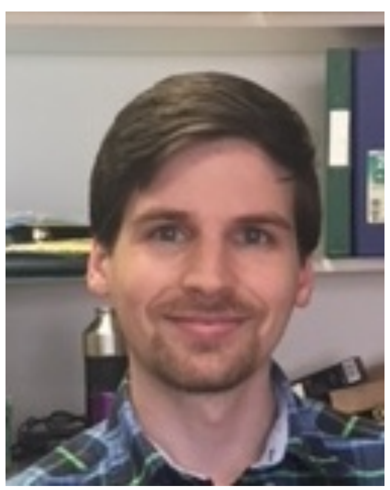
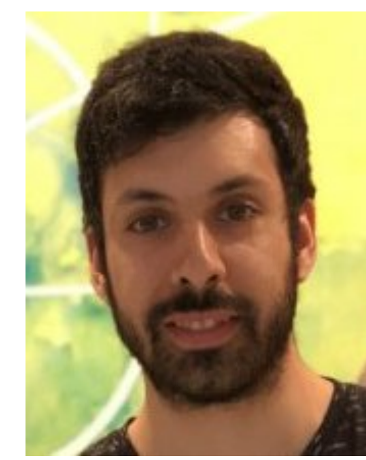
Implementing the three-particle quantization condition for 2+1 systems: theoretical issues

Steve Sharpe
University of Washington

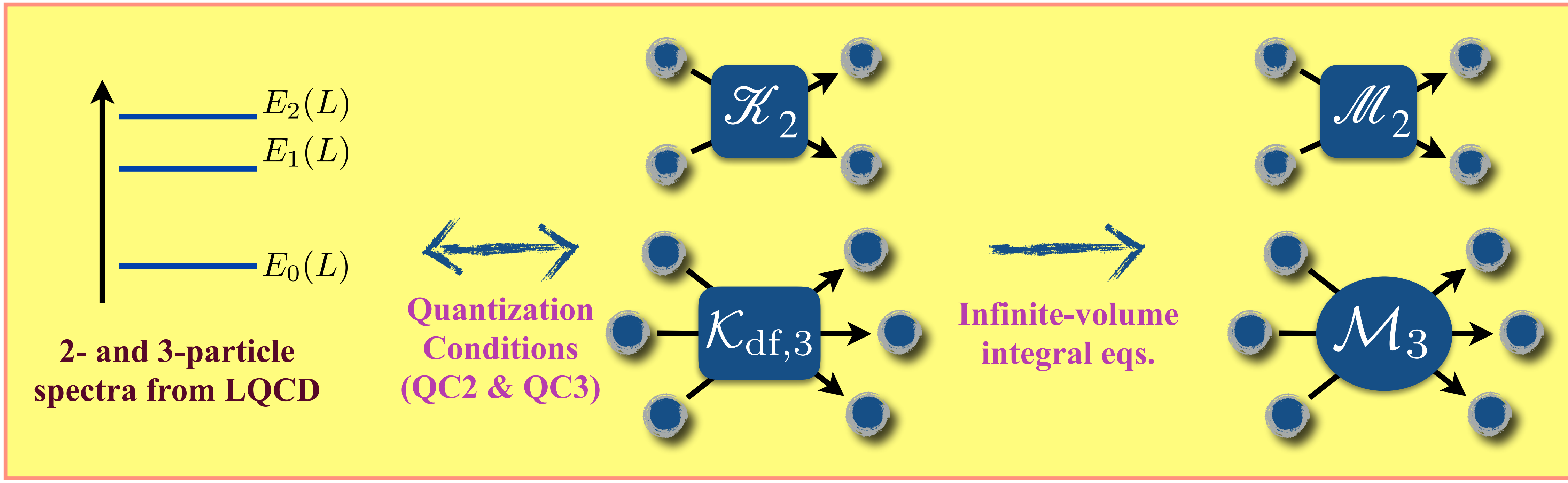


Primarily based on work with Tyler Blanton & Fernando Romero-López: 2111.12734 [hep-lat] (JHEP)



For numerical results, see talk by Zack Draper

Motivation



- Formalism exists for arbitrary choices of spinless particles [References in backup slides]
  - Implemented for 3 identical scalars $(3\pi^+, 3K^+)$ & $3\pi(I=1)$ & ϕ^4 theory
- Many systems of interest involve nondegenerate particles, e.g. "2+1" systems such as $\pi\pi N$
  - Simplest 2+1 systems contain spinless particles that do not allow $2 \leftrightarrow 3$ transitions, e.g. $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$
- **We discuss issues that arise in implementing the QC for such simple 2+1 systems, and provide useful ancillary results**

Outline

- Summary of 2+1 QC3
- Cutoff/transition function for nondegenerate particles
- Threshold expansion for $\mathcal{K}_{\rm df,3}$
- $\mathcal{K}_{\rm df,3}$ in chiral perturbation theory
- Expansion of threshold energy in powers of 1/L
- Python implementation

Summary of QC for 2+1 systems

[Blanton & SRS, 2105.12904 (PRD)]

- Use RFT formalism, with symmetric form of the QC3

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

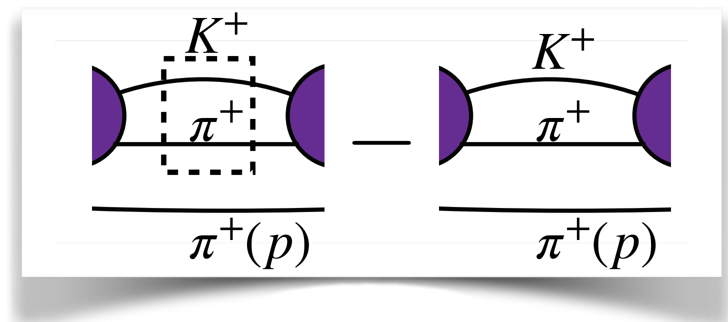
$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\overline{\mathcal{K}}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$

- QC has standard form, but matrices have an additional spectator-flavor index: $p\ell mi$
    - E.g., for $\pi^+\pi^+K^+$, spectator is π^+ ($i=1\Rightarrow\pi^+K^+$ scattering) or K^+ ($i=2\Rightarrow\pi^+\pi^+$ scattering)
    - All partial waves contribute to π^+K^+ scattering ($i=1$), while only even waves contribute to $\pi^+\pi^+$ scattering ($i=2$)
- Using symmetric QC3 implies that $\hat{\mathscr{K}}_{\mathrm{df},3}$ has the full symmetries of $\mathscr{M}_3$

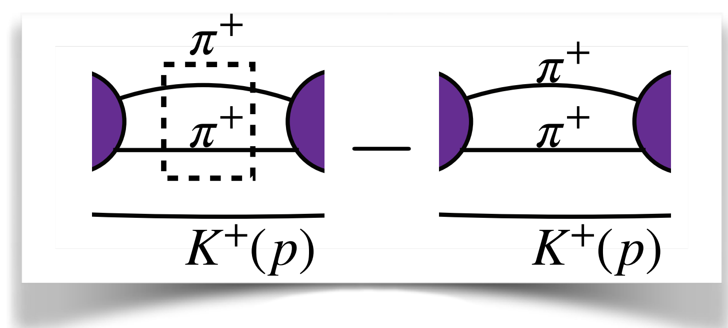
Details on matrices (for $\pi^+\pi^+K^+$)

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



$$\widehat{F}=\begin{pmatrix}\tilde{F}^{(1)} & 0\\ 0 & \tilde{F}^{(2)}\end{pmatrix}$$



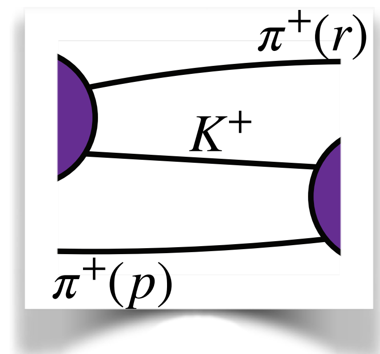
$$\left[\widetilde{F}^{(i)}\right]_{p'\ell'm';p\ell m}=\delta_{\boldsymbol{p}'\boldsymbol{p}}\frac{H^{(i)}(\boldsymbol{p})}{2\omega_p^{(i)}L^3}\left[\frac{1}{L^3}\sum_{\boldsymbol{a}}^{\mathrm{UV}}-\mathrm{PV}\int^{\mathrm{UV}}\frac{d^3a}{(2\pi)^3}\right]$$
$$\times\left[\frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{a}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell'}}\frac{1}{4\omega_a^{(j)}\omega_b^{(k)}(E-\omega_p^{(i)}-\omega_a^{(j)}-\omega_b^{(k)})}\frac{\mathcal{Y}_{\ell m}(\boldsymbol{a}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell}}\right]$$

- $H^{(i)}(p)$ is transition/cutoff function
- Only even ℓ contribute if $i=2$

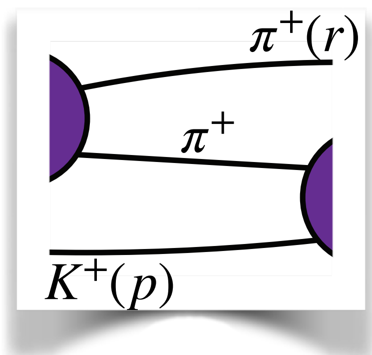
More details on matrices

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



$$\widehat{G}=\begin{pmatrix}\widetilde{G}^{(11)} & \sqrt{2}P_L\widetilde{G}^{(12)}\\ \sqrt{2}\widetilde{G}^{(21)}P_L & 0\end{pmatrix}$$



Basis-change factor of $(-1)^{\ell}$

Symmetry factor

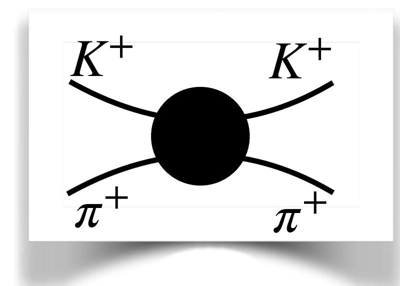
$$\left[\widetilde{G}^{(ij)}\right]_{p\ell'm';r\ell m}=\frac{1}{2\omega_p^{(i)}L^3}\frac{\mathcal{Y}_{\ell'm'}(\boldsymbol{r}^{*(i,j,p)})}{(q_{2,p}^{*(i)})^{\ell'}}\frac{H^{(i)}(\boldsymbol{p})H^{(j)}(\boldsymbol{r})}{b_{ij}^2-m_k^2}\frac{\mathcal{Y}_{\ell m}(\boldsymbol{p}^{*(j,i,r)})}{(q_{2,r}^{*(j)})^{\ell}}\frac{1}{2\omega_r^{(j)}L^3}$$

- Same $H^{(i)}(p)$ as in $\tilde{F}^{(i)}$

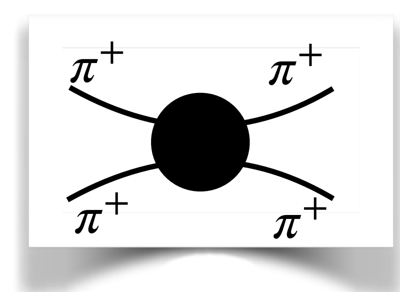
More details on matrices

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right]=0$$

$$\widehat{F}_3=\frac{\widehat{F}}{3}-\widehat{F}\frac{1}{\widehat{\overline{\mathcal{K}}}_{2,L}^{-1}+\widehat{F}+\widehat{G}}\widehat{F}$$



$$\widehat{\overline{\mathscr{K}}}_{2,L}=\begin{pmatrix}\overline{\mathscr{K}}_{2,L}^{(1)} & 0\\ 0 & \frac{1}{2}\overline{\mathscr{K}}_{2,L}^{(2)}\end{pmatrix}$$



$$\left[\overline{\mathcal{K}}_{2,L}^{(i)}\right]_{p\ell'm';r\ell m}=\delta_{\boldsymbol{p}\boldsymbol{r}}2\omega_r^{(i)}L^3\left[\mathcal{K}_2^{(i)}(\boldsymbol{r})\right]_{\ell'm';\ell m},$$

$$\left[\mathcal{K}_2^{(i)}(\boldsymbol{r})^{-1}\right]_{\ell'm';\ell m}=\delta_{\ell'\ell}\delta_{m'm}\frac{\eta_i}{8\pi\sqrt{\sigma_i}}\left\{q_{2,r}^{*(i)}\cot\delta_\ell^{(i)}(q_{2,r}^{*(i)})+|q_{2,r}^{*(i)}|[1-H^{(i)}(\boldsymbol{r})]\right\}$$

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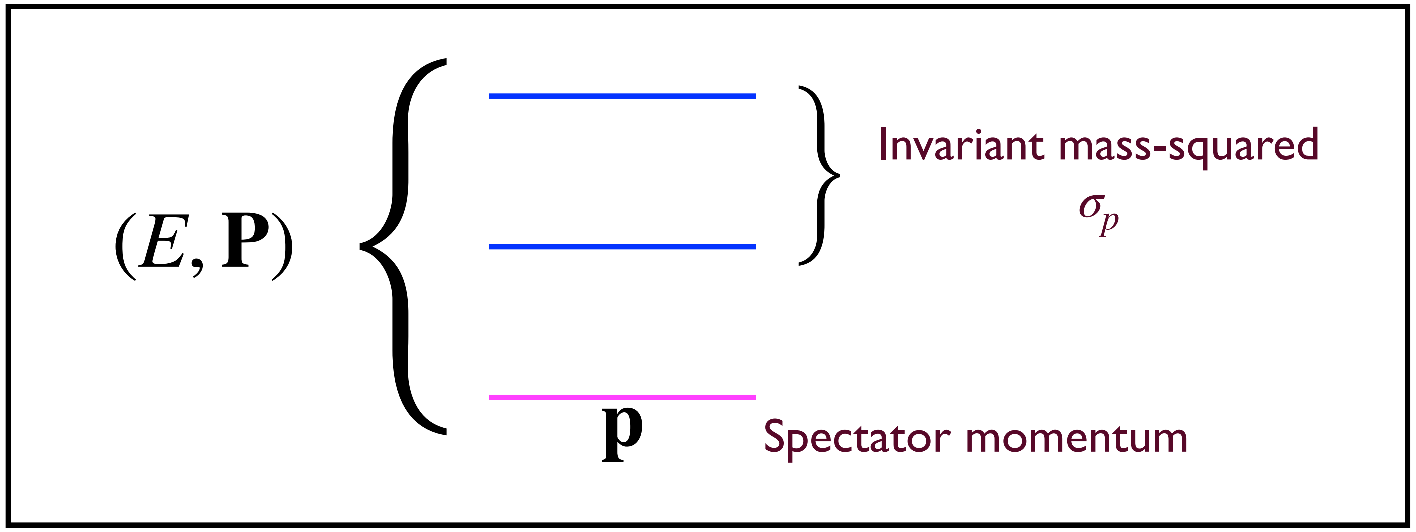
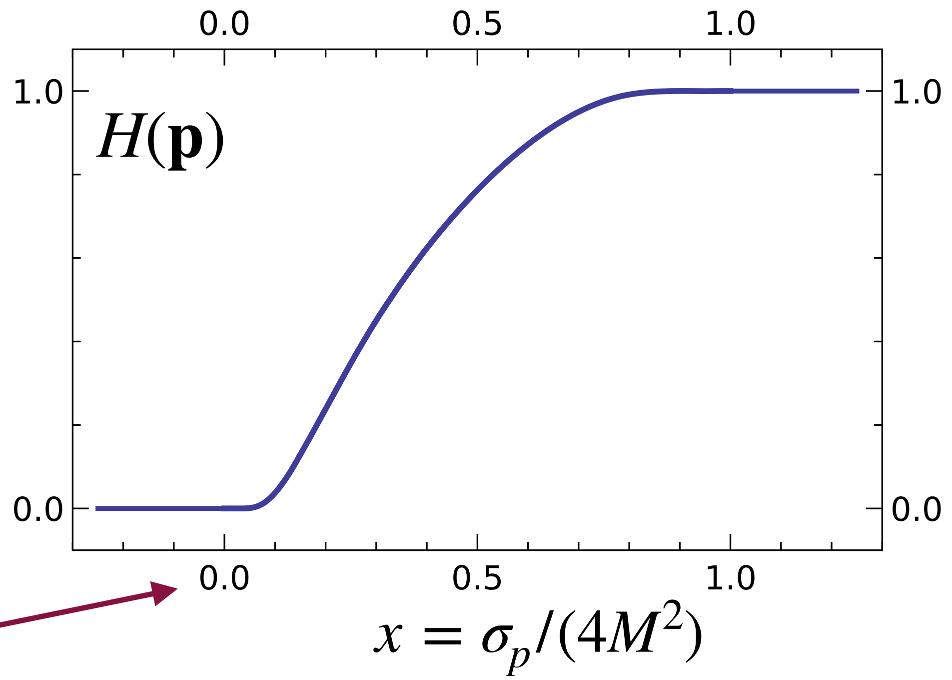
Cutoff/transition function

- In RFT derivation, need cutoff function to truncate matrix indices and to avoid LH cut
  - Must be smooth to avoid power-law finite-volume (FV) effects

Form for degenerate particles





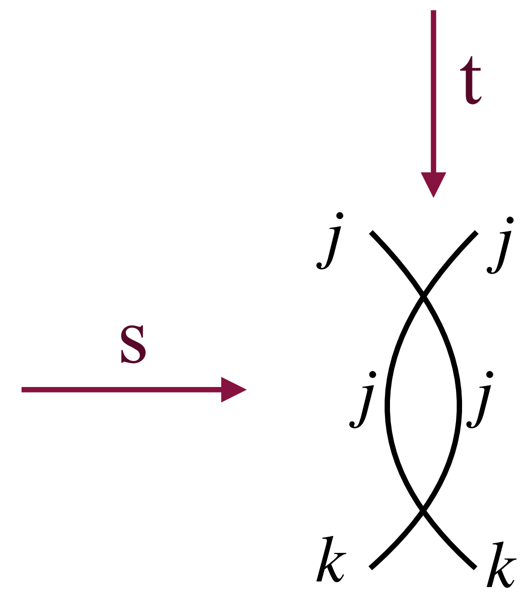
Position of left-hand cut in pair interaction ($\mathscr{K}_2$ or $\mathscr{M}_2$): $s = u = 0,\ t = 4M^2$

- May be possible to raise the cutoff, following the arguments used to relativize the NREFT approach [F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP]

Cutoff/transition function

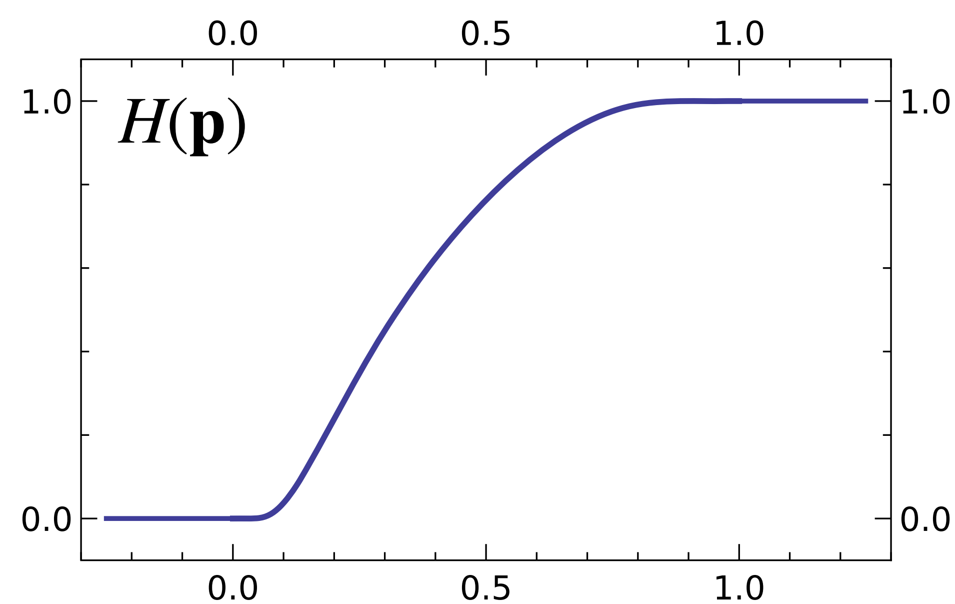
- For nondegenerate particles, LH cut moves, and must change cutoff function accordingly

Assume $M_j < M_k$:



$$t = 4M_j^2 = 4M_{\min,jk}^2$$

$$s = u = |M_k^2 - M_j^2| > 0$$



$$x = (1+\epsilon_H)\frac{\sigma_p - |M_k^2 - M_j^2|}{(M_j + M_k)^2 - |M_k^2 - M_j^2|}$$

- Same functional form, but argument adjusted so $H(\mathbf{p})$ vanishes at position of left-hand cut
- Strictly speaking, to avoid power-law FV effects, need $\epsilon_H > 0$ (though in practice set to zero)

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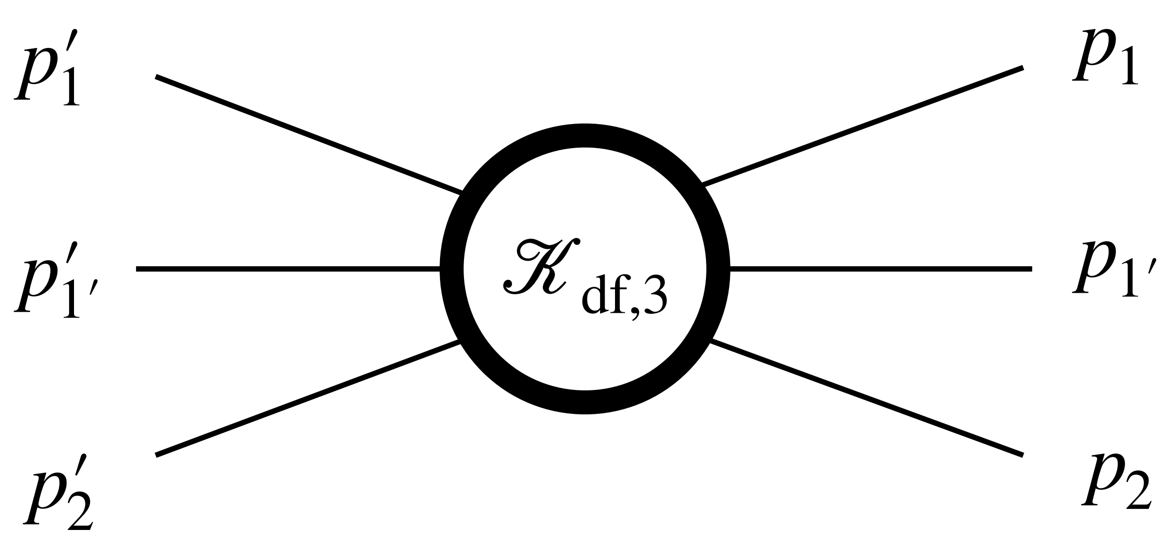
Threshold expansion for $\mathcal{K}_{\rm df,3}$

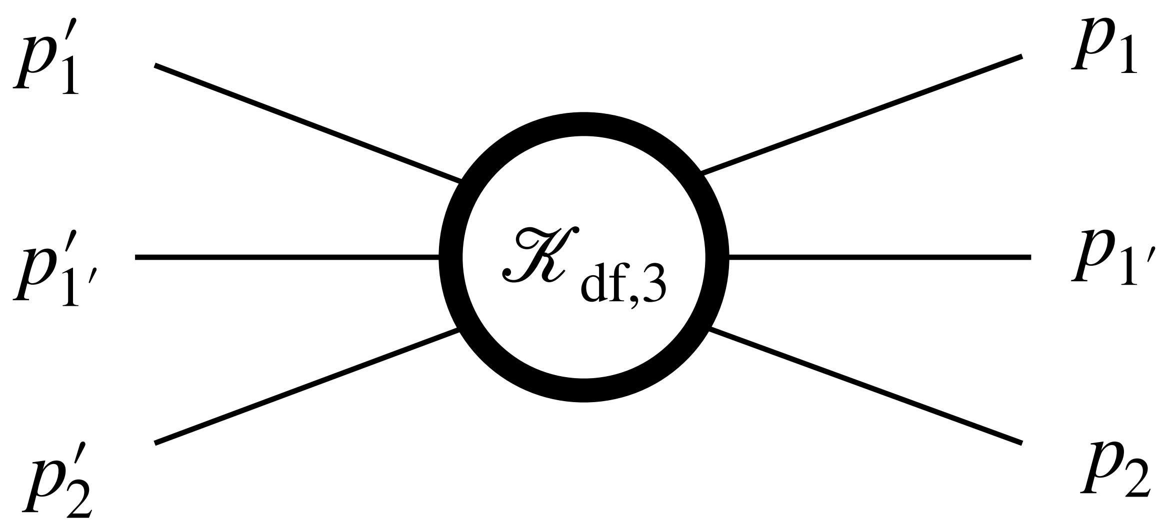
$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L)+\widehat{\mathcal{K}}_{\rm df,3}(E^*)\right]=0$$

$$\widehat{\mathcal{K}}_{\rm df,3}=\begin{pmatrix}[\mathcal{K}_{\rm df,3}]_{p\ell'm'1;k\ell m1} & [\mathcal{K}_{\rm df,3}]_{p\ell'm'1;k\ell m2}/\sqrt{2}\\ [\mathcal{K}_{\rm df,3}]_{p\ell'm'2;k\ell m1}/\sqrt{2} & [\mathcal{K}_{\rm df,3}]_{p\ell'm'2;k\ell m2}/2\end{pmatrix}$$

- Each entry involves the ***same*** infinite-volume amplitude, decomposed in different coords
- Infinite-volume amplitude $\mathcal{K}_{\rm df,3}(p_1',p_{1'}',p_2';p_1,p_{1'},p_2)$ is smooth (no cuts or two-particle poles) aside from three-particle poles, and is invariant under Lorentz transformations, T, P, and interchange of identical particles in initial and/or final states
- For nonresonant system, e.g. $\pi^+\pi^+K^+$, can use expansion about threshold analogous to effective-range expansion for $\mathcal{K}_2$



Threshold expansion for $\mathscr{K}_{\mathrm{df},3}$



Useful invariants:

$$\Delta = \frac{s-M}{M^2}\,, \qquad s = (p_1 + p_{1'} + p_2)^2 = P^2\,,$$

$$\Delta_2^S = \Delta_2 + \Delta_2'\,, \quad \Delta_2 = \frac{(p_1+p_{1'})^2 - 4m_1^2}{M^2}\,, \quad \Delta_2' = \frac{(p_1'+p_{1'}')^2 - 4m_1^2}{M^2}\,,$$

$$\tilde{t}_{22} = \frac{t_{22}}{M^2} = \frac{(p_2 - p_2')^2}{M^2}\,, \quad M = 2m_1 + m_2\,.$$

- Expand in powers of $\Delta \sim \Delta_2^S \sim \tilde{t}_{22}$
- 1 term of $\mathscr{O}(\Delta^0)$, 3 terms of $\mathscr{O}(\Delta)$, 9 terms of $\mathscr{O}(\Delta^2)$
- In practice, work to linear order, so that there are 4 undetermined constants:

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{B,1}\Delta_2^S + \mathcal{K}_{\mathrm{df},3}^{E,1}\tilde{t}_{22}$$

Properties of the terms

$$\mathcal{K}_{\rm df,3} = \mathcal{K}_{\rm df,3}^{\rm iso,0} + \mathcal{K}_{\rm df,3}^{\rm iso,1}\Delta + \mathcal{K}_{\rm df,3}^{B,1}\Delta_2^S + \mathcal{K}_{\rm df,3}^{E,1}\widetilde{t}_{22}$$

Independent of relative momenta & angles

Depends on relative momenta & angles, but only contains $J_{\rm tot}=0$, and only appears in trivial FV irreps

Depends on relative momenta & angles, contains $J_{\rm tot}=0$ & 1, and appears in some nontrivial FV irreps

- Even though $\mathscr{K}_{\rm df,3}^{E,1}$ term is of higher order than $\mathscr{K}_{\rm df,3}^{\rm iso,0}$ term, it can be easier to determine as it appears in more FV irreps
- When decompose into p, ℓ, m, i basis (a straightforward but very tedious exercise)
  - Isotropic terms lead only to terms with $\ell' = \ell = 0$
  - $\mathscr{K}_{\rm df,3}^{B,1}$ & $\mathscr{K}_{\rm df,3}^{E,1}$ contain $\ell', \ell = 0,1$ terms
  - Only $\mathscr{K}_{\rm df,3}^{E,1}$ contains $\ell' = \ell = 1$ terms
- For consistency, truncate effective-range expansion of $\mathscr{K}_2$ at linear order in q^2

FV-irrep projections

$\boldsymbol{d}_{\text{ref}}$	LG($\boldsymbol{P}$)	irreps
$(0,0,0)$	O_h	$A_{1g}[1], A_{2g}[1], E_g[2], T_{1g}[3], T_{2g}[3], A_{1u}[1], A_{2u}[1], E_u[2], T_{1u}[3], T_{2u}[3]$
$(0,0,n)$	C_{4v}	$A_1[1],\ A_2[1],\ B_1[1],\ B_2[1],\ E[2]$
$(n,n,0)$	C_{2v}	$A_1[1],\ A_2[1],\ B_1[1],\ B_2[1]$
(n,n,n)	C_{3v}	$A_1[1],\ A_2[1],\ E[2]$
$(n_1,n_2,0)$	C_2	$A_1[1],\ A_2[1]$
(n_1,n_1,n_2)	C_2	$A_1[1],\ A_2[1]$
(n_1,n_2,n_3)	C_1	$A_1[1]$

Table 1. Little group LG($\boldsymbol{P}$) for each type of frame, along with its irreps, each with its dimension listed in square brackets. Frames are denoted by $\boldsymbol{d}_{\text{def}} = \boldsymbol{P}L/(2\pi)$, taking a canonical choice for each type of frame. The integers n, n_1, n_2, and n_3 are nonzero, with n_1, n_2, and n_3 being distinct.

$\boldsymbol{d}_{\text{ref}}$	$\mathcal{K}_2^{(1)}$ (QC2)	$\mathcal{K}_{\text{df},3}^{B,1}$ (QC3)	$\mathcal{K}_{\text{df},3}^{E,1}$ (QC3)	$\widehat{\overline{\mathcal{K}}}_{2,L}$ (QC3)
$(0,0,0)$	$A_{1g}(1),\ T_{1u}(3)$	$A_{1u}(2)$	$A_{1u}(2),\ T_{1g}(3)$	all
$(0,0,n)$	$A_1(2),\ E(2)$	$A_2(2)$	$A_2(3),\ E(2)$	all
$(n,n,0)$	$A_1(2),\ B_1(1),\ B_1(1)$	$A_2(2)$	$A_2(3),\ B_1(1),\ B_2(1)$	all
(n,n,n)	$A_1(2),\ E(2)$	$A_2(2)$	$A_2(3),\ E(2)$	all
$(n_1,n_2,0)$	$A_1(3),\ A_2(1)$	$A_2(2)$	$A_2(4),\ A_1(1)$	all
(n_1,n_1,n_2)	$A_1(3),\ A_2(1)$	$A_2(2)$	$A_2(4),\ A_1(1)$	all
(n_1,n_2,n_3)	$A_1(4)$	$A_1(2)$	$A_1(5)$	all

Table 2. Irrep decomposition of eigenvalues, assuming $\ell_{\max} = 1$, as $L \to \infty$

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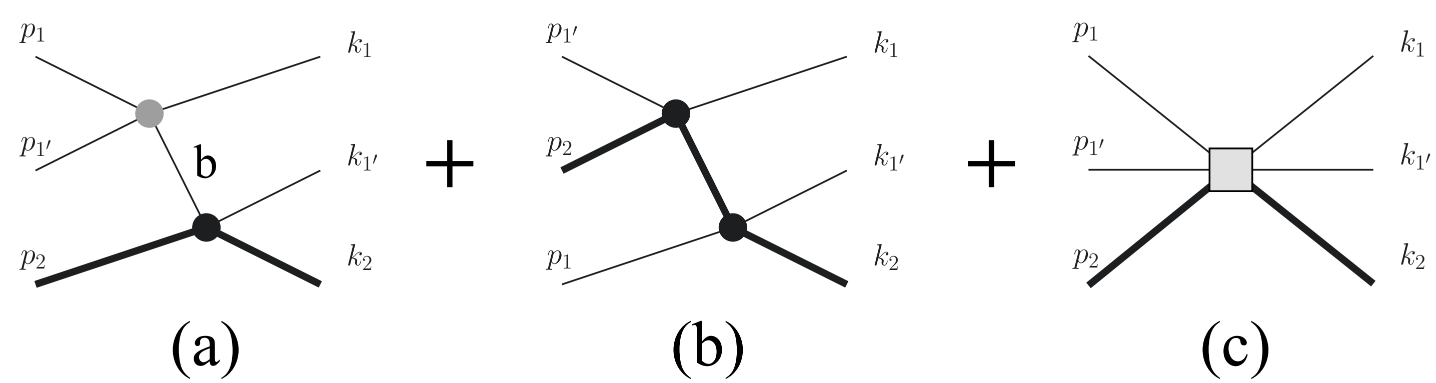
$\mathscr{K}_{\text{df},3}$ in ChPT

- Important to have an approximate prediction for $\mathscr{K}_{\text{df},3}$, so as to provide a reality check on numerical determinations (which are challenging)
- Previous work determined LO ChPT prediction for $3\pi^+$ and $3K^+$ [Blanton, Romero-López & SRS, 1909.02973 (PRL)]
  - Here extend to $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ (using SU(3) ChPT)
- Key result: at LO in ChPT $\mathscr{K}_{\text{df},3} = \mathscr{M}_{\text{df},3}$
  - At higher-order, relation requires inverting an integral equation
  - Although $\mathscr{K}_{\text{df},3}$ and $\mathscr{M}_{\text{df},3}$ in general depend on the cutoff function $H(\mathbf{p})$, they do not at LO in ChPT
- $\mathscr{M}_{\text{df},3}$ is related to $\mathscr{M}_3$ by the subtraction of on-shell divergences, and is finite (see below)
- Results for $\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ simply related by interchanging masses

$\mathscr{M}_{\mathrm{df},3}$ in ChPT

$$\mathcal{K}_{\mathrm{df},3}^{\mathrm{LO}} = \mathcal{M}_{\mathrm{df},3}^{\mathrm{LO}} \equiv \mathcal{M}_{3}^{\mathrm{LO}} - \mathcal{D}^{\mathrm{LO}}$$

$\mathscr{M}_3^{\mathrm{LO}} =$ 

+ relabelings & reorderings

$$\mathcal{D}^{(a)} = -\mathcal{M}_2^{(2),\mathrm{on}}(p_1, p_{1'}) \frac{1}{b_{(a)}^2 - m_1^2 + i\epsilon} \mathcal{M}_2^{(1),\mathrm{on}}(k_{1'}, k_2)$$ [and similarly for (b)]

- Results for individual types of diagram include all four allowed forms at $\mathscr{O}(\Delta)$

$$F^4 \mathcal{M}_{\mathrm{df},3}^{(a),\mathrm{all}} = \frac{M^2}{18}\left(6\Delta + 3\Delta_2^S - 2\tilde{t}_{22}\right) + \frac{4}{3}(m_1 m_2 + m_1^2)\,,$$
$$F^4 \mathcal{M}_{\mathrm{df},3}^{(b),\mathrm{all}} = \frac{M^2}{36}\left(12\Delta - 5\Delta_2^S + \tilde{t}_{22}\right) + \frac{4}{3} m_1 m_2\,.$$
$$F^4 \mathcal{M}_3^{(c)} = \frac{1}{3} M^2 \Delta - \frac{1}{36} M^2 \Delta_2^S + \frac{1}{12} M^2 \tilde{t}_{22} + \frac{2}{3}\left(2 m_1 m_2 + m_1^2\right)$$

(For $\pi^+\pi^+K^+$, masses are $m_1 = M_\pi$, $m_2 = M_K$, and vice versa; while $F = F_\pi$ or F_K; $M = 2m_1 + m_2$)

- But total includes only isotropic terms (for reasons we don't understand)

$$F^4 \mathscr{M}_{\mathrm{df},3}^{\mathrm{LO}} = 4 m_1 m_2 + 2 m_1^2 + M^2 \Delta$$

Summary for $\mathscr{K}_{\rm df,3}$

$$F^4 \mathscr{K}^{\rm LO}_{\rm df,3} = F^4 \mathscr{M}^{\rm LO}_{\rm df,3} = 4m_1 m_2 + 2m_1^2 + M^2 \Delta$$

$$\Rightarrow \mathscr{K}^{\rm iso,0}_{\rm df,3} = \frac{4m_1 m_2 + 2m_1^2}{F^4} \text{ and } \mathscr{K}^{\rm iso,1}_{\rm df,3} = \frac{M^2}{F^4}$$

- Terms with nontrivial angular dependence are of NLO in ChPT, and thus suppressed by an additional power of m^2/F^2

$$M_\pi^2 \mathcal{K}^{B,E}_{\rm df,3}(\pi\pi K) = c_{BE} r_\pi^4 r_K^2 \,,$$

$$M_\pi^2 \mathcal{K}^{B,E}_{\rm df,3}(KK\pi) = c_{BE} r_K^4 r_\pi^2 \,.$$

$$r_\pi = \frac{M_\pi}{F_\pi} \quad \text{and} \quad r_K = \frac{M_K}{F_K}\,,$$

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Background

- Historically, the first step in the finite-volume formalism was the determination of the shift in the energy of the ground ("threshold") state of two particles with $\mathbf{P} = 0$

Scattering length for jk pair

$$\Delta E_2^{(i)} = E_2^{(i)} - m_j - m_k = \frac{2\pi a_0^{(i)}}{\mu_i L^3} + \mathcal{O}(L^{-4})$$

[Huang & Yang, 57; Lüscher, 86]

Use "i" to label jk pair, with 3-particle system in mind

$$\mu_i = \frac{m_j m_k}{m_j + m_k}$$

- [Smigielski & Wasem, 0811.4392 (PRD)] determined the first 3 terms in the expansion for systems with $n\pi^+ + mK^+$, using NREFT

- Using the RFT formalism, we have (re-)derived the result for 2+1 systems, and extended it to three nondegenerate particles
  - We find agreement with NREFT, which provides a nontrivial check of our formalism, since non-relativistic corrections do not enter until the 4th $(1/L^6)$ term
  - We can use the results to check our numerical implementations of 2+1 and 1+1+1 QC3s

Sketch of method

- Up to third order, can set $\mathscr{K}_{\mathrm{df},3} = 0$, simplifying the QC3

$$\det\left[\widehat{F}_3^{-1}(E,\boldsymbol{P},L) + \widehat{\mathcal{K}}_{\mathrm{df},3}(E^*)\right] = 0$$

$$\widehat{F}_3 = \frac{\widehat{F}}{3} - \widehat{F}\frac{1}{\widehat{\overline{\mathcal{K}}}_{2,L}^{-1} + \widehat{F} + \widehat{G}}\widehat{F}$$

$$\longrightarrow \quad \det\left(\hat{\overline{\mathscr{K}}}_{2,L}^{-1} + \hat{F} + \hat{G}\right) = 0$$

- Need to keep only $\ell = 0$ terms at threshold (s-wave scattering length), and can use the standard $1/L$ expansion for $\hat{F}$ & $\hat{G}$, plus some determinant tricks

Results

- For nondegenerate case, with masses m_1, m_2, m_3

$$\Delta E = E - m_1 - m_2 - m_3 \; = \frac{c_3}{L^3} + \frac{c_4}{L^4} + \frac{c_5}{L^5} + \mathcal{O}(L^{-6})$$

$$L^3 \Delta E = \sum_{i=1}^{3} \frac{2\pi a_0^{(i)}}{\mu_i} \left[1 - \frac{a_0^{(i)}}{\pi L} \mathcal{I} + \left(\frac{a_0^{(i)}}{\pi L} \right)^2 (\mathcal{I}^2 - \mathcal{J}) + \frac{a_0^{(j)} a_0^{(k)}}{(\pi L)^2} 2\mathcal{J} \right] + O(L^{-6})$$

Known numerical constants

Sum over pairs (determined cyclically)

Pair energy shift à la Lüscher

3-particle term

- Terms of the form $a_0^{(i)}[a_0^{(j)}]^2$ appear at intermediate stages but cancel in the end: why?
- Get correct result for 2+1 system by setting $m_2 = m_1$; identical particle factors cancel

Checking numerical implementation

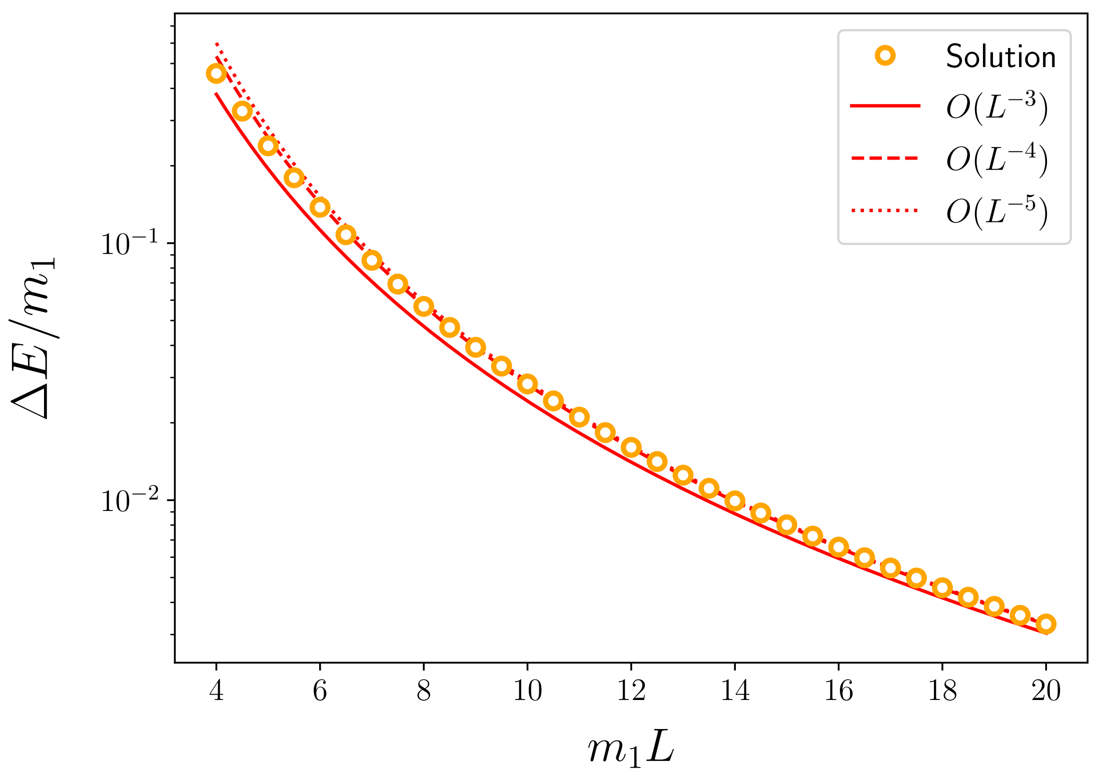
$$L^3 \Delta E^{\text{th}} = \sum_{i=1}^{3} \frac{2\pi a_0^{(i)}}{\mu_i} \left[1 - \frac{a_0^{(i)}}{\pi L} \mathcal{I} + \left(\frac{a_0^{(i)}}{\pi L} \right)^2 (\mathcal{I}^2 - \mathcal{J}) + \frac{a_0^{(j)} a_0^{(k)}}{(\pi L)^2} 2\mathcal{J} \right] + O(L^{-6})$$

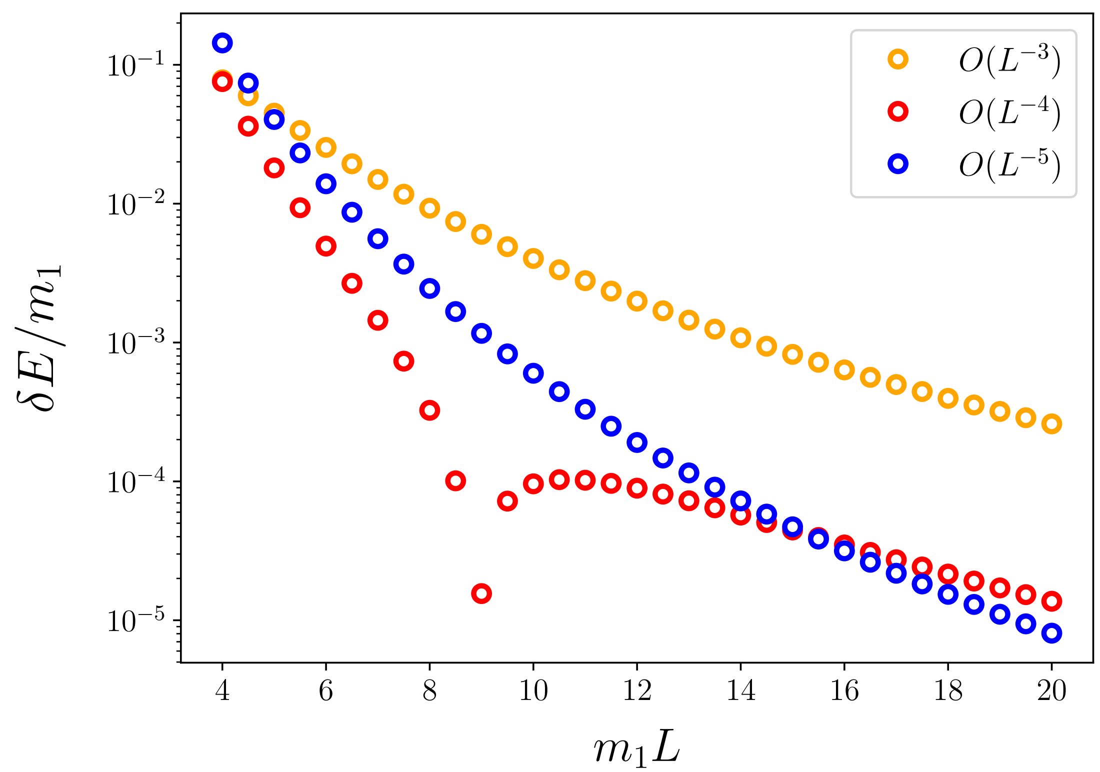
- Solve QC3 with significantly nondegenerate masses and scattering lengths (and $\mathcal{K}_{\text{df},3} = 0$)

$$\frac{m_2}{m_1} = 1.5, \quad \frac{m_3}{m_1} = 0.5\,. \qquad m_1 a_0^{(1)} = 0.7, \quad m_1 a_0^{(2)} = 0.5, \quad m_1 a_0^{(3)} = 0.3.$$

- Observe expected agreement at large L

$$\delta E \equiv |\, (\Delta E)^{\text{QC3}} - (\Delta E)^{\text{th}} \,|$$





Outline

- Summary of 2+1 QC3
- Cutoff/transition function for nondegenerate particles
- Threshold expansion for $\mathscr{K}_{\rm df,3}$
- $\mathscr{K}_{\rm df,3}$ in chiral perturbation theory
- Expansion of threshold energy in powers of 1/L
- Python implementation

Github repository

- Python codes available at https://github.com/ferolo2/QC3_release
  - Substantially debugged by cross-checking with Mathematica code
  - Both 2+1 (with $\ell_{\max} = 1$) and 1+1+1 (with $\ell_{\max} = 0$) available
  - We provide example results from running the code in our paper
  - Code is not optimized, nor parallelized, so serves only as a starting point for someone wishing to do serious fitting with the RFT formalism
    - Efficient implementation requires adding "numba" functionality
    - Fitting in practice requires medium-sized clusters
  - However, all the nasty algebra for implementing $\mathcal{K}_{df,3}$, and for doing the projections onto finite-volume irreps is included

Summary & Outlook

- Implementing 2+1 QC3 is straightforward generalization of that for identical particles
  - In practice, requires finding eigenvalues of larger matrix
  - We have provided a basic python implementation for 2+1 and 1+1+1 systems
  - Numerical results will be presented by Zack Draper
- 2+1 implementation remains valid for nonmaximal isospin
  - If there are resonances (2- and/or 3-particle), only forms of K matrices change
- Future theoretical work on 2+1 formalism
  - Solving integral equations (second step of formalism)
  - Calculate $\mathcal{K}_{\rm df,3}$ at NLO in ChPT

Any questions?

Backup slides

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

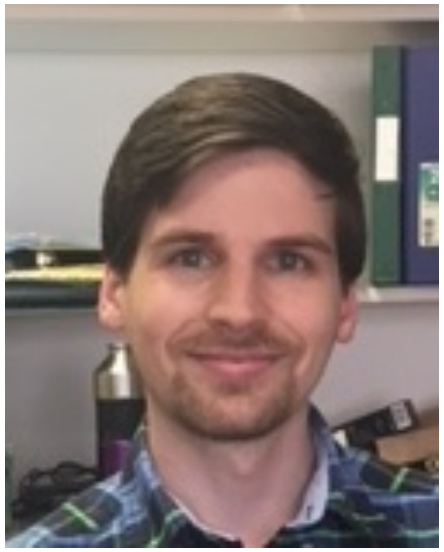
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

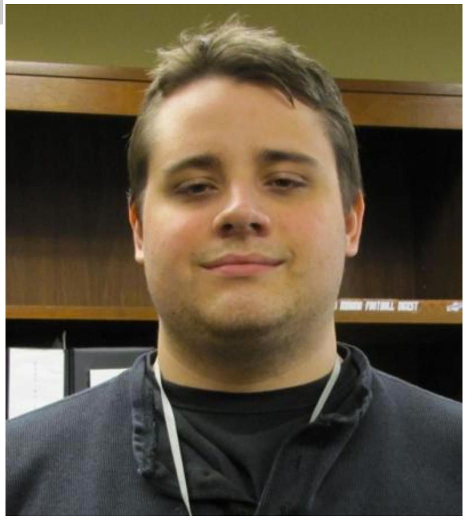
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)

Tyler Blanton & SRS:

“Alternative derivation of the relativistic three-particle quantization condition,” arXiv:2007.16188 (PRD) [BS20a]

“Equivalence of relativistic three-particle quantization conditions,” arXiv:2007.16190 (PRD) [BS20b]

“Relativistic three-particle quantization condition for nondegenerate scalars,” arXiv:2011.05520 (PRD)

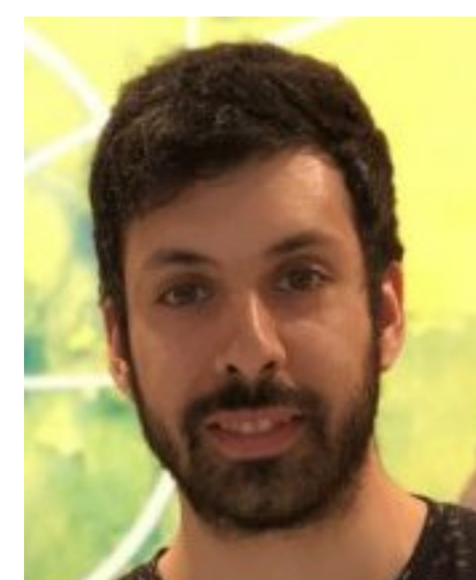
“Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems,” arXiv:2105.12904 (PRD)

Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS

“$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,” arXiv:2106.05590 (JHEP)

“$\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ interactions from lattice QCD,” in progress



Other work

★ Implementing RFT integral equations

- A. Jackura et al., 2010.09820 [Solving s-wave RFT integral equations in presence of bound states]
- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]

★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ Numerical simulations in scalar theories

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, [2- & 3-body interactions in φ^4 theory]

Other work

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807, JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP, [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
- J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, 2205.11316, [Resonance form factors from finite-volu correlation functions with the external field method]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749,PRD [applying FVU approach to $3\pi^+$spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144 [determining $3\pi^+$ interaction from LQCD spectrum]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]