Relativistic Invariance of the NREFT Three-Particle Quantization Condition

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Outline

- Introduction
- Problems in non-covariant framework
  - Parametrization of interactions
  - Non-covariance of three-particle propagator
- Relativistic invariant formulation
  - Construction
  - Three-particle quantization condition
- Test of invariance with synthetic data
- Conclusion and Outlook

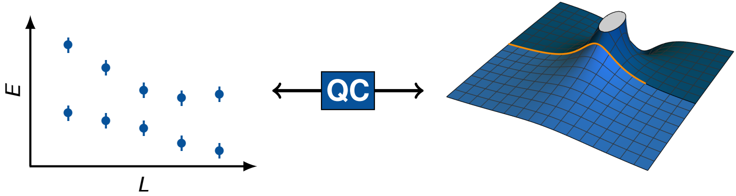
Introduction

finite volume (FV):

- two-, three-particle energy levels

infinite volume (IV):

- bound states, elastic scattering



- (NREFT) Quantization condition:
  FV energy spectrum $\longleftrightarrow$ IV observables
  - use QC to fit parameters of EFT Lagrangian to spectrum
  - use parameters to calculate IV observables
- Lorentz invariance broken in finite volume

Lorentz invariance constraints the form of interactions

Example: $2 \to 2$ scattering at tree level

underlying model: invariant

on shell 4-vector

$$\#dof. = 3 \times (2+2) - 10 = 2$$

Poincare inv.

$$\to \quad p, \cos\theta$$

$$K_r^{(2)} = -32\pi a_0 + \tfrac{1}{2}\tilde{r}_0 p^2 + \mathcal{O}(p^4)$$

NREFT: non-invariant

3-vector

$$\#dof. = 3 \times (2+2) - 6 = 6$$

total momentum cons. + rotation inv.

$$\to \quad \mathbf{P}^2, \mathbf{p}^2, \mathbf{q}^2, \mathbf{Pp}, \mathbf{Pq}, \mathbf{pq}$$

$$K_{nr}^{(2)} = c_0 + c_1(\mathbf{p}^2 + \mathbf{q}^2) + c_2\mathbf{P}^2 + \mathcal{O}(p^4)$$

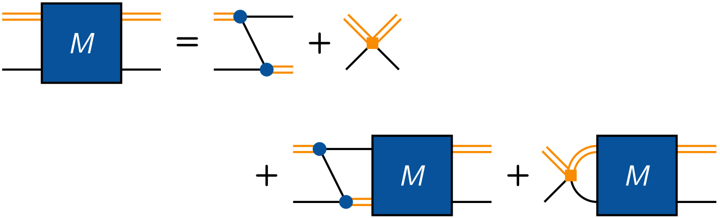
redundant / frame-depended parameters!

or inconvenient matching:

$$\prod_{i=1}^{4} \sqrt{2w(\mathbf{p}_i)}\, K_{nr}^{(2)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = K_r^{(2)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4)$$

non-covariant factors $\sqrt{2w(\mathbf{p}_i)}$ mixing different orders

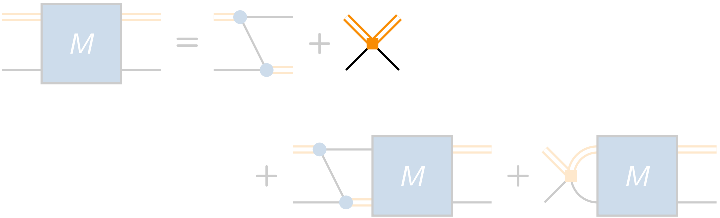
Particle-dimer amplitude obeys Faddeev equation:



$$M(p,q;P) = Z(p,q;P) + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(p,k;P)\tau(k;P)M(k,q;P)$$

Problems
Non-Covariance of Three-Particle Propagator

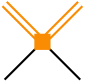
Particle-dimer amplitude obeys Faddeev equation:



$$M(p,q;P) = Z(p,q;P) + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(p,k;P)\tau(k;P)M(k,q;P)$$

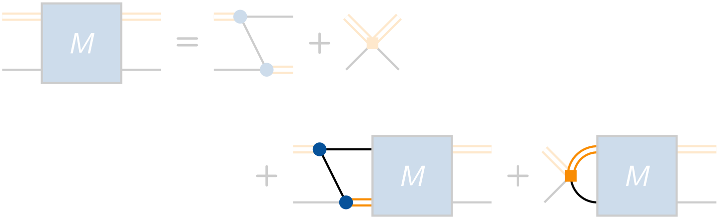
Kernel Z contains three-particle short-range interaction:

can be parametrized Lorentz invariant



$$\sim K^{(3)} = h_0 + h_1(P^2 - 9m^2) + \dots$$

Particle-dimer amplitude obeys Faddeev equation:



$$M(p,q;P) = Z(p,q;P) + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(p,k;P)\tau(k;P)M(k,q;P)$$

Dimer propagator τ is Lorentz invariant two-particle amplitude

 $\sim T^{(2)}$

Problems

Non-Covariance of Three-Particle Propagator

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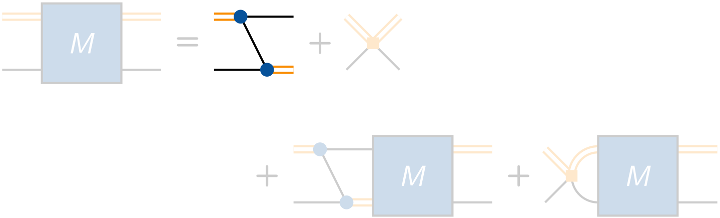
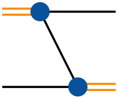


$$M(p,q;P) = Z(p,q;P) + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(p,k;P)\tau(k;P)M(k,q;P)$$

Kernel Z contains three-particle propagator:

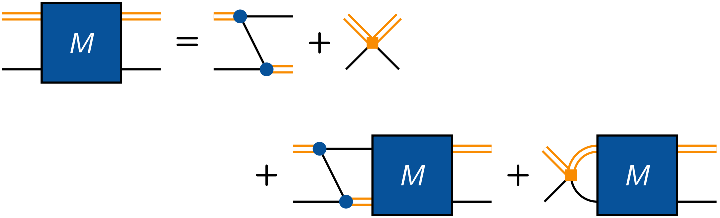
not Lorentz invariant



$$\sim \frac{1}{2w(\mathbf{P}-\mathbf{p}-\mathbf{q})(w(\mathbf{P}-\mathbf{p}-\mathbf{q})+w(\mathbf{p})+w(\mathbf{q})-P^0-i\varepsilon)}$$

Problems
Non-Covariance of Three-Particle Propagator

Particle-dimer amplitude obeys Faddeev equation:



$$M(p,q;P) = Z(p,q;P) + \int \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(p,k;P)\tau(k;P)M(k,q;P)$$

Amplitude M is not Lorentz invariant!

$\Rightarrow$ IV observables not the same in different frames

Attempt:

add *anti-particle-contribution*

$$\frac{1}{2w(\mathbf{K})(-w(\mathbf{K})+w(\mathbf{p})+w(\mathbf{q})-P^0)}$$

$$\frac{1}{2w(\mathbf{K})(w(\mathbf{K})+w(\mathbf{p})+w(\mathbf{q})-P^0)} \quad \rightarrow \quad \frac{1}{K^2-m^2}$$

$K^\mu = (P-p-q)^\mu$

Problem:

additional poles in *anti-particle-contribution*

$\Rightarrow$ loss of decoupling of low- and high-momentum regimes

$\Rightarrow$ breaking of unitarity (IV) & spurious subthreshold poles (FV)

cure by adjusting cutoff $\Rightarrow$ cutoff not removable anymore

we need:

- Lorentz-invariant matching condition → covariant NREFT

$$\phi^\dagger(i\partial_t - M + \ldots)\phi \;\to\; \phi^\dagger 2w(i\partial_t - w)\phi\,,\;\; w = \sqrt{m^2 - \nabla^2}$$

new normalization → cancels $(2w(\mathbf{p}_i))^{1/2}$ in matching

- formally invariant propagator → arbitrary frame v^μ

$$\phi^\dagger 2w(i\partial_t - w)\phi \to \phi^\dagger 2w_v(i(v\cdot\partial) - w_v)\phi\,,\;\; w_v = \sqrt{m^2 + \partial_\perp^2}$$

$$\partial_\perp^\mu = \partial^\mu - v^\mu(v\cdot\partial)$$

$$D_v(k) = \frac{1}{2w_v(k)(w_v(k) - v\cdot k - i\varepsilon)}\,,\;\; w_v(k) = \sqrt{m^2 - k^2 + (v\cdot k)^2}$$

- propagator formally Lorentz-invariant
- antiparticle pole absent (integrated out in EFT)

NREFT:

particle number conservation $\Rightarrow$ separation of n-particle sectors

Modified matching:

$$K_{nr}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = K_r(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4) = -32\pi a_0 + \frac{1}{2}\tilde{r}_0 p^2 + \ldots$$

Two-particle sector: (tree level)

$$\mathcal{L}^{(2)} = C_0 \phi^\dagger \phi^\dagger \phi\phi + C_2 \left[(w_\mu \phi)^\dagger (w^\mu \phi)^\dagger \phi\phi - m^2 \phi^\dagger \phi^\dagger \phi\phi + \text{h.c.} \right] + \ldots$$

$$w^\mu = v^\mu w_v + i\partial_\perp^\mu$$

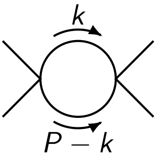
for on-shell momenta:

$$w_v(k) = \sqrt{m^2 - k^2 + (v \cdot k)^2} = (v \cdot k)$$

$$\Rightarrow \tilde{k}^\mu = w^\mu(k) = v^\mu w_v(k) + k_\perp^\mu = v^\mu (v \cdot k) + k_\perp^\mu = k^\mu$$

$$\Rightarrow K_{nr}^{(2)} = 4C_0 + 4C_2(s - 4m^2) + \ldots \quad \longleftrightarrow \quad -32\pi a_0 + \frac{1}{2}\tilde{r}_0 p^2$$

Two-particle loop:

$I(P) =$ 

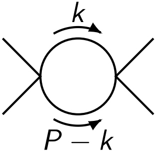
$$I(P) = \int \frac{d^4k}{(2\pi)^4 i} \frac{1}{2w_v(k)(w_v(k) - vk)} \frac{1}{2w_v(P-k)(w_v(P-k) - v(P-k))}$$

hard scale m in integrand $\rightarrow$ change renorm. prescription

Threshold expansion:

$$\frac{1}{2w_v(k)(w_v(k) - vk)} = \frac{1}{m^2 - k^2} - \overbrace{\frac{1}{2w_v(k)(w_v(k) + vk)}}^{\text{expand in } k_\perp^\mu \text{ at } vk = w_v(k)}$$

Two-particle loop:

$I(P) =$ 

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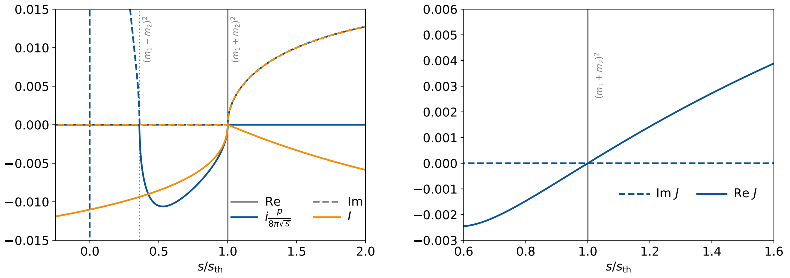
hard scale m in integrand $\rightarrow$ change renorm. prescription

Threshold expansion:

$$I(P) = \int \frac{d^4k}{(2\pi)^4 i} \left(\frac{1}{m^2 - k^2} \frac{1}{m^2 - (P-k)^2} + \cancel{\text{low energy polynomial}} \right)$$

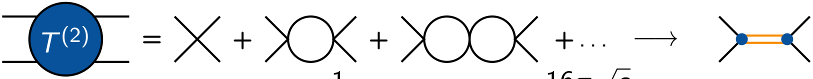
$$= J(s) + \frac{ip(s)}{8\pi\sqrt{s}}$$



- imaginary parts above threshold coincide (unitarity!)
- $\frac{ip}{8\pi\sqrt{s}}$: left-hand cut from $s = (\Delta m)^2$ & pole at $s = 0$
- $I(s)$: real and analytic in sub-threshold region
- $J(s)$: analytic and real around threshold

$\frac{ip(s)}{8\pi\sqrt{s}} \to J(s) + \frac{ip(s)}{8\pi\sqrt{s}}$: different renorm. prescirpition

Two-particle scattering:



$$T_S^{(2)}(P) = \frac{1}{(K_S^{(2)})^{-1} - I/2} = \frac{16\pi\sqrt{s}}{p\cot\delta_0(s) - ip(s)}$$

$= 16\pi\sqrt{s}((K_S^{(2)})^{-1} - J(s)/2)$

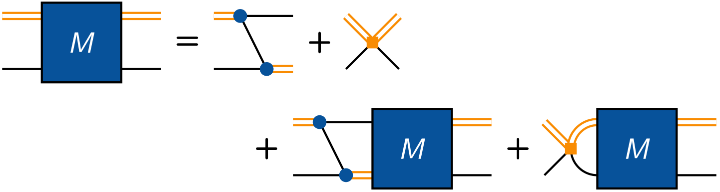
v^μ-indep. function of $s = P^2$ only

Introduction of dimers:

describe two-particle sector introducing dimer field T

- for S-wave: $T \sim \phi\phi$
  $$\mathcal{L}_S^{(2)} = T^\dagger \left[f_0\phi\phi + f_2\left((w_\mu\phi)(w^\mu\phi) - m^2\,\phi\phi \right) + \dots \right] + \text{h.c.}$$
- For higher partial waves:
  $$\mathcal{L}_L^{(2)} = T^\dagger_{\mu_1,\dots,\mu_L} O^{\mu_1,\dots,\mu_L} + \text{h.c.}$$
  rel cov. two-particle operators in L-rep. of SO(3)

Particle-dimer scattering:



$\Lambda^2 + k^2 - (vk)^2 \geq 0$

$$M(p,q) = Z(p,q) + \int^{\Lambda_v(k)} \frac{d^4k}{(2\pi)^4 i} \, Z(p,k) \frac{\tau((P-k)^2)}{2w_v(k)(w_v(k) - vk)} M(k,q)$$

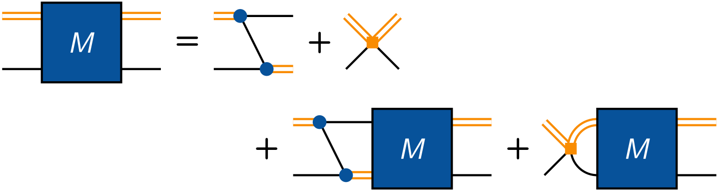
Lorentz invariance:

boost into the frame where $vk = v'k' = k_0'$, $w_v(k) = w(\mathbf{k}')$

$$M(p,q) = Z(p,q) + \int^{\Lambda_v(k)} \frac{d^3\mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} \, Z(p,k)\tau((P-k)^2)M(k,q)$$

$k^2 = k'^2 = m^2$

Particle-dimer scattering:



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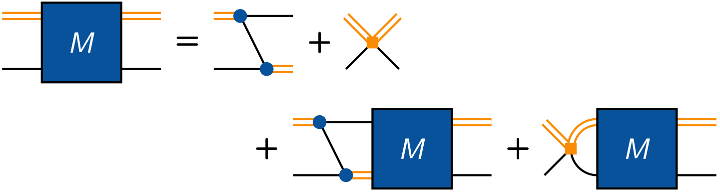
Particle-dimer scattering:



$\Lambda^2 + k^2 - (vk)^2 \geq 0$

$$M(p,q) = Z(p,q) + \int^{\Lambda_v(k)} \frac{d^4 k'}{(2\pi)^4 i} \, Z(p,k) \frac{\tau((P-k)^2)}{2w(\mathbf{k}')(w(\mathbf{k}') - k'_0)} M(k,q)$$

Lorentz invariance:

boost into the frame where $vk = v'k' = k'_0$, $w_v(k) = w(\mathbf{k}')$

$$M(p,q) = Z(p,q) + \int^{\Lambda_v(k)} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} \, Z(p,k)\tau((P-k)^2) M(k,q)$$

$k^2 = k'^2 = m^2$

$$\int^{\Lambda_v(k)} \frac{d^3\mathbf{k}'}{(2\pi)^3 2w(\mathbf{k}')} \longrightarrow \int^{\Lambda_v(k)} \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})}$$

Perform discretization in the lab frame and **set** $v^\mu = P^\mu$:

$$\Lambda^2 + m^2 - (vk)^2 \geq 0$$

$$M_L(p,q) = Z(p,q) + \frac{1}{L^3} \sum_{\mathbf{k}}^{\Lambda_v} \frac{1}{2w(\mathbf{k})} Z(p,k)\tau_L((P-k)^2) M_L(k,q)$$

Quantization Condition

$$\det(A) = 0$$

$$A_{\mathbf{pq}} = L^3 2w(\mathbf{p})\delta_{\mathbf{pq}}\tau_L^{-1}((P-p)^2) - Z(p,q)$$

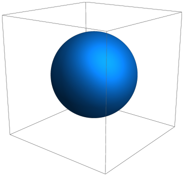
p, q on-shell and obey $\Lambda^2 + m^2 - (vk)^2 \geq 0$

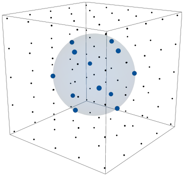
Partial diagonalization:

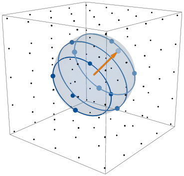
$$A_{\mathbf{pq}} = L^3 2w(\mathbf{p})\delta_{\mathbf{pq}}\tau_L^{-1}((P-p)^2) - Z(p,q;P)$$

IV	$\longrightarrow$	FV	$\longrightarrow$	moving frame
O(3)	$\longrightarrow$	O_h	$\longrightarrow$	$g \in O_h : g\mathbf{P} = \mathbf{P}$

shells:







similar to partial wave expansion:

$$A^{\Gamma}_{\rho\sigma}(r,s) = \delta_{\rho\sigma}\delta_{rs}L^3 2w(s)\tau_L^{-1}(s) - \frac{\sqrt{\nu(r)\nu(s)}}{G} Z^{\Gamma}_{\rho\sigma}(r,s)$$

$$Z^{\Gamma}_{\rho\sigma}(r,s) = \sum_g (T^{\Gamma}_{\sigma\rho}(g))^* Z(g\,p_0(r), q_0(s))$$

Test of Invariance

Setup:

consider EFT at LO in S-wave $\Rightarrow$ 2 LECs:

- $\tau(s) = \dfrac{16\pi\sqrt{s}}{-a^{-1} - 8\pi\sqrt{s}J(s) - ip}$ (FV: $ip \to \frac{2}{\sqrt{\pi}L\gamma} Z^{\mathbf{d}}_{00}(1; q_0^2)$)
- $K^{(3)} = \dfrac{H_0(\Lambda)}{\Lambda^2} \subset Z(p, k)$

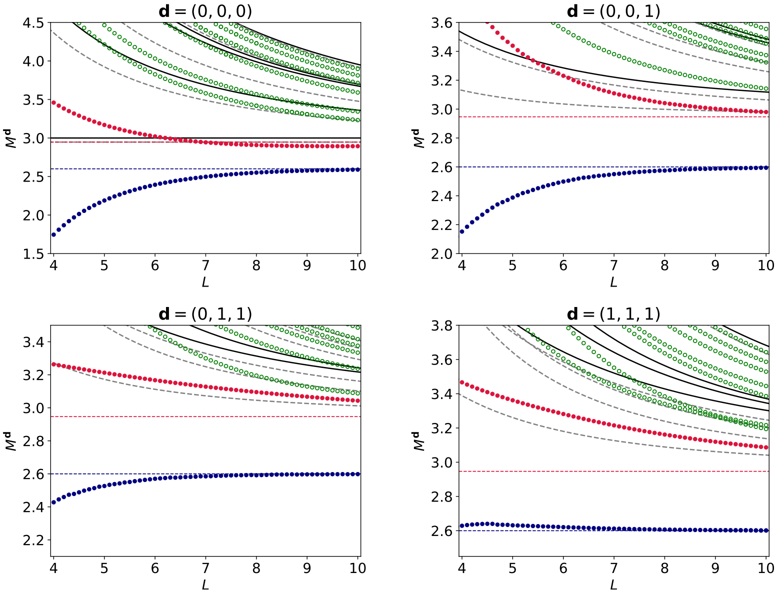
Procedure:

- IV ($m = 1$) in lab frame only:
  - fix $\Lambda = 3$ and $a = 5$
  - fix bound state at $E_1 = 2.6 \Rightarrow H_0 = -0.118$
  - find shallow bound state at $E_2 = 2.9467$ ($E_{PD} = 2.9473$)
- FV:

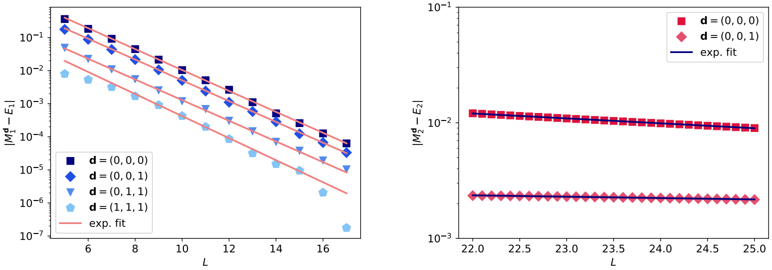
  Calculate spectra in various frames $\mathbf{P} = 2\pi/L\mathbf{d}$

exp. decay of bound state energy $\Rightarrow$ Lorentz invariance

Test of Invariance



Test of Invariance



- deep bound state:
  exponential decay with same slope
- shallow bound state:
  exponential decay but different slope

slope not fixed (Lorentz invariance broken in a finite volume)

Conclusion and Outlook

Conclusion:

- proposed a manifestly relativistically invariant formulation:
  - no frame depended three-particle couplings
    $\Rightarrow$ allows to analyze data from different moving frames
  - IV observables invariant
- no violation of unitarity or spurious poles
- tested invariance using a toy model

Outlook:

- relativistically invariant formulation of three particle decays:
  three particle Lellouche-Lüscher equation at higher orders