University of Southampton School of Physics and Astronomy

elusives invisiblesPlus

neutrinos, dark matter & dark energy physics

Flavour Symmetries and their Modular Origin

Steve King, 2nd May 2022

Bethe Forum

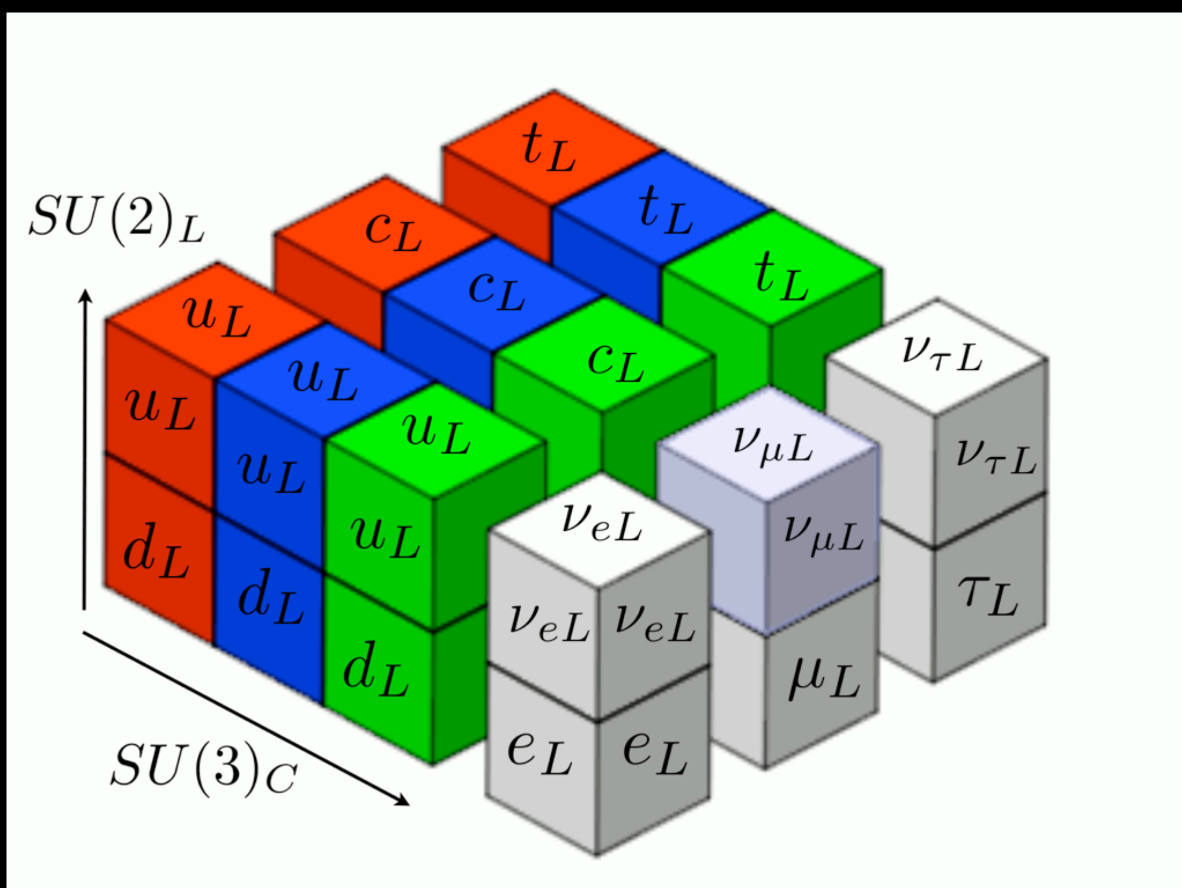
Modular Flavor Symmetries

Universität Bonn

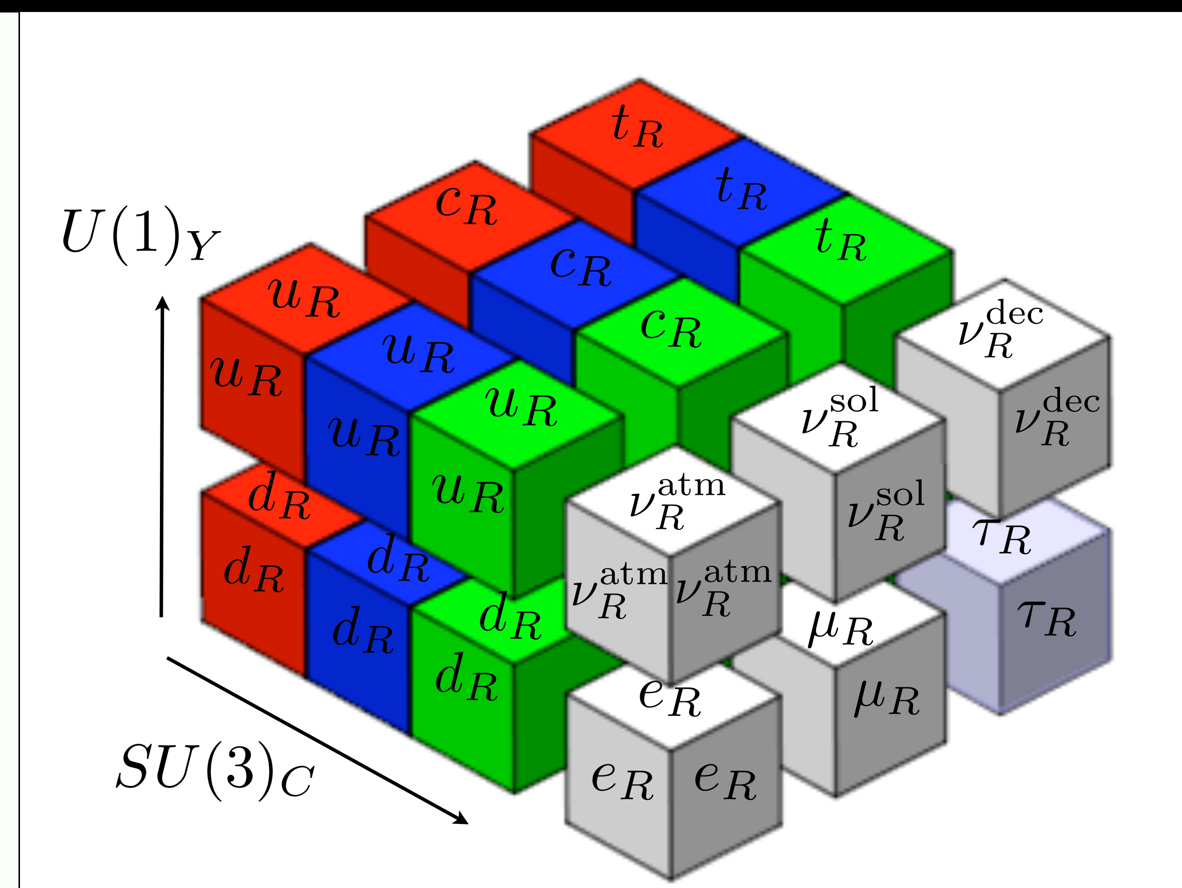
bctp Bethe Center for Theoretical Physics

The Standard Model

Left-handed

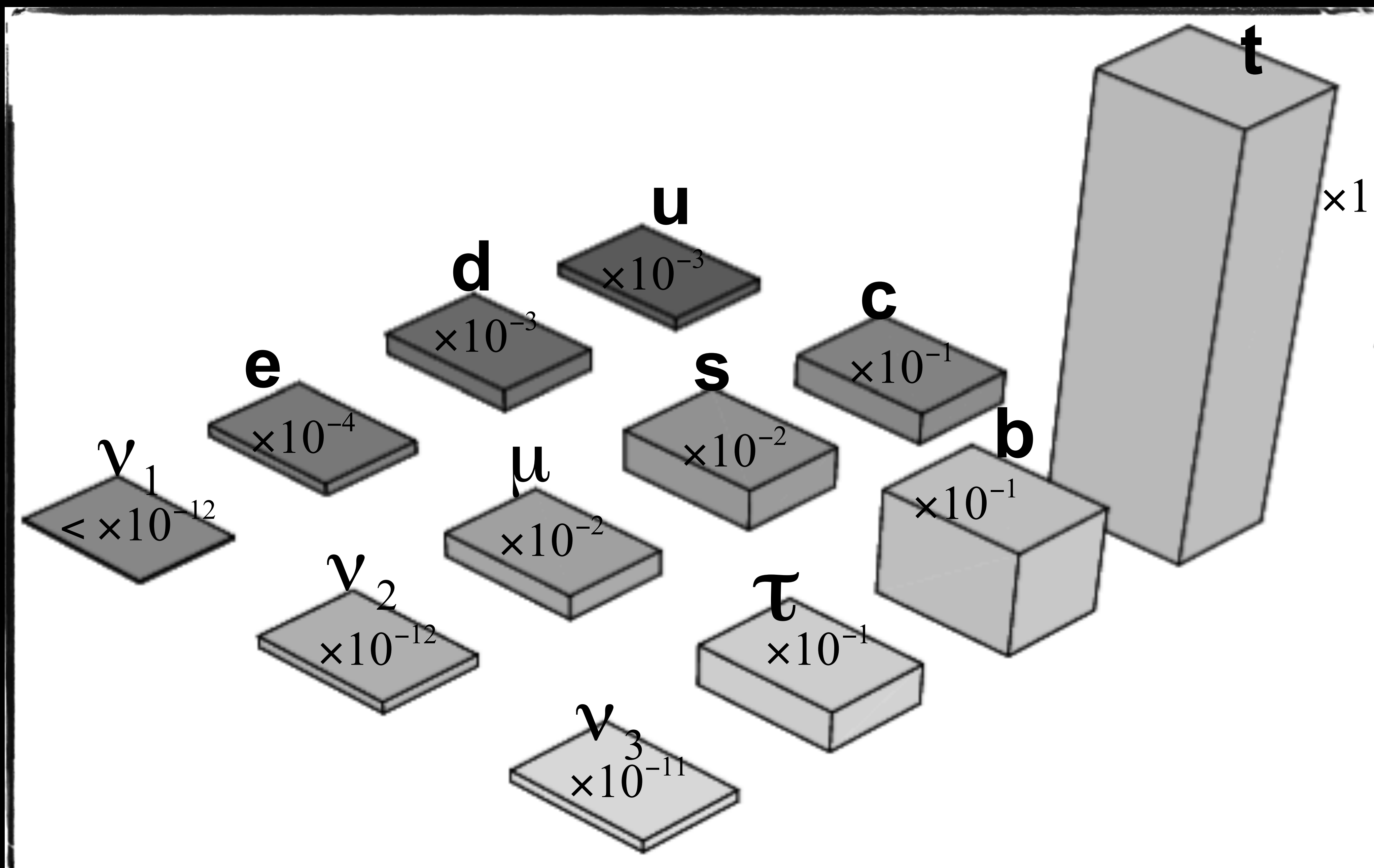


Right-handed

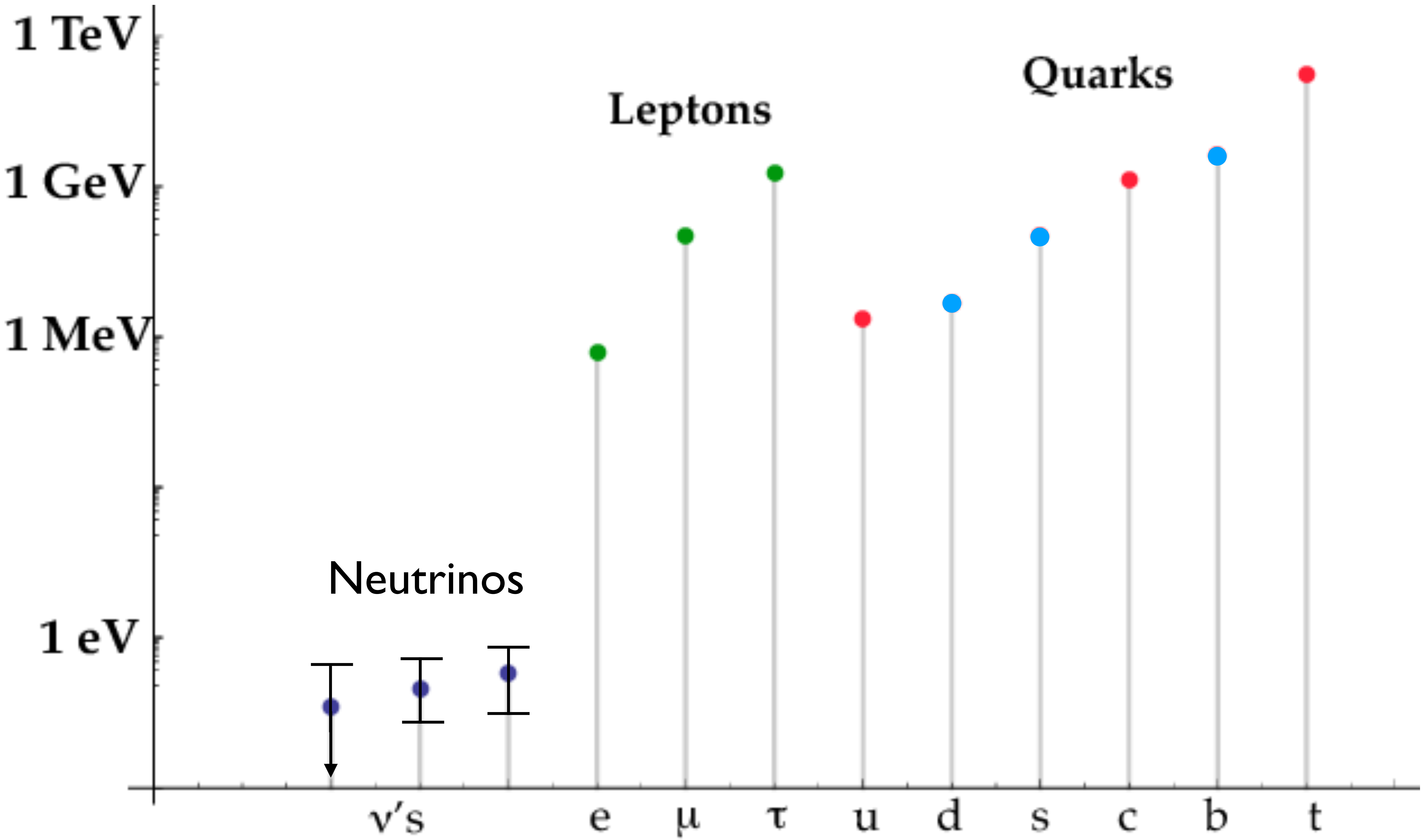


(Including three right-handed neutrinos)

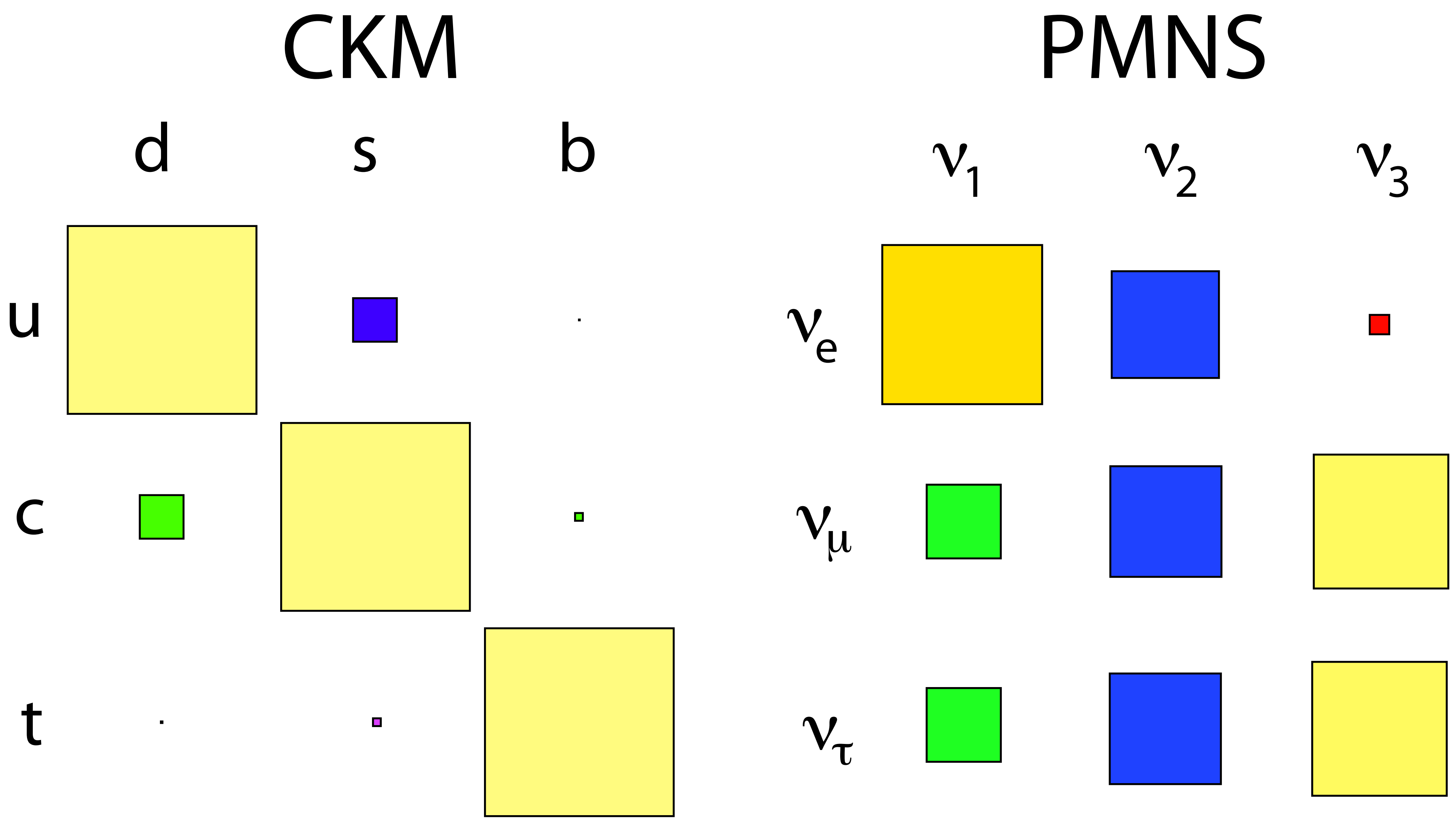
The Flavour Problem



Masses

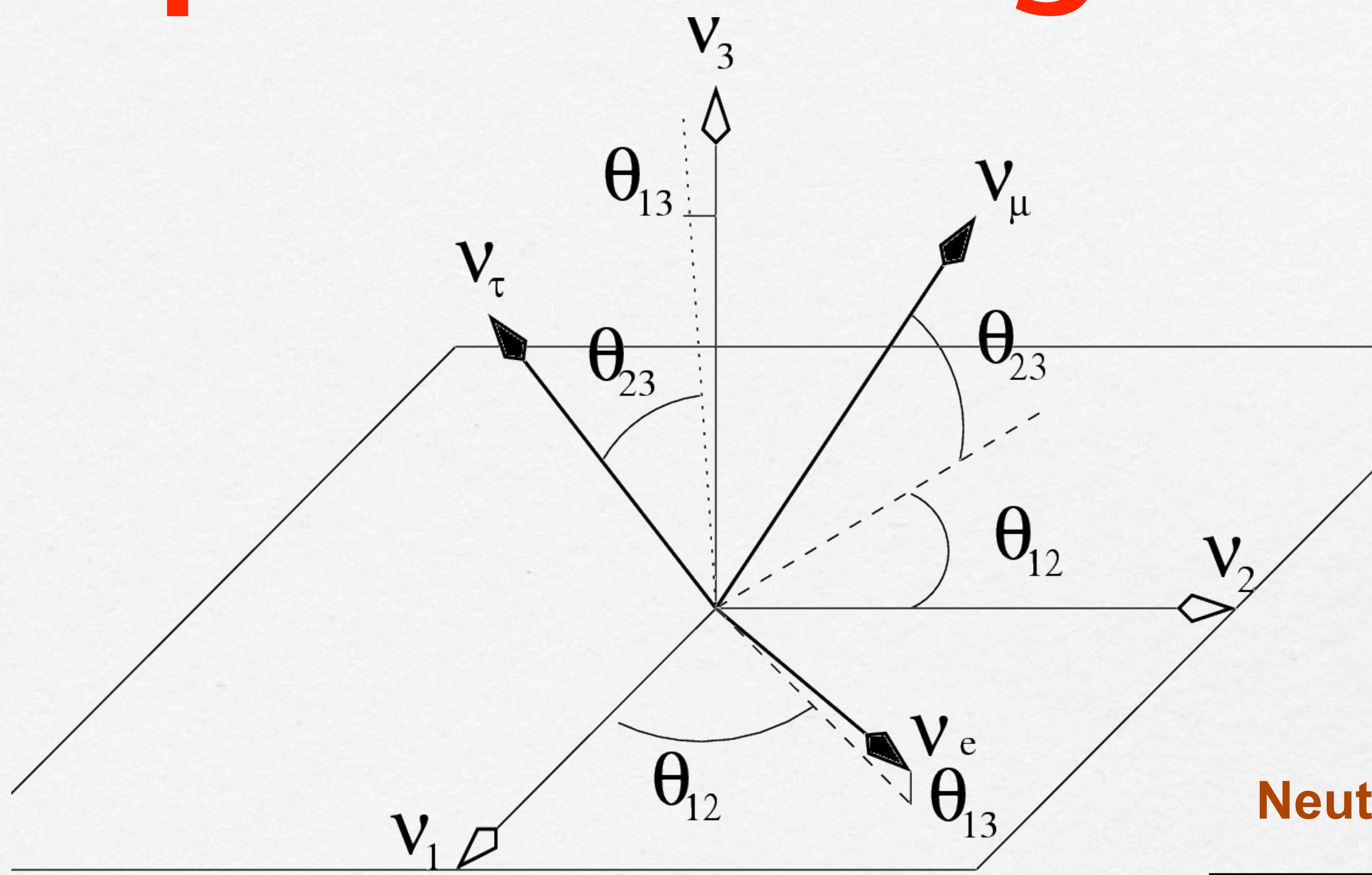


Mixing



PMNS Lepton mixing matrix

Pontecorvo
Maki
Nakagawa
Sakata

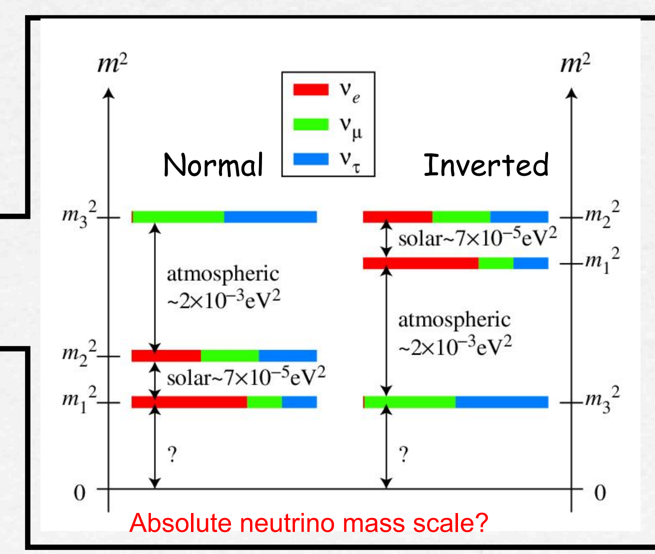


Standard Model states

$$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L$$

Neutrino mass states

$$\begin{pmatrix}\nu_e \\ \nu_\mu \\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix} \begin{pmatrix}\nu_1 \\ \nu_2 \\ \nu_3\end{pmatrix}$$



PMNS Lepton mixing matrix

$$U_{PMNS}=\begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0\\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}}\end{pmatrix}$$

Atmospheric | Reactor | Solar | Majorana

$$=\begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23}\\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}\end{pmatrix}$$

$$\times\ \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$$

NuFIT 5.1 (2021)

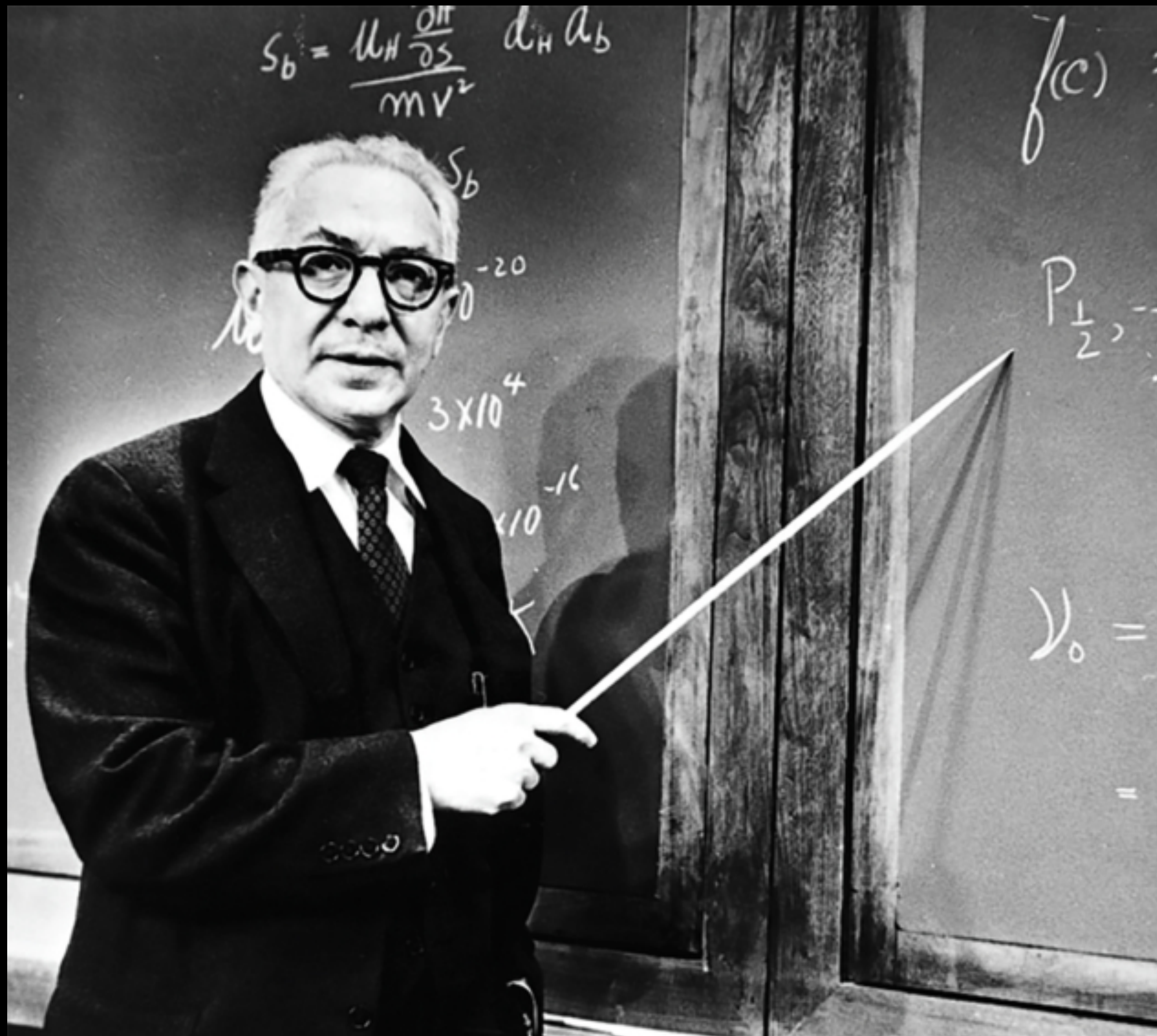
PMNS parameters

with SK atmospheric data

	Normal Ordering (best fit)	
	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$
$\theta_{12}/^\circ$	$33.45^{+0.77}_{-0.75}$	$31.27 \to 35.87$
$\sin^2\theta_{23}$	$0.450^{+0.019}_{-0.016}$	$0.408 \to 0.603$
$\theta_{23}/^\circ$	$42.1^{+1.1}_{-0.9}$	$39.7 \to 50.9$
$\sin^2\theta_{13}$	$0.02246^{+0.00062}_{-0.00062}$	$0.02060 \to 0.02435$
$\theta_{13}/^\circ$	$8.62^{+0.12}_{-0.12}$	$8.25 \to 8.98$
$\delta_{CP}/^\circ$	230^{+36}_{-25}	$144 \to 350$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.510^{+0.027}_{-0.027}$	$+2.430 \to +2.593$

Inverted Ordering ($\Delta\chi^2 = 7.0$)

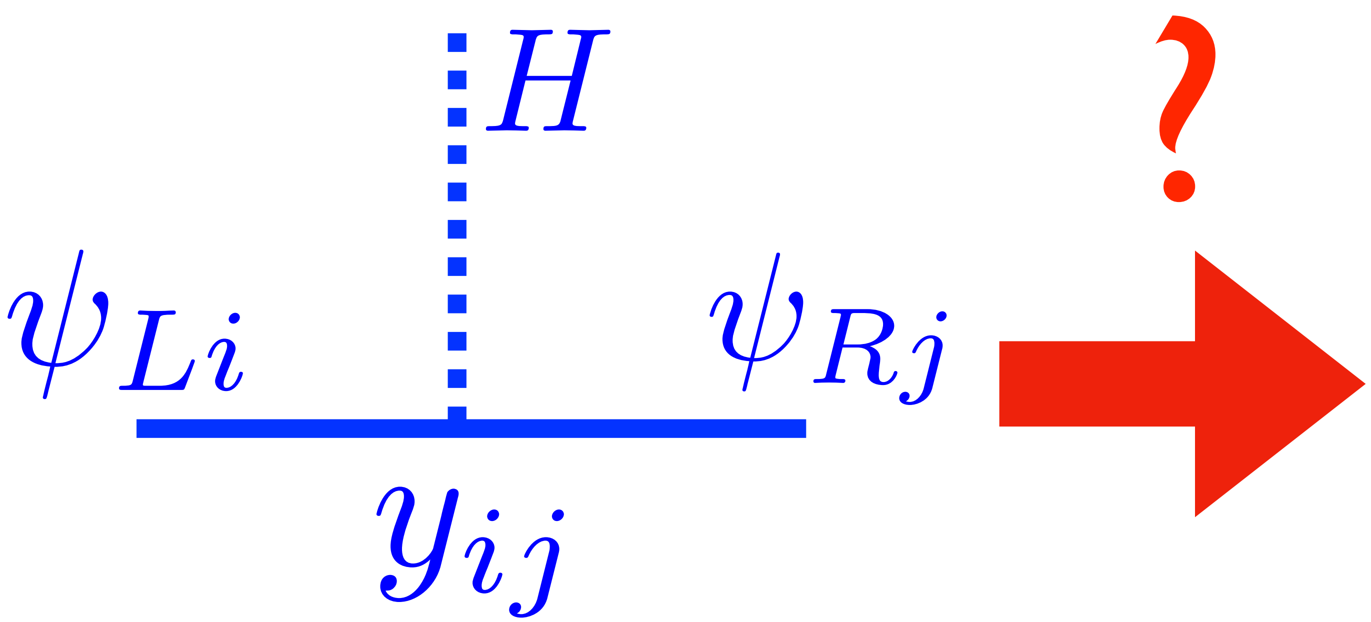
Who ordered all of that?

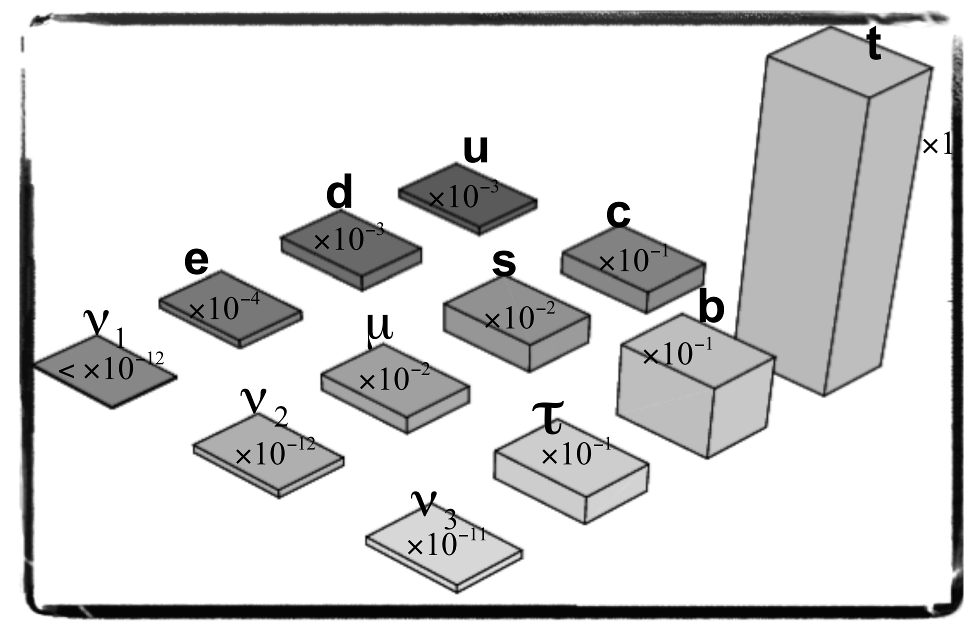


Isidor Issac Rabi

SM Yukawa couplings

$$y_{ij} H \overline{\psi}_{Li} \psi_{Rj}$$



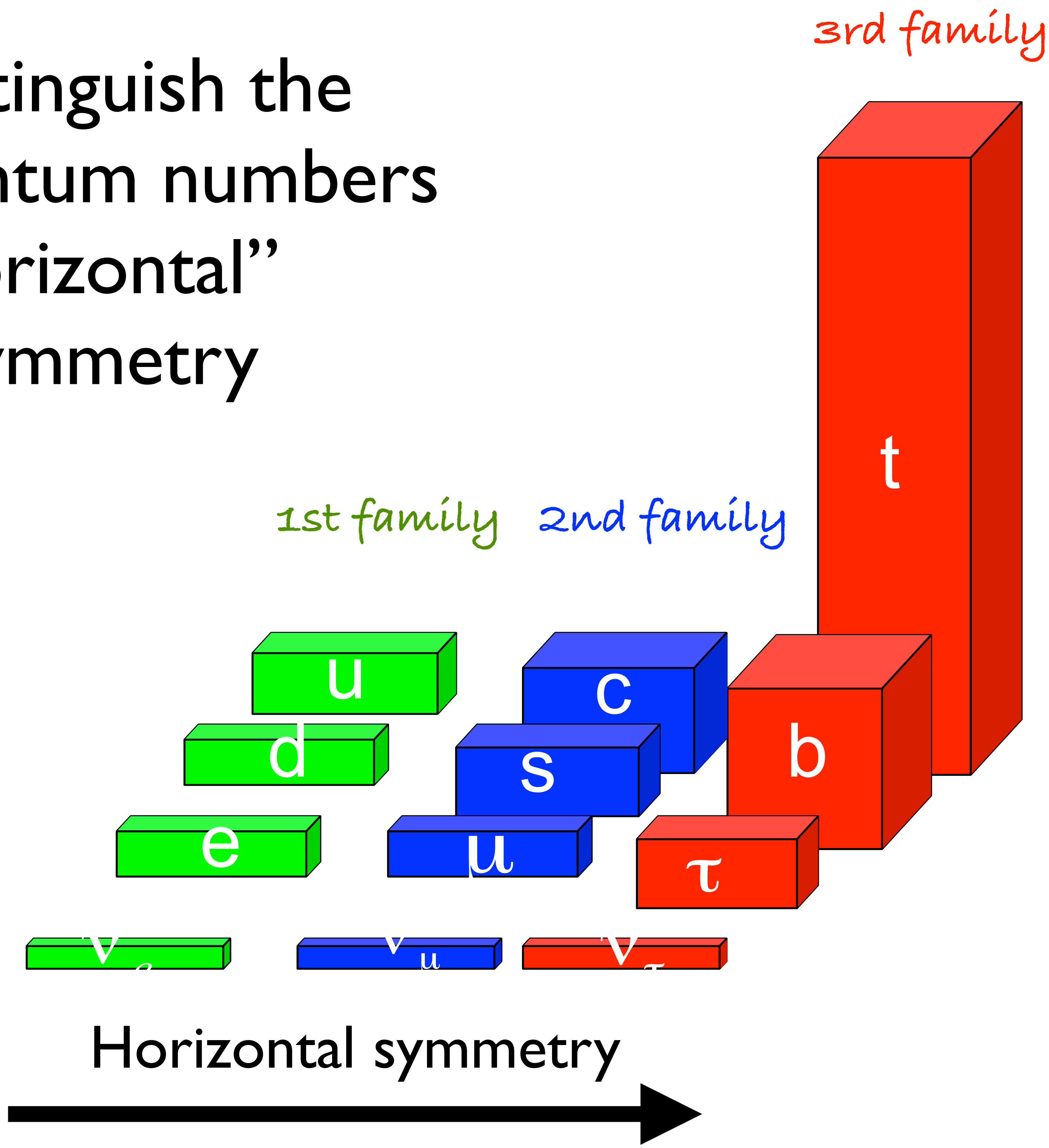


Is there a symmetry at work?

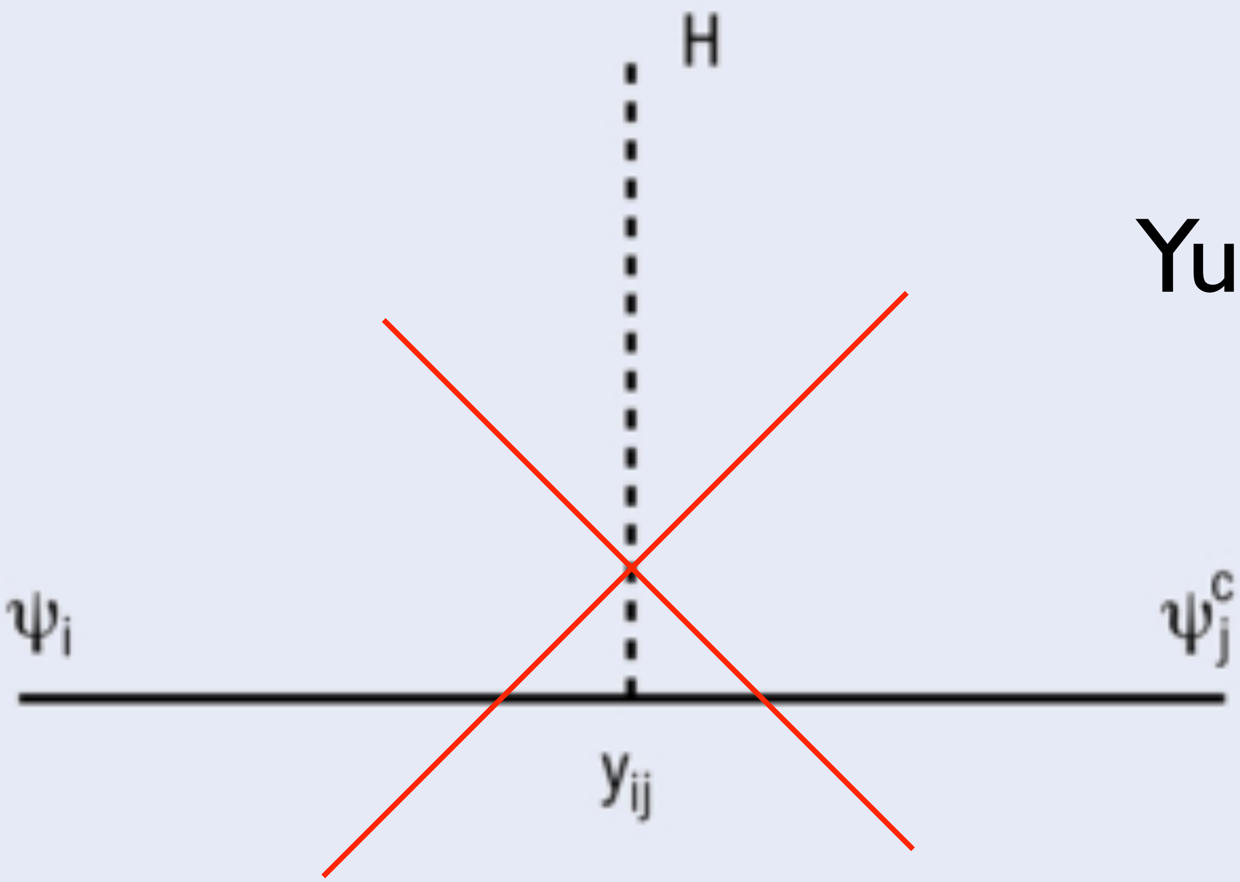
Family/Flavour Symmetry

Basic idea is to distinguish the families by some quantum numbers under a new "horizontal" family/flavour symmetry

The symmetry is assumed to be spontaneously broken by "flavons"

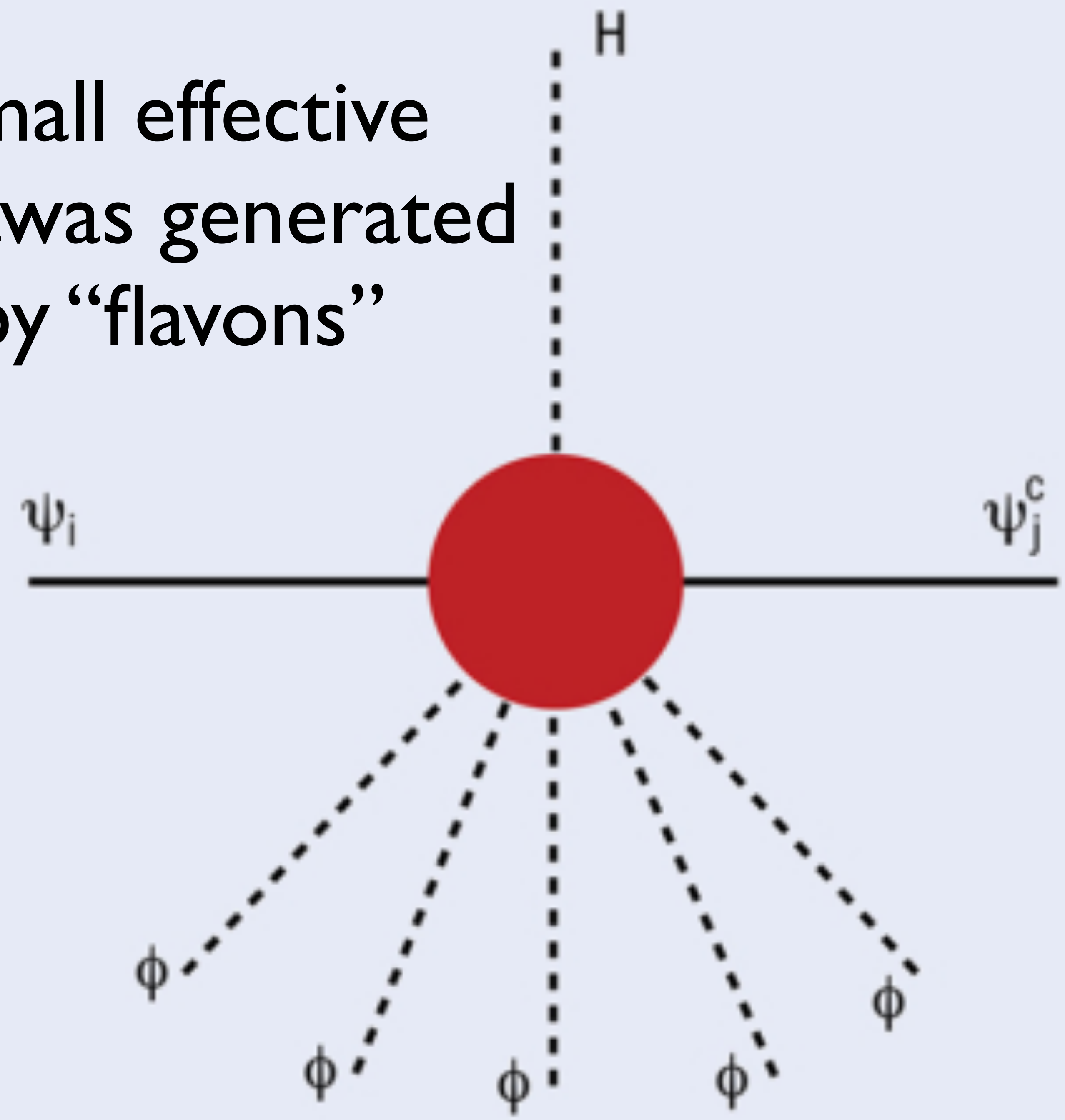


Family/Flavour Symmetry



Yukawas forbidden by the symmetry (apart from third family couplings)

Small effective Yukawas generated by "flavons"



Example: U(1) Family/Flavour Symmetry

Consider a U(1) family symmetry spontaneously broken by a flavon vev $\langle \phi \rangle \neq 0$

Suppose U(1) charges are Q (ψ_3)=0, Q (ψ_2)=1, Q (ψ_1)=3, Q(H)=0, Q(ϕ)=-1

Then the lowest order allowed Yukawa coupling is H ψ_3 ψ_3

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The other Yukawa couplings are generated from higher order operators which respect U(1) family symmetry due to flavon ϕ insertions:

$$\boxed{-1+0+1+0=0}$$

$$\frac{\phi}{M} H\psi_2\psi_3 + \left(\frac{\phi}{M}\right)^2 H\psi_2\psi_2 + \left(\frac{\phi}{M}\right)^3 H\psi_1\psi_3 + \left(\frac{\phi}{M}\right)^4 H\psi_1\psi_2 + \left(\frac{\phi}{M}\right)^6 H\psi_1\psi_1$$

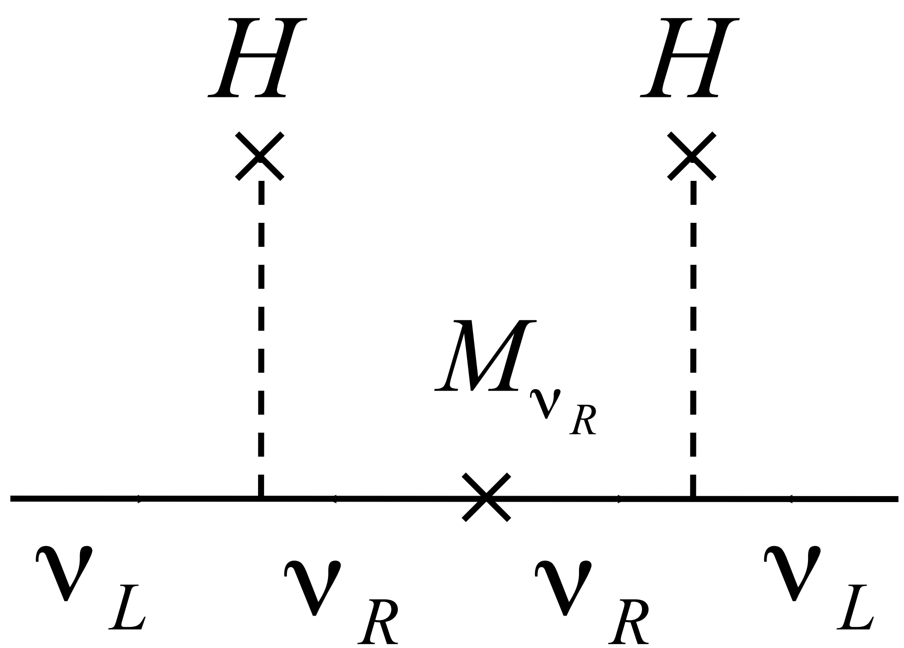
When the flavon gets its VEV it generates small effective Yukawa couplings in terms of an expansion parameter

$$\varepsilon = \frac{\langle \phi \rangle}{M}$$

Approximate texture zero

$$Y = \begin{pmatrix} \varepsilon^6 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^4 & \varepsilon^2 & \varepsilon \\ \varepsilon^3 & \varepsilon & 1 \end{pmatrix}$$

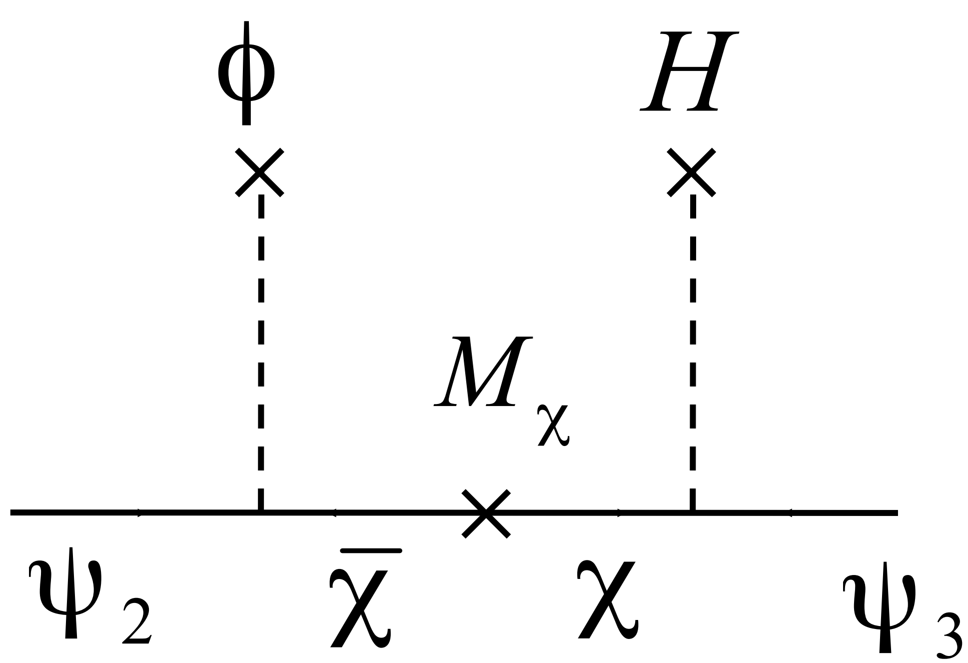
Froggatt-Nielsen Mechanism (1979)



What is the origin of the higher order operators?

Froggatt and Nielsen took their inspiration from the see-saw mechanism

$$\frac{H^2}{M_{\nu_R}}\nu_L \nu_L$$

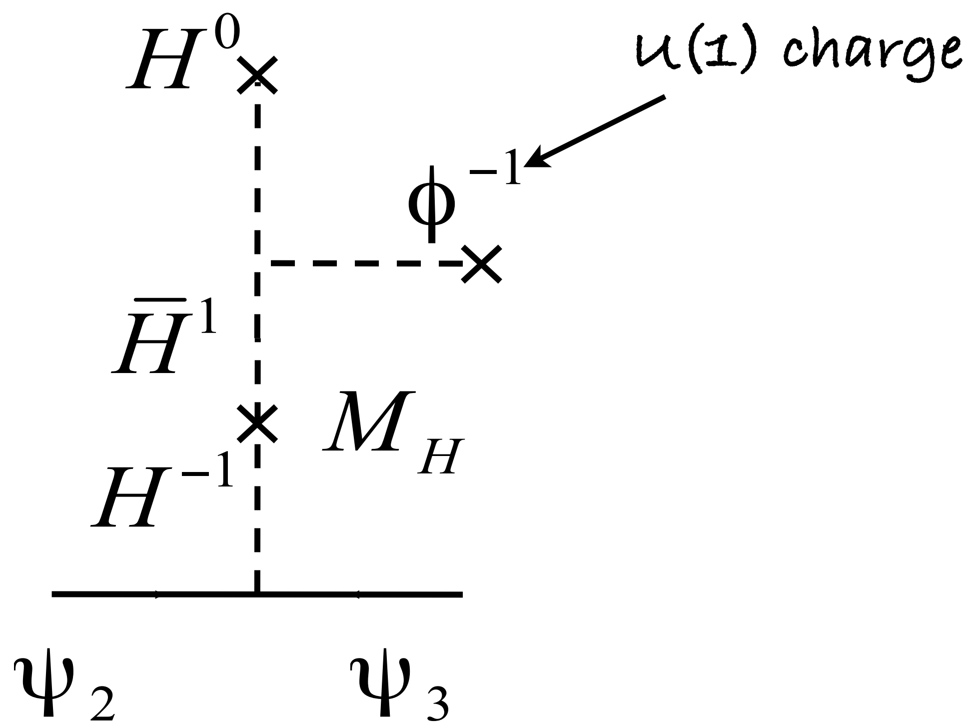


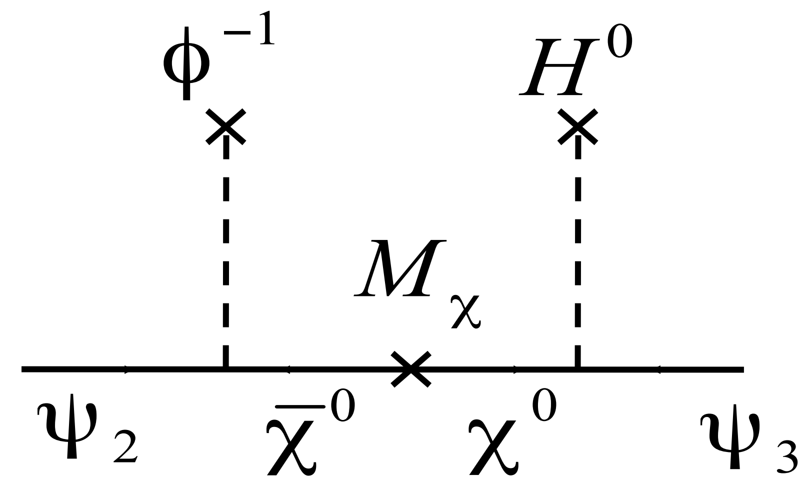
$$\frac{\phi}{M_\chi} H \psi_2 \psi_3$$

Where χ are heavy fermion messengers c.f. heavy RH neutrinos

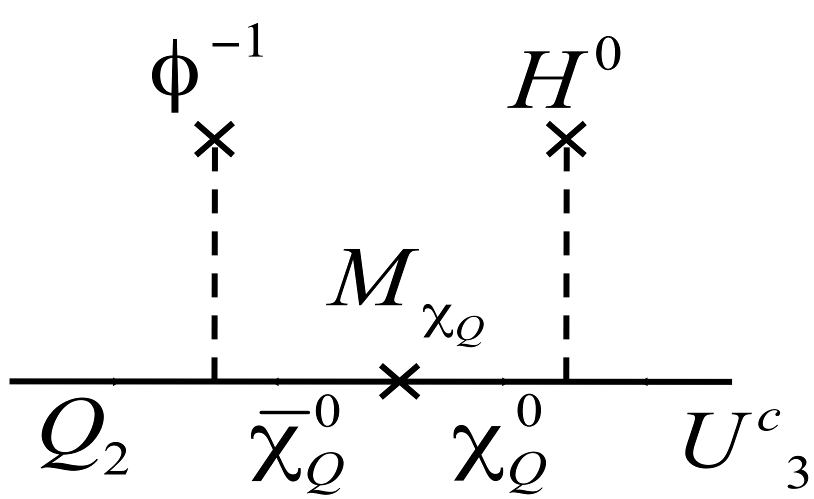
Froggatt-Nielsen Mechanism (1979)

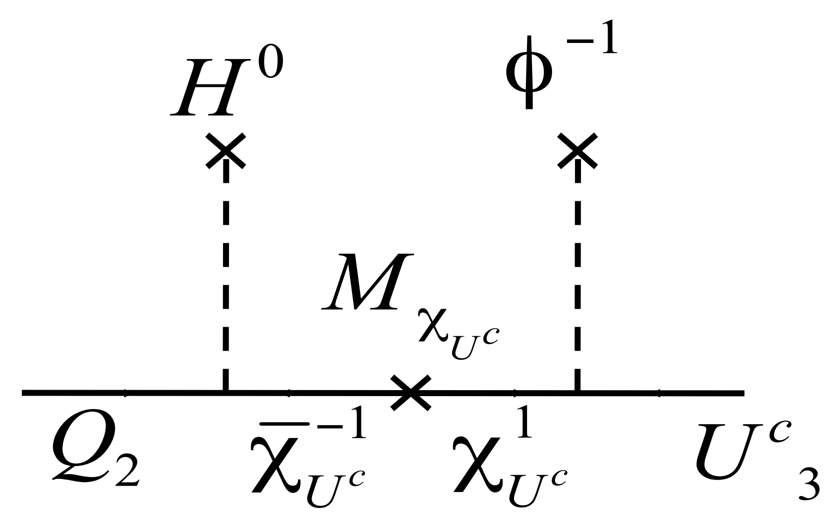
There may be Higgs messengers or fermion messengers





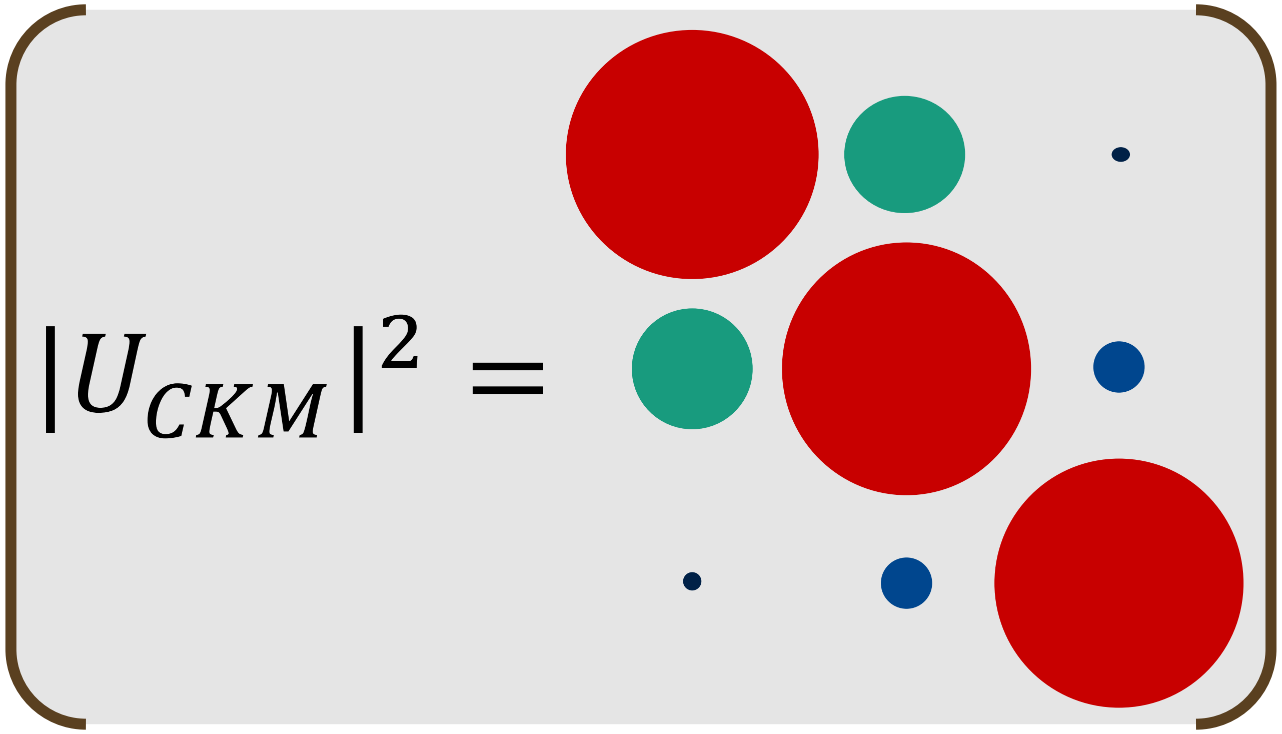
Fermion messengers may be $SU(2)_L$ doublets or singlets





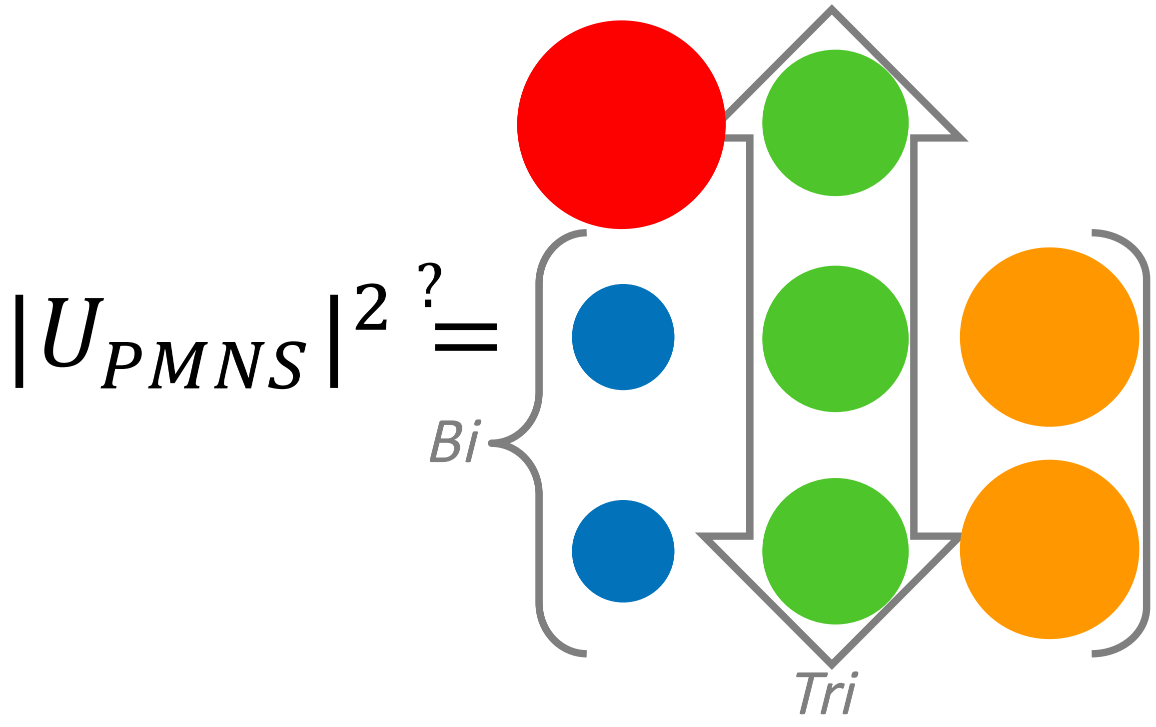
Neutrinos motivate new family/flavour symmetries

CKM Matrix



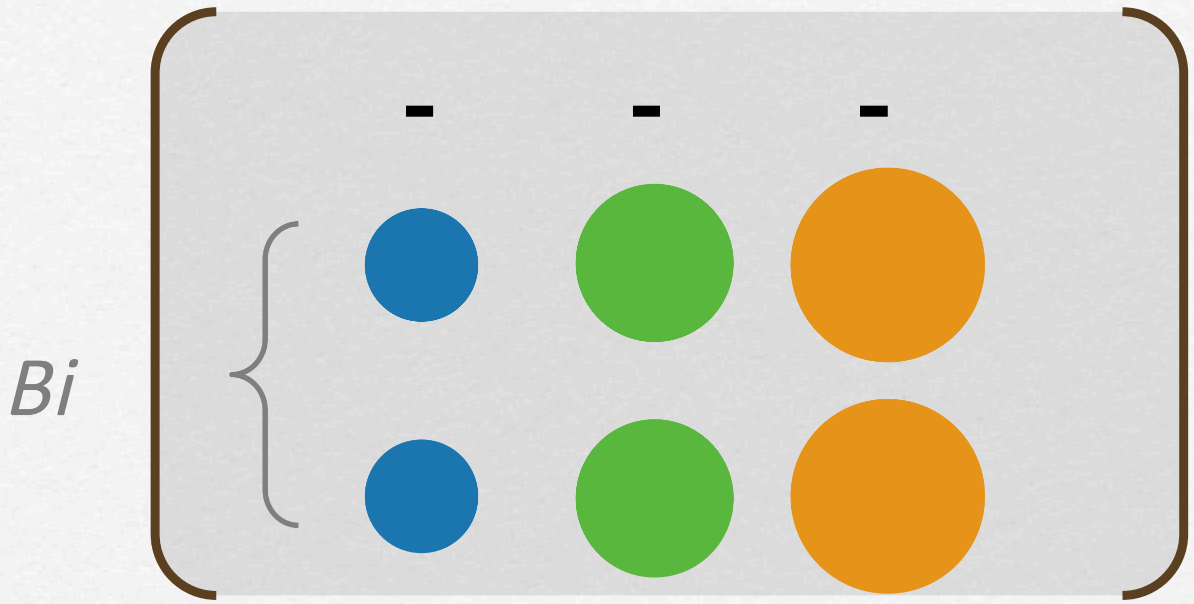
Froggatt-Nielsen tends to predict small mixing

PMNS Matrix



What symmetry gives this?

Mu-Tau Symmetry $\nu_\mu \leftrightarrow \nu_\tau^*$



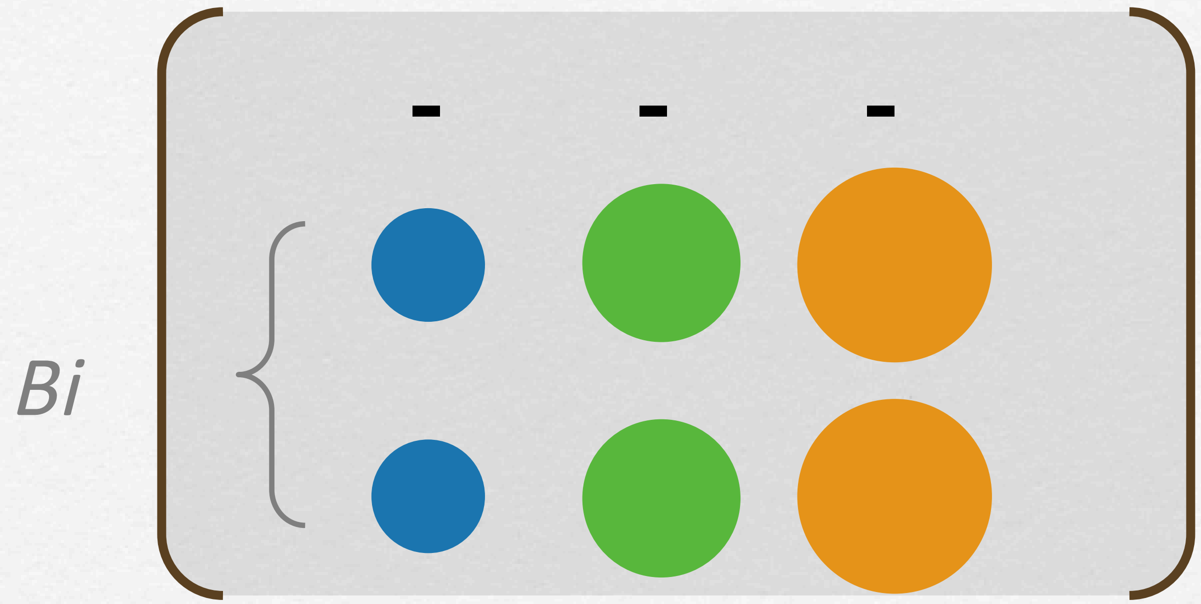
Basic Idea:

→ Two rows have
→ equal magnitudes

Z.z.Xing and S.Zhou, 0804.3512

→ $$\theta_{13} \neq 0\,, \quad \theta_{23} = 45^\circ, \quad \delta_{\rm CP} = \pm 90^\circ$$

Mu-Tau Symmetry $\nu_\mu \leftrightarrow \nu_\tau^*$



Basic Idea:

→ Two rows have
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Z.z.Xing and S.Zhou, 0804.3512

→ $\theta_{13} \neq 0\,, \quad \theta_{23} = 45^\circ, \quad \delta_{\rm CP} = \pm 90^\circ$

$$V_0 = \begin{pmatrix} |V_{e1}| & |V_{e2}| & |V_{e3}| \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\mu 1}^* & V_{\mu 2}^* & V_{\mu 3}^* \end{pmatrix}$$

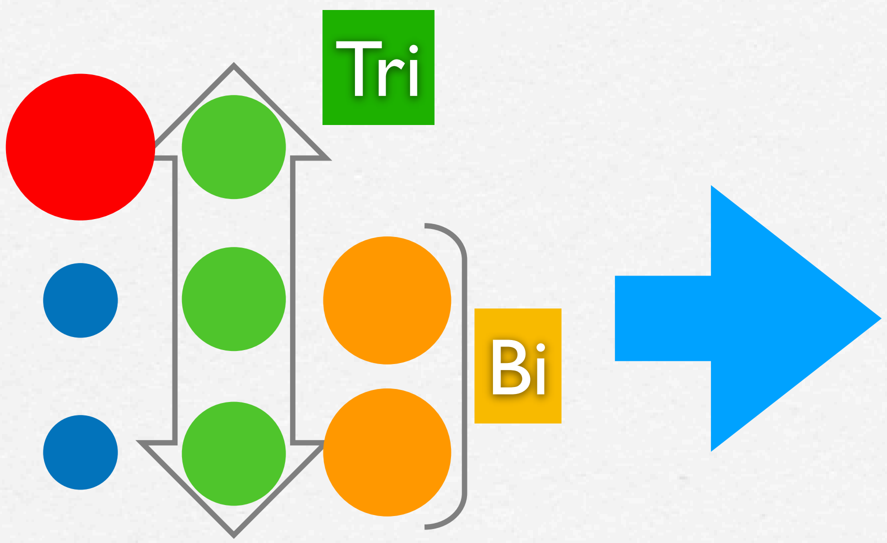
Generalisation of:

Mu-tau reflection symmetry

P.F.Harrison and W.G.Scott, hep-ph/0210197

P.F.Harrison, D.H.Perkins and W.G.Scott, hep-ph/0202074

Tri-Bimaximal Mixing



$$\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}$$

Allowed at 3 sigma

$$\sin\theta_{23} = \frac{1}{\sqrt{2}}$$

Allowed at 3 sigma

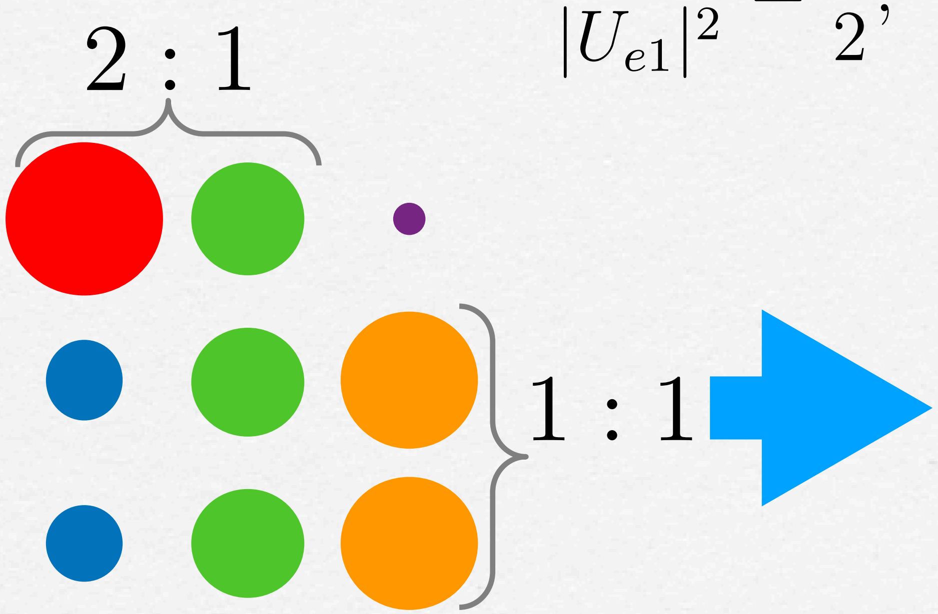
$$\sin\theta_{13} = 0$$

Excluded at many sigma

SFK 0903.3199, 1205.0506

Tri-Bimaximal-Reactor

$$\frac{|U_{e2}|^2}{|U_{e1}|^2} = \frac{1}{2}, \quad \frac{|U_{\mu 3}|^2}{|U_{\tau 3}|^2} = 1.$$



$$\begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{2}}\lambda e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\lambda e^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \\ \frac{1}{\sqrt{6}}(1-\lambda e^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \end{pmatrix}$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}$$

Allowed at 3 sigma

$$\sin\theta_{23} = \frac{1}{\sqrt{2}}$$

Allowed at 3 sigma

$$\sin\theta_{13} = \frac{\lambda}{\sqrt{2}}$$

Allowed ✓

Huge literature e.g. Antusch and SFK, hep-ph/0508044; I.Girardi, S.T.Petcov and A.V.Titov,1410.8056, ..

Charged lepton corrections

Charged lepton rotation Tri-bimaximal neutrinos

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & \cdots & \frac{s^e_{12}}{\sqrt{2}}e^{-i\delta^e_{12}} \\ \cdots & \cdots & \frac{c^e_{12}}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\rightarrow s_{13} = \frac{s^e_{12}}{\sqrt{2}}$$

Suggests $\theta^e_{12} \approx \theta_C$

$$\rightarrow c_{23}c_{13} = \frac{1}{\sqrt{2}} \rightarrow s^2_{23} < \frac{1}{2}$$

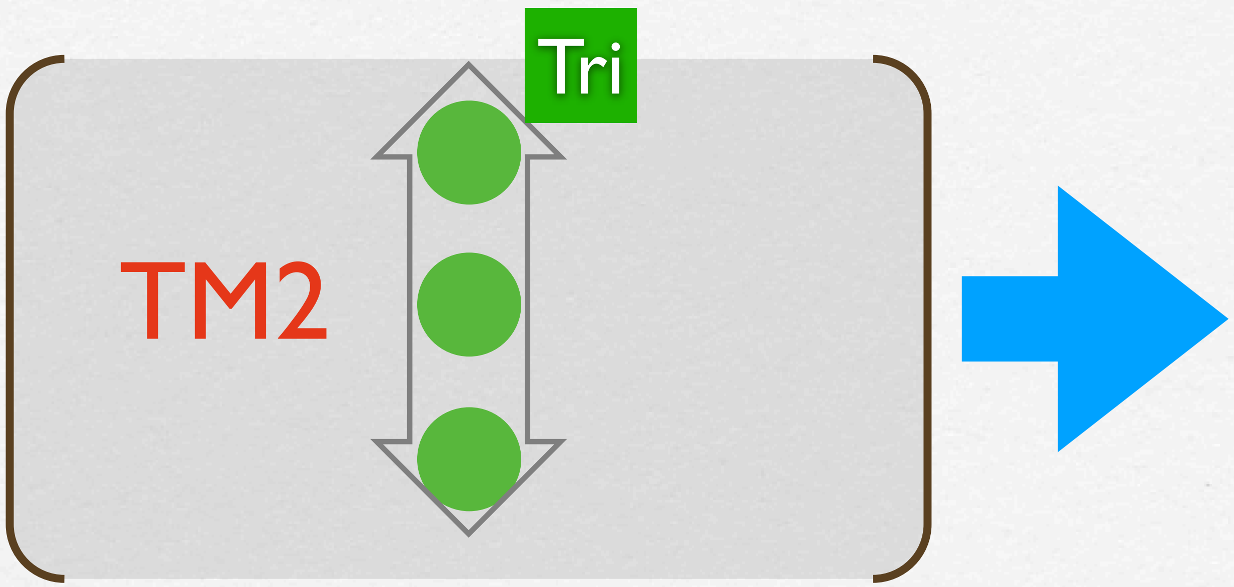
Prediction for CP phase

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = \frac{|s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta}|}{|-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}|} = \frac{1}{\sqrt{2}} \rightarrow \cos\delta = \frac{t_{23}s^2_{12} + s^2_{13}c^2_{12}/t_{23} - \frac{1}{3}(t_{23} + s^2_{13}/t_{23})}{\sin 2\theta_{12}s_{13}}$$

This derivation: P.Ballett, S.F.K., C.Luhn, S.Pascoli and M.A.Schmidt, 1410.7573

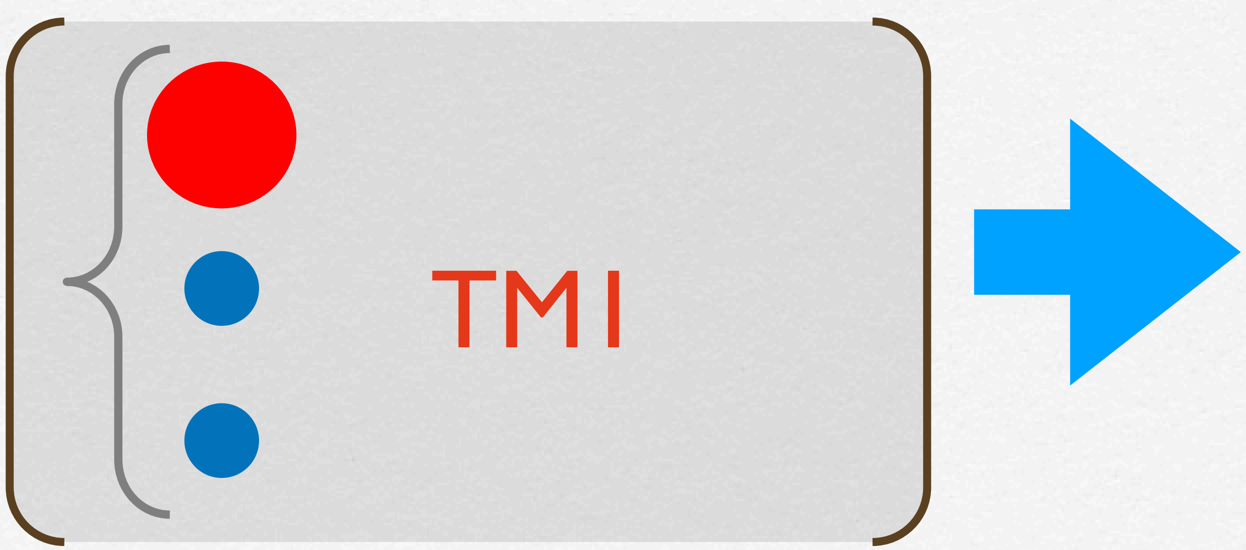
C.H.Albright and W.Rodejohann, 0812.0436; C.H.Albright, A.Dueck and W.Rodejohann, 1004.2798

Tri-maximal Mixing



Second column of TBM

$$U_{\rm TM2} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$



First column of TBM

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

C.H.Albright and W.Rodejohann, 0812.0436; C.H.Albright, A.Dueck and W.Rodejohann, 1004.2798

Tri-maximal Mixing

$$U_{\rm TM2} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$

$$|U_{e2}| = s_{12}c_{13} = \sqrt{\frac{1}{3}} \rightarrow s_{12}^2 > \frac{1}{3}$$

Disfavoured

$$|U_{\mu 2}| = |c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}| = \sqrt{\frac{1}{3}}$$

$$|U_{\tau 2}| = |-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}| = \sqrt{\frac{1}{3}}$$

$$\rightarrow \cos\delta = \frac{2c_{13}\cot 2\theta_{23}\cot 2\theta_{13}}{\sqrt{2-3s_{13}^2}}$$

C.H.Albright and W.Rodejohann, 0812.0436; C.H.Albright, A.Dueck and W.Rodejohann, 1004.2798

Tri-maximal Mixing

Disfavoured

$$U_{\mathrm{TM2}} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$

$$\to |U_{e2}| = s_{12}c_{13} = \sqrt{\frac{1}{3}} \to s_{12}^2 > \frac{1}{3}$$

$$\to |U_{\mu 2}| = |c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}| = \sqrt{\frac{1}{3}}$$

$$\to |U_{\tau 2}| = |-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}| = \sqrt{\frac{1}{3}}$$

$$\to \cos\delta = \frac{2c_{13}\cot 2\theta_{23}\cot 2\theta_{13}}{\sqrt{2-3s_{13}^2}}$$

Favoured

$$U_{\mathrm{TM1}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

$$\to |U_{e1}| = c_{12}c_{13} = \sqrt{\frac{2}{3}} \to s_{12}^2 < \frac{1}{3}$$

$$\to |U_{\mu 1}| = |-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta}| = \sqrt{\frac{1}{6}}$$

$$\to |U_{\tau 1}| = |s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta}| = \sqrt{\frac{1}{6}}$$

$$\to \cos\delta = -\frac{\cot 2\theta_{23}(1-5s_{13}^2)}{2\sqrt{2}s_{13}\sqrt{1-3s_{13}^2}}$$

Littlest Seesaw

SFK,1304.6264; 1512.07531
SFK, Molina Sedgwick, Rowley, 1808.01005

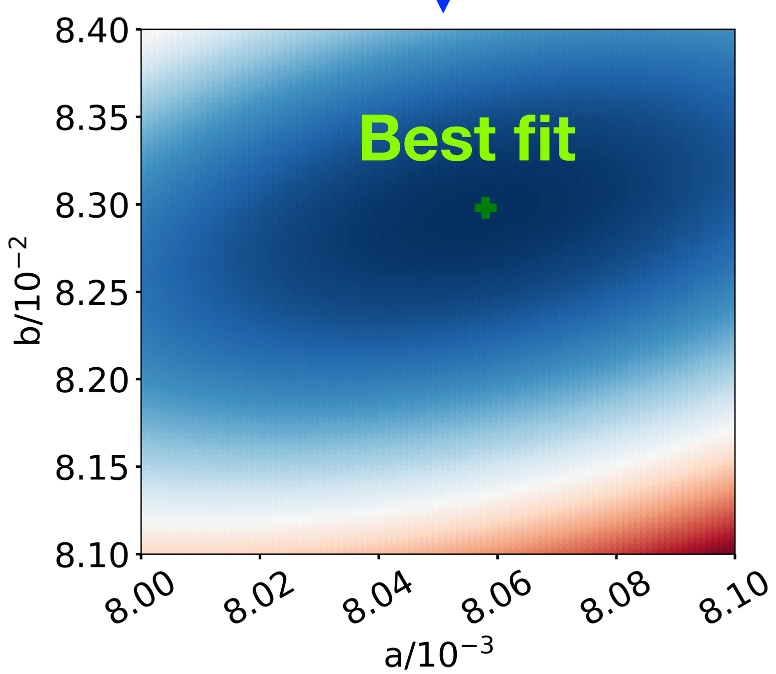
Dirac texture zero

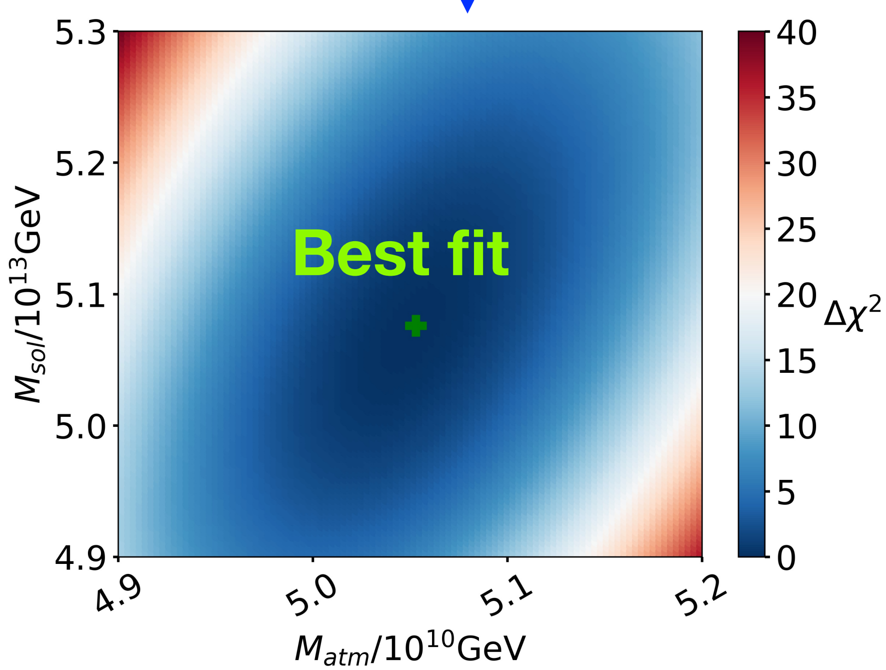
2 RHNs

$$Y^\nu = \begin{pmatrix} 0 & be^{i\pi/3} \\ a & 3be^{i\pi/3} \\ a & be^{i\pi/3} \end{pmatrix} \qquad M_R = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}$$

Constrained couplings

4 real input parameters





- **Fit includes effects of RG corrections**
- **Determines the RHN masses!**

4 real input parameters

Describes:

3 neutrino masses (m_1=0),
3 mixing angles,
1 Dirac CP phase,
2 Majorana phases (1 zero)
1 BAU parameter Y_B
= 10 observables
of which 7 are constrained

Predictions	1σ **range**
$\theta_{12}/^\circ$	$34.254 \to 34.350$
$\theta_{13}/^\circ$	$8.370 \to 8.803$
$\theta_{23}/^\circ$	$45.405 \to 45.834$
$\Delta m_{12}{}^2/10^{-5}\mathrm{eV}^2$	$7.030 \to 7.673$
$\Delta m_{31}{}^2/10^{-3}\mathrm{eV}^2$	$2.434 \to 2.561$
$\delta/^\circ$	$-88.284 \to -86.568$
$Y_B/10^{-10}$	$0.839 \to 0.881$

Also predicts NO and m_1=0

Non-Abelian Family Symmetry

Traditionally used for TB mixing

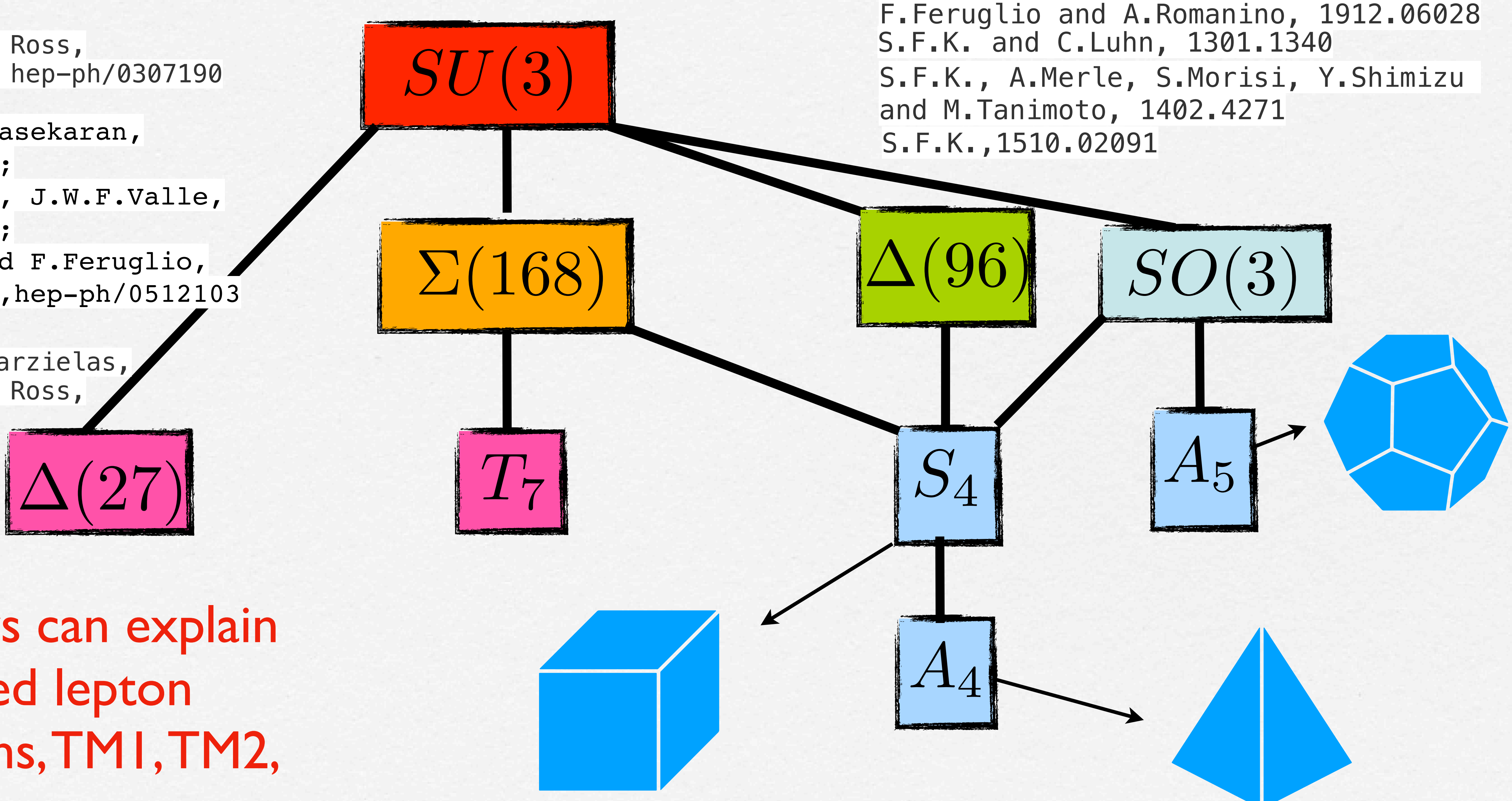
S.F.K. and G.G. Ross,
hep-ph/0108112; hep-ph/0307190

E.Ma and G.Rajasekaran,
hep-ph/0106291;
K.S.Babu, E.Ma, J.W.F.Valle,
hep-ph/0206292;
G.Altarelli and F.Feruglio,
hep-ph/0504165,hep-ph/0512103

I.de Medeiros Varzielas,
S.F.K. and G.G. Ross,
hep-ph/0512313;
hep-ph/0607045

Reviews

F.Feruglio and A.Romanino, 1912.06028
S.F.K. and C.Luhn, 1301.1340
S.F.K., A.Merle, S.Morisi, Y.Shimizu and M.Tanimoto, 1402.4271
S.F.K.,1510.02091



These days can explain charged lepton corrections, TM1, TM2, Littlest seesaw,…

A_4 and S_4 Group Theory

S.F.K., C.Luhn, 1301.1340

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$$

S_4	A_4	S	T	U
$\mathbf{1}, \mathbf{1}'$	$\mathbf{1}$	1	1	± 1
$\mathbf{2}$	$\begin{pmatrix} \mathbf{1}'' \\ \mathbf{1}' \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3}, \mathbf{3}'$	$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Diagonalised by TB matrix

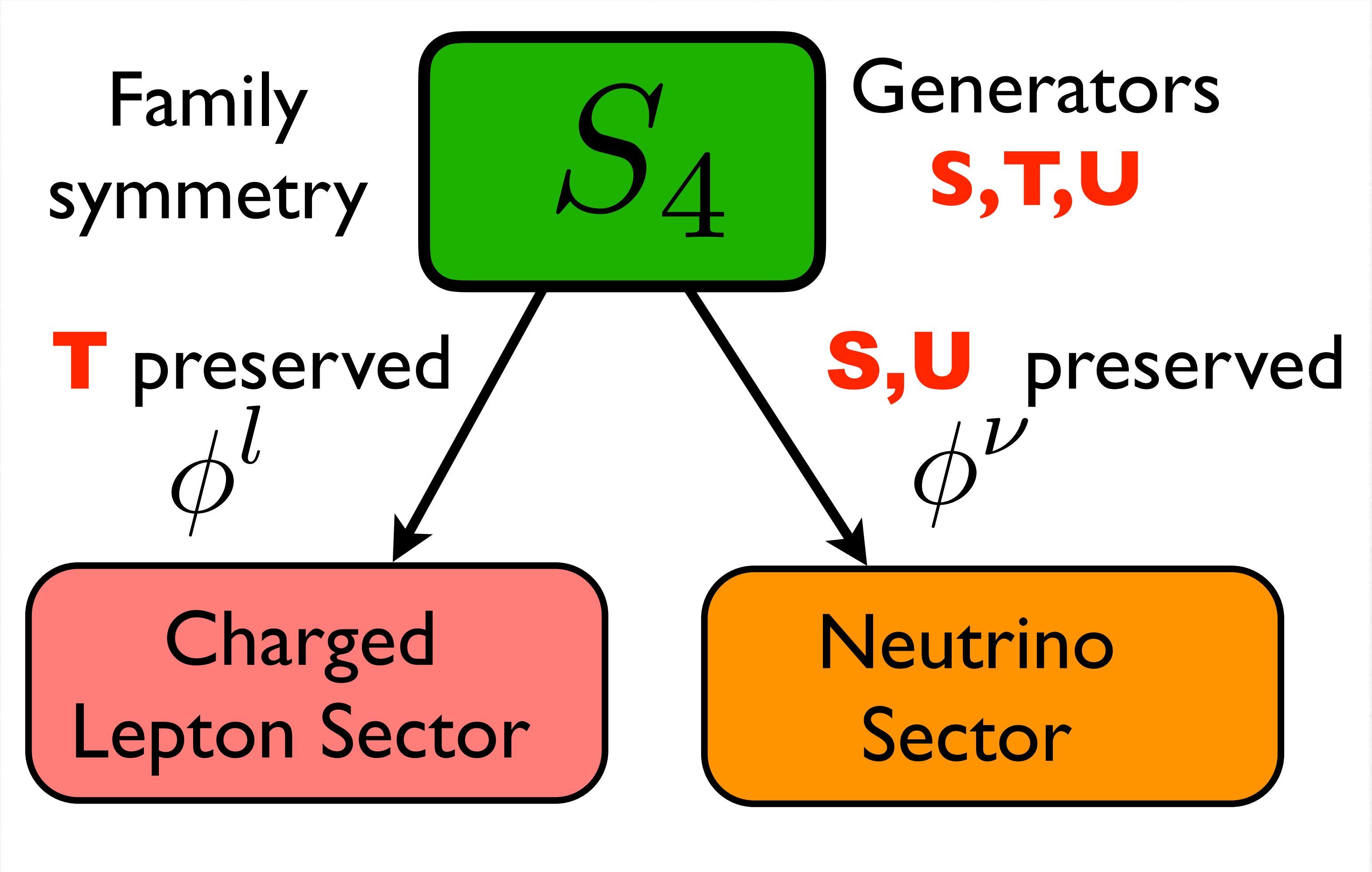
S_4 vacuum alignments

C.Luhn, 1306.2358

$\langle \phi^{\nu}_{3'} \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ preserves S,U $\quad$ $\langle \phi^{l}_{3'} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ preserves T

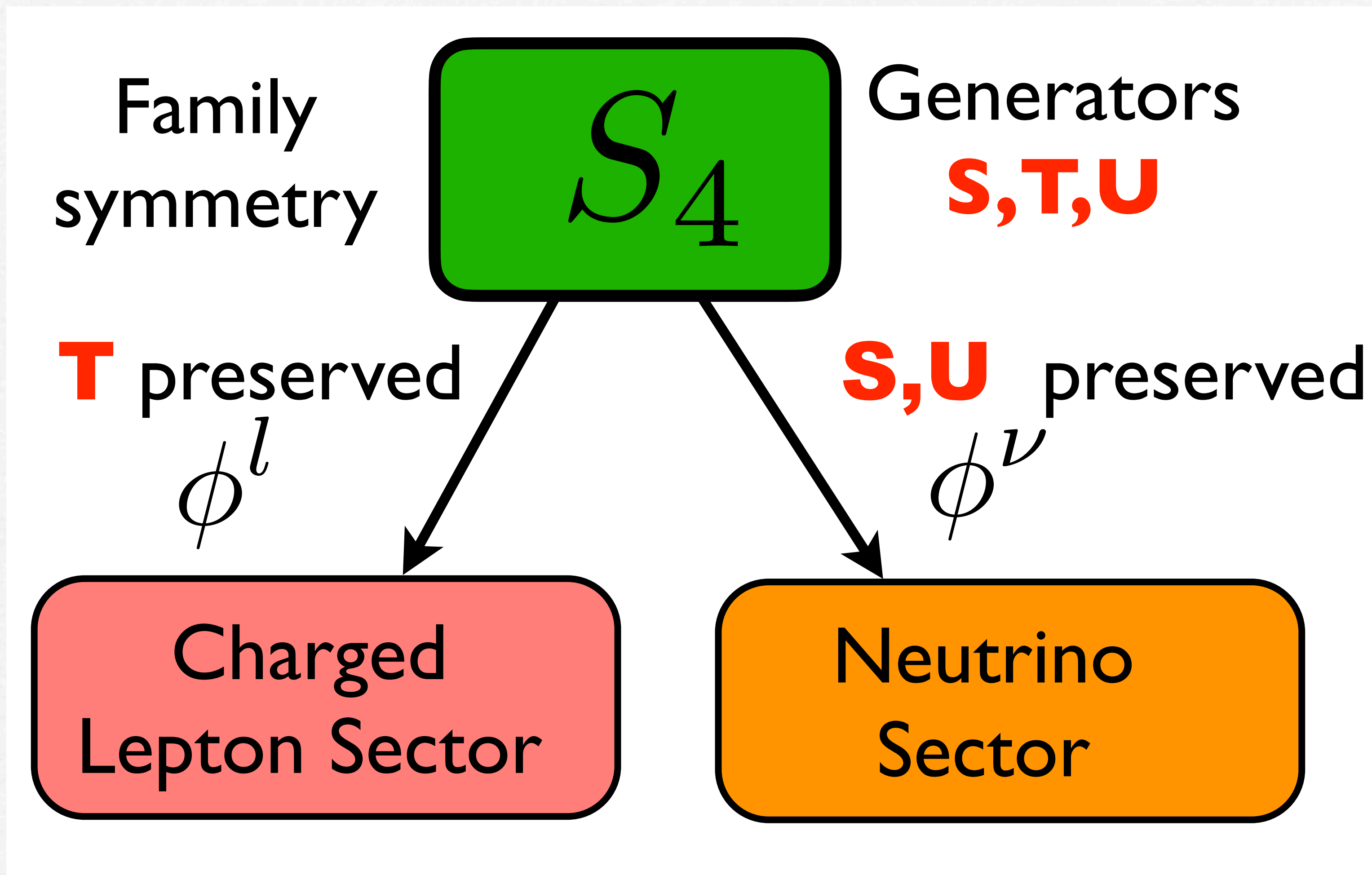
residual symmetry	U	S	SU
$\mathbf{1}$	$-$	$-$	$-$
$\mathbf{1'}$	$-$	1	$-$
$\mathbf{2}$	$-$	$(1\,,\,-1)^T$	$-$
$\mathbf{3}$	$(0\,,\,1\,,\,-1)^T$	$(1\,,\,1\,,\,1)^T$	$(2\,,\,-1\,,\,-1)^T$
$\mathbf{3'}$	$(1\,,\,0\,,\,0)^T$	$-$	$(0\,,\,1\,,\,-1)^T$

S_4 flavour symmetry



S.F.K., C.Luhn, 1301.1340

S_4 flavour symmetry



S.F.K., C.Luhn, 1301.1340

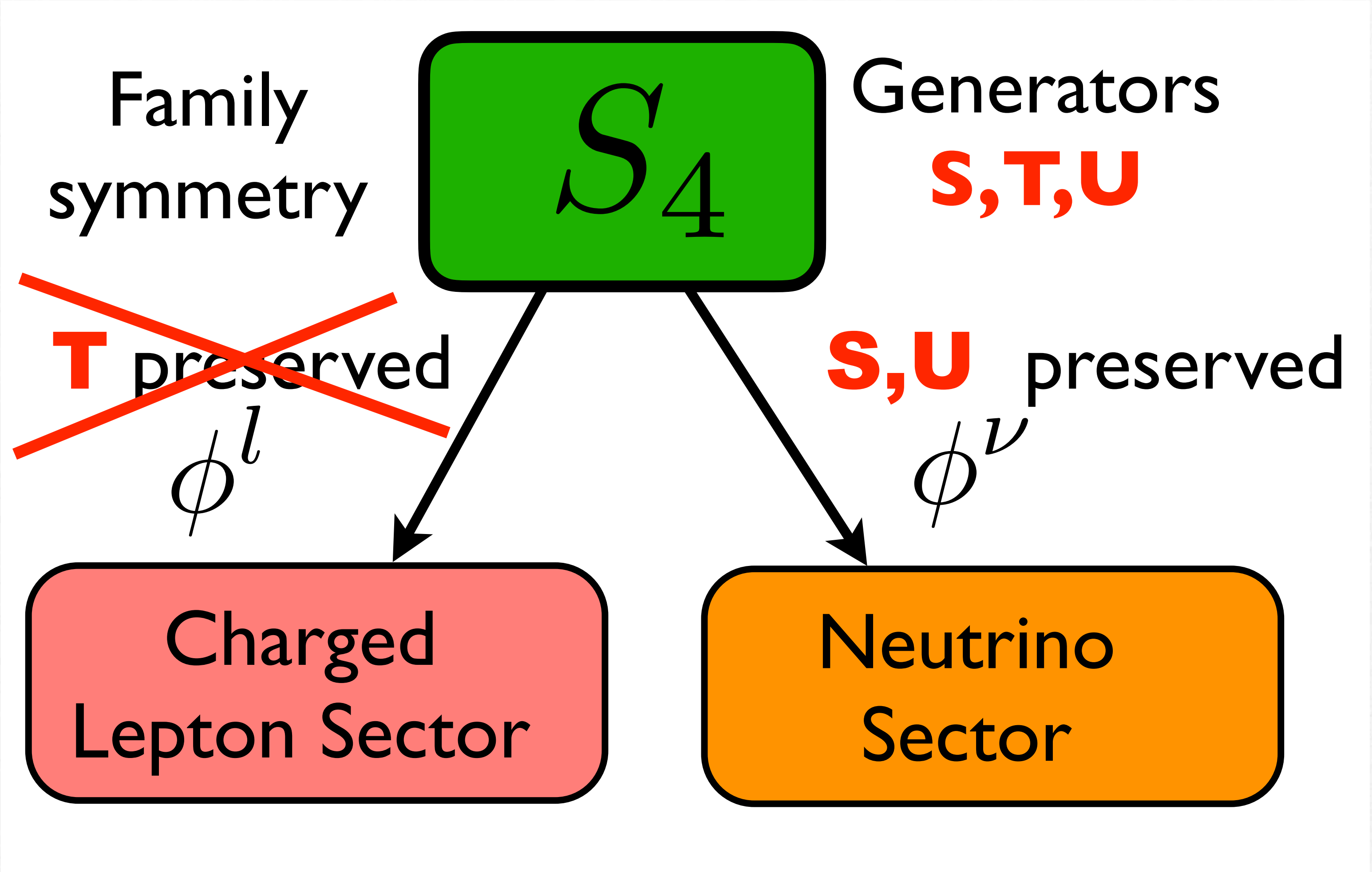
$$TM^ET = M^E$$
$$SM^\nu S = M^\nu$$
$$UM^\nu U = M^\nu$$

$$\Rightarrow \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

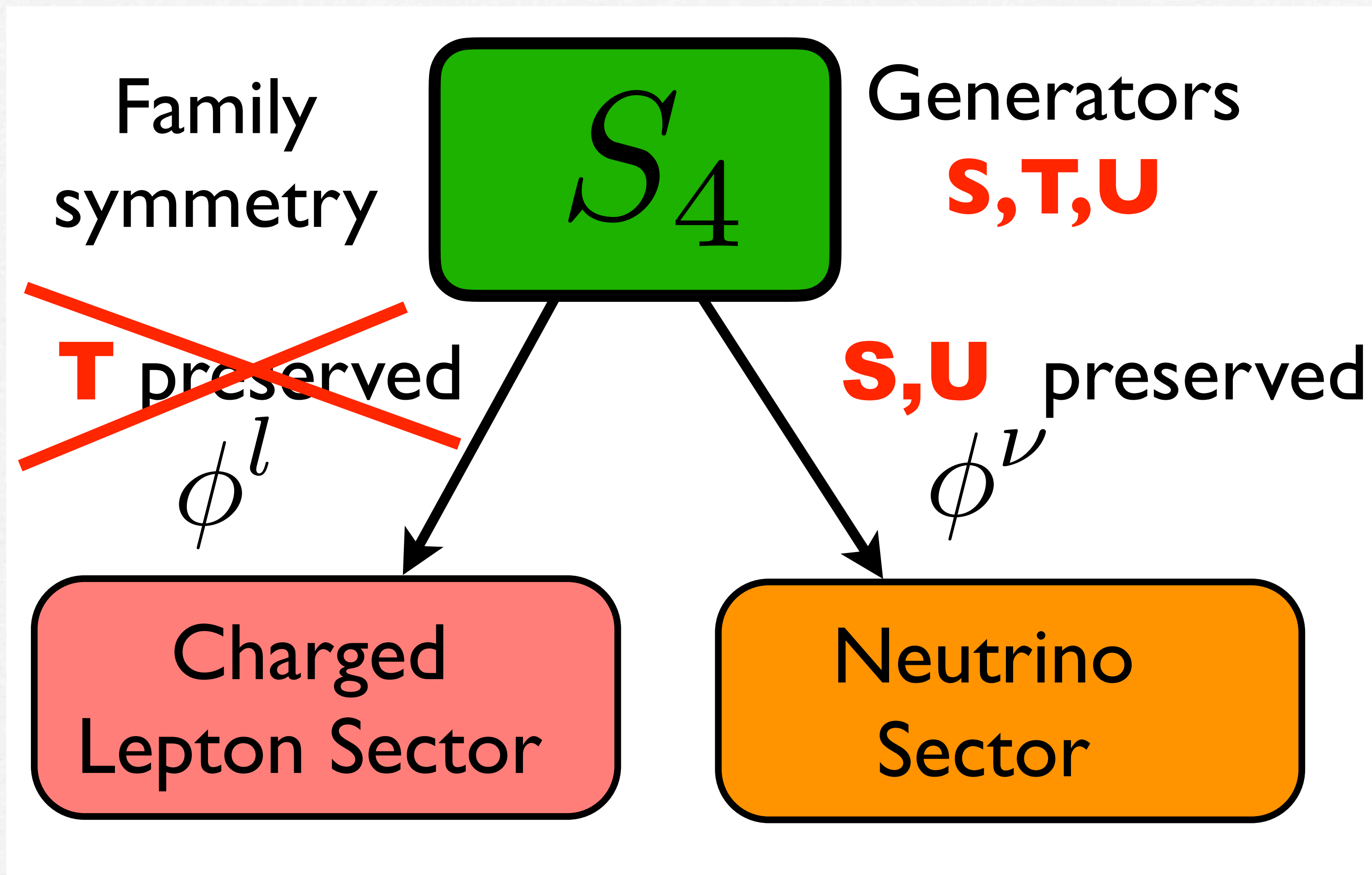
TB mixing excluded so need to break S,T,U

S_4 flavour symmetry

break T



S_4 flavour symmetry

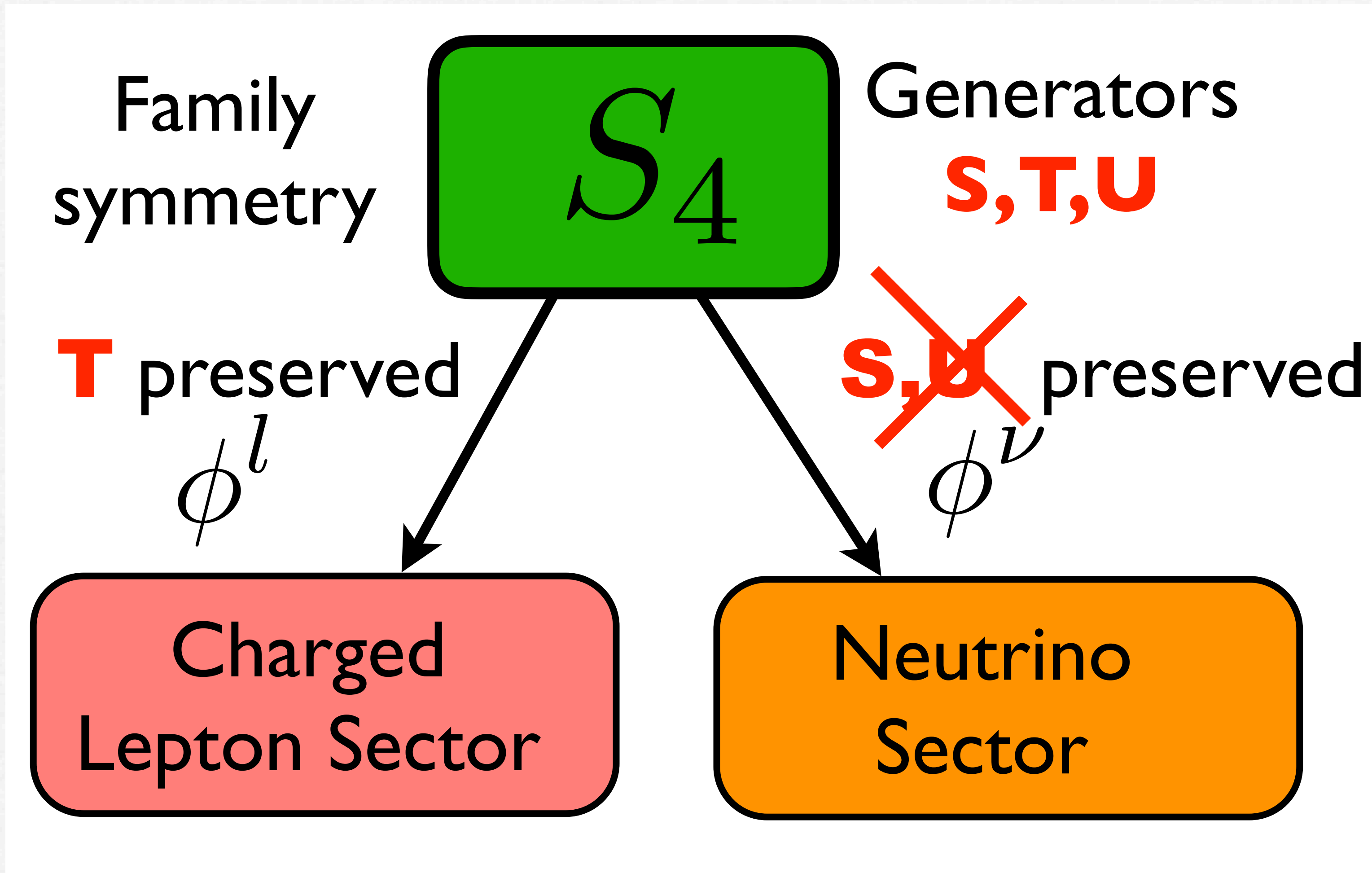


break T

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Charged lepton rotation

S_4 flavour symmetry

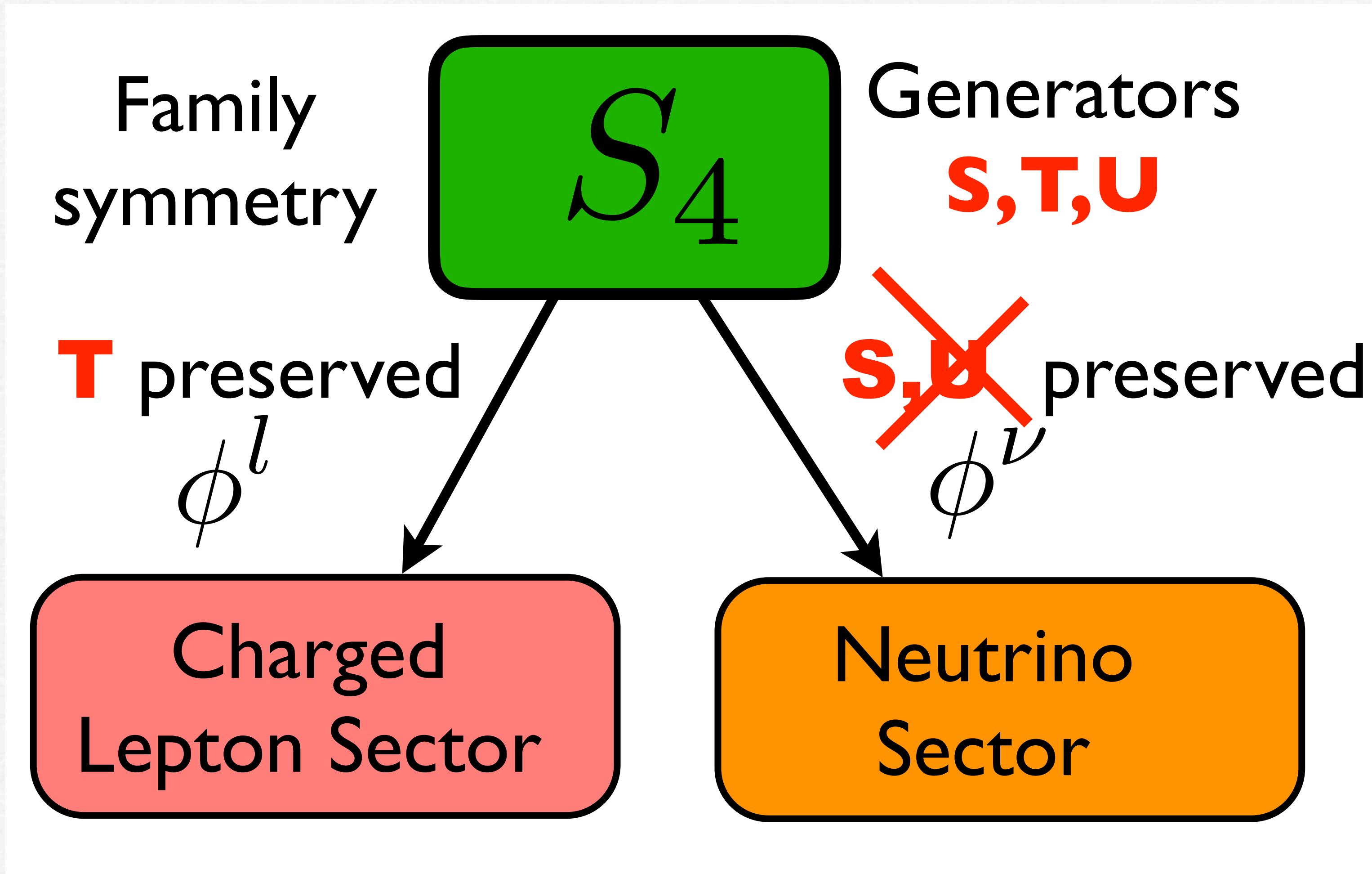


S.F.K., C.Luhn, 1301.1340

Y.Shimizu, M.Tanimoto, A.Watanabe, 1105.2929; S.F.K.,C.Luhn,1107.5332

break U

S_4 flavour symmetry



S.F.K., C.Luhn, 1301.1340

Y.Shimizu, M.Tanimoto, A.Watanabe, 1105.2929; S.F.K.,C.Luhn,1107.5332

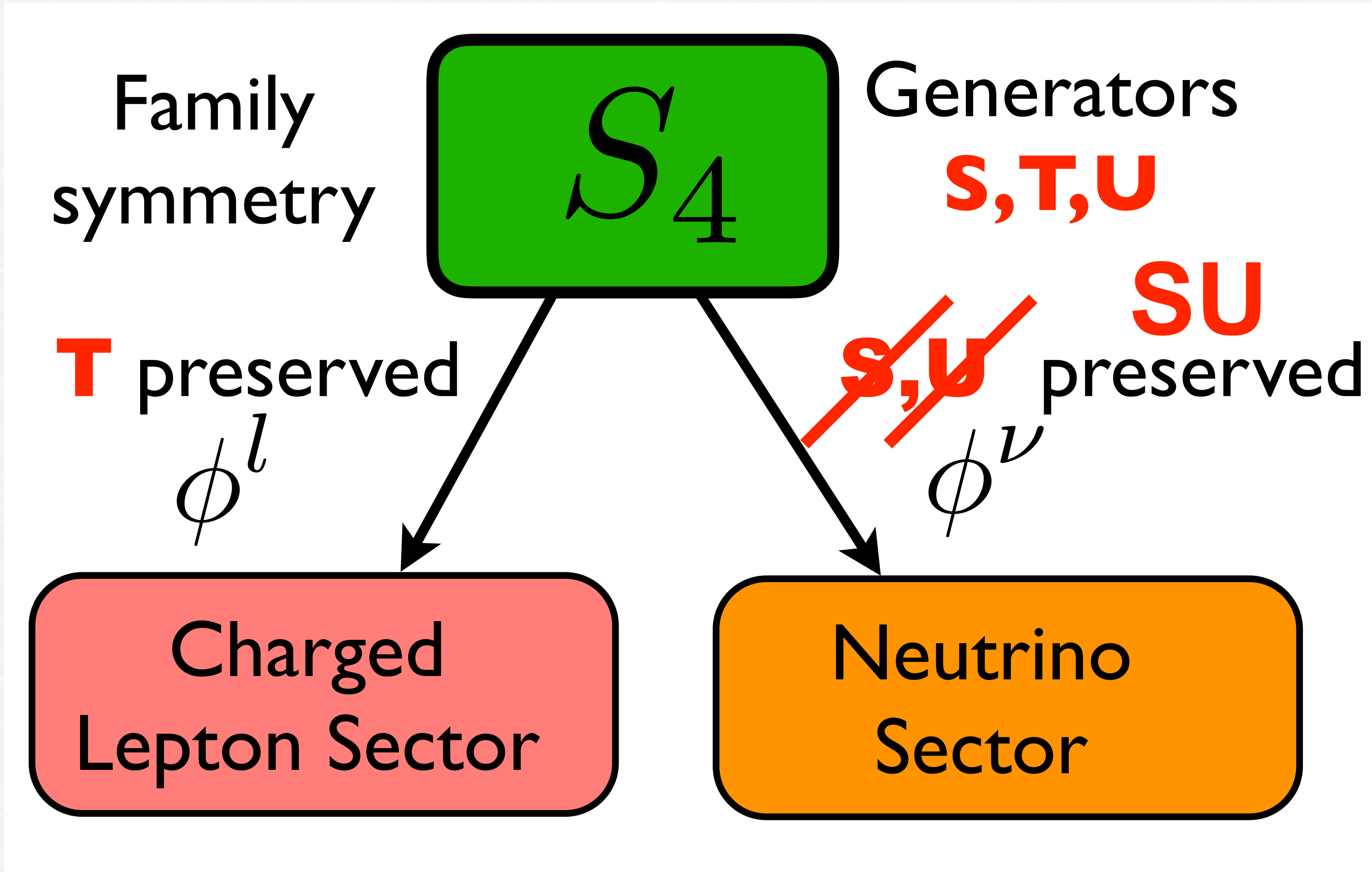
break U

Alternatively A4 with just S and T

$$T M^E T = M^E$$
$$S M^\nu S = M^\nu$$

$$U_{\mathrm{TM2}} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$

S_4 flavour symmetry

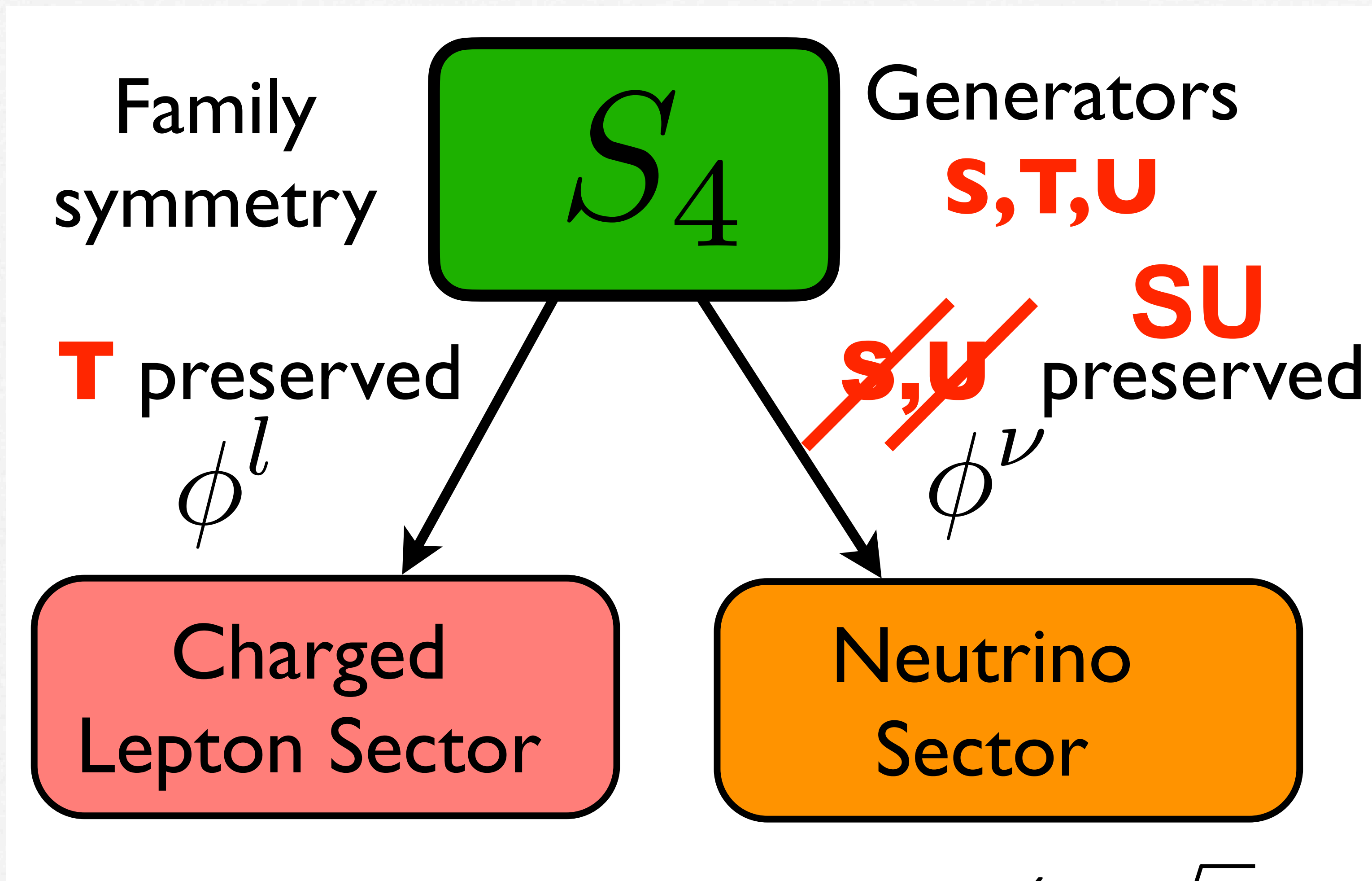


S.F.K., C.Luhn, 1301.1340

break S,U separately preserve SU

S_4 flavour symmetry

D.Hernandez and A.Y.Smirnov 1204.0445,1212.2149,1304.7738; C.Luhn, 1306.2358 S.F.K.,C.Luhn 1607.05276



break S,U separately preserve SU

$$T M^E T = M^E$$
$$SU M^\nu SU = M^\nu$$

$$U_{\mathrm{TM1}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

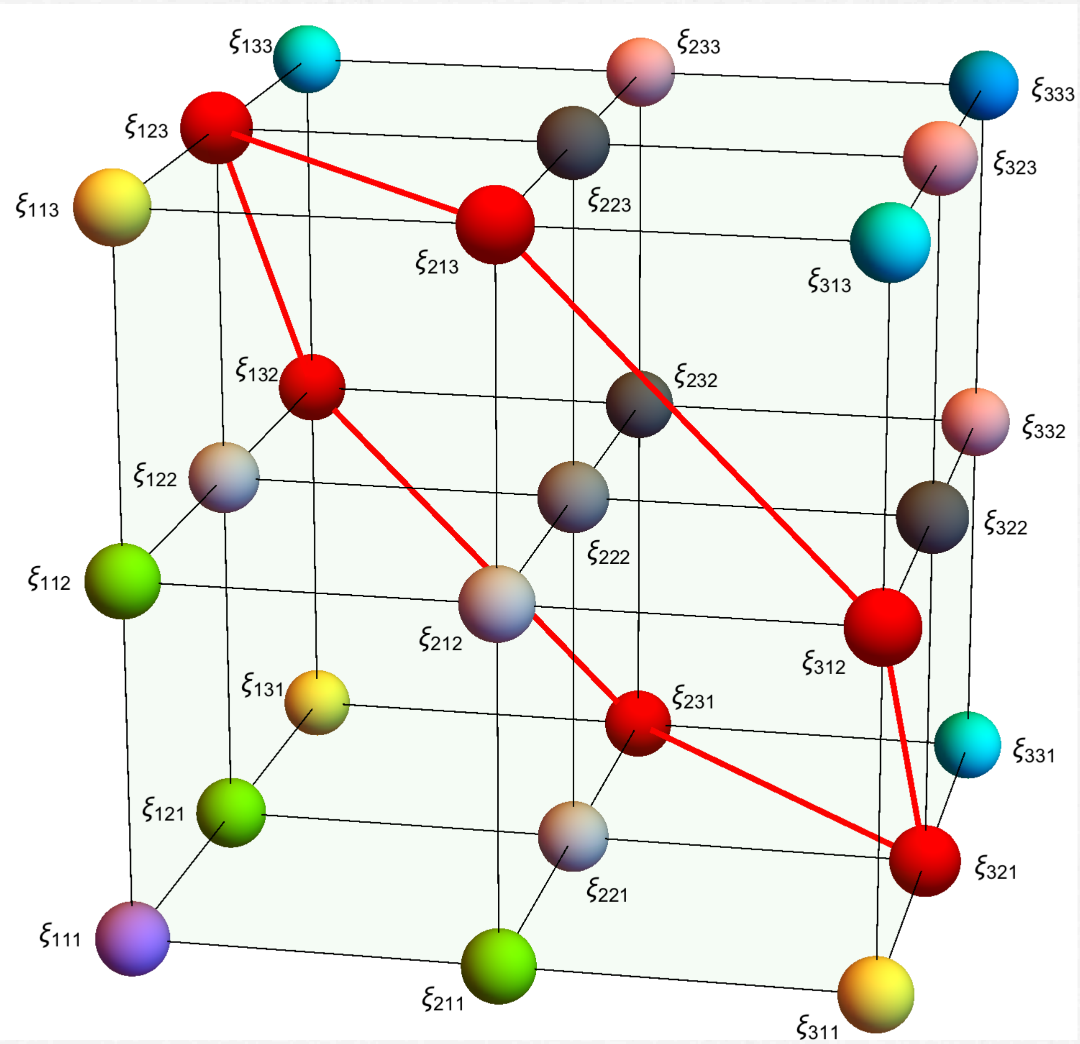
Y.Koide,0705.2275; T.Banks and N.Seiberg,1011.5120;
Y.L.Wu,1203.2382; A.Merle and R.Zwicky,1110.4891;
B.L.Rachlin and T.W.Kephart,1702.08073; C. Luhn, 1101.2417

Origin of flavour symmetry

Break SO(3) using large Higgs reps E.g. 7-plet

S.F.K. and Ye-Ling Zhou, 1809.10292

irrep	$\underline{1}$	$\underline{3}$	$\underline{5}$	$\underline{7}$
subgroups	$SO(3)$	$SO(2)$	$Z_2 \times Z_2$	$\mathbf{1}$
		$SO(3)$	$SO(2)$	A_4
			$SO(3)$	Z_3
				D_4
				$SO(2)$
				$SO(3)$

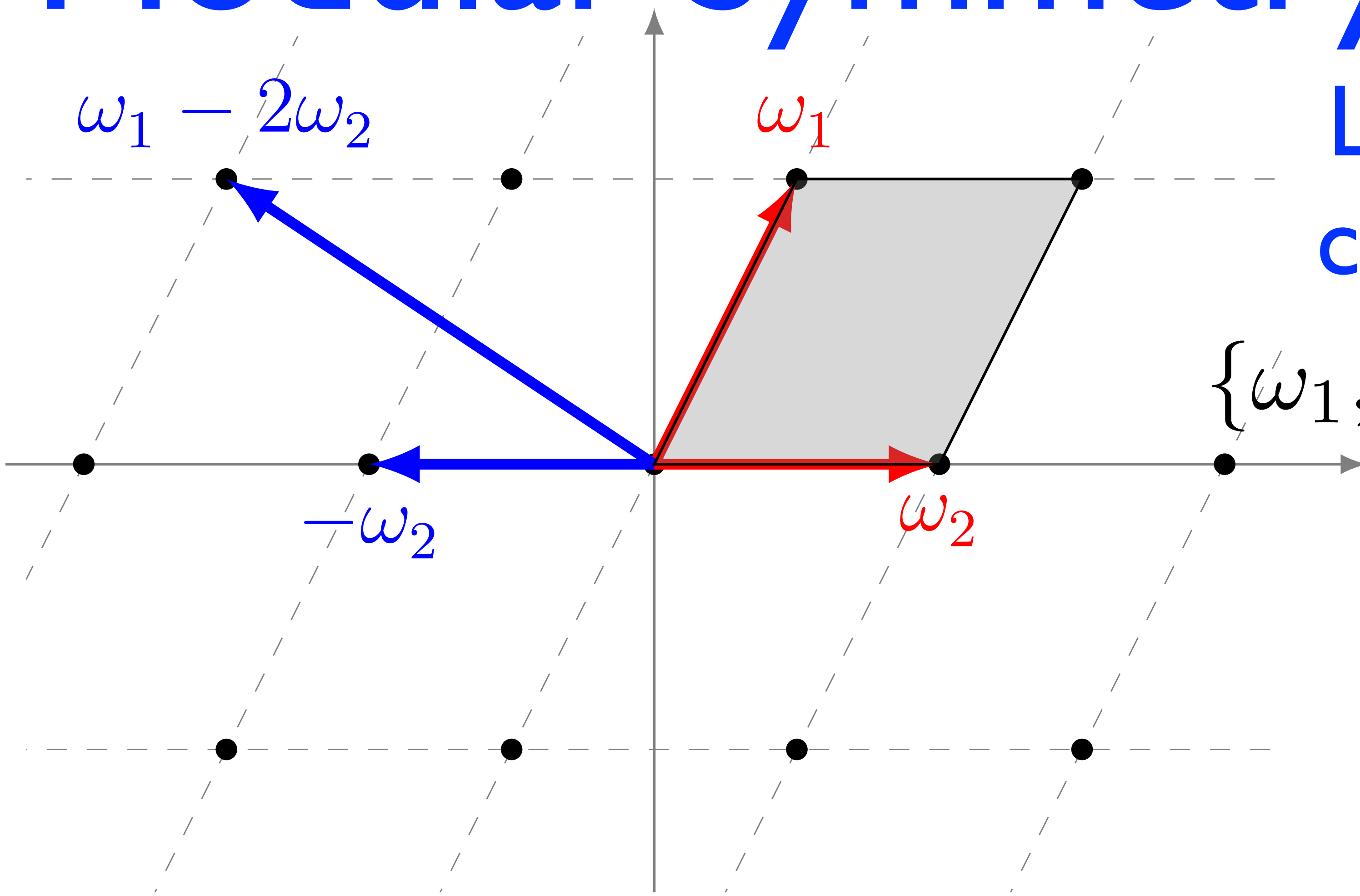


A4 preserving direction of 7-plet VEV

$$\langle \xi_{123} \rangle \equiv \frac{v_\xi}{\sqrt{6}}, \quad \langle \xi_{111} \rangle = \langle \xi_{112} \rangle = \langle \xi_{113} \rangle = \langle \xi_{133} \rangle = \langle \xi_{233} \rangle = \langle \xi_{333} \rangle = 0$$

G.Altarelli and F.Feruglio, hep-ph/0512103
R.de Adelhart Toorop, F.Feruglio and C.Hagedorn, 1112.1340

Modular Symmetry



Lattice invariant under change of basis vectors

$$\{\omega_1, \omega_2\} \longrightarrow \{-\omega_2, \omega_1 - 2\omega_2\}$$

$$\tau \equiv \omega_1/\omega_2$$

$$\tau \to -1/(\tau - 2)$$

General modular transformation

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Integers a,b,c,d

$$ad - bc = 1$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Infinite group

$$\Gamma \equiv SL(2, \mathbb{Z})$$

$$S: \ \tau \mapsto -\frac{1}{\tau}, \qquad T: \ \tau \mapsto \tau + 1$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

See Ferruccio's talk F.Feruglio, 1706.08749

From Infinite to Finite Modular Symmetry

$$\Gamma \equiv SL(2,\mathbb{Z}) \qquad S^2 = -\mathbb{1}_2, \qquad S^4 = (ST)^3 = \mathbb{1}_2, \qquad S^2T = TS^2$$

$\longrightarrow$ $\overline{\Gamma} \equiv PSL(2,\mathbb{Z}) \qquad S^2 = (ST)^3 = (TS)^3 = 1$ Infinite

$\longrightarrow$ Finite level N $\Gamma_N \qquad S^2 = (ST)^3 = (TS)^3 = 1$ and $T^N = 1$

$\Gamma_2 \approx S_3$
$\Gamma_3 \approx A_4$
$\Gamma_4 \approx S_4$
$\Gamma_5 \approx A_5$
$\Gamma_7 \approx \Sigma(168)$

Yukawa coupling transforms as an irrep of Γ_N and as a modular form

$$Y(\tau) \to Y(\gamma\tau) = (c\tau + d)^{k_Y} \rho_{\mathbf{r}_Y}(\gamma) Y(\tau)$$

$$Y(\tau)\phi_1\phi_2\phi_3 \qquad \phi_1 \to (c\tau+d)^{k_1}\rho_1(\gamma)\phi_1$$

$k_Y = k_1 + k_2 + k_3$ modular weights balance

$\rho_{\mathbf{r}_Y} \times \rho_1 \times \rho_2 \times \rho_3 = 1 + \ldots$ contains singlet

	Leptons	Quarks	SU(5)	SO(10)
N=2, S_3	T.Kobayashi, K.Tanaka and T.H.Tatsuishi, 1803.10391,…		T.Kobayashi, Y.Shimizu, K.Takagi, M.Tanimoto and T.H.Tatsuishi 1906.10341,…	
N=3, A_4	F.Feruglio,1706.08749 J.C.Criado and F.Feruglio,1807.01125 G.J.Ding,S.F.King and X.G.Liu, 1907.11714,…	H.Okada, M.Tanimoto, 1812.09677; 1905.13421; S.J.D. King,S.F.King, 2002.00969,…	F.J.de Anda, S.F.King, E.Perdomo, 1812.05620; P.Chen, G.J.Ding and S.F.King, 2101.12724, …	G.J.Ding, S.F.King, J.N.Lu, 2108.09655
N=4, S_4	J.T.Penedo,S.T.Petcov, 1806.11040;P.P.Novichkov J.T.Penedo,S.T.Petcov, A.V.Titov,1811.04933, J.C.Criado,F.Feruglio, S.J.D.King, 1908.11867,…		Y.Zhao and H.H.Zhang, 2101.02266; G.J.Ding, S.F.King and C.Y.Yao, 2103.16311 ,…	
N=5, A_5	P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov, 1812.02158; G.J.Ding, S.F.King, X.G.Liu, 1903.12588,…			
N=6, $S_3 x A_4$				
N=7, $\Sigma(168)$	G.J.Ding, S.F.King, C.C.Li, Y.L.Zhou, 2004.12662			

For integer/fractional/CP/eclectic/stabilisation… see other talks

See Ferruccio's talk F.Feruglio, 1706.08749

Example: Level N=3 ~ A_4

Yukawa couplings involving twisted states whose modular weights do not add up to zero are <u>modular forms</u>

A_4 triplet **3**
Weight k_Y=2
Notation $Y=Y_{\mathbf{3}}^{(2)}$

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \dots) \end{pmatrix}$$

$$q \equiv e^{i2\pi\tau}$$ ← modulus vev

Weinberg operator $\frac{1}{\Lambda}(H_u H_u \; LL \; Y) \longrightarrow m_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$

A_4 rep: **3 3 3**
Modular weights k : 1 1 2

no flavons (apart from tau)

S.J.D. King and S.F.K, 2002.00969

Example with weighton: Level N=3 ~ A4

	L	e_3^c	e_2^c	e_1^c	N^c	$H_{u,d}$	ϕ
A_4	$\mathbf{3}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	0	−1	−3	1	0	1

$$W_{driv} = \chi(Y_{\mathbf{1}}^{(4)} \frac{\phi^4}{M_{fl}^2} - M^2)$$

$$\tilde{\phi} \equiv \frac{\langle \phi \rangle}{M_{fl}} \sim (M/M_{fl})^{1/2}$$

small parameter

$$W_e = \alpha_e e_1^c \tilde{\phi}^4 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta_e e_2^c \tilde{\phi}^2 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1'}} H_d + \gamma_e e_3^c \tilde{\phi} (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1''}} H_d$$

$$Y_e = \begin{pmatrix} \alpha_e \tilde{\phi}^4 Y_1 & \alpha_e \tilde{\phi}^4 Y_3 & \alpha_e \tilde{\phi}^4 Y_2 \\ \beta_e \tilde{\phi}^2 Y_2 & \beta_e \tilde{\phi}^2 Y_1 & \beta_e \tilde{\phi}^2 Y_3 \\ \gamma_e \tilde{\phi} Y_3 & \gamma_e \tilde{\phi} Y_2 & \gamma_e \tilde{\phi} Y_1 \end{pmatrix}$$

Natural explanation of charged lepton hierarchy c.f. FN

Unlike the FN flavon, the weighton phi does not break the flavour symmetry

P.P.Novichkov, S.T.Petcov and M.Tanimoto, 1812.11289
I.de Medeiros Varzielas, S.F.K. and Y.L.Zhou, 1906.02208

Stabilizers and Fixed points

$$\gamma_0 \tau_0 = \tau_0$$

e.g. $S\tau_S = \tau_S \longrightarrow \tau_S = i$

Invariant under $S: \ \tau \mapsto -\frac{1}{\tau}$

Alignments from fixed points

Modular transformation

$$Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_{I_Y}(\gamma) Y_{I_Y}(\tau)$$

Fixed point relations

$$\gamma\tau_\gamma = \tau_\gamma \qquad Y_I(\gamma\tau_\gamma) = Y_I(\tau_\gamma)$$

Eigenvalue equation gives alignments directly

$$\rho_I(\gamma) Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k} Y_I(\tau_\gamma)$$

Example

Eigenvalue equation

$$\rho(S) Y(\tau_S) = Y(\tau_S)$$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

Eigenvector

$$Y(\tau_S) \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Level 3 fixed points and alignments

G.J.Ding, S.F.K., X.G.Liu
and J.N.Lu, 1910.03460

A_4

The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3}'}(\gamma\tau_S)$ of level 3 up to weight 6

γ	$\gamma\tau_S$	$Y_{\mathbf{3}}^{(2)}(\gamma\tau_S)$, $Y_{\mathbf{3},I}^{(6)}(\gamma\tau_S)$	$Y_{\mathbf{3}}^{(4)}(\gamma\tau_S)$	$Y_{\mathbf{3},II}^{(6)}(\gamma\tau_S)$
$\{1,S\}$	i	$(1,1-\sqrt{3},\sqrt{3}-2)$	$(1,1,1)$	$(1,-2-\sqrt{3},1+\sqrt{3})$
$\{T,TS\}$	$1+i$	$(1,(1-\sqrt{3})\omega,(\sqrt{3}-2)\omega^2)$	$(1,\omega,\omega^2)$	$(1,(-2-\sqrt{3})\omega,(1+\sqrt{3})\omega^2)$
$\{ST,STS\}$	$\frac{-1+i}{2}$	$(1,(1+\sqrt{3})\omega,(-2-\sqrt{3})\omega^2)$	$(1,\omega,\omega^2)$	$(1,(\sqrt{3}-2)\omega,(1-\sqrt{3})\omega^2)$
$\{T^2,T^2S\}$	$2+i$	$(1,(1-\sqrt{3})\omega^2,(-2+\sqrt{3})\omega)$	$(1,\omega^2,\omega)$	$(1,(-2-\sqrt{3})\omega^2,(1+\sqrt{3})\omega)$
$\{ST^2,ST^2S\}$	$\frac{-2+i}{5}$	$(1,(1+\sqrt{3})\omega^2,(-2-\sqrt{3})\omega)$	$(1,\omega^2,\omega)$	$(1,(\sqrt{3}-2)\omega^2,(1-\sqrt{3})\omega)$
$\{T^2ST,TST^2\}$	$\frac{3+i}{2}$	$(1,1+\sqrt{3},-2-\sqrt{3})$	$(1,1,1)$	$(1,\sqrt{3}-2,1-\sqrt{3})$

The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3}'}(\gamma\tau_{ST})$ of level 3 up to weight 6

γ	$\gamma\tau_{ST}$	$Y_{\mathbf{3}}^{(2)}(\gamma\tau_{ST})$, $Y_{\mathbf{3},I}^{(6)}(\gamma\tau_{ST})$	$Y_{\mathbf{3}}^{(4)}(\gamma\tau_{ST})$	$Y_{\mathbf{3},II}^{(6)}(\gamma\tau_{ST})$
$\{1,ST,T^2S\}$	$\frac{-1+i\sqrt{3}}{2}$	$(1,\omega,\frac{-1}{2}\omega^2)$	$(1,\frac{-1}{2}\omega,\omega^2)$	$(1,-2\omega,-2\omega^2)$
$\{T,ST^2S,S\}$	$\frac{1+i\sqrt{3}}{2}$	$(1,\omega^2,-\frac{1}{2}\omega)$	$(1,-\frac{1}{2}\omega^2,\omega)$	$(1,-2\omega^2,-2\omega)$
$\{TS,T^2,T^2ST\}$	$2+\omega$	$(1,1,-\frac{1}{2})$	$(1,-\frac{1}{2},1)$	$(1,-2,-2)$
$\{STS,ST^2,TST^2\}$	$\frac{-3+i\sqrt{3}}{6}$	$(0,0,1)$	$(0,1,0)$	$(1,0,0)$

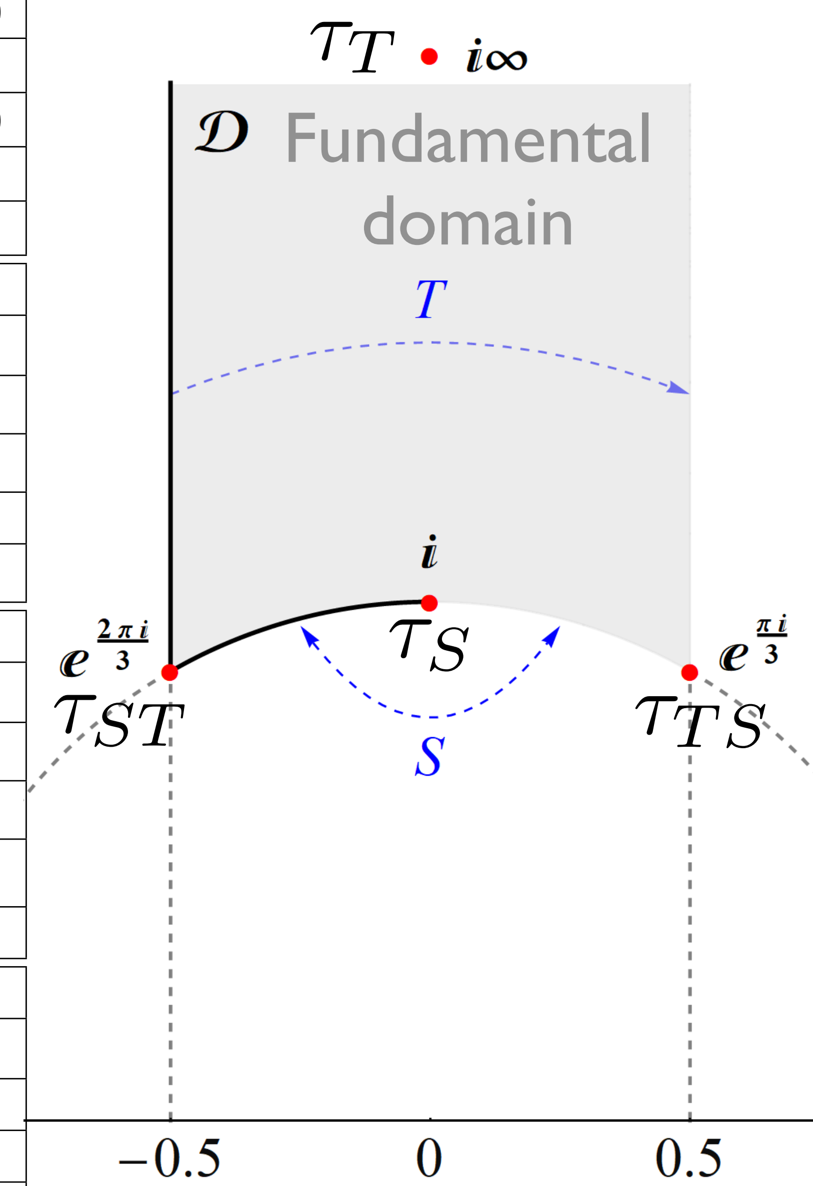
The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3}'}(\gamma\tau_{TS})$ of level 3 up to weight 6

γ	$\gamma\tau_{TS}$	$Y_{\mathbf{3}}^{(2)}(\gamma\tau_{TS})$, $Y_{\mathbf{3},I}^{(6)}(\gamma\tau_{TS})$	$Y_{\mathbf{3}}^{(4)}(\gamma\tau_{TS})$	$Y_{\mathbf{3},II}^{(6)}(\gamma\tau_{TS})$
$\{1,TS,ST^2\}$	$\frac{1+i\sqrt{3}}{2}$	$(1,\omega^2,-\frac{1}{2}\omega)$	$(1,-\frac{1}{2}\omega^2,\omega)$	$(1,-2\omega^2,-2\omega)$
$\{T,T^2S,TST^2\}$	$\frac{3+i\sqrt{3}}{2}$	$(1,1,-\frac{1}{2})$	$(1,-\frac{1}{2},1)$	$(1,-2,-2)$
$\{ST,ST^2S,T^2ST\}$	$\frac{(-1)^{5/6}}{\sqrt{3}}$	$(0,0,1)$	$(0,1,0)$	$(1,0,0)$
$\{STS,T^2,S\}$	$2+\omega$	$(1,\omega,\frac{-1}{2}\omega^2)$	$(1,\frac{-1}{2}\omega,\omega^2)$	$(1,-2\omega,-2\omega^2)$

The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3}'}(\gamma\tau_T)$ of level 3 up to weight 6

γ	$\gamma\tau_T$	$Y_{\mathbf{3}}^{(2)}(\gamma\tau_T)$, $Y_{\mathbf{3},I}^{(6)}(\gamma\tau_T)$, $Y_{\mathbf{3}}^{(4)}(\gamma\tau_T)$	$Y_{\mathbf{3},II}^{(6)}(\gamma\tau_T)$
$\{1,T,T^2\}$	$i\infty$	$(1,0,0)$	$(0,1,0)$
$\{ST,ST^2,S\}$	0	$(1,-2,-2)$	$(1,-\frac{1}{2},1)$
$\{TS,ST^2S,TST^2\}$	1	$(1,-2\omega,-2\omega^2)$	$(1,-\frac{1}{2}\omega,\omega^2)$
$\{STS,T^2S,T^2ST\}$	-1	$(1,-2\omega^2,-2\omega)$	$(1,-\frac{1}{2}\omega^2,\omega)$

Fixed points in fundamental domain



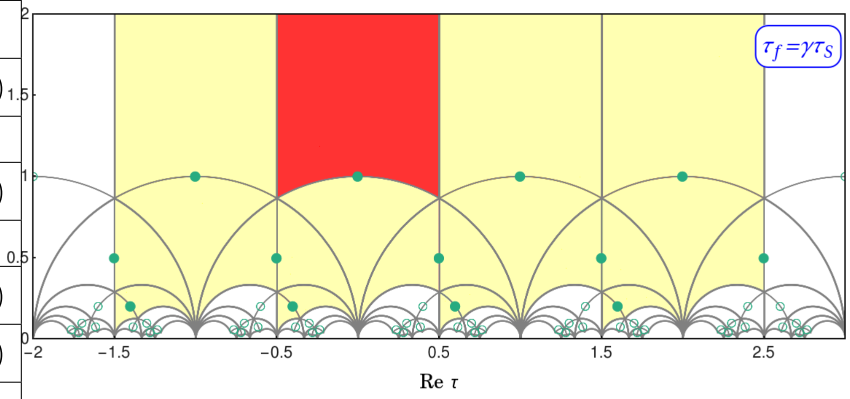
Level 4 fixed points and alignments

G.J.Ding, S.F.K., X.G.Liu and J.N.Lu, 1910.03460

S_4

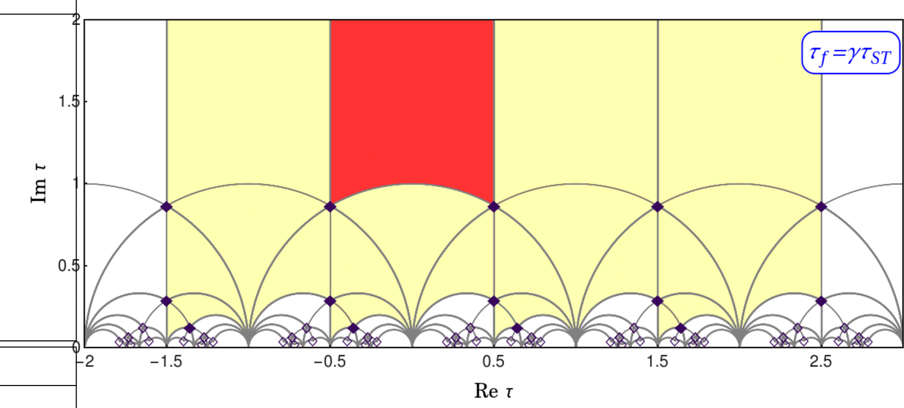
The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3'}}(\gamma\tau_S)$ of level 4 up to weight 6

γ	$\gamma\tau_S$	$Y^{(2)}_{\mathbf{3}}(\gamma\tau_S)$, $Y^{(6)}_{\mathbf{3,I}}(\gamma\tau_S)$	$Y^{(4)}_{\mathbf{3}}(\gamma\tau_S)$, $Y^{(6)}_{\mathbf{3'}}(\gamma\tau_S)$	$Y^{(4)}_{\mathbf{3'}}(\gamma\tau_S)$, $Y^{(6)}_{\mathbf{3,II}}(\gamma\tau_S)$
$\{1,S\}$	i	$(1,1+\sqrt{6},1-\sqrt{6})$	$(1,-\frac{1}{2},-\frac{1}{2})$	$(1,1-\sqrt{\frac{3}{2}},1+\sqrt{\frac{3}{2}})$
$\{T^2,T^2S\}$	$2+i$	$(1,\frac{1}{3}(-1+i\sqrt{2}),\frac{1}{3}(-1+i\sqrt{2}))$	$(0,1,-1)$	$(1,-\frac{i}{\sqrt{2}},-\frac{i}{\sqrt{2}})$
$\{ST^2S,ST^2\}$	$-\frac{2}{5}+\frac{i}{5}$	$(1,-\frac{1}{3}(1+i\sqrt{2}),-\frac{1}{3}(1+i\sqrt{2}))$	$(0,1,-1)$	$(1,\frac{i}{\sqrt{2}},\frac{i}{\sqrt{2}})$
$\{(ST^2)^2,T^2ST^2\}$	$-\frac{8}{13}+\frac{i}{13}$	$(1,1-\sqrt{6},1+\sqrt{6})$	$(1,-\frac{1}{2},-\frac{1}{2})$	$(1,1+\sqrt{\frac{3}{2}},1-\sqrt{\frac{3}{2}})$
$\{ST,STS\}$	$-\frac{1}{2}+\frac{i}{2}$	$(1,\omega^2(1+\sqrt{6}),\omega(1-\sqrt{6}))$	$(1,-\frac{\omega^2}{2},-\frac{\omega}{2})$	$(1,\omega^2(1-\sqrt{\frac{3}{2}}),\omega(1+\sqrt{\frac{3}{2}}))$
$\{TS,T\}$	$1+i$	$(1,-\frac{\omega}{3}(1+i\sqrt{2}),-\frac{\omega^2}{3}(1+i\sqrt{2}))$	$(0,1,-\omega)$	$(1,\frac{i\omega}{\sqrt{2}},\frac{i\omega^2}{\sqrt{2}})$
$\{(ST)^2,T^3\}$	$-1+i$	$(1,\omega(1+\sqrt{6}),\omega(1-\sqrt{6}))$	$(1,-\frac{\omega}{2},-\frac{\omega^2}{2})$	$(1,\omega(1-\sqrt{\frac{3}{2}}),\omega^2(1+\sqrt{\frac{3}{2}}))$
$\{(TS)^2,TST\}$	$\frac{1}{2}+\frac{i}{2}$	$(1,\frac{\omega^2}{3}(-1+i\sqrt{2}),\frac{\omega}{3}(-1+i\sqrt{2}))$	$(0,1,-\omega^2)$	$(1,-\frac{i\omega^2}{\sqrt{2}},-\frac{i\omega}{\sqrt{2}})$
$\{(T^2ST,TST^3\}$	$\frac{3}{2}+\frac{i}{2}$	$(1,\omega^2(1-\sqrt{6}),\omega(1+\sqrt{6}))$	$(1,-\frac{\omega^2}{2},-\frac{\omega}{2})$	$(1,\omega^2(1+\sqrt{\frac{3}{2}}),\omega(1-\sqrt{\frac{3}{2}}))$
$\{(TST^2,TST^2S\}$	$\frac{3}{5}+\frac{i}{5}$	$(1,\omega(1-\sqrt{6}),\omega^2(1+\sqrt{6}))$	$(1,-\frac{\omega}{2},-\frac{\omega^2}{2})$	$(1,\omega(1+\sqrt{\frac{3}{2}}),\omega^2(1-\sqrt{\frac{3}{2}}))$
$\{T^3ST^2,ST^2ST\}$	$\frac{13}{5}+\frac{i}{5}$	$(1,\frac{\omega}{3}(-1+i\sqrt{2}),\frac{\omega^2}{3}(-1+i\sqrt{2}))$	$(0,1,-\omega)$	$(1,-\frac{i\omega}{\sqrt{2}},-\frac{i\omega^2}{\sqrt{2}})$
$\{T^2ST^3,T^3ST\}$	$\frac{17}{10}+\frac{i}{10}$	$(1,-\frac{\omega^2}{3}(1+i\sqrt{2}),-\frac{\omega}{3}(1+i\sqrt{2}))$	$(0,1,-\omega^2)$	$(1,\frac{i\omega^2}{\sqrt{2}},\frac{i\omega}{\sqrt{2}})$



The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3'}}(\gamma\tau_{ST})$ of level 4 up to weight 6

γ	$\gamma\tau_{ST}$	$Y^{(2)}_{\mathbf{3}}(\gamma\tau_{ST})$	$Y^{(4)}_{\mathbf{3}}(\gamma\tau_{ST})$, $Y^{(4)}_{\mathbf{3'}}(\gamma\tau_{ST})$	$Y^{(6)}_{\mathbf{3},II}(\gamma\tau_{ST})$, $Y^{(6)}_{\mathbf{3'}}(\gamma\tau_{ST})$	$Y^{(6)}_{\mathbf{3},I}(\gamma\tau_{ST})$
$\{1,ST,(ST)^2\}$	ω	$(0,1,0)$	$(0,0,1)$	$(1,0,0)$	$(0,0,0)$
$\{T^2,TS,T^2ST\}$	$\omega+2$	$(1,-\frac{\omega^2}{2},\omega)$	$(1,\omega^2,-\frac{\omega}{2})$	$(1,1+i\sqrt{3},1-i\sqrt{3})$	
$\{ST^2S,(TS)^2,TST^2\}$	$\frac{-5+i\sqrt{3}}{14}$	$(1,-\frac{\omega}{2},\omega^2)$	$(1,\omega,-\frac{\omega^2}{2})$	$(1,1-i\sqrt{3},1+i\sqrt{3})$	
$\{(ST^2)^2,T^3ST^2,T^2ST^3\}$	$\frac{-9+i\sqrt{3}}{14}$	$(1,-\frac{1}{2},1)$	$(1,1,-\frac{1}{2})$	$(1,-2,-2)$	
$\{(S,T,TST\}$	$-\omega^2$	$(1,1,-\frac{1}{2})$	$(1,-\frac{1}{2},1)$	$(1,-2,-2)$	
$\{T^3ST,T^3,T^2S\}$	$\omega+3$	$(1,\omega,-\frac{\omega^2}{2})$	$(1,-\frac{\omega}{2},\omega^2)$	$(1,-2\omega,-2\omega^2)$	
$\{TST^3,T^2ST^2,TST^2S\}$	$\frac{9+i\sqrt{3}}{14}$	$(0,0,1)$	$(0,1,0)$	$(1,0,0)$	
$\{ST^2ST,ST^2,STS\}$	$\frac{-3+i\sqrt{3}}{6}$	$(1,\omega^2,-\frac{\omega}{2})$	$(1,-\frac{\omega^2}{2},\omega)$	$(1,-2\omega^2,-2\omega)$	



The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3'}}(\gamma\tau_{TS})$ of level 4 up to weight 6

γ	$\gamma\tau_{TS}$	$Y^{(2)}_{\mathbf{3}}(\gamma\tau_{TS})$	$Y^{(4)}_{\mathbf{3}}(\gamma\tau_{TS})$, $Y^{(4)}_{\mathbf{3'}}(\gamma\tau_{TS})$	$Y^{(6)}_{\mathbf{3},II}(\gamma\tau_{TS})$, $Y^{(6)}_{\mathbf{3'}}(\gamma\tau_{TS})$	$Y^{(6)}_{\mathbf{3},I}(\gamma\tau_{TS})$
$\{1,TS,(TS)^2\}$	$-\omega^2$	$(1,1,-\frac{1}{2})$	$(1,-\frac{1}{2},1)$	$(1,-2,-2)$	$(0,0,0)$
$\{T^2,(ST)^2,T^2ST^3\}$	$\frac{5+i\sqrt{3}}{2}$	$(1,\omega,-\frac{1}{2}\omega^2)$	$(1,-\frac{1}{2}\omega,\omega^2)$	$(1,-2\omega,-2\omega^2)$	
$\{ST^2S,ST,T^3ST^2\}$	$\frac{\sqrt{3}i-3}{6}$	$(1,\omega^2,-\frac{1}{2}\omega)$	$(1,-\frac{1}{2}\omega^2,\omega)$	$(1,-2\omega^2,-2\omega)$	
$\{(ST^2)^2,T^2ST,TST^2\}$	$\frac{\sqrt{3}i-23}{38}$	$(0,0,1)$	$(0,1,0)$	$(1,0,0)$	
$\{S,T^3,STS\}$	ω	$(0,1,0)$	$(0,0,1)$	$(1,0,0)$	
$\{T^3ST,T^2ST^2,ST^2ST\}$	$3+\frac{(-1)^{5/6}}{\sqrt{3}}$	$(1,-\frac{1}{2},1)$	$(1,1,-\frac{1}{2})$	$(1,-2,-2)$	
$\{TST^3,T,T^2S\}$	$\frac{19+i\sqrt{3}}{26}$	$(1,-\frac{1}{2}\omega^2,\omega)$	$(1,\omega^2,-\frac{1}{2}\omega)$	$(1,-2\omega^2,2\omega)$	
$\{TST^2S,ST^2,TSTS\}$	$\frac{3+i\sqrt{3}}{6}$	$(1,-\frac{1}{2}\omega,\omega^2)$	$(1,\omega,-\frac{1}{2}\omega^2)$	$(1,-2\omega,-2\omega^2)$	

The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3'}}(\gamma\tau_T)$ of level 4 up to weight 6

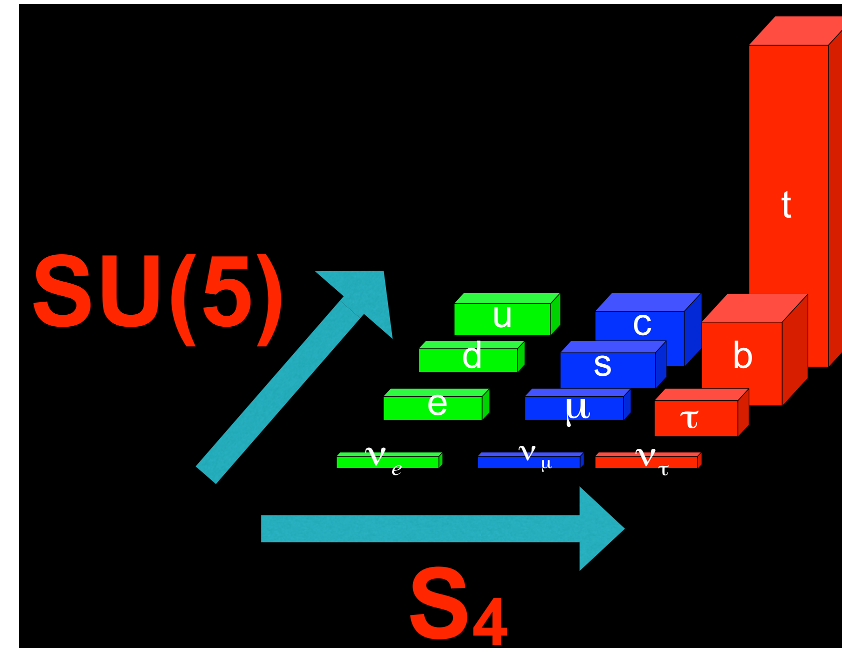
γ	$\gamma\tau_T$	$Y^{(2)}_{\mathbf{3}}(\gamma\tau_T)$, $Y^{(4)}_{\mathbf{3}}(\gamma\tau_T)$, $Y^{(6)}_{\mathbf{3,I}}(\gamma\tau_T)$, $Y^{(6)}_{\mathbf{3,II}}(\gamma\tau_T)$	$Y^{(4)}_{\mathbf{3'}}(\gamma\tau_T)$, $Y^{(6)}_{\mathbf{3'}}(\gamma\tau_T)$
$\{1,T,T^2,T^3\}$	$i\infty$	$(1,\omega^2,\omega)$	$(0,0,0)$
$\{ST^2S,ST^2ST,(ST^2)^2,TST^2S\}$	$-\frac{1}{2}$		
$\{ST,(TS)^2,S,ST^2\}$	0	$(1,\omega,\omega^2)$	
$\{T^2ST,T^2ST^3,T^2ST^2,T^2S\}$	2		
$\{TS,TST^2,TST^3,TST\}$	1	$(1,1,1)$	
$\{(ST)^2,T^3ST^2,T^3ST,STS\}$	-1		

G.J.Ding, S.F.K. and C.Y.Yao, 2103.16311

Example with SU(5) GUT: Level N=4 ~ S4

weighton

Field	T_3	$T=(T_2,T_1)^T$	F	N_a	N_s	H_5	$H_{\bar{5}}$	$H_{\overline{45}}$	ϕ	χ^0
$SU(5)$	$\mathbf{10}$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{5}$	$\bar{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$
S_4	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1'}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
k_I	4	1	3	4	-1	-2	1	1	1	0



$$\alpha_u\tilde{\phi}^4 Y_{\mathbf{2}}^{(4)}(TT)_{\mathbf{2}}H_5+\beta_u\tilde{\phi}^2 Y_{\mathbf{2}}^{(2)}(TT)_{\mathbf{2}}H_5+\gamma_u Y_{\mathbf{1'}}^{(6)}T_3T_3H_5+\epsilon_u\tilde{\phi}T_3(TY_{\mathbf{2}}^{(4)})_{\mathbf{1'}}H_5$$

$$\mathcal{Y}^u_{\rm GUT}\approx\begin{pmatrix}\alpha_u\tilde{\phi}^4 & 0 & 0\\ 0 & \beta_u\tilde{\phi}^2 & \epsilon_u\tilde{\phi}\\ 0 & \epsilon_u\tilde{\phi} & \gamma_u\end{pmatrix}$$

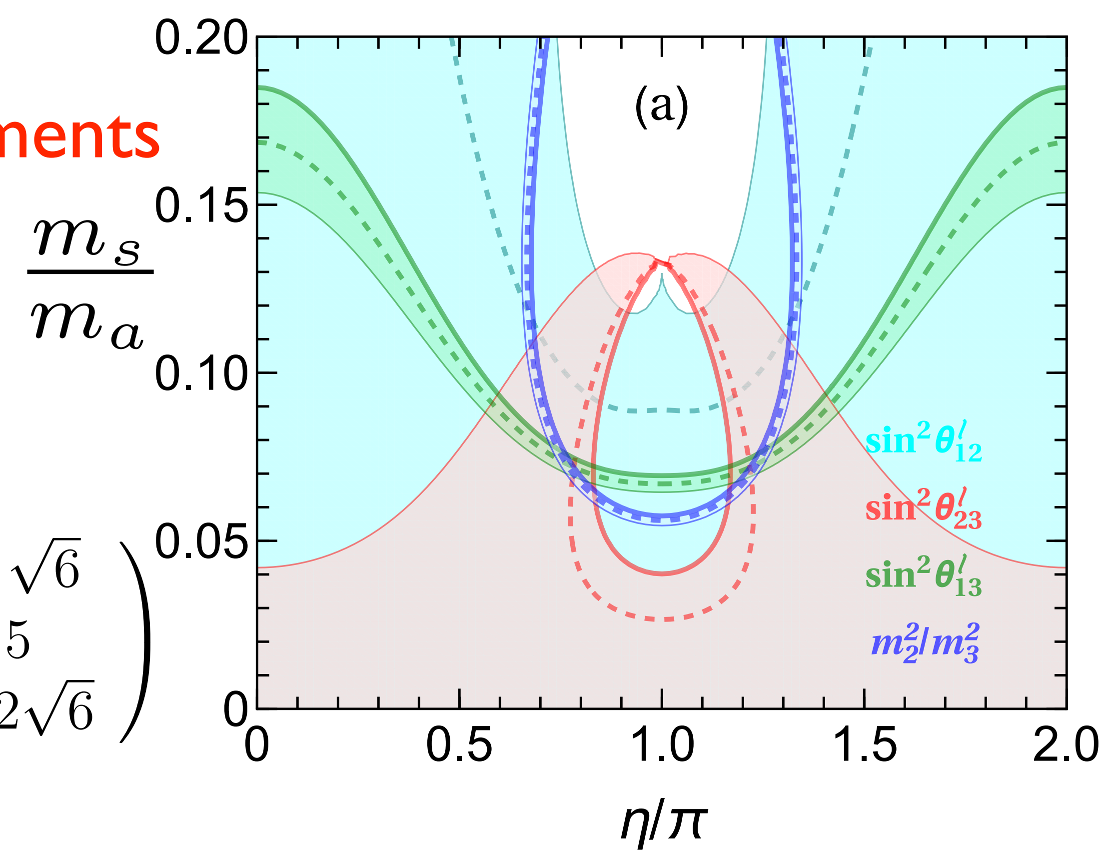
$$\mathcal{Y}^d_{\rm GUT}=(\mathcal{Y}_{\bar{5}}+\mathcal{Y}_{\overline{45}})^T$$
$$\mathcal{Y}^e_{\rm GUT}=\mathcal{Y}_{\bar{5}}-3\mathcal{Y}_{\overline{45}}$$

Littlest Modular Seesaw from fixed point alignments

$$Y_{\mathbf{3'}}^{(6)}\propto\begin{pmatrix}0\\1\\-1\end{pmatrix},\quad Y_{\mathbf{3}}^{(2)}\propto\begin{pmatrix}1\\1+\sqrt{6}\\1-\sqrt{6}\end{pmatrix}$$

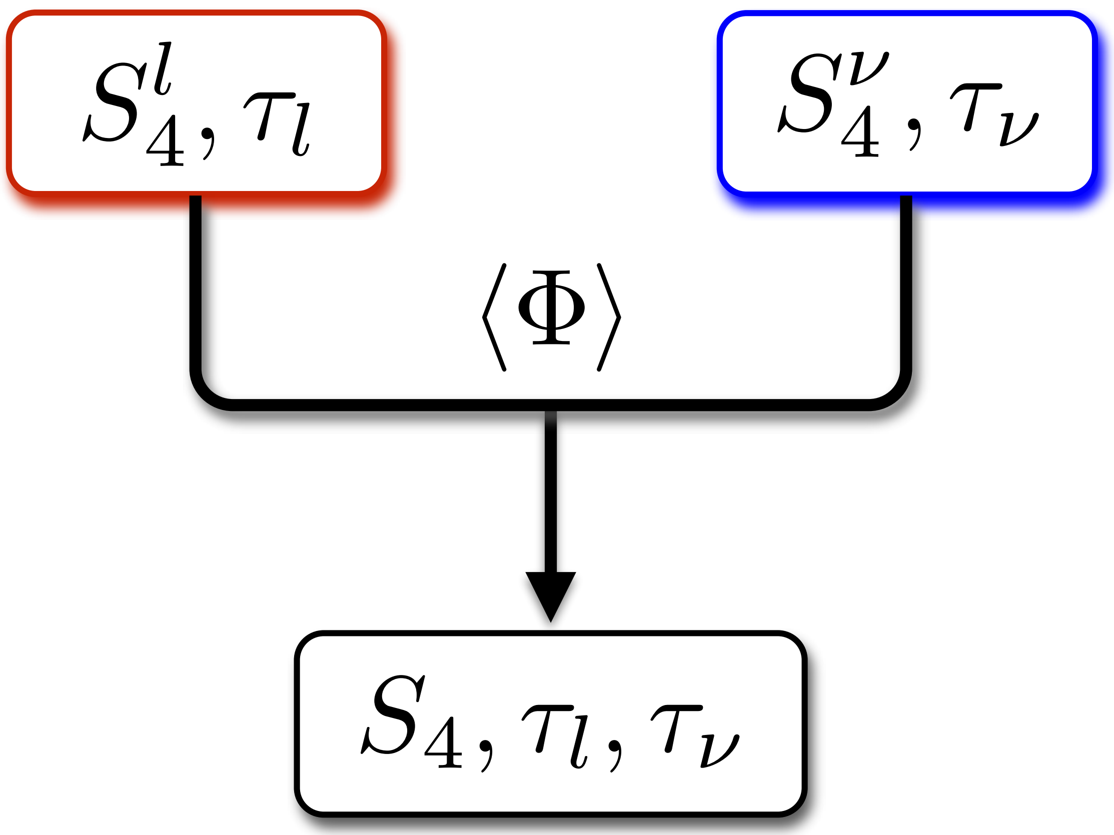
$$m_\nu=m_a\begin{pmatrix}0&0&0\\0&1&-1\\0&-1&1\end{pmatrix}+m_s e^{i\eta}\begin{pmatrix}1&1-\sqrt{6}&1+\sqrt{6}\\1-\sqrt{6}&7-2\sqrt{6}&-5\\1+\sqrt{6}&-5&7+2\sqrt{6}\end{pmatrix}$$

$$Y_{\mathbf{3'}}^{(6)}Y_{\mathbf{3'}}^{(6)T}\qquad\qquad Y_{\mathbf{3}}^{(2)}Y_{\mathbf{3}}^{(2)T}$$

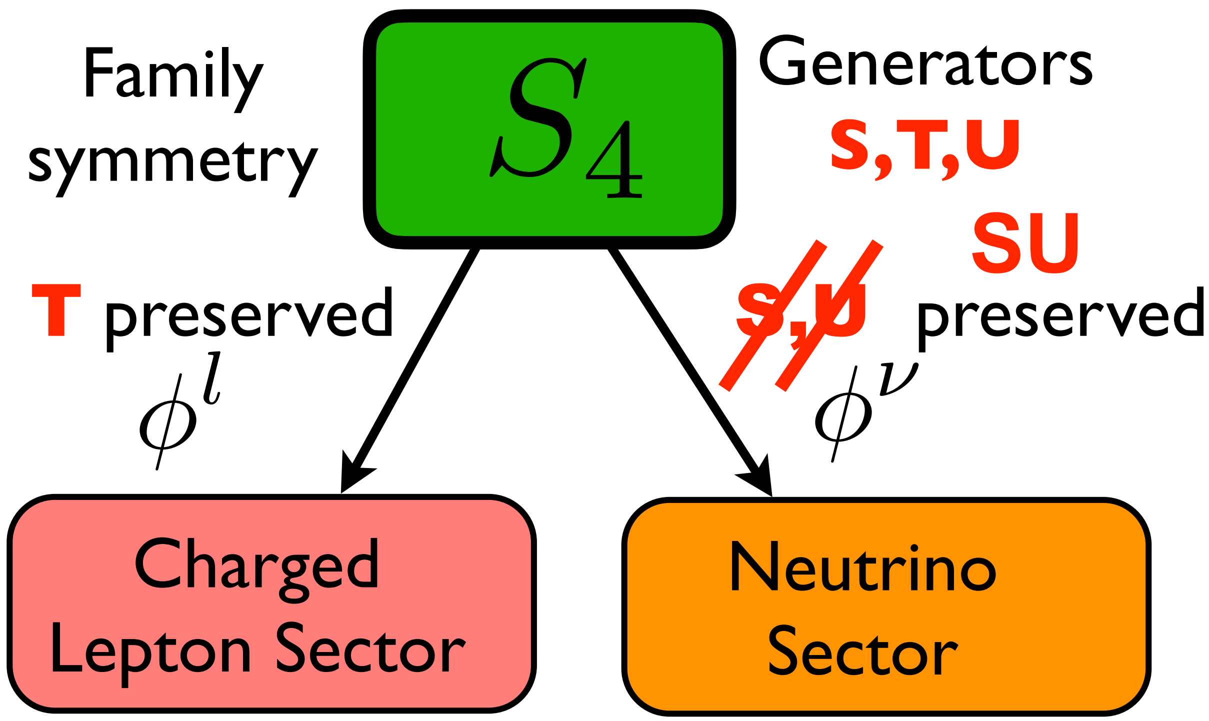


S.F.K. and Y.L. Zhou, 1908.02770

Example with two groups: Level N=4 ~ S_4



c.f.



Use S_4 basis:

$$T = S_\tau T_\tau\,, \qquad S = T_\tau^2\,, \qquad U = T_\tau S_\tau T_\tau^2 S_\tau$$

Fixed points:

$$\langle \tau_\nu \rangle = \tau_{SU} = -\frac{1}{2} + \frac{i}{2}$$

$$\langle \tau_l \rangle = \tau_T = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$Y(\tau_{SU}) \propto \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$U_{\mathrm{TM}_1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \end{pmatrix}$$

Summary

- Flavour problem motivates family/flavour symmetry
- U(1) with FN for hierarchies and small mixing
- Neutrino mass and mixing motivates non-Abelian
- TBM, TM1/TM2, Littlest Seesaw…enforced by S_4 and flavon alignments…gauged or modular origin
- Large literature on bottom-up modular models
- Weightons for charged fermion hierarchies
- Stabilizers/fixed points for Yukawa alignments
- SU(5) GUT with S_4 and Littlest Modular Seesaw
- Twin modular S_4 symmetries for TM1