Resonance form factors from finite-volume correlation functions with the external field method

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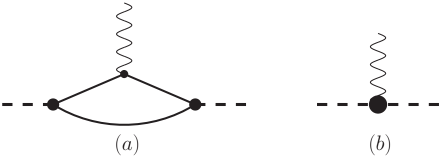
Motivation

- The straightforward analytic continuation of resonance form factors (RFF) is hindered by the presence of the triangle diagram.
- Subtracting this contribution turns the extraction of the RFF into a challenging task. D. Hoja et al. 2010, V. Bernard et al. 2012, R. Briceño and M.T. Hansen 2016, A. Baroni 2019, R. Briceño et al. 2020. Talks by Ortega-Gama, Briceño, ... at LATTICE22



a) The triangle diagram, and b) contact term

Motivation

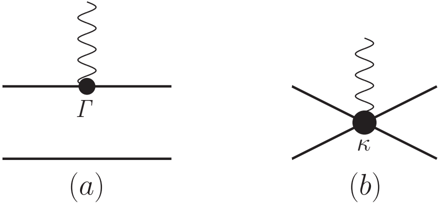
- Recently, the Feynman-Hellmann theorem has been used to extract form factors of stable particles in the Breit frame.

  A.J. Chambers et al. 2017, Briceño et al. 2020

- A natural question is: Can we extend this to composite particles?

- Aim: Calculate the form factor of a resonance using lattice QCD.
- Standard method: Measure the three-point function on the lattice.
- Our proposal: A method for the extraction of RFF by using a generalization of the Lüscher's method in the presence of an external source is presented.

Our proposal



Advantages:

- It suffices to determine the local contribution, proportional to κ.
- Finite-volume corrections in κ are suppressed.
- The subtraction of the triangle diagram is not required.

The findings

- Our two main results of this project are:

$\longrightarrow$ Lüscher equation in an external field:

$$\boxed{\det\left(X^{-1} - \frac{1}{2}\Pi\right) = 0}, \tag{1}$$

where X^{-1} is a counterpart of the inverse K-matrix, $p\cot\delta(p)$, and the loop function Π is the Lüscher zeta-function with an external source.

$\longrightarrow$ Generalized Feynman-Hellmann theorem for the case of resonances:

$$\boxed{\left.\frac{dP_R^0(e)}{de}\right|_{e=0} \propto F}, \tag{2}$$

where P_R^0 is the complex pole position, e the coupling of the external field and F the form factor.

Framework

What are the tools required for this derivation?

- The system is placed in a spatially periodic external field.
  $\rightarrow$Inject momentum transfer in the form factor.
- We make use of a non-relativistic effective field theory.
- Include only linear terms in the coupling constant e of the external field:

$$\begin{aligned}\mathcal{L} =& \phi^\dagger\left(i\partial_t - m + eA^0 + \frac{eC_R}{6m^2}\triangle A^0 + \frac{\nabla^2}{2m}\right)\phi + C_0\phi^\dagger\phi^\dagger\phi\phi \\ &+ C_2\left(\phi^\dagger\phi^\dagger(\phi\overset{\leftrightarrow}{\nabla}{}^2\phi) + \text{h.c.}\right) + \frac{e\kappa}{4}\phi^\dagger\phi^\dagger\phi\phi\triangle A^0 \\ &+ \quad \text{higher order terms with derivative couplings}\end{aligned}$$

Two-particle scattering amplitude

- The two-particle scattering amplitude for the process $q_1 + q_2 \to p_1 + p_2$ at $e = 0$:

$$T(q_0) = \frac{8\pi/m}{K^{-1}(q_0, q_0) - iq_0} = \frac{8\pi/m}{-1/a + rq_0^2/2 + \cdots - iq_0} \tag{3}$$

$q_0^2 = \mathbf{p}^2 = \mathbf{q}^2$, $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$, $\mathbf{q} = \frac{1}{2}(\mathbf{q}_1 - \mathbf{q}_2)$.

- Only S-wave is considered.
- The resonance pole position is determined from the equation:

$$-\frac{1}{a} + \frac{1}{2}rq_R^2 - \sqrt{-q_R^2} = 0. \tag{4}$$

Form factor of a single particle

- The two-point function up to $O(e)$:

$$S(\mathbf{p},\mathbf{q};p^0) = i\int dt\, d^3\mathbf{x} d^3\mathbf{y}\, e^{ip^0 t - i\mathbf{p}\mathbf{x} + i\mathbf{q}\mathbf{y}} \langle 0|\, T\phi(\mathbf{x},t)\phi^\dagger(\mathbf{y},0)|0\rangle$$
$$= \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}{m+\frac{\mathbf{p}^2}{2m}-p^0} + \frac{e\,\boxed{\Gamma(\mathbf{p},\mathbf{q})}\,\tilde{A}^0(\mathbf{p}-\mathbf{q})}{\left(m+\frac{\mathbf{p}^2}{2m}-p^0\right)\left(m+\frac{\mathbf{q}^2}{2m}-p^0\right)} + O(e^2)\,, \tag{5}$$

where

$$\tilde{A}^0(\mathbf{p}-\mathbf{q}) = \int d^3\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}} A^0(\mathbf{x}). \tag{6}$$

- The one-particle form factor:

$$\boxed{\Gamma(\mathbf{p},\mathbf{q})} = 1 - \frac{C_R}{6m^2}(\mathbf{p}-\mathbf{q})^2 \tag{7}$$

The Resonance Form Factor I

- The four-point function in the external field:

$$\tilde{G}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) = i\int dt\, d^3\mathbf{x}_1 d^3\mathbf{x}_2 d^3\mathbf{y}_1 d^3\mathbf{y}_2\, e^{iP^0t - i\mathbf{p}_1\mathbf{x}_1 - i\mathbf{p}_2\mathbf{x}_2 + i\mathbf{q}_1\mathbf{y}_1 + i\mathbf{q}_2\mathbf{y}_2}$$
$$\times \langle 0|T\phi(\mathbf{x}_1,t)\phi(\mathbf{x}_2,t)\phi^\dagger(\mathbf{y}_1,0)\phi^\dagger(\mathbf{y}_2,0)|0\rangle\,. \qquad (8)$$

- The RFF can be defined through the expansion of the four-point function, $\tilde{G}$, in the external field.

$$\tilde{G} = \tilde{G}_0 + e\underbrace{\tilde{G}_1}_{RFF} + O(e^2).$$

The Resonance Form Factor II

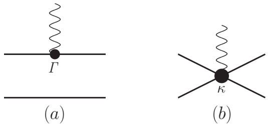
- At order $O(e^0)$:

$$\tilde{G}_0(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) \to (2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q})\frac{\Psi(\mathbf{P},\mathbf{p})\bar{\Psi}(\mathbf{Q},\mathbf{q})}{P^0-P^0_R} \tag{9}$$

where

$$\Psi(\mathbf{P},\mathbf{p}) = \frac{1}{2m+\frac{\mathbf{P}^2}{4m}+\frac{\mathbf{p}^2}{m}-P_0}\sqrt{\frac{Z}{m}\frac{K(p,q_R)}{K(q_R,q_R)}}\,,$$

$$\bar{\Psi}(\mathbf{Q},\mathbf{q}) = \sqrt{\frac{Z}{m}\frac{K(q,q_R)}{K(q_R,q_R)}}\,\frac{1}{2m+\frac{\mathbf{Q}^2}{4m}+\frac{\mathbf{q}^2}{m}-P_0}\,. \tag{10}$$



- At order e:

$$\tilde{G}_1(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) = \frac{1}{(2!)^2}\int \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{d^3\mathbf{P}'}{(2\pi)^3}\frac{d^3\mathbf{q}'}{(2\pi)^3}\frac{d^3\mathbf{Q}'}{(2\pi)^3}\tilde{G}_0(\mathbf{p},\mathbf{P};\mathbf{p}',\mathbf{P}';P^0)$$

$$\times \boxed{\tilde{\Gamma}(\mathbf{p}',\mathbf{P}';\mathbf{q}',\mathbf{Q}')}\,\tilde{G}_0(\mathbf{q}',\mathbf{Q}';\mathbf{q},\mathbf{Q};P^0)\,. \qquad (11)$$

- $\tilde{\Gamma}$ is a sum of the diagrams *(a)* and *(b)*:

$$\boxed{\tilde{\Gamma}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q})} = \bar{\Gamma}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q})\tilde{A}^0(\mathbf{P}-\mathbf{Q})\,, \qquad (12)$$

The Resonance Form Factor IV

- The vertex:

$$\bar{\Gamma}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q}) = -\boxed{\kappa}(\mathbf{P}-\mathbf{Q})^2$$
$$+\left\{(2\pi)^3\delta^3\left(\frac{(\mathbf{P}-\mathbf{Q})}{2}+(\mathbf{p}-\mathbf{q})\right)\boxed{\Gamma}\left(\frac{\mathbf{P}}{2}-\mathbf{p},\frac{\mathbf{Q}}{2}-\mathbf{q}\right)+\left(\begin{array}{c}\mathbf{p}\to-\mathbf{p}\\ \mathbf{q}\to-\mathbf{q}\\ \mathbf{p}\to-\mathbf{p},\ \mathbf{q}\to-\mathbf{q}\end{array}\right)\right\} \tag{13}$$

- The quantity $\tilde{G}_1$ has a double pole in P^0 and Q^0. The residue at the double pole defines the RFF:

$$\boxed{F(\mathbf{P},\mathbf{Q}) = \frac{1}{(2!)^2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\bar{\Psi}(\mathbf{P},\mathbf{p})\bar{\Gamma}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q})\Psi(\mathbf{Q},\mathbf{q})},$$

- Energy is fixed at the resonance pole.
- Only unknown parameter: κ (at leading order).

Single Particle in an External Field I

- Consider a single particle in an spatially periodic field:

$$A^0(\mathbf{x}) = A_0 \cos(\omega \mathbf{x}), \quad \omega = (0, 0, \omega), \quad \mathbf{x} = (\mathbf{x}_\perp, x_\parallel) \tag{14}$$

- Quantization of the external field: $\omega = \frac{2\pi}{L} N$
- E.O.M (Mathieu Equation):

$$\left(i\partial_t + e\Gamma A_0 \cos(\omega x_\parallel) - m + \frac{\nabla^2}{2m} \right) \Phi(\mathbf{x}, t) = 0 \,, \tag{15}$$

where

$$\Gamma = \Gamma(\boldsymbol{\omega}) = 1 - \frac{C_R}{6m^2} \omega^2. \tag{16}$$

Single Particle in an External Field II

- The solutions of the e.o.m:

$$\Phi(\mathbf{x}, t) = e^{-iEt + i\mathbf{p}_\perp \mathbf{x}_\perp} \boxed{\mathrm{me}_{\nu_i+2n}(z, q)}\,,$$

$$z = \frac{\omega x_\parallel}{2}\,, \qquad q = -\frac{4me\Gamma A_0}{\omega^2}\,. \tag{17}$$

- $\mathrm{me}_{\nu_i+2n}(z, q)$ are the Mathieu functions.
- Eigenvalues: $\lambda_{\nu_i+2n}(q)$, $n \in \mathbb{Z}$, $i = 1, \cdots, N$.
- Full propagator in the external field:

$$S(\mathbf{x}, \mathbf{y}; E) = \frac{1}{L^3} \sum_{\mathbf{p}_\perp} \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp - \mathbf{y}_\perp)}}{m + \frac{\mathbf{p}_\perp^2}{2m} + \frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q) - E} \times \mathrm{me}_{\nu_i+2n}\left(\frac{\omega x_\parallel}{2}, q\right) \mathrm{me}_{\nu_i+2n}\left(-\frac{\omega y_\parallel}{2}, q\right). \tag{18}$$

Energy shift in the Periodic Field I

- The spectrum of the particle in the external field is determined by the poles of the propagator.
- Propagator in momentum space:

$$S(\mathbf{p},\mathbf{q};E) = L^3 \delta^2_{\mathbf{p}_\perp,\mathbf{q}_\perp} \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} \sum_{a,b=-\infty}^{\infty} C_{2a}^{\nu_i+2n}(q) C_{2b}^{\nu_i+2n}(q)$$

$$\times \frac{\delta_{-p_\parallel,\frac{\omega}{2}(\nu_i+2n+2a)} \delta_{-q_\parallel,\frac{\omega}{2}(\nu_i+2n+2b)}}{m + \frac{\mathbf{q}_\perp^2}{2m} + \frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q) - E} . \tag{19}$$

- The propagator contains a tower of poles, which are determined by

$$E = m + \frac{\mathbf{p}_\perp^2}{2m} + \frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q) . \tag{20}$$

Energy shift in the Periodic Field II

- $\lambda_{\nu_i+2n}(q) = (\nu_i + 2n)^2 + O(q^2)$ for all values of $\nu_i + 2n$ except $(\nu_i + 2n) = \pm 1$.
  $\hookrightarrow$ $\boxed{\lambda_1(q) = 1 + q + O(q^2)\,, \qquad \lambda_{-1}(q) = 1 - q + O(q^2)\,.}$
- PBC in the Mathieu function:
  $\hookrightarrow$ $e^{i\nu\pi N} = 1$ and $-1 < \nu \leq 1$.
- Examine different N.

$$N = 1: \quad \nu = 0 \longrightarrow \text{Energy shift} = O(e^2)$$
$$N = 2: \quad \nu = 0, 1 \longrightarrow \text{Energy shift} = O(e) : \text{Breit frame}$$
$$\text{and so on} \cdots$$

Energy shift in the Periodic Field III

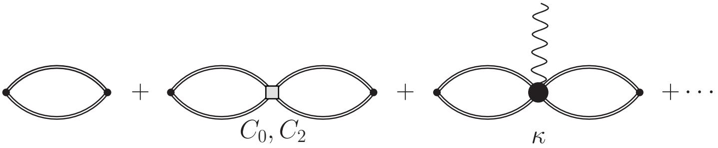
- The lowest energy level:

$$E = m + \frac{\omega^2 \lambda_1(q)}{8m} = m + \frac{\omega^2}{8m} - \frac{1}{2}\, eA_0 \Gamma\,. \tag{21}$$

- Differentiating the pole shift with respect to e, one obtains the particle form factor $\Gamma(\omega)$.
- The result of A.J. Chambers et al. 2017 are confirmed and extended.

- Up to now:
  - The RFF was derived from the four-point function.
    $\rightarrow$ Only unknown parameter: κ (at lowest order).
  - The periodic solutions of the E.O.M were introduced
    $\rightarrow$ Mathieu functions.
  - The energy levels of a single particle in the external field were discussed.
    $\rightarrow$ The form factor was derived by taking the derivative of the pole shift.
- Next:
  - Derive the Lüscher equation in an external field.
  - Find κ.
  - Determine the pole position in the complex plane.
  - Verify the Feynman-Hellmann theorem for resonances.

Lüscher equation I



- Two-point function of the composite field

$$D(\mathbf{P},\mathbf{Q};t) = i\int^L d^3\mathbf{x}d^3\mathbf{y}e^{-i\mathbf{P}\mathbf{x}+i\mathbf{Q}\mathbf{y}}\langle 0|T\phi^2(\mathbf{x},t)[\phi^\dagger(\mathbf{y},0)]^2|0\rangle\,.$$

- Note that:
  - The propagators are the full ones that include the summation of all external field insertions.
  - There are vertices with the external field attached (proportional to κ).
  - Three-momentum is not conserved in the external field:
    $\rightarrow$ Two-point function no longer diagonal in **P** and **Q**.

Lüscher equation II

The two-point function obeys the equation

$$\frac{1}{4}\, D_{\mathbf{PQ}}(E) = \frac{1}{2}\, \Pi_{\mathbf{PQ}}(E) + \frac{1}{L^6} \sum_{\mathbf{P}'\mathbf{Q}'} \frac{1}{2}\, \Pi_{\mathbf{PP}'}(E) X_{\mathbf{P}'\mathbf{Q}'}(E) \frac{1}{4}\, D_{\mathbf{Q}'\mathbf{Q}}(E)\,, \quad (22)$$

where (if no derivative couplings present)

$$\Pi(\mathbf{P}, \mathbf{Q}; E) = \int \frac{dp^0}{2\pi i} \int^L d^3\mathbf{x} d^3\mathbf{y}\, e^{-i\mathbf{Px}+i\mathbf{Qy}} S(\mathbf{x}, \mathbf{y}; p^0) S(\mathbf{x}, \mathbf{y}; E - p^0)\,. \quad (23)$$

Lüscher equation III

- The kernel X

$$X_{\mathbf{PQ}}(E) = L^3\delta^3_{\mathbf{PQ}}\boxed{X^{(0)}_{\mathbf{P}}(E)} + \frac{e}{2}L^3(\delta^3_{\mathbf{P}+\boldsymbol{\omega},\mathbf{Q}} + \delta^3_{\mathbf{P}-\boldsymbol{\omega},\mathbf{Q}})\boxed{X^{(1)}_{\mathbf{PQ}}(E)} + O(e^2)\,, \tag{24}$$

where

$$\boxed{X^{(0)}_{\mathbf{P}}(E)} = 4C_0 - 8mC_2\left(E - 2m - \frac{\mathbf{P}^2}{4m}\right) + \cdots\,,$$
$$\boxed{X^{(1)}_{\mathbf{PQ}}(E)} = -\kappa\omega^2 A_0 - 16mC_2\Gamma A_0 + \cdots\,. \tag{25}$$

- The energy levels are determined by the equation

$$\det \mathcal{M} = 0\,, \qquad \boxed{\mathcal{M}_{\mathbf{PQ}}(E) = [X_{\mathbf{PQ}}(E)]^{-1} - \frac{1}{2}\,\Pi_{\mathbf{PQ}}(E)}\,. \tag{26}$$

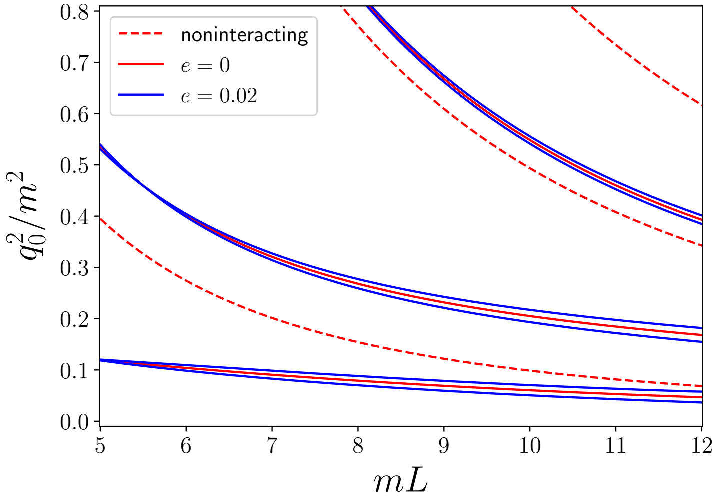
- At order e:

$$\begin{aligned}[X_{\mathbf{PQ}}(E)]^{-1} = & L^3 \delta^3_{\mathbf{PQ}} \boxed{k(\mathbf{P};E)} \\ & - \frac{e}{2} L^3 (\delta^3_{\mathbf{P}+\boldsymbol{\omega},\mathbf{Q}} + \delta^3_{\mathbf{P}-\boldsymbol{\omega},\mathbf{Q}}) \boxed{k(\mathbf{P};E)} X^{(1)}_{\mathbf{PQ}}(E) \boxed{k(\mathbf{Q};E)}\end{aligned} \tag{27}$$

where

$$\begin{aligned} k(\mathbf{P};E) = & \frac{m}{8\pi}\left(-\frac{1}{a} + \frac{1}{2}\, r q_0^2(\mathbf{P};E) + \cdots\right) = \frac{m}{8\pi} q_0 \cot\delta_0(q_0) + \cdots\,, \\ q_0^2(\mathbf{P};E) = & m\left(E - 2m - \frac{\mathbf{P}^2}{4m}\right). \end{aligned} \tag{28}$$

Energy levels



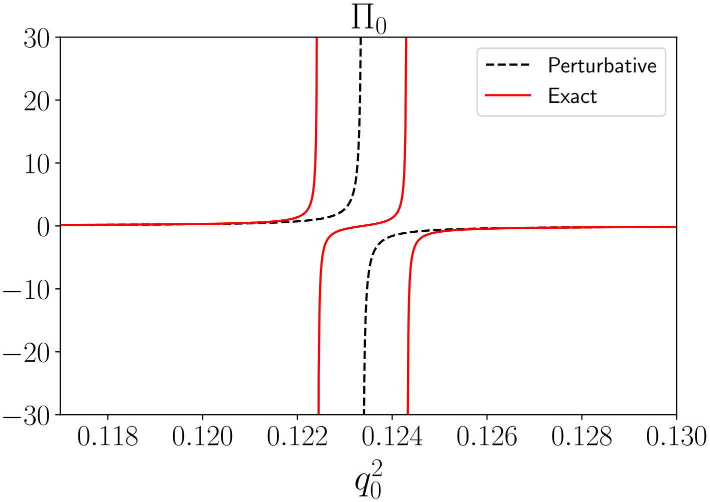
Lüscher zeta-function: Perturbative expansion I

- At $O(e)$, Π is a 2×2 matrix:
  - Diagonal components $\rightarrow \Pi_0$.
  - Off-diagonal $\rightarrow \Pi_1$.
- Perform an expansion to first order in e.
- The denominator:

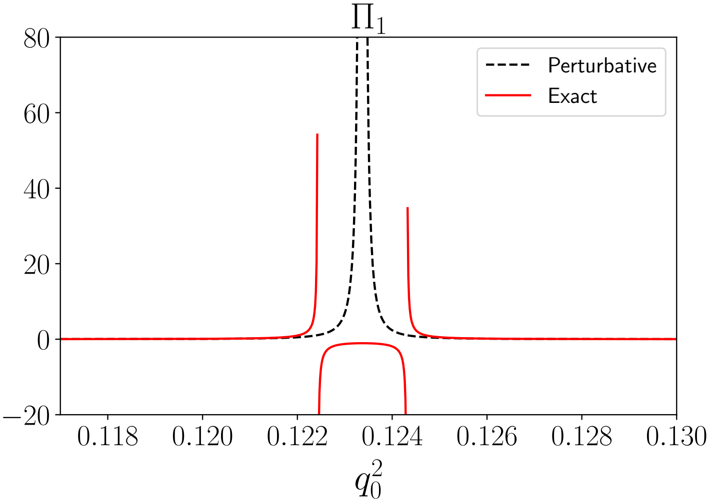
$$\frac{1}{2m + \frac{\mathbf{p}_\perp^2}{2m} + \frac{(\mathbf{P}-\mathbf{p})_\perp^2}{2m} + \frac{\omega^2}{8m}\left(\lambda_{\nu_i+2n}(q) + \lambda_{\nu_j+2m}(q)\right) - E} \tag{29}$$

- Expanding the denominator up to first order in q: the perturbative expression.
- Leaving the denominator intact: the exact expression.

Lüscher zeta-function: Perturbative expansion II



Lüscher zeta-function: Perturbative expansion II



Extracting the resonance pole I

- To determine the complex pole position, the Lüscher equation in the infinite volume is solved.
  - Momenta no longer quantized.
  - From conservation of three-momentum, $\mathbf{P} = \mathbf{Q} \pm l\omega$, two-point function still obeys matrix equation.
- The perturbative expansion is allowed.
  $\rightarrow$ Energy denominator not singular in the complex plane.
- Up to $O(e)$:

$$\Pi(\mathbf{P}, \mathbf{Q}; E) = (2\pi)^3 \delta^3(\mathbf{P} - \mathbf{Q}) \Pi_0(\mathbf{P}; E) + eA_0 \Gamma (2\pi)^3 \left[\delta^3(\mathbf{P} + \boldsymbol{\omega} - \mathbf{Q}) + \delta^3(\mathbf{P} + \boldsymbol{\omega} - \mathbf{Q})\right] \Pi_1(\mathbf{P}, \mathbf{Q}; E) + \dots, \quad (30)$$

Extracting the resonance pole II

- Π_0 and Π_1 in the infinite volume:

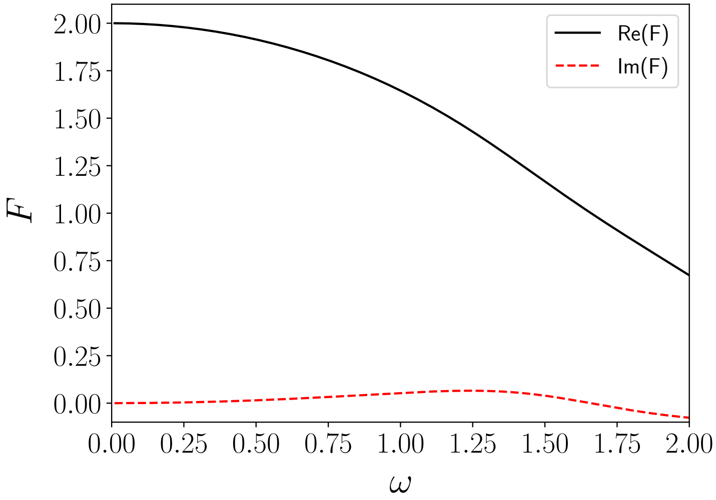
$$\mathit{\Pi}_0(\mathbf{P};E)=\frac{m}{4\pi}\left[-m\left(E-2m-\frac{\mathbf{P}^2}{4m}\right)\right]^{1/2},$$

$$\mathit{\Pi}_1(\mathbf{P},\mathbf{Q};E)=-\frac{m^2}{2\pi\omega}\arcsin\frac{\omega}{\sqrt{16m(2m+\mathbf{P}^2/(4m)-E)+\omega^2}}\,. \qquad (31)$$

- The pole position, $P_R^0(e)$, is determined by solving the Lüscher equation in an external field in an infinite volume.
- It can be explicitly shown that:

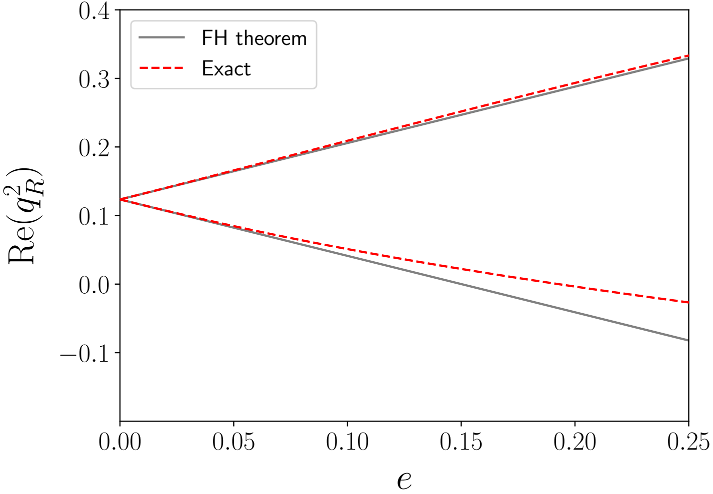
$$\left.\frac{dP_R^0(\mathbf{P})}{de}\right|_{e=0}\propto F(\mathbf{P},\mathbf{Q}). \qquad (32)$$

Results



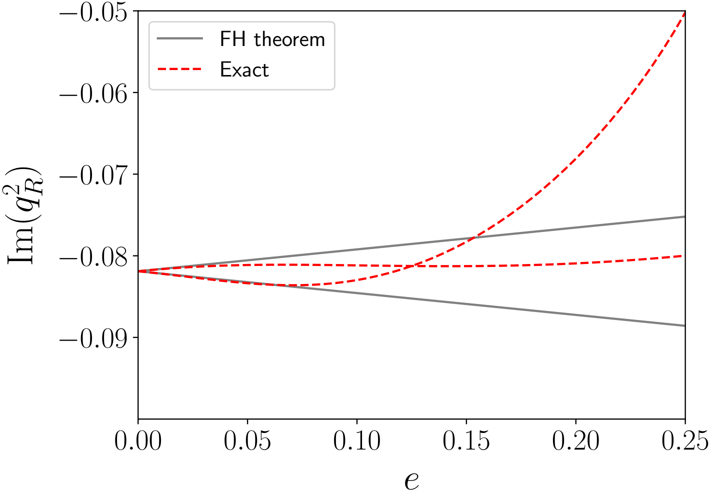
$$a = -1.5,\ \ r = -9,\ \ \kappa = 10,\ \ C_R = 0.9$$

Results



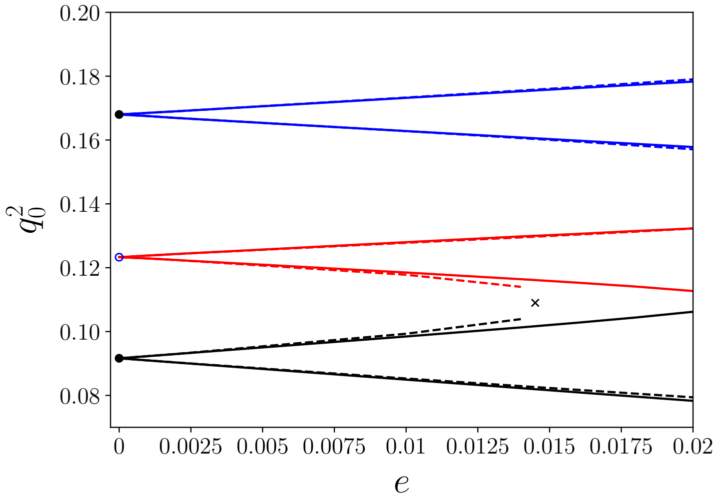
$$a = -1.5, \; r = -9, \; \kappa = 10, \; C_R = 0.9, \; \omega = 1$$

Results



$$a = -1.5,\ r = -9,\ \kappa = 10,\ C_R = 0.9,\ \omega = 1$$

Results



$a = -1.5, \; r = -9, \; \kappa = 10, \; C_R = 0.9, \; L = 20$

Summary

- A novel method for the measurement of the resonance form factors on the lattice has been proposed.
- A generalization of the Lüscher equation in the presence of an external periodic field is obtained:

$$\boxed{\mathcal{M}_{\mathbf{PQ}}(E) = [X_{\mathbf{PQ}}(E)]^{-1} - \frac{1}{2}\,\Pi_{\mathbf{PQ}}(E)}. \tag{33}$$

- Additionally, the Feynman-Hellmann theorem is extended to the case of resonances:

$$\boxed{\left.\frac{dP_R^0(\mathbf{P})}{de}\right|_{e=0} = \frac{1}{2}A_0 F(\mathbf{P}, -\mathbf{P}).} \tag{34}$$

Outlook

- Numerical implementation should be considered for realistic values of existing resonances.
- Partial-wave mixing should be addressed.
- Take into account relativistic corrections to all orders.
- Include higher orders in the effective theory.

Backup I

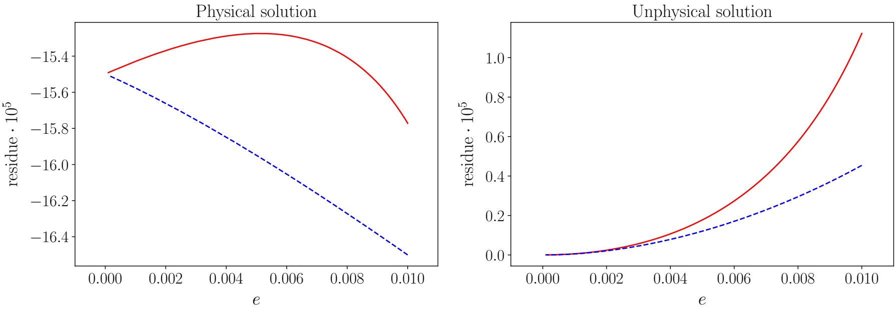
- The explicit form of the RFF (in dimensional regularization) is given by:

$$F(\omega) = \frac{\sqrt{-q_R^2}}{4\pi\left(1 + r\sqrt{-q_R^2}\right)} \left\{ -\kappa\omega^2 q_R^2 + 8\pi\Gamma\left(r + \frac{4}{\omega} \arcsin\frac{\omega}{\sqrt{\omega^2 - 16 q_R^2}} \right) \right\}.$$

- For instance, with $\omega = 1$, we have:

$$F(\omega) = 1.6454 + i0.0535\,, \qquad \left.\frac{dP_R^0}{de}\right|_{e=0} = 1.6455 + i0.0534 \qquad \checkmark$$

Backup II


Physical solution
residue · 10^5
−15.4
−15.6
−15.8
−16.0
−16.2
−16.4
0.000
0.002
0.004
0.006
0.008
0.010
e
Unphysical solution
residue · 10^5
1.0
0.8
0.6
0.4
0.2
0.0
0.000
0.002
0.004
0.006
0.008
0.010
e

Backup III

- Expanding the Mathieu function in q

$$\mathrm{me}_\nu(z,q) = e^{i\nu z} - \frac{q}{4}\left(\frac{1}{\nu+1}\, e^{i(\nu+2)z} - \frac{1}{\nu-1}\, e^{i(\nu-2)z}\right) + O(q^2)\,,$$

$$\mathrm{me}_k(z,q) = \sqrt{2}\left\{\cos kz - \frac{q}{4}\left(\frac{1}{k+1}\cos(k+2)z - \frac{1}{k-1}\cos(k-2)z\right) + O(q^2)\right\}$$

$$\mathrm{me}_{-k}(z,q) = -i\sqrt{2}\left\{\sin kz - \frac{q}{4}\left(\frac{1}{k+1}\sin(k+2)z - \frac{1}{k-1}\sin(k-2)z\right) + O(q^2)\right. \tag{35}$$

- If $k = 1, 0, -1$

$$\mathrm{me}_1(z,q) = \sqrt{2}\left\{\cos z - \frac{q}{8}\cos 3z + O(q^2)\right\},$$

$$\mathrm{me}_0(z,q) = 1 - \frac{q}{2}\cos 2z + O(q^2)\,,$$

$$\mathrm{me}_{-1}(z,q) = -i\sqrt{2}\left\{\sin z - \frac{q}{8}\sin 3z + O(q^2)\right\}. \tag{36}$$