c\,T+d)^{k_Y}\,\rho_Y(\gamma)\,Y(T)$.

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- **R symmetries** for non-Abelian discrete R flavor symmetries see [Chen, Ratz, AT '13]

$$\Phi(x,\theta)=\boldsymbol{\phi}(x)+\sqrt{2}\boldsymbol{\theta}\,\boldsymbol{\psi}(x)+\theta\theta F(x)\;,\;\Longrightarrow\;\phi\mapsto e^{iq_\Phi\alpha}\phi,\;\psi\mapsto e^{i(q_\Phi-q_\theta)\alpha}\psi\,.$$

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- **general CP(-like) symmetries** [Novichkov, Penedo et al. '19],[Baur et al. '19]

$$\Phi \overset{\overline{\gamma}}{\longmapsto} (c\overline{T}+d)^n \rho(\overline{\gamma})\overline{\Phi} \;, \quad T \overset{\overline{\gamma}}{\longmapsto} \frac{a\overline{T}+b}{c\overline{T}+d} \;, \quad \det\left[\,\overline{\gamma} \in \mathrm{GL}(2,\mathbb{Z})\,\right] = -1 \;.$$

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From the bottom-up: All kinds known, individually!

$\rightarrow$ See talks by Penedo, Feruglio, de Medeiros Varzielas.

for an up-to-date review see [Feruglio&Romanino '19]

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From the top-down: *all, at the same time*!

$$\boxed{G_{\textbf{eclectic}} \ = \ G_{\text{traditional}} \ \cup \ G_{\text{modular}} \ \cup \ G_{\text{R}} \ \cup \ \mathcal{CP}\ ,}$$

see works by [Baur, Nilles, AT, Vaudrevange '19; Nilles, Ramos-Sánchez, Vaudrevange '20]

$\rightarrow$ See also talk by Ramos-Sánchez.

Top down flavor symmetries

- We identify points $Y \sim gY$ with $g \in S_{\text{Narain}} \Rightarrow$ fixed points.
- g constitutes boundary condition for closed strings

$\Rightarrow$ "Strings are localized at fixed points." [Dixon, Harvey, Vafa, Witten '85,'86]

- Each fixed point corresponds to a whole conjugacy class $[g] = \{f\, g\, f^{-1} \mid f \in S_{\text{Narain}}\}$ of space group elements
- each c.c. corresponds to a different fixed point.

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  $\Rightarrow$ non-trivial maps between strings at different f.p.'s!

New insight: Flavor symmetries are given by outer automorphisms of the Narain space group! [Baur, Nilles, AT, Vaudrevange '19]

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New insight: Flavor symmetries are given by outer automorphisms of the Narain space group! [Baur, Nilles, AT, Vaudrevange '19]

- The thus derived flavor symmetries automatically contain the so-called "space-group selection rules". [Hamidi and Vafa '86]
- They agree with previously derived non-Abelian flavor symmetries. [Kobayashi, Nilles, Plöger, Raby, Ratz '06]

Narain vielbein

The Narain vielbein can be parameterized as (in absence of Wilson lines)

$$E \;:=\; \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G-B) & -\sqrt{\alpha'}\,e^{-\mathrm{T}} \\ \dfrac{e^{-\mathrm{T}}}{\sqrt{\alpha'}}\,(G+B) & \sqrt{\alpha'}\,e^{-\mathrm{T}} \end{pmatrix} \;.$$

In this definition of the Narain vielbein, e denotes the vielbein of the D-dimensional geometrical torus $\mathbb{T}^D$ with metric $G := e^{\mathrm{T}}e$, $e^{-\mathrm{T}}$ corresponds to the inverse transposed matrix of e, B is the anti-symmetric background B-field ($B = -B^{\mathrm{T}}$), and α' is called the Regge slope.

World-sheet modular invariance requires E to span even, self-dual lattice $\Gamma = \{E\,\hat{N} \mid \hat{N} \in \mathbb{Z}^{2D}\}$ with metric η of signature (D, D). Consequently, one can always choose E such that

$$E^{\mathrm{T}}\eta\,E \;=\; \hat{\eta}\;, \quad \text{where} \quad \eta \;:=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \hat{\eta} \;:=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \;.$$

Transformation of moduli

To compute the transformation properties of the moduli T and U we use the generalized metric $\mathcal{H} = E^{\mathrm{T}}E$. As the Narain vielbein depends on the moduli $E = E(T,U)$ so does the generalized metric $\mathcal{H} = \mathcal{H}(T,U)$. It transforms as

$$\mathcal{H}(T,U) \xmapsto{\hat{\Sigma}} \mathcal{H}(T',U') = \hat{\Sigma}^{-\mathrm{T}}\mathcal{H}(T,U)\hat{\Sigma}^{-1} .$$

This equation can be used to read off the transformations of the moduli

$$T \xmapsto{\hat{\Sigma}} T' = T'(T,U) \quad \text{and} \quad U \xmapsto{\hat{\Sigma}} U' = U'(T,U) .$$

For a two-torus $\mathbb{T}^2$, the generalized metric in terms of the torus moduli reads

$$\mathcal{H}(T,U) = \frac{1}{\mathrm{Im}\,T\,\mathrm{Im}\,U}\begin{pmatrix} |T|^2 & |T|^2\,\mathrm{Re}\,U & \mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,T \\ |T|^2\,\mathrm{Re}\,U & |T\,U|^2 & |U|^2\,\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U \\ \mathrm{Re}\,T\,\mathrm{Re}\,U & |U|^2\,\mathrm{Re}\,T & |U|^2 & -\mathrm{Re}\,U \\ -\mathrm{Re}\,T & -\mathrm{Re}\,T\,\mathrm{Re}\,U & -\mathrm{Re}\,U & 1 \end{pmatrix} .$$

Explicit generators of $\Omega(2)$ for $\mathbb{T}^2/\mathbb{Z}_3$

$\mathrm{SL}(2,\mathbb{Z})_T$ modular generators S and T arise from rotational outer automorphisms and act on the modulus via

$$\mathrm{S} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{T} \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} ,$$

Reflectional outer automorphism coresponding to $\mathbb{Z}_2^{\mathcal{CP}}$ $\mathcal{CP}$-like transformation:

$$\mathrm{K}_* \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\rho(\mathrm{S}) = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \qquad \text{and} \qquad \rho(\mathrm{T}) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

The traditional flavor symmetry $\Delta(54)$ is generated by two translational outer automorphisms of the Narain space group A and B, together with the $\mathbb{Z}_2$ rotational outer automorphism $\mathrm{C} := \mathrm{S}^2$.

$$\rho(\mathrm{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ,\ \rho(\mathrm{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and } \rho(\mathrm{C}) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;=\; \rho(\mathrm{S})^2 ,$$

Example toy model:

“CP violation with an unbroken CP transformation”

[Ratz, AT ’16]

An interesting observation

Observation:

Type I groups can arise as subgroups of type II groups.

For example: small finite subgroups of simple Lie groups.

$$\mathrm{SU}(3) \;\supset\; \mathrm{T}_7$$

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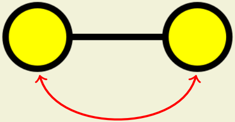
Structure of outer automorphisms:

$$\mathrm{Out}(\,\mathfrak{su}(3)\,) \;\cong\; \mathbb{Z}_2$$



$$\boldsymbol{r}_i \;\leftrightarrow\; \boldsymbol{r}_i^{\,*} \quad \forall i \;✓$$

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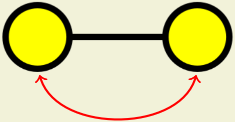

$$\boldsymbol{r}_i \leftrightarrow \boldsymbol{r}_i^{\,*} \quad \forall i \checkmark$$

T_7	C_{1a}	C_{3a}	C_{3b}	C_{7a}	C_{7b}
$\mathbf{1}_0$	1	1	1	1	1
$\mathbf{1}_1$	1	ω	ω^2	1	1
$\overline{\mathbf{1}}_1$	1	ω^2	ω	1	1
$\mathbf{3}_1$	3	0	0	η	η^*
$\overline{\mathbf{3}}_1$	3	0	0	η^*	η

$$\boldsymbol{r}_i \cancel{\leftrightarrow} \boldsymbol{r}_i^{\,*} \quad \forall i \;✗$$

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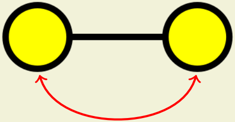

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Note: $\mathrm{Out}(\,\mathfrak{su}(3)\,)$ acts on the $\mathrm{T}_7 \subset \mathrm{SU}(3)$ subgroup as $\mathrm{Out}(\,\mathrm{T}_7\,)$!

Toy model overview

Facts:

- $\mathrm{SU}(3)$ is **consistent** with a physical CP transformation.
- The T_7 subgroup of $\mathrm{SU}(3)$ is **inconsistent** with a physical CP transformation.

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$$\mathscr{L} \;=\; (D_\mu \phi)^\dagger \, (D^\mu \, \phi) - \frac{1}{4} \, G^a_{\mu\nu} \, G^{\mu\nu,a} - V(\phi) \;,$$

$$V(\phi) \;=\; -\,\mu^2 \phi^\dagger \phi + \sum_{i=1}^{5} \lambda_i \, \mathcal{I}_i^{(4)}(\phi) \;. \qquad \text{with } \lambda_i \in \mathbb{R}$$

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- VEV of the $\mathbf{15}$-plet $\langle \phi \rangle$ breaks $\mathrm{SU}(3) \to \mathrm{T}_7$. [Luhn, '11], [Merle, Zwicky '11]
- $\mathrm{Out}(\,\mathfrak{su}(3)\,) \cong \mathbb{Z}_2 \;\to\; \mathrm{Out}(\,\mathrm{T}_7\,) \cong \mathbb{Z}_2$; Out **unbroken** by VEV.

$$\mathrm{SU}(3) \rtimes \mathbb{Z}_2 \;\xrightarrow{\langle \phi \rangle}\; \mathrm{T}_7 \rtimes \mathbb{Z}_2; .$$

CP violation in $\mathbf{SU(3)} \rightarrow \mathbf{T_7}$ toy model

[Ratz, AT '16]

Name	SU(3)	$\xrightarrow{\langle\phi\rangle}$ Name	T_7	mass
A_μ	$\mathbf{8}$	Z_μ	$\mathbf{1_1}$	$m_Z^2 = 7/3\, g^2\, v^2$
		W_μ	$\mathbf{3}$	$m_W^2 = g^2\, v^2$
ϕ	$\mathbf{15}$	$\mathrm{Re}\,\sigma_0$	$\mathbf{1_0}$	$m^2_{\mathrm{Re}\,\sigma_0} = 2\,\mu^2$
		$\mathrm{Im}\,\sigma_0$	$\mathbf{1_0}$	$m^2_{\mathrm{Im}\,\sigma_0} = 0$
		σ_1	$\mathbf{1_1}$	$m^2_{\sigma_1} = -\mu^2 + \sqrt{15}\,\lambda_5\, v^2$
		τ_1	$\mathbf{3}$	$m^2_{\tau_1} = m^2_{\tau_1}(\mu, \lambda_i)$
		τ_2	$\mathbf{3}$	$m^2_{\tau_2} = m^2_{\tau_2}(\mu, \lambda_i)$
		τ_3	$\mathbf{3}$	$m^2_{\tau_3} = m^2_{\tau_3}(\mu, \lambda_i)$

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			τ_3	$\mathbf{3}$	$m_{\tau_3}^2 = m_{\tau_3}^2(\mu, \lambda_i)$

The action is invariant under the $\mathbb{Z}_2 - \mathrm{Out}$ transformation:

SU(3)	T_7
$A_\mu^a(x) \mapsto R^{ab}\,\mathcal{P}_\mu^{\ \nu}\,A_\nu^b(\mathcal{P}\,x)$,	$W_\mu(x) \mapsto \mathcal{P}_\mu^{\ \nu}\,W_\nu^*(\mathcal{P}\,x)$,
$\phi_i(x) \mapsto U_{ij}\,\phi_j^*(\mathcal{P}\,x)$.	$\sigma_0(x) \mapsto \sigma_0(\mathcal{P}\,x)$,
	$\tau_i(x) \mapsto \tau_i^*(\mathcal{P}\,x)$,
	$Z_\mu(x) \mapsto -\mathcal{P}_\mu^{\ \nu}\,Z_\nu(\mathcal{P}\,x)$,
	$\sigma_1(x) \mapsto \sigma_1(\mathcal{P}\,x)$.
physical CP ✓	physical CP ✗

CP violation in $\mathbf{SU(3)} \rightarrow \mathbf{T_7}$ toy model

- The VEV does **not** break the CP transformation, $U\langle\phi\rangle^* = \langle\phi\rangle$.
- However, at the level of T_7, the $\mathrm{SU}(3)$-CP transformation merges to $\mathrm{Out}(\,\mathrm{T}_7\,)$:

$$
\mathbb{Z}_2 - \mathrm{Out}: \quad
\begin{array}{ccccccccccccccccc}
\mathbf{15} & \rightarrow & \mathbf{1_0} & \oplus & \mathbf{1_1} & \oplus & \overline{\mathbf{1}}_\mathbf{1} & \oplus & \mathbf{3} & \oplus & \mathbf{3} & \oplus & \overline{\mathbf{3}} & \oplus & \overline{\mathbf{3}} \\
\big\downarrow & & & & & & & & & & & & & & \\
\overline{\mathbf{15}} & \rightarrow & \mathbf{1_0} & \oplus & \overline{\mathbf{1}}_\mathbf{1} & \oplus & \mathbf{1_1} & \oplus & \overline{\mathbf{3}} & \oplus & \overline{\mathbf{3}} & \oplus & \mathbf{3} & \oplus & \mathbf{3}
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$\Rightarrow$ The $\mathbb{Z}_2$-Out is conserved at the level of T_7, but it is **not** interpreted as a physical CP trafo,

$$\mathrm{SU}(3) \rtimes \mathbb{Z}_2^{(\mathrm{CP})} \xrightarrow{\langle\phi\rangle} \mathrm{T}_7 \rtimes \mathbb{Z}_2^{\cancel{(\mathrm{CP})}}\,.$$

- There is no other possible allowed CP transformation at the level of T_7 (type I).
- Imposing a transformation $\boldsymbol{r}_{\mathrm{T}_7,i} \leftrightarrow \boldsymbol{r}_{\mathrm{T}_7,i}{}^*$ enforces decoupling, $g = \lambda_i = 0$.

CP violation in $\mathbf{SU(3)} \to \mathbf{T_7}$ toy model

Explicit crosscheck: compute decay asymmetry.

$$\varepsilon_{\sigma_1 \to W\,W^*} \;:=\; \frac{\left|\mathscr{M}(\sigma_1 \to W\,W^*)\right|^2 - \left|\mathscr{M}(\sigma_1^* \to W\,W^*)\right|^2}{\left|\mathscr{M}(\sigma_1 \to W\,W^*)\right|^2 + \left|\mathscr{M}(\sigma_1^* \to W\,W^*)\right|^2}\;.$$

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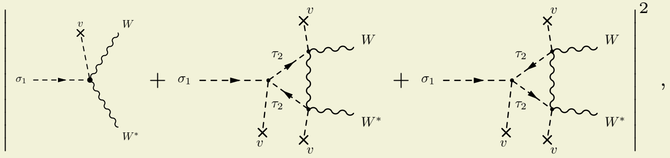
Contribution to $\varepsilon_{\sigma_1 \to W\,W^*}$ from interference terms, e.g.



corresponding to non-vanishing CP-odd basis invariants

$$\begin{aligned}
\mathcal{I}_1 &= \left[Y^\dagger_{\sigma_1 W W^*}\right]_{k\ell} \left[Y_{\sigma_1 \tau_2 \tau_2^*}\right]_{ij} \left[Y_{\tau_2^* W W^*}\right]_{imk} \left[\left(Y_{\tau_2^* W W^*}\right)^*\right]_{jm\ell}\;,\\
\mathcal{I}_2 &= \left[Y^\dagger_{\sigma_1 W W^*}\right]_{k\ell} \left[Y_{\sigma_1 \tau_2 \tau_2^*}\right]_{ij} \left[Y_{\tau_2^* W W^*}\right]_{i\ell m} \left[\left(Y_{\tau_2^* W W^*}\right)^*\right]_{jkm}\;.
\end{aligned}$$

- ✓ Contribution to $\varepsilon_{\sigma_1 \to W\,W^*}$ is proportional to $\operatorname{Im}\mathcal{I}_{1,2} \neq 0$.
- ✓ All CP odd phases are geometrical, $\mathcal{I}_1 = \mathrm{e}^{2\,\pi\,\mathrm{i}/3}\,\mathcal{I}_2$.
- ✓ $\left(\varepsilon_{\sigma_1 \to W\,W^*}\right) \to 0$ for $v \to 0$, i.e. CP is restored in limit of vanishing VEV.

Natural protection of $\theta = 0$

Topological vacuum term of the gauge group

$$\mathscr{L}_\theta \;=\; \theta\,\frac{g^2}{32\pi^2}\,G^a_{\mu\nu}\,\widetilde{G}^{\mu\nu,a}\;,$$

is forbidden by $\mathbb{Z}_2 - \mathrm{Out}$ (the $\mathrm{SU}(3)$-CP transformation).

The unbroken Out

$$\mathbb{Z}_2 - \mathrm{Out}\;:\; W_\mu(x)\;\mapsto\;\mathcal{P}_\mu^{\;\nu}\,W_\nu^*(\mathcal{P}\,x)\;,\quad Z_\mu(x)\;\mapsto\;-\mathcal{P}_\mu^{\;\nu}\,Z_\nu(\mathcal{P}\,x)\;,$$

still enforces $\theta = 0$ even though CP is violated for the physical T_7 states.

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still enforces $\theta = 0$ even though CP is violated for the physical T_7 states.

Physical scalars (T_7 singlets and triplets):

$$\mathrm{Re}\,\sigma_0 \;=\; \frac{1}{\sqrt{2}} \left(\phi_1 + \phi_1^*\right) \;, \qquad \mathrm{Im}\,\sigma_0 \;=\; -\,\frac{\mathrm{i}}{\sqrt{2}} \left(\phi_1 - \phi_1^*\right) \;,$$

$$\sigma_1 \;=\; \phi_2 \;,$$

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \;=\; \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \begin{pmatrix} T_2 \\ \overline{T}_3^* \\ T_1 \end{pmatrix} \;.$$

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Possible application to strong CP problem?

- Starting point: CP conserving theory based on

$$[G_{\mathrm{SM}} \times G_{\mathrm{F}}] \rtimes \mathrm{CP}\;.$$

- break $G_{\mathrm{F}} \rtimes \mathrm{CP} \;\longrightarrow\; \mathrm{Type\,I} \rtimes \mathrm{Out}.$

↷ CP broken in flavor sector but not in strong interactions.

- Main problem: finding realistic model based on Type I group allowing for outer automorphism.