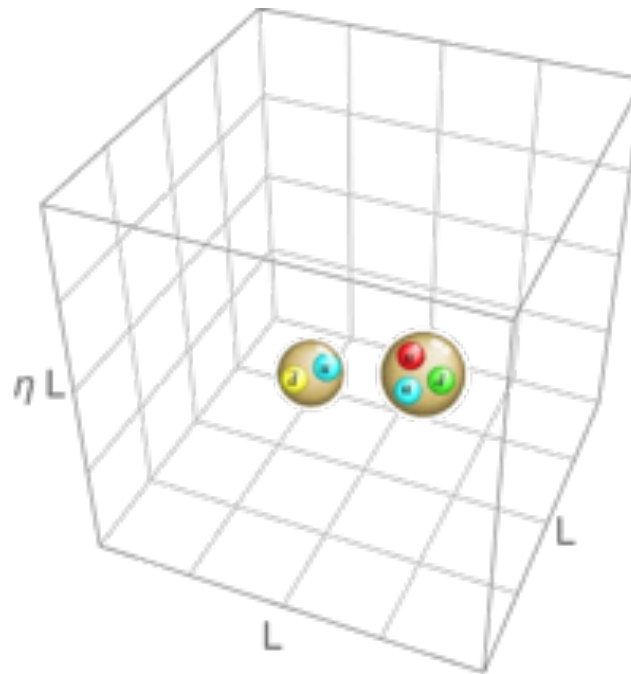


Multi-hadron scattering in elongated boxes

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Outline

- 1) Validation of two-body QC
- 2) $\pi\pi$ scattering (For 3-body systems like $\pi\pi\pi$, KKK , a_1 , see talks by C. Culver and M. Döring)
- 3) Compton scattering

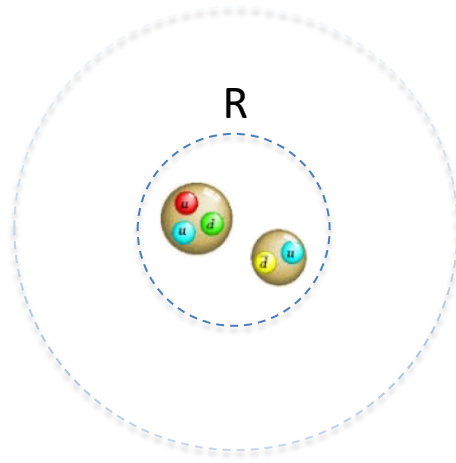
Part 1

Validation of quantization conditions via quantum mechanics

Lüscher method (or FV-IV Paradigm)

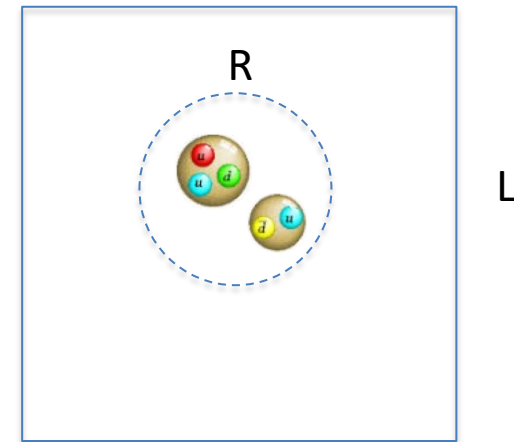
Infinite volume,

- Finite interaction range R
- Phaseshift is introduced in the asymptotic region ($r > R$) to characterize the interaction



Finite box of size L ,

- Periodic boundary conditions
- Require $L > 2R$



Lüscher established an exact relation between elastic **scattering phaseshift** in the infinite volume and **discrete interaction energy levels** in the box.

It does not matter how the interaction energy levels are obtained in the box,

- Quantum mechanics
- Lattice QCD (pi-pi scattering)
- Effective field theories

Quantization Condition (QC)

$$\prod_{\Gamma} \det \left[M_{J,J'}^{\Gamma}(k, L) - \delta_{JJ'} \cot \delta_J(k) \right] = 0$$

Γ = irreducible representations of a symmetry group

L = box size

k = discrete box levels

$\delta_J(k)$ = infinite-volume phaseshifts

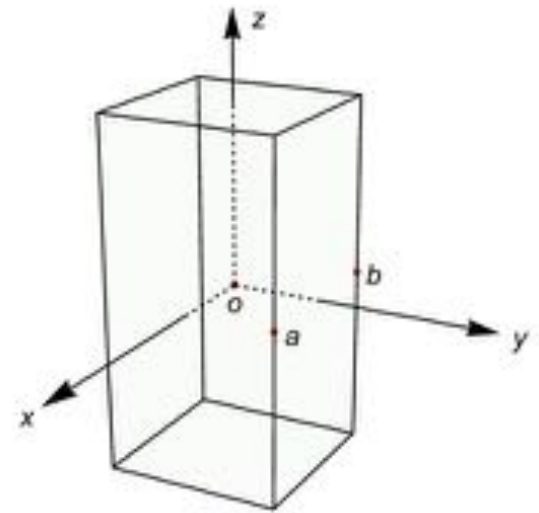
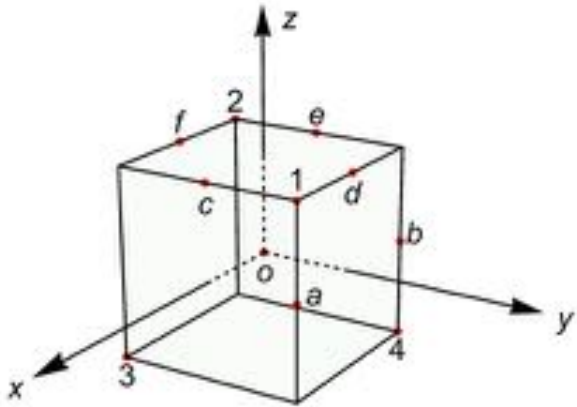
Each QC is a single condition that couples to an infinite tower of angular momenta. Only the lowest phaseshift can be predicted.

$$\det \begin{pmatrix} M_{11} - \cot \delta_{J_1} & M_{12} & \dots \\ M_{21} & M_{22} - \cot \delta_{J_2} & \dots \\ \dots & \dots & \ddots \end{pmatrix} = 0$$

How to validate ?

- 1) Compute $\delta_J(k)$ from an interaction model
- 2) Compute k in a box from the same interaction
- 3) Use $\delta_J(k)$ as input, solve for k and compare with box levels
- 4) Enlarge QC order by order: order 1= lowest J , order 2= lowest 2 J s, order 3= lowest 3 J s, etc
- 5) Convergence is expected as the order is increased.

Cubic vs. Elongated symmetry

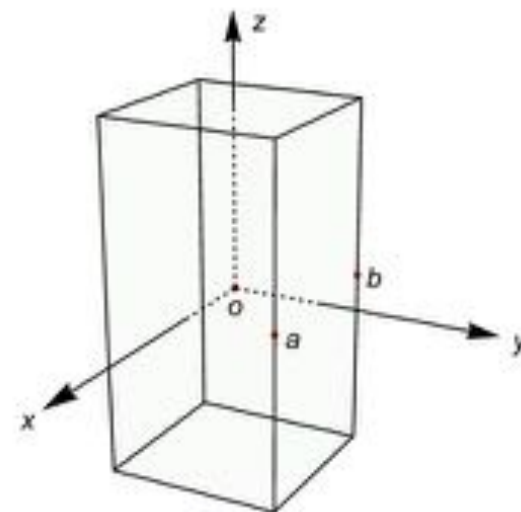
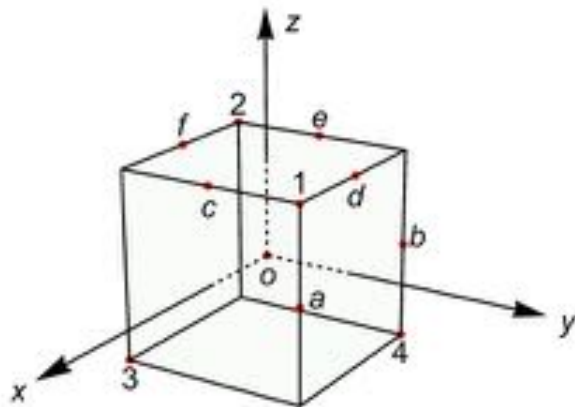


Angular momentum decomposition in box geometry for rest and moving frames

O_h		D_{4h}		C_{4v}		C_{3h}		C_{2v}		C_{1v}	
Γ	l	Γ	l	Γ	l	Γ	l	Γ	l	Γ	l
A_{1g}	0, 4, 6	A_{1g}	0, 2, 4(2), 5	A_1	0, 1, 2, 3, 4(2)	A_1	0, 1, 2, 3(2), 4(2)	A_1	0, 1, 2(2), 3(2), 4(3)	A_1	0, 1(2), 2(3), 3(4)
A_{1u}	9, 13, 15	A_{1u}	5, 7, 9(2)								
A_{2g}	6, 10, 12	A_{2g}	4, 6, 8(2)	A_2	4, 5, 6	A_2	3, 4, 5	A_2	2, 3, 4(2), 5(2)	A_2	1, 2(2), 3(3), 4(4)
A_{2u}	3, 7, 9	A_{2u}	1, 3, 5(2), 7(2)								
E_g	2, 4, 6	E_g	2, 4(2), 6(3)	E	1, 2, 3(2), 4(2)	E	1, 2(2), 3(2), 4(3)				
E_u	5, 7, 9	E_u	1, 3(2), 5(3)								
T_{1g}	4, 6, 8(2)	B_{1g}	2, 4, 6(2), 8(2)	B_1	2, 3, 4, 5			B_1	1, 2, 3(2), 4(2), 5(3)		
T_{1u}	1, 3, 5(2)	B_{1u}	3, 5, 7(2), 9(2)								
T_{2g}	2, 4, 6(2)	B_{2g}	2, 4, 6(2), 8(2)	B_2	2, 3, 4, 5			B_2	1, 2, 3(2), 4(2), 5(3)		
T_{2u}	3, 5, 7(2)	B_{2u}	3, 5, 7(2), 9(2)								

Lowest partial wave in each irrep can be predicted by the Lüscher method.

Moving frames $\mathbf{d}=(n_x, n_y, n_z)$: Cubic vs. Elongated



	Cubic (O_h)	z-elongated (D_{4h})
C_{4v}	(0, 0, 1)	(0, 0, 1)
C_{3v}	(1, 1, 1)	None
C_{2v}	(1, 1, 0)	(1, 0, 0), (1, 1, 0)
C_{1v}	(0, 1, 2), (1, 1, 2), (1, 2, 2)	(0, 1, 1), (1, 1, 1), (1, 2, 0), (0, 1, 2), (0, 2, 1), (1, 2, 1), (2, 2, 1)

Angular momentum decomposition in box geometry for rest and moving frames

Example:

Matrix elements up to $l=5$ for
rest frame $\mathbf{d}=(0,0,0)$ in
elongated box (group D4h).

$$\prod_{\Gamma} \det [\mathcal{M}_{J,J'}^{\Gamma} - \delta_{JJ'} \cot \delta_J] = 0$$

$$W_{lm} = \frac{\mathcal{Z}_{lm}(q^2, L)}{\pi^{3/2} q^{l+1}}, \quad q = k \frac{L}{2\pi}$$

$$\mathcal{Z}_{lm} = \sum_{n=\{n_1, n_2, n_3\} \in \mathbb{Z}^3} \frac{n^l Y_{lm}(\theta, \phi)}{n^2 - q^2} \quad (\text{zeta function})$$

Γ	l	m	l'	m'	$M_{ln,l'm'}^{\Gamma}$
A_{1g}	0	1	0	1	w_{00}
	0	1	2	1	$-w_{20}$
	0	1	4	1	$\sqrt{2}w_{41}$
	0	1	4	2	w_{40}
	2	1	2	1	$w_{00} + \frac{2\sqrt{5}w_{20}}{7} + \frac{6w_{40}}{7}$
	2	1	4	1	$\frac{5\sqrt{10}w_{41}}{11} - \frac{15}{11}\sqrt{\frac{2}{13}}w_{61}$
	2	1	4	2	$-\frac{6w_{20}}{7} - \frac{20\sqrt{5}w_{41}}{77} - \frac{15}{11}\sqrt{\frac{5}{13}}w_{60}$
	4	1	4	1	$w_{00} - \frac{4\sqrt{5}w_{20}}{11} + \frac{54w_{40}}{143} - \frac{4w_{60}}{11\sqrt{13}} + \frac{7w_{80}}{143\sqrt{17}} + 21\sqrt{\frac{10}{2431}}w_{88}$
	4	1	4	2	$\frac{55\sqrt{2}w_{41}}{143} - \frac{12}{11}\sqrt{\frac{10}{13}}w_{61} + \frac{21}{13}\sqrt{\frac{10}{187}}w_{84}$
	4	2	4	2	$w_{00} + \frac{20\sqrt{5}w_{20}}{77} + \frac{486w_{40}}{1001} + \frac{20w_{60}}{11\sqrt{13}} + \frac{400w_{80}}{143\sqrt{17}}$
A_{1u}	5	1	5	1	$w_{00} + \frac{30\sqrt{21}w_{100}}{4199} - \frac{6}{19}\sqrt{\frac{2310}{221}}w_{108} - \frac{2\sqrt{5}w_{20}}{13} - \frac{6w_{40}}{13} + \frac{32w_{60}}{17\sqrt{13}} - \frac{217w_{80}}{247\sqrt{17}} - \frac{21}{19}\sqrt{\frac{110}{221}}w_{88}$
A_{2g}	4	1	4	1	$w_{00} - \frac{4\sqrt{5}w_{20}}{11} + \frac{54w_{40}}{143} - \frac{4w_{60}}{11\sqrt{13}} + \frac{7w_{80}}{143\sqrt{17}} - 21\sqrt{\frac{10}{2431}}w_{88}$
A_{2u}	1	1	1	1	$w_{00} + \frac{2w_{20}}{\sqrt{5}}$
	1	1	3	1	$-3\sqrt{\frac{3}{35}}w_{20} - \frac{4w_{40}}{\sqrt{21}}$
	1	1	5	1	$\sqrt{\frac{5}{11}}w_{41} + 2\sqrt{\frac{30}{143}}w_{61}$
	1	1	5	2	$\frac{5w_{40}}{\sqrt{33}} + 6\sqrt{\frac{3}{143}}w_{60}$
	3	1	3	1	$w_{00} + \frac{4w_{20}}{3\sqrt{5}} + \frac{6w_{40}}{11} + \frac{100w_{60}}{33\sqrt{13}}$
	3	1	5	1	$\frac{12}{13}\sqrt{\frac{18}{11}}w_{41} + \sqrt{\frac{70}{143}}w_{61} - \frac{4}{13}\sqrt{\frac{70}{17}}w_{81}$
	3	1	5	2	$-\frac{10}{3}\sqrt{\frac{3}{77}}w_{20} - \frac{60w_{40}}{13\sqrt{77}} - \frac{7}{3}\sqrt{\frac{7}{143}}w_{60} - \frac{56}{13}\sqrt{\frac{7}{187}}w_{80}$
	5	1	5	1	$w_{00} + \frac{30\sqrt{21}w_{100}}{4199} + \frac{6}{19}\sqrt{\frac{2310}{221}}w_{108} - \frac{2\sqrt{5}w_{20}}{13} - \frac{6w_{40}}{13} + \frac{32w_{60}}{17\sqrt{13}} - \frac{217w_{80}}{247\sqrt{17}} + \frac{21}{19}\sqrt{\frac{110}{221}}w_{88}$
	5	1	5	2	$\frac{120}{323}\sqrt{\frac{22}{13}}w_{104} + \frac{6\sqrt{2}w_{41}}{13} - \frac{8}{17}\sqrt{\frac{10}{13}}w_{61} - \frac{77}{247}\sqrt{\frac{110}{17}}w_{84}$
	5	2	5	2	$w_{00} + \frac{756\sqrt{21}w_{100}}{4199} + \frac{10\sqrt{5}w_{20}}{39} + \frac{6w_{40}}{13} + \frac{80w_{60}}{51\sqrt{13}} + \frac{400w_{80}}{247\sqrt{17}}$
B_{1g}	2	1	2	1	$w_{00} - \frac{2\sqrt{5}w_{20}}{7} + \frac{w_{40}}{7} + \sqrt{\frac{10}{7}}w_{61}$
	2	1	4	1	$-\frac{1}{7}\sqrt{15}w_{20} + \frac{30\sqrt{3}w_{41}}{77} + \frac{3}{11}\sqrt{\frac{30}{7}}w_{41} - \frac{5}{11}\sqrt{\frac{3}{13}}w_{60} - \frac{5}{11}\sqrt{\frac{82}{13}}w_{64}$
	4	1	4	1	$w_{00} + \frac{8\sqrt{5}w_{20}}{77} - \frac{27w_{40}}{91} + \frac{81}{143}\sqrt{\frac{10}{7}}w_{41} - \frac{2w_{60}}{\sqrt{13}} + \frac{6}{11}\sqrt{\frac{18}{13}}w_{61} + \frac{196w_{80}}{143\sqrt{17}} + \frac{42}{13}\sqrt{\frac{18}{187}}w_{84}$
	3	1	3	1	$w_{00} - \frac{7w_{20}}{11} - \frac{\sqrt{70}w_{41}}{11} + \frac{10w_{60}}{11\sqrt{13}} - \frac{10}{11}\sqrt{\frac{18}{13}}w_{61}$
B_{1u}	3	1	5	1	$-\sqrt{\frac{3}{11}}w_{20} + \frac{10w_{40}}{13\sqrt{11}} - \frac{2}{13}\sqrt{\frac{70}{11}}w_{41} + \frac{7w_{60}}{13\sqrt{143}} + \sqrt{\frac{18}{143}}w_{61} - \frac{56w_{80}}{13\sqrt{187}} + \frac{12}{13}\sqrt{\frac{18}{187}}w_{84}$
	5	1	5	1	$w_{00} + \frac{360\sqrt{21}w_{100}}{4199} - \frac{30}{323}\sqrt{\frac{270}{13}}w_{104} + \frac{2\sqrt{5}w_{20}}{13} - \frac{w_{40}}{13} - \frac{\sqrt{70}w_{41}}{13} - \frac{2w_{60}}{17\sqrt{13}} - \frac{8}{17}\sqrt{\frac{18}{13}}w_{61} - \frac{28\sqrt{17}w_{80}}{247} - \frac{42}{247}\sqrt{\frac{18}{17}}w_{84}$
	2	1	2	1	$w_{00} - \frac{2\sqrt{5}w_{20}}{7} + \frac{w_{40}}{7} - \sqrt{\frac{10}{7}}w_{61}$
B_{2g}	2	1	4	1	$-\frac{1}{7}\sqrt{15}w_{20} + \frac{30\sqrt{3}w_{41}}{77} - \frac{3}{11}\sqrt{\frac{30}{7}}w_{41} - \frac{5}{11}\sqrt{\frac{3}{13}}w_{60} + \frac{5}{11}\sqrt{\frac{82}{13}}w_{64}$
	4	1	4	1	$w_{00} + \frac{8\sqrt{5}w_{20}}{77} - \frac{27w_{40}}{91} + \frac{81}{143}\sqrt{\frac{10}{7}}w_{41} - \frac{2w_{60}}{\sqrt{13}} - \frac{6}{11}\sqrt{\frac{18}{13}}w_{61} + \frac{196w_{80}}{143\sqrt{17}} - \frac{42}{13}\sqrt{\frac{18}{187}}w_{84}$
	4	1	4	1	$w_{00} + \frac{8\sqrt{5}w_{20}}{77} - \frac{27w_{40}}{91} - \frac{81}{143}\sqrt{\frac{10}{7}}w_{41} - \frac{2w_{60}}{\sqrt{13}} - \frac{6}{11}\sqrt{\frac{18}{13}}w_{61} + \frac{196w_{80}}{143\sqrt{17}} - \frac{42}{13}\sqrt{\frac{18}{187}}w_{84}$

A toy model

Phaseshift can be obtained in the CM frame in infinite volume,

$$E = \frac{\hbar^2 k^2}{2\mu}$$

$$H = -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \quad \left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) \right] u_l(r) = E u_l(r)$$

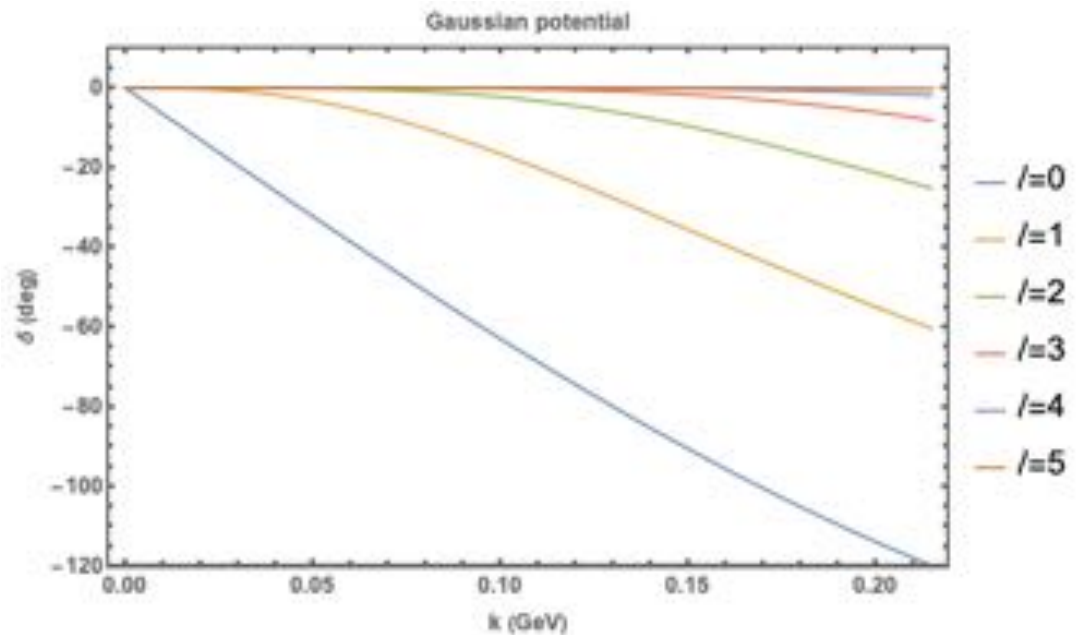
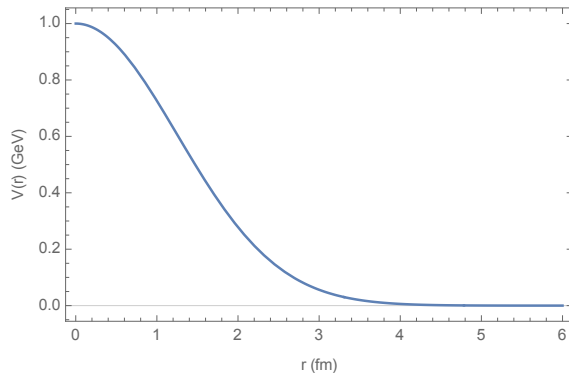
$$\lim_{r \rightarrow 0} u_l(r) = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} u_l(r) \propto e^{i\delta_l(k)} \sin \left[kr - \frac{l\pi}{2} + \delta_l(k) \right]$$

Simple test potential,

$$m_1 = 0.138 \text{ GeV}, m_2 = 0.94 \text{ GeV}$$

$$V(r) = C e^{-\frac{1}{2} \left(\frac{r}{R_0} \right)^2}$$

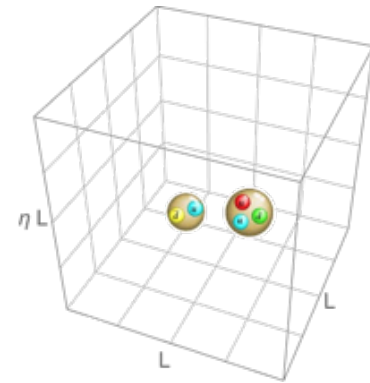
$$C = 1.0 \text{ GeV}, R_0 = 1.25 \text{ fm}$$



Since the range $R=4$ fm, we use a box of $L=24$ fm.

Lattice Hamiltonian

$$H|\psi\rangle = E|\psi\rangle \quad H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(|\vec{r}_1 - \vec{r}_2|)$$



Total momentum of the two-particle system in the lab frame: $P = p_1 + p_2$.

Periodic boundary conditions $\Psi(\mathbf{r}_1 + \mathbf{n}_1 L, \mathbf{r}_2 + \mathbf{n}_2 L) = \Psi(\mathbf{r}_1, \mathbf{r}_2)$.

It can be solved directly by discretizing it on the lattice
($V = N_x \times N_y \times N_z$, spacing a)

- 6-dimensional problem ($x_1, y_1, z_1, x_2, y_2, z_2$)
- Laplacian operator is approximated by finite differences
- **$N^6 \times N^6$ eigensystem**
- Only feasible for small lattices
- Useful for checking purposes

$$P = \frac{2\pi}{L}d = \frac{2\pi}{a} \left(\frac{i}{N_x}, \frac{j}{N_y}, \frac{k}{N_z} \right)$$

$$i = 0, 1, 2, \dots, N_x - 1$$

$$j = 0, 1, 2, \dots, N_y - 1$$

$$k = 0, 1, 2, \dots, N_z - 1$$

For $N=4$ lattice,

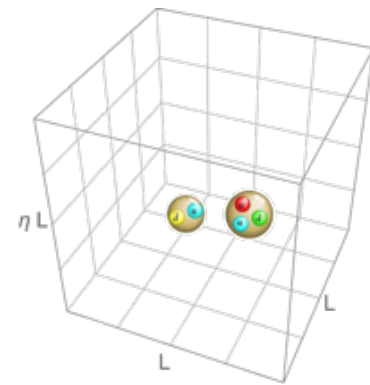
- Full spectrum has 4096 eigenvalues and eigenvectors.
- They can be separated into 64 sectors of P , each sector having 64 states.
- The separation can be realized uniquely by constructing common eigenvectors of commuting operators $\{H, p_{x\sin}, p_{x\cos}, p_{y\sin}, p_{y\cos}, p_{z\sin}, p_{z\cos}\}$.

$$\text{Sin}(P_x a) = \frac{T_x - T_x^\dagger}{2i}$$

$$\text{Cos}(P_x a) = \frac{T_x + T_x^\dagger}{2}$$

$$\text{Translation operator } T_x = e^{iP_x a}$$

Reduced Lattice Hamiltonian



$$H|\psi\rangle = E|\psi\rangle \quad H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(|\vec{r}_1 - \vec{r}_2|)$$

It can be projected onto a basis of fixed P and relative coordinate r, given in 3-stencil form

$$|P, r\rangle = \sum_m e^{iPm} |m, m+r\rangle$$

$$H|P, r\rangle = -\frac{\hbar^2}{2a^2} \left[\left(\frac{e^{iP}}{m_1} + \frac{1}{m_2} \right) |P, r+1\rangle + \left(\frac{e^{-iP}}{m_1} + \frac{1}{m_2} \right) |P, r-1\rangle - 2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) |P, r\rangle \right] + V(r)|P, r\rangle + O(a^2)$$

If $P=0$, it reduces to the standard form,

$$H|r\rangle = -\frac{\hbar^2}{2\mu a^2} [|r+1\rangle + |r-1\rangle - 2|r\rangle] + V(r)|r\rangle$$

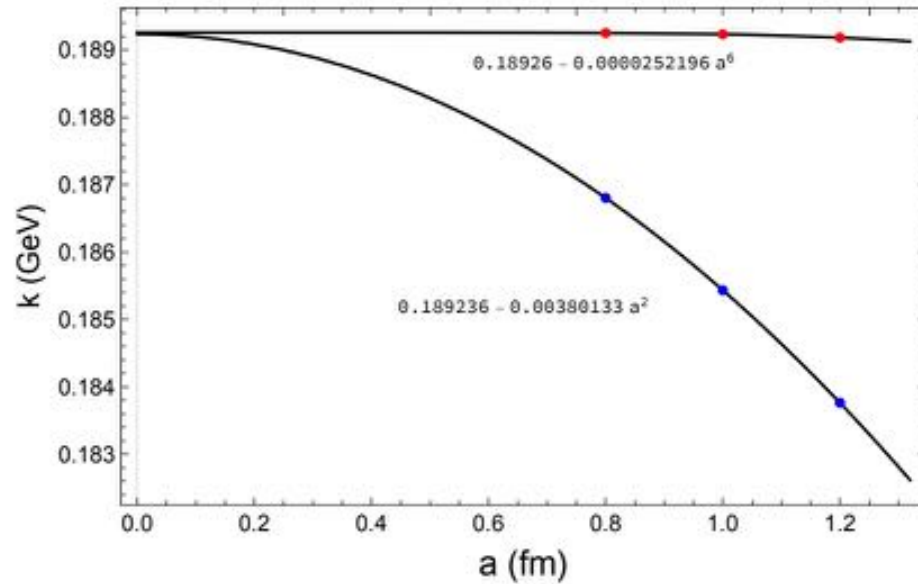
which coincides with the form in CM frame.

Main advantages of this formalism:

- 1) A N^6 eigenvalue problem is reduced to a N^3 one. We can obtain the full spectrum quickly for lattices up to 32^3 on a laptop.
- 2) Working in the lab frame
- 3) Moving frames are naturally described (P is an input)

In practice, we use an improved version from a centered **7-point stencil** for precision,

$$H = -\frac{\hbar^2}{2} \sum_{j=x,y,z} \frac{-1}{180a^2} \left[-2 \left(\frac{e^{3iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j+3\rangle + 27 \left(\frac{e^{2iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j+2\rangle - 270 \left(\frac{e^{iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j+1\rangle \right. \\ \left. - 2 \left(\frac{e^{-3iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j-3\rangle + 27 \left(\frac{e^{-2iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j-2\rangle - 270 \left(\frac{e^{-iP_j}}{m_1} + \frac{1}{m_2} \right) |P_j, j-1\rangle + 490 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) |P_j, j\rangle \right] \\ + V(x, y, z) |P, x, y, z\rangle + O(a^6)$$



3-point stencil,

$$H|P, r\rangle = -\frac{\hbar^2}{2a^2} \left[\left(\frac{e^{iP}}{m_1} + \frac{1}{m_2} \right) |P, r+1\rangle + \left(\frac{e^{-iP}}{m_1} + \frac{1}{m_2} \right) |P, r-1\rangle - 2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) |P, r\rangle \right] + V(r) |P, r\rangle + O(a^2)$$

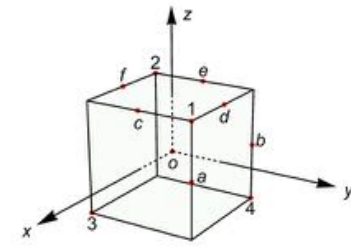
Energy levels in a box

TABLE V. Number of distinct and total noninteracting levels for two spinless particles of unequal masses in a box of geometry $L \times L \times \eta L$ with $L = 24$ fm and cutoff $k < 0.2$ GeV.

d	Cubic ($\eta = 1$)		z -elongated ($\eta = 1.5$)	
	Distinct	Total	Distinct	Total
(0, 0, 0)	14	251	36	359
(0, 0, 1)	54	252	79	367
(1, 1, 0)	73	245	108	355
(1, 1, 1)	54	342	204	363
(0, 1, 2)	107	240	218	366

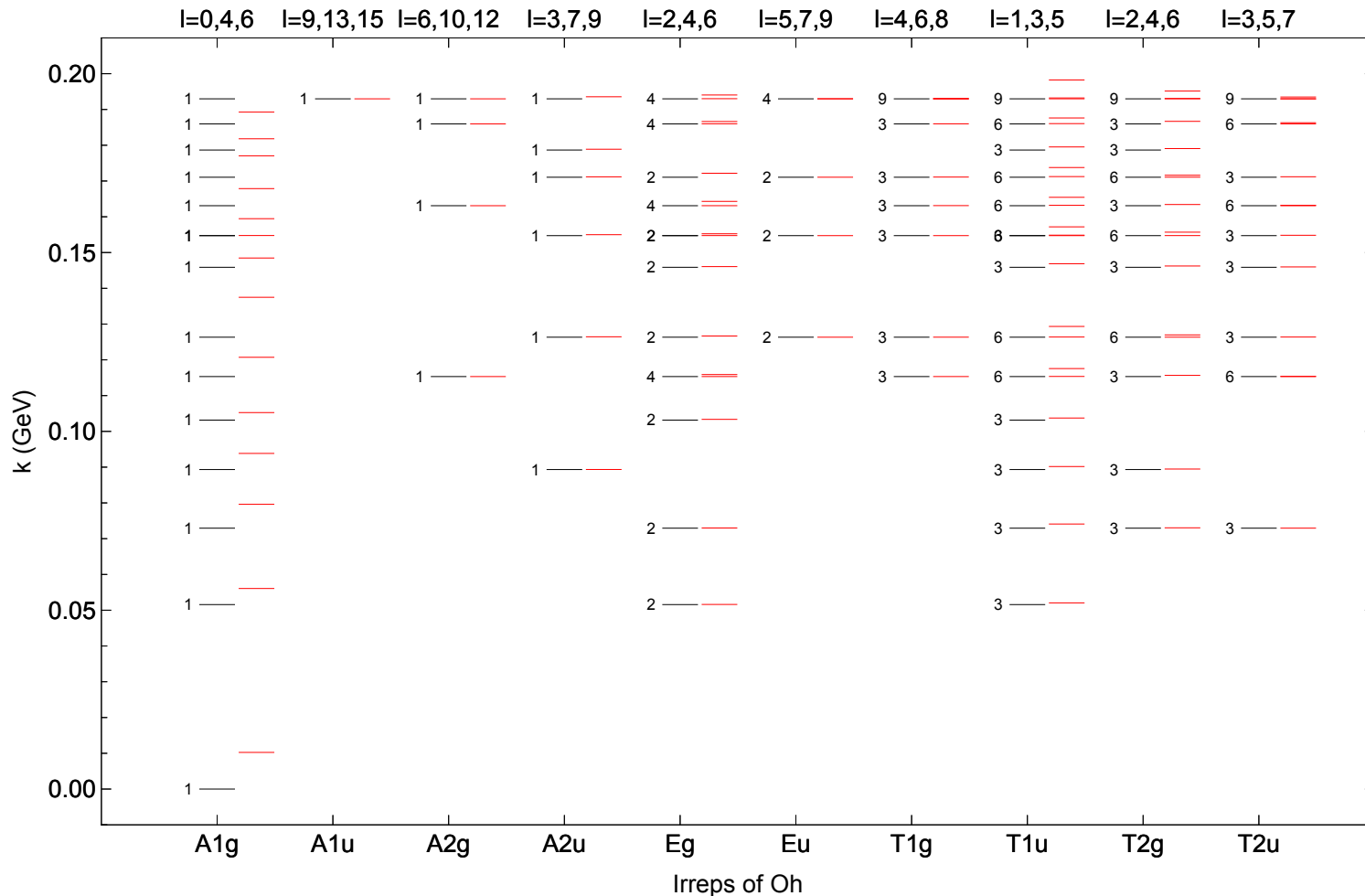
We use cutoff of $k < 0.2$ GeV or 40 distinct levels.

Energy levels in a cubic box of $L=24$ fm, projected into irreps of Oh.



$$\lim_{\substack{a \rightarrow 0 \\ N \rightarrow \infty}} Na = L$$

Box levels are obtained by continuum extrapolation using lattices 20^3 at $a=1.2$ fm, 24^3 at $a=1.0$ fm, 30^3 at $a=0.8$ fm.



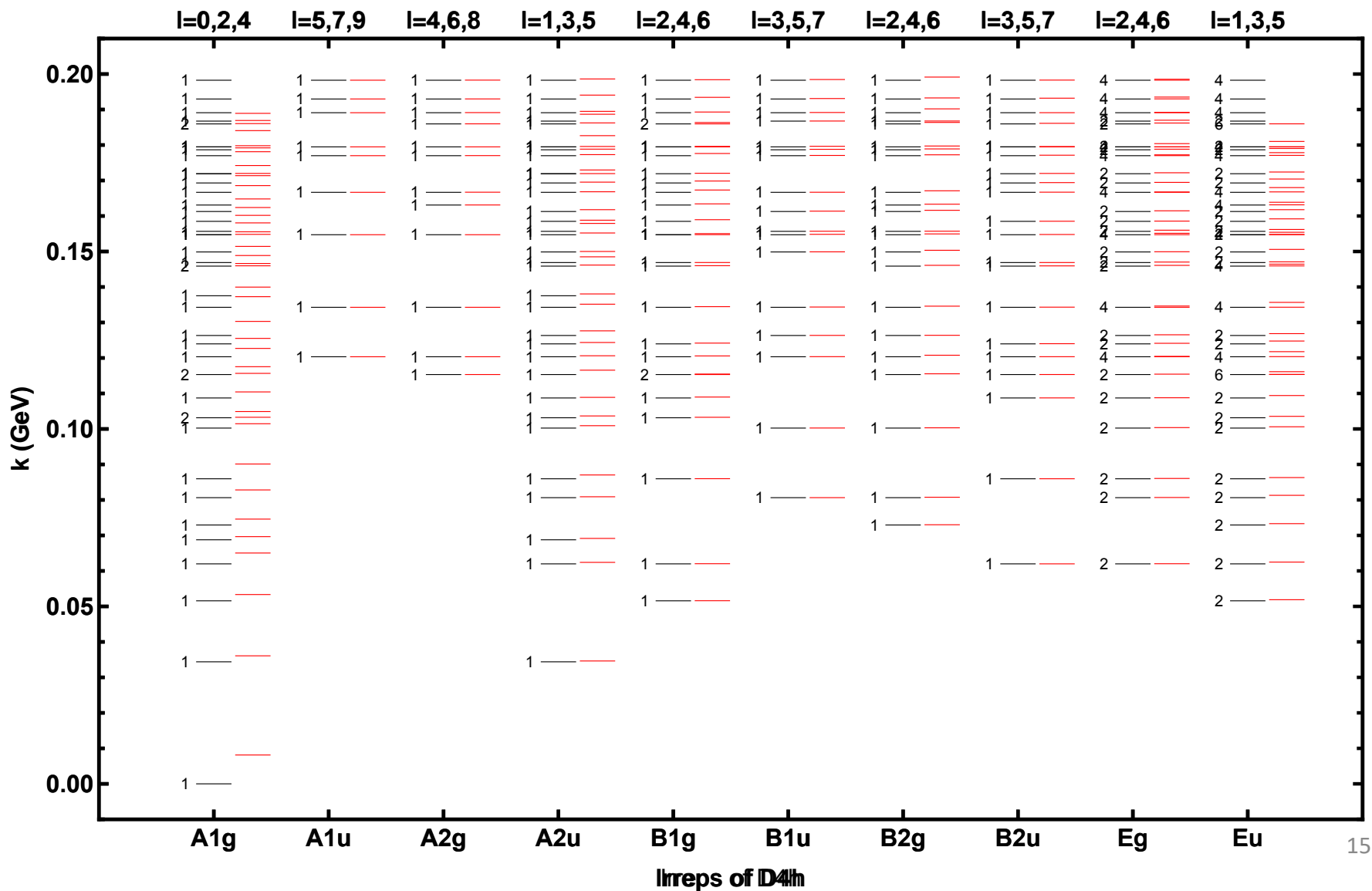
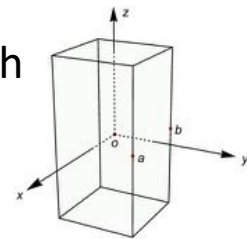
$$E = \frac{\hbar^2 k^2}{2\mu}$$

black: free particle levels (with degeneracies)

red: interacting levels

Energy levels in a z-elongated box of L=24 fm and $\eta=1.5$, projected into irreps of D_{4h}

Box levels are obtained by continuum extrapolation using lattices
20x20x30 at a=1.2 fm, 24x24x36 at a=1.0 fm, 30x30x45 at a=0.8 fm.



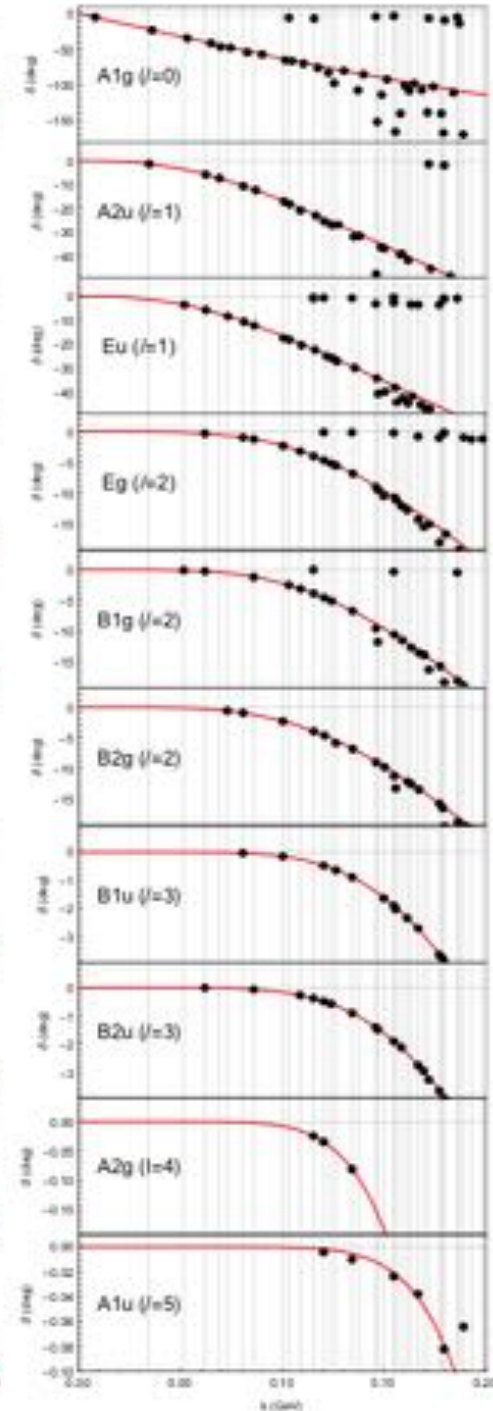
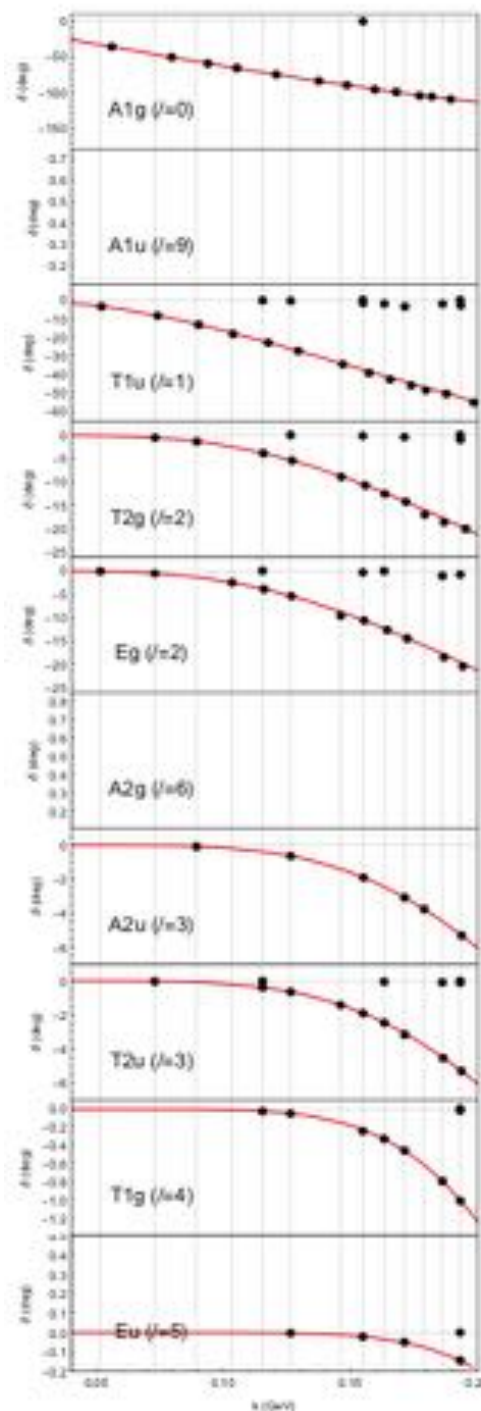
Lowest partial wave phaseshift reconstruction in cubic (left) and elongated (right) boxes.

Red curves: infinite-volume $\delta_J(k)$

Black points: $\delta_J(k)$ from finite volume k

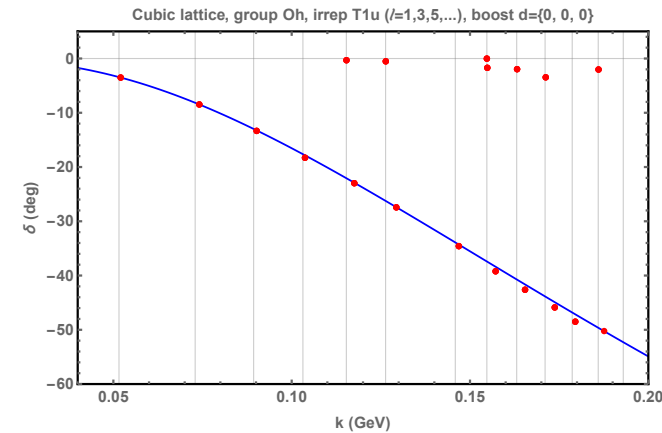
Faint vertical lines: free-particle poles

- 1) Feed the box levels k to QC to predict lowest δ and compare with infinite-volume values.
- 2) Phaseshift largely reconstructed up to $k=0.2$ GeV.
- 3) Exceptions are observed. They happen at free-particle poles (pinching).
- 4) Higher partial waves?



Convergence of QC

$$T_1^- : \det \begin{pmatrix} M_{11} - \cot \delta_1(k) & M_{13} & \dots \\ M_{31} & M_{33} - \cot \delta_3(k) & \dots \\ \dots & \dots & \ddots \end{pmatrix} = 0$$



chi-square measure:

$$\chi^2 = \frac{\left(k_{\text{box}} - k_{\text{QC}}\right)^2}{\left(k_{\text{box}} - k_{\text{lat}}\right)^2}$$

k_{box} = box levels

k_{lat} = levels from largest lattice

k_{QC} = roots from QC at each order.

* = free-particle poles

Level	Box L	Lattice	Order 1	Order 2	Order 3	χ^2 Order 1	χ^2 Order 2	χ^2 Order 3
22	0.198249	0.198241	0.198146	0.198245	0.19825	167.097	0.249283	0.0101593
21	0.193217	0.193213	*0.192974	0.193209	0.193221	2998.88	3.71223	0.470423
20	0.192982	0.192978	*0.192974	*0.192974	0.192975	3.26057	3.26057	2.49916
19	0.187618	0.187613	0.187614	0.187615	0.187619	0.747838	0.42273	0.00535232
18	0.186043	0.186039	*0.185954	0.186041	0.186045	389.699	0.356124	0.194373
17	0.179522	0.179521	0.179489	0.179517	0.179522	807.378	17.3011	0.0125204
16	0.173768	0.173763	0.173698	0.173767	0.173769	206.536	0.0359251	0.00582583
15	0.171233	0.171228	*0.171053	0.171231	0.171233	1447.64	0.209609	0.00394721
14	0.165438	0.165434	0.165382	0.165438	0.165439	128.455	0.0154137	0.00520841
13	0.163206	0.163201	*0.163093	0.163204	0.163206	526.933	0.147073	0.0123243
12	0.157179	0.157177	0.157134	0.157178	0.157179	551.783	0.156363	0.00460883
11	0.154856	0.154853	*0.154723	0.154856	0.154857	1267.98	0.0467443	0.00685293
10	0.154726	0.154725	*0.154723	*0.154723	0.154726	6.59657	6.59657	0.0176498
9	0.146868	0.146867	0.14686	0.146867	0.146868	172.894	2.71263	0.0020146
8	0.129339	0.129339	0.129335	0.129339	0.129339	58.3417	0.000653416	0.0113939
7	0.126379	0.126379	*0.126331	0.126379	0.126379	30130.9	0.976735	0.00253806
6	0.11756	0.117559	0.117552	0.11756	0.11756	323.653	0.0133041	0.00803915
5	0.115359	0.115359	*0.115324	0.115359	0.115359	13893.6	0.272455	0.002095
4	0.103734	0.103734	0.10372	0.103734	0.103734	1539.04	0.0089603	0.0024343
3	0.0901719	0.0901719	0.0901658	0.0901719	0.0901719	11790.7	0.0587094	0.0204841
2	0.0740985	0.0740985	0.0740982	0.0740985	0.0740985	27.9224	0.011686	0.0285701
1	0.0520501	0.0520501	0.0520494	0.0520501	0.0520501	836.425	0.0286906	0.0308669
Total χ^2						67276.5	36.5936	3.35684

Convergence indicates high-precision validation of QC.

45 quantization conditions up to J=5

					Cubic box					
No.	d	Group	QC	I(n)	N	Order 1	Order 2	Order 3	Order 4	Order 5
1	(0,0,0)	O_h	A_{1g}	0, 4, 6, ...	14	277.206	3.7296			
			A_{1u}	9, 13, 15, ...						
			A_{2g}	6, 10, 12, ...						
2			A_{2u}	3, 7, 9, ...	6	0.0458813				
3			E_g	2, 4, 6, ...	16	2123.07	5.44639			
4			E_u	5, 7, 9, ...	5	0.0546959				
5			T_{1g}	4, 6, 8(2), ...	9	0.243897				
6			T_{1u}	1, 3, 5(2), ...	22	67276.5	36.5936	3.35684		
7		C_{4v}	T_{2g}	2, 4, 6(2), ...	16	2485.36	2.70125			
8			T_{2u}	3, 5, 7(2), ...	14	4.55575	0.267262			
10	(0,0,1)		A_1	0, 1, 2, 3, 4(2), ...	40	4.08008×10^8	1.06106×10^{10}	10175.7	284.022	6.92792
11			A_2	4, 5, 6, ...	15	7.46202	0.145375			
12			B_1	2, 3, 4, 5, ...	34	276889.	402.025	5.9668		
13			B_2	2, 3, 4, 5, ...	27	350956.	779.183	3.84012		
14			E	1, 2, 3(2), 4(2), ...	40	1.32751×10^7	72379.5	514.979	8.21687	
14	(1,1,0)		C_{2v}	A_1	0, 1, 2(2), 3(2), 4(3), ...	40	1.96654×10^8	4.77126×10^6	17298.1	190.94
15		A_2		2, 3, 4(2), 5(2), ...	40	54211.3	2998.61	3.01327		
16		B_1		1, 2, 3(2), 4(2), ...	40	2.84806×10^6	47754.8	241.517	4.62961	
17		B_2		1, 2, 3(2), 4(2), ...	40	9.36978×10^6	88719.7	308.013	6.28652	
18	(1,1,1)	A_1		0, 1, 2, 3(2), 4(2), ...	40	9.16556×10^7	1.09895×10^6	22260.	115.412	3.53446
19		A_2		3, 4, 5, ...	26	111.322	0.44229			
20		E		1, 2(2), 3(2), 4(3), ...	40	8.06257×10^6	23959.	271.289	4.13508	
21	(0,1,2)	C_{1v}		A_1	0, 1(2), 2(3), 3(4), 4(5), ...	40	1.01852×10^8	993629.	6858.3	17.3429
22			A_2	1, 2(2), 3(3), 4(4), ...	40	8.98645×10^6	37314.8	293.484	3.18387	
Elongated box										
No.	d	Group	QC	I(n)	N	Order 1	Order 2	Order 3	Order 4	Order 5
23	(0,0,0)	D_{4h}	A_{1g}	0, 2, 4(2), 5, ...	40	4.06688×10^6	573.449	8.64225		
24			A_{1u}	5, 7, 9(2), ...	9	0.134224				
25			A_{2g}	4, 6, 8(2), ...	3	0.00828583				
26			A_{2u}	1, 3, 5(2), 7(2), ...	34	131946.	16.2795	10.7998		
27			E_g	2, 4(2), 6(3), ...	38	2009.93	0.319341			
28			E_u	1, 3(2), 5(3), ...	40	221693.	41.2595	8.52709		
29			B_{1g}	2, 4, 6(2), 8(2), ...	27	703.291	11.0395			
30			B_{1u}	3, 5, 7(2), 9(2), ...	17	7.07159	0.168506			
31		C_{2v}	B_{2g}	2, 4, 6(2), 8(2), ...	22	2093.46	0.192			
32			B_{2u}	3, 5, 7(2), 9(2), ...	21	2.06599	0.488025			
33	(0,0,1)		A_1	0, 1, 2, 3, 4(2), ...	40	5.70458×10^8	9.64404×10^6	14025.4	166.777	11.0137
34			A_2	4, 5, 6, ...	21	40.092	0.21509			
35			B_1	2, 3, 4, 5, ...	40	161200.	274.945	5.1077		
36			B_2	2, 3, 4, 5, ...	39	318459.	845.868	2.82798		
37			E	1, 2, 3(2), 4(2), ...	40	1.66366×10^7	79648.3	353.966	7.28492	
38	(1,1,0)		C_{4v}	A_1	0, 1, 2(2), 3(2), 4(3), ...	40	2.19334×10^8	3.60527×10^6	9669.56	99.3024
39		A_2		2, 3, 4(2), 5(2), ...	40	34211.4	295.311	0.804797		
40		B_1		1, 2, 3(2), 4(2), ...	40	2.97904×10^6	28179.	115.601	6.80754	
41		B_2		1, 2, 3(2), 4(2), ...	40	9.05592×10^6	71558.8	146.885	18.7042	
42	(1,1,1)	A_1		0, 1(2), 2(3), 3(4), 4(5), ...	40	5.45564×10^8	2.89417×10^6	15983.3	39.1208	8.00598
43		A_2		1, 2(2), 3(3), 4(4), ...	40	6.85779×10^6	23813.9	69.434	3.24328	
44	(0,1,2)	C_{1v}		A_1	0, 1(2), 2(3), 3(4), 4(5), ...	40	3.79138×10^8	3.16621×10^6	6213.69	17.0296
45			A_2	1, 2(2), 3(3), 4(4), ...	40	1.27375×10^7	45256.8	146.374	2.67257	

Sensitivity to second partial wave

$$\cot \delta_{2nd} = M_{22} + \frac{|M_{12}|^2}{\cot \delta_{1st} - M_{11}}$$

Since the 'pinched' points have $|M_{11}| \gg 1$, we can predict 2nd partial wave by ignoring $\cot \delta_{1st}$.

'Pinched' points can yield predictions for the 2nd partial wave.

Cubic

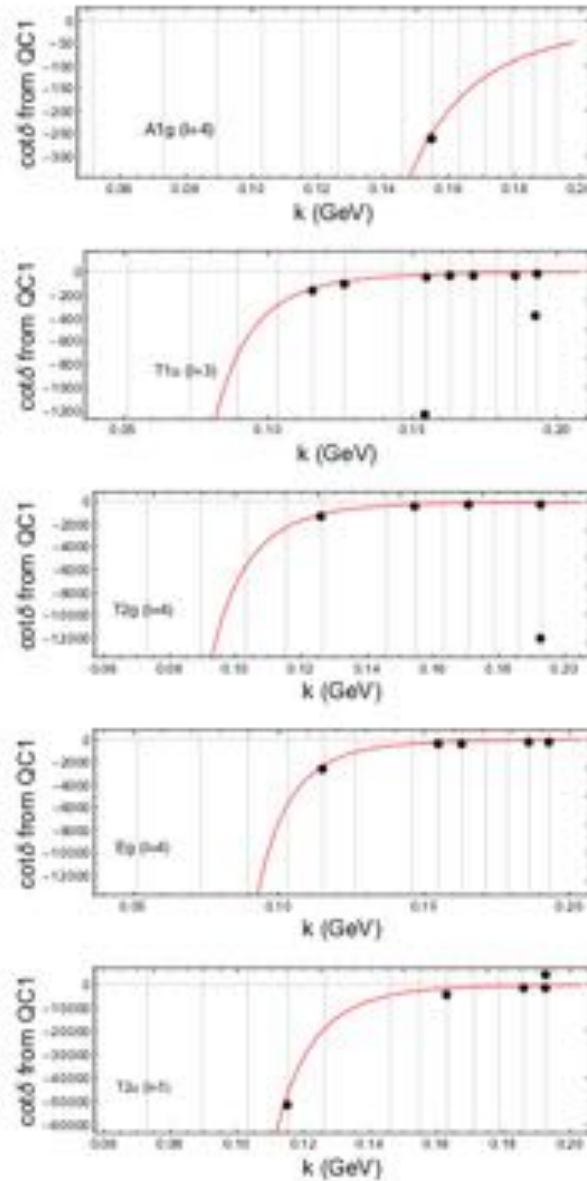


FIG. 14. Second partial waves in rest frame of cubic box. The black points are $\cos \delta_{2nd}$ in Eq.(63) with $\cos \delta_{1st}$ neglected, evaluated at the pinched box levels from order 1. The red curves are the infinite-volume values for $\cos \delta_{2nd}$. The faint vertical lines are the free-particle levels.

Elongated

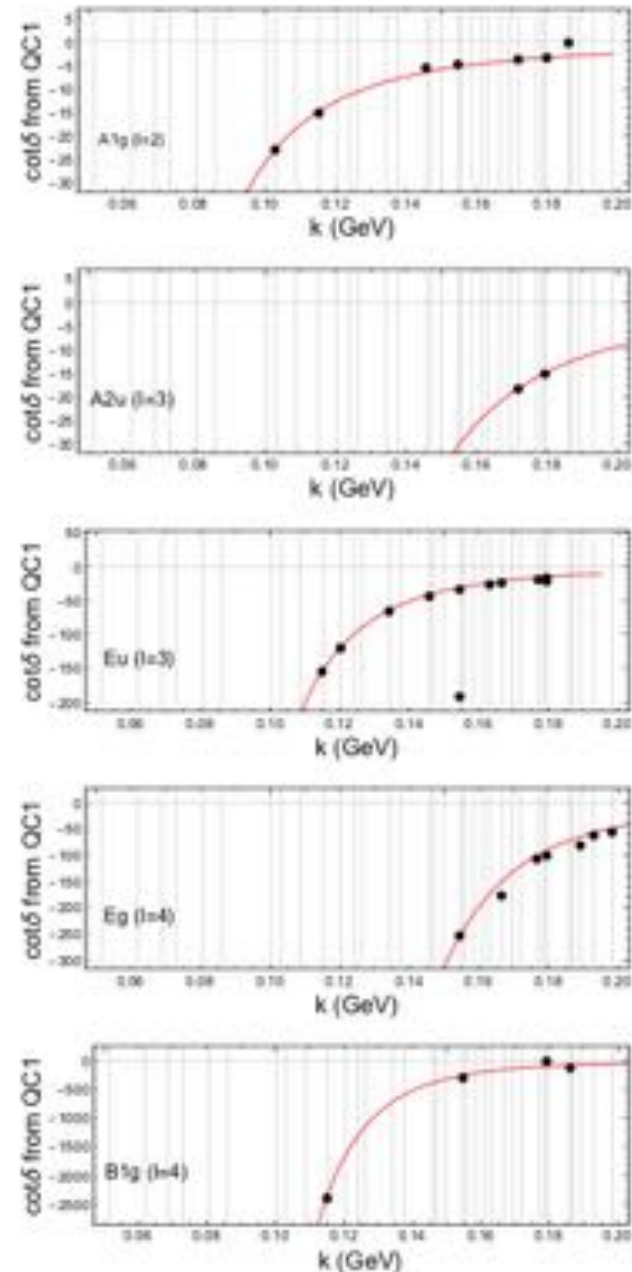


FIG. 15. Second partial waves in rest frame of elongated box.

Conclusion for validation of QC

- We validated with high precision 45=22+23 quantization conditions up to $J=5$ for two spinless particles.
 - Cubic and elongated boxes, unequal masses, moving frames
 - They are safe to use 😊
- We found sensitivity to the 2nd partial wave at the ‘pinched’ points.
- Details can be found in
 - “Higher order finite volume quantization conditions for two spinless particles”
F.X. Lee, A. Alexandru, and R. Brett
Phys. Rev. D 105, 054517 (2022), arXiv:2110.03750
 - “Scattering phaseshift formulas for mesons and baryons in elongated boxes”
F.X. Lee, A. Alexandru
Phys. Rev. D 5, 054508, e-Print: 1908.01847
 - “Energy spectrum of two-particle scattering in a periodic box”
F.X. Lee, C. Morningstar, A. Alexandru
Int.J.Mod.Phys.C 31 (2020) 09, 2050131

Outlook 1

$$V(r) = (C + c_{ls} \mathbf{L} \cdot \mathbf{S}) e^{-0.5(r/R_0)^2}$$

- How to extend the formalism to half-integer spin?
 - Example application: pion-nucleon scattering
 - Group theory is done: 25=14+11 quantization conditions are (re)-derived up to $J=9/2$ in cubic and elongated boxes and moving frames.
 - Infinite volume phaseshifts can be obtained in the $\{J^2, J_z, L^2, S^2\}$ basis
 - To do: **obtain energy levels with spin-orbit interaction on the lattice**

No.	d	Group	QC	$j(n)$
1	(0,0,0)	2O_h	G_{1g}	$\frac{1}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \dots$
2			G_{1u}	$\frac{1}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \dots$
3			G_{2g}	$\frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}, \dots$
4			G_{2u}	$\frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}, \dots$
5			H_g	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}(2), \frac{11}{2}(2), \dots$
6			H_u	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}(2), \frac{11}{2}(2), \dots$
7	(0,0,1)	$^2C_{4v}$	G_1	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}(2), \dots$
8			G_2	$\frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(2), \frac{9}{2}(2), \dots$
9	(1,1,0)	$^2C_{2v}$	G_1	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \frac{7}{2}(4), \dots$
10	(1,1,1)		K_1	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}(2), \dots$
11		$^2C_{3v}$	K_2	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}(2), \dots$
12			G_1	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(3), \dots$
13	(0,1,2)		K_1	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$
14		$^2C_{1v}$	K_2	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$

No.	d	Group	QC	$j(n)$
15	(0,0,0)	$^2D_{4h}$	G_{1g}	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}(2), \dots$
16			G_{1u}	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}(2), \dots$
17			G_{2g}	$\frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(2), \frac{9}{2}(2), \dots$
18			G_{2u}	$\frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(2), \frac{9}{2}(2), \dots$
19	(0,0,1)	$^2C_{4v}$	G_1	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}(2), \dots$
20			G_2	$\frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(2), \frac{9}{2}(2), \dots$
21	(1,1,0)	$^2C_{2v}$	G_1	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \frac{7}{2}(4), \dots$
22	(1,1,1)		K_1	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$
23		$^2C_{1v}$	K_2	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$
24	(0,1,2)		K_1	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$
25			K_2	$\frac{1}{2}, \frac{3}{2}(2), \frac{5}{2}(3), \dots$

Outlook 2

- Formalism may be useful for validating 3-body QC in a toy model?
 - Need a potential model $V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ for energy levels
 - Exact infinite volume observables
 - Reduction to a dimer + spectator model?

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 - \frac{\hbar^2}{2m_3}\nabla_3^2 + V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$$

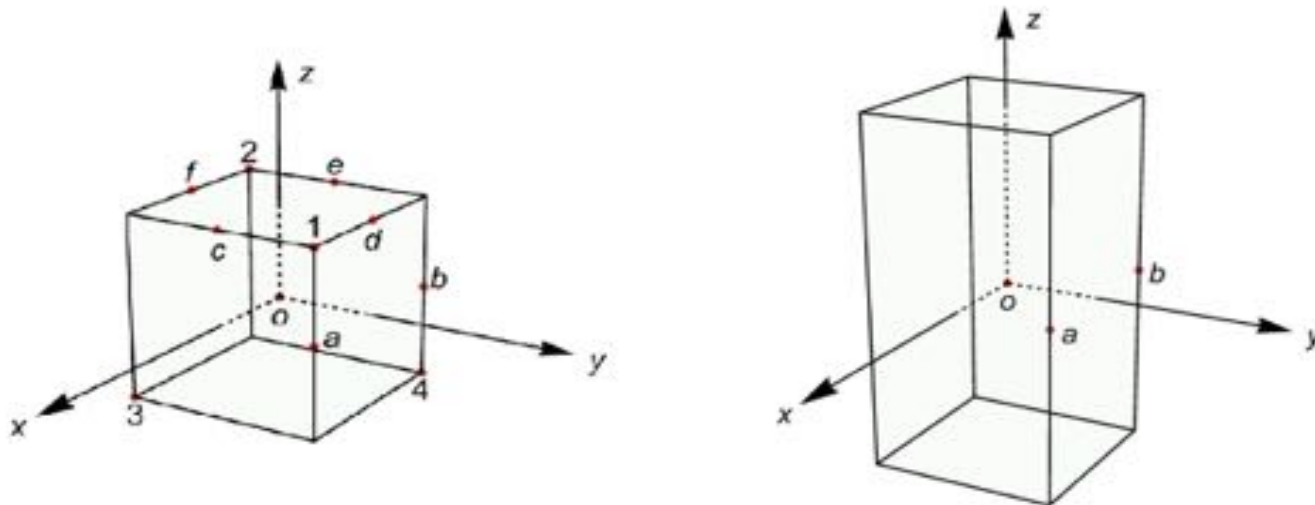
Part 2

Pion-pion scattering from Lattice QCD

Simulation parameters

Two-flavor nHYP clover fermion + Lüscher-Weisz gauge action.

Scale for this action set in PRD102, 094506 (2020), Niyazi, Alexandru, Lee



Ensemble	$N_t \times N_{x,y}^2 \times N_z$	η	$a[\text{fm}]$	N_{cfg}	am_π	$am_{u/d}^{\text{pcac}}$	af_π
$\mathcal{E}_1$ $m_\pi=315$ MeV	$48 \times 24^2 \times 24$	1.00	0.1210(2)(24)	300	0.1931(4)	0.01226(5)	0.0648(8)
$\mathcal{E}_2$	$48 \times 24^2 \times 30$	1.25			0.1944(3)	0.01239(4)	0.0651(6)
$\mathcal{E}_3$	$48 \times 24^2 \times 48$	2.00			0.1932(3)	0.01227(5)	0.0663(6)
$\mathcal{E}_4$ $m_\pi=227$ MeV	$64 \times 24^2 \times 24$	1.00	0.1215(3)(24)	400	0.1378(6)	0.00612(5)	0.0600(10)
$\mathcal{E}_5$	$64 \times 24^2 \times 28$	1.17		378	0.1374(5)	0.00620(4)	0.0600(8)
$\mathcal{E}_{66}$	$64 \times 24^2 \times 32$	1.33		400	0.1380(5)	0.00619(4)	0.0599(10)

Elongated lattices for wide kinematical coverage

$I=2$ ($\pi^+\pi^+$ s-wave scattering)

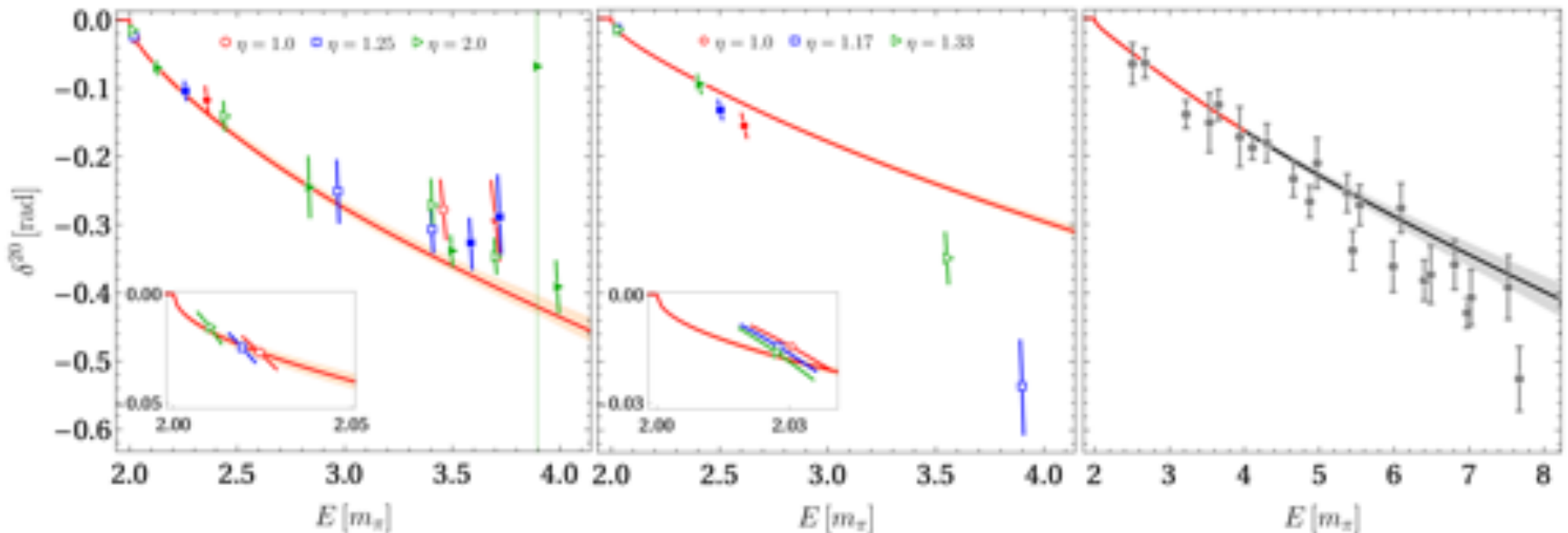
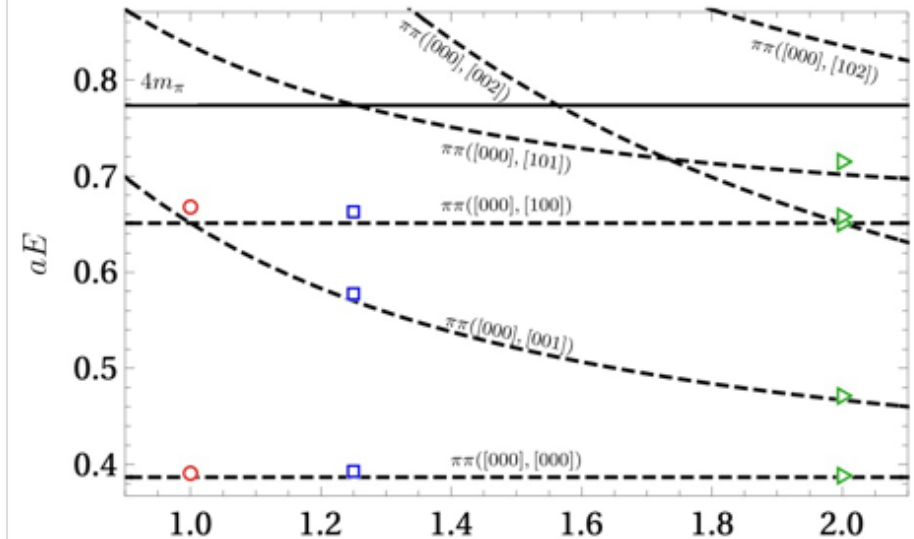
No disconnected diagrams.

Operator basis transforms as A_1+ irrep of O_h and D_{4h}

$$\pi\pi(\mathbf{P}, \mathbf{p}) = \frac{1}{|G|} \sum_{g \in G} \chi_{A_1^+}(g) \pi\pi(R(g)\mathbf{P}, R(g)\mathbf{p})$$

Phaseshifts

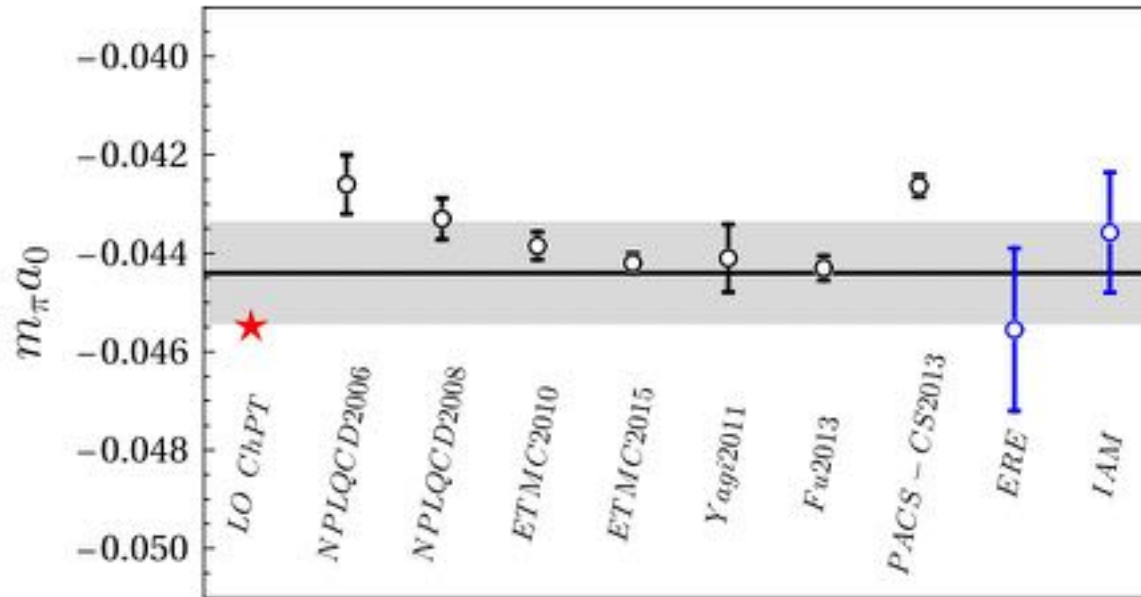
Energy levels



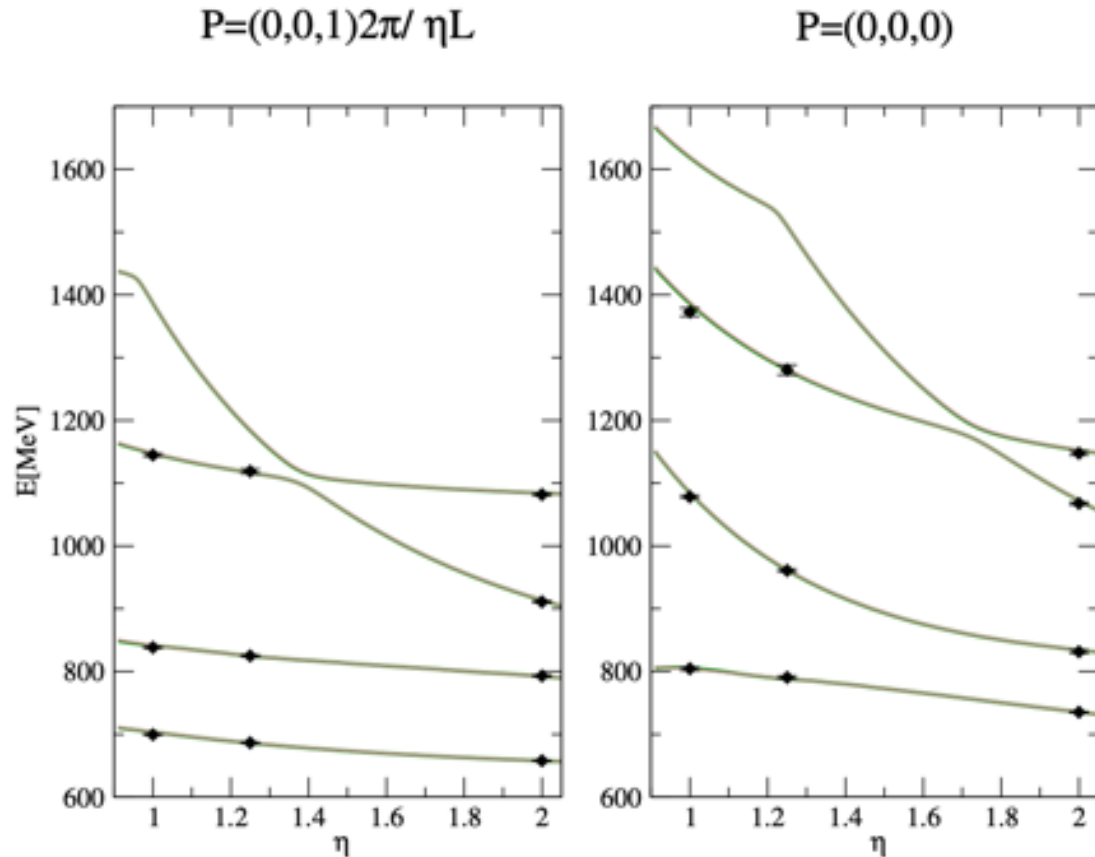
$l=2$ ($\pi^+\pi^+$ scattering length)

Effective range expansion $p \cot \delta(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2.$

m_π	Heavy 315 MeV	Light 226 MeV	Phys 139 MeV
$m_\pi a_0$	$-0.1673^{+0.0177}_{-0.0191}$	$-0.1001^{+0.0067}_{-0.0064}$	$-0.0436^{+0.0013}_{-0.0012}$



$l=1$ energy levels as function of elongation



For $m_\pi=315$ MeV, we extract 3 levels from 6x6 variational basis for $\eta=1$ and $\eta=1.25$; 4 levels from 7x7 variational basis for $\eta=2$, using double exponential fits.

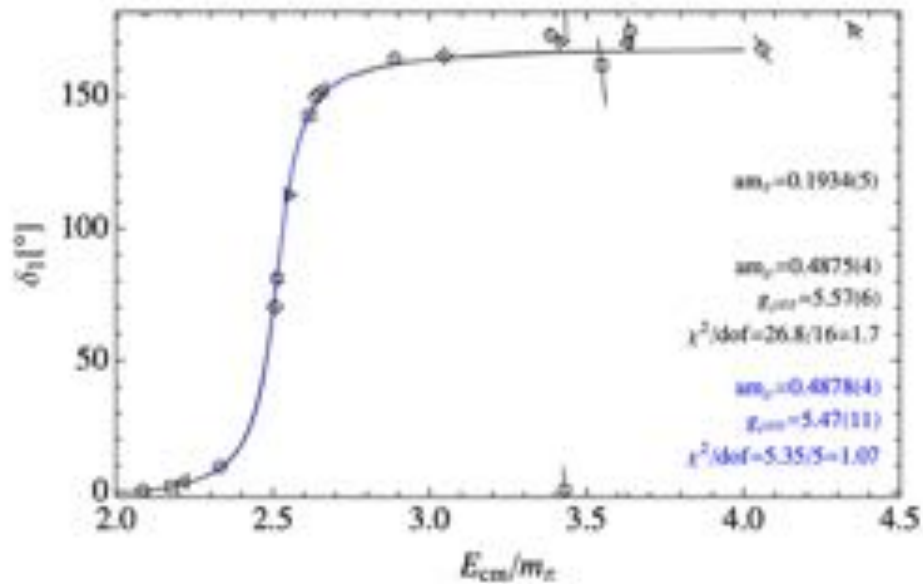
For $m_\pi=227$ MeV, we extract 19 levels from $\eta=1, 1.17, 1.33$.

C. Pelissier and AA, Phys.Rev. D87 (2013) 014503, [[arXiv:1211.0092](https://arxiv.org/abs/1211.0092)].

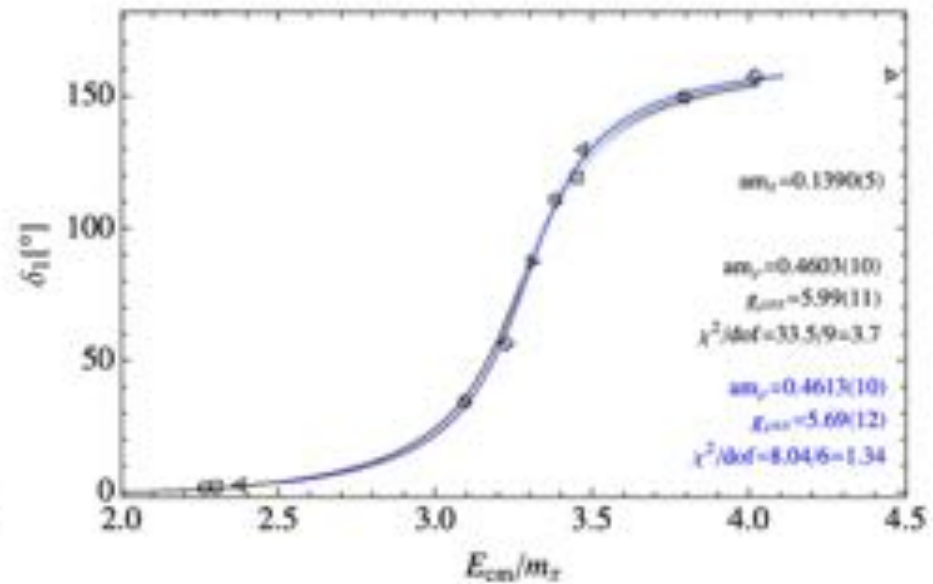
D. Guo, AA, R. Molina, and M. Doering, Phys. Rev. D94 (2016), no. 3 034501, [[arXiv:1605.03993](https://arxiv.org/abs/1605.03993)].

$l=1$ phaseshift

$m_\pi=315$ MeV



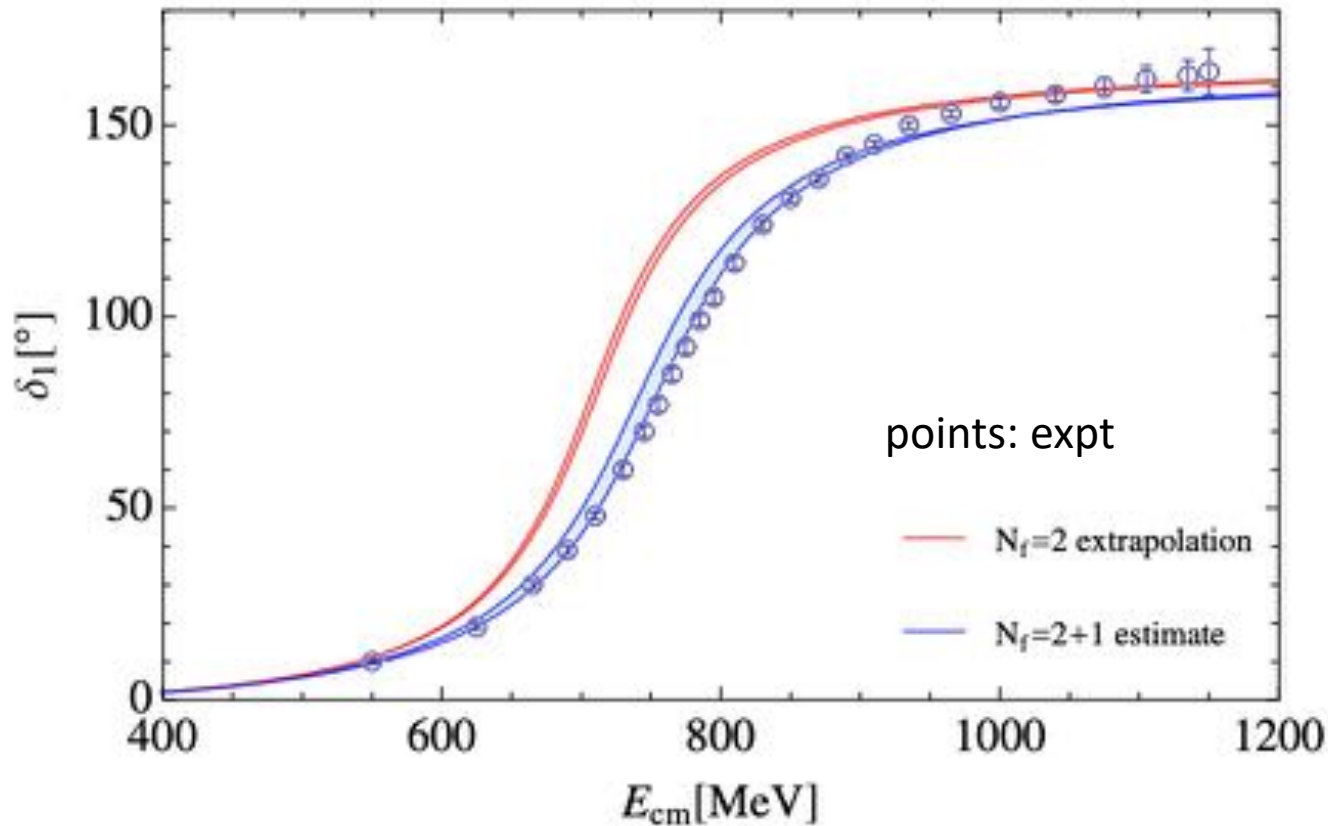
$m_\pi=227$ MeV



Points: high precision lattice data.
 Black curve: Breit-Wigner fit
 Blue curve: $U\chi$ PT fit

Triangles: cubic
 Squares: elongation 1
 Hexagons: elongation 2

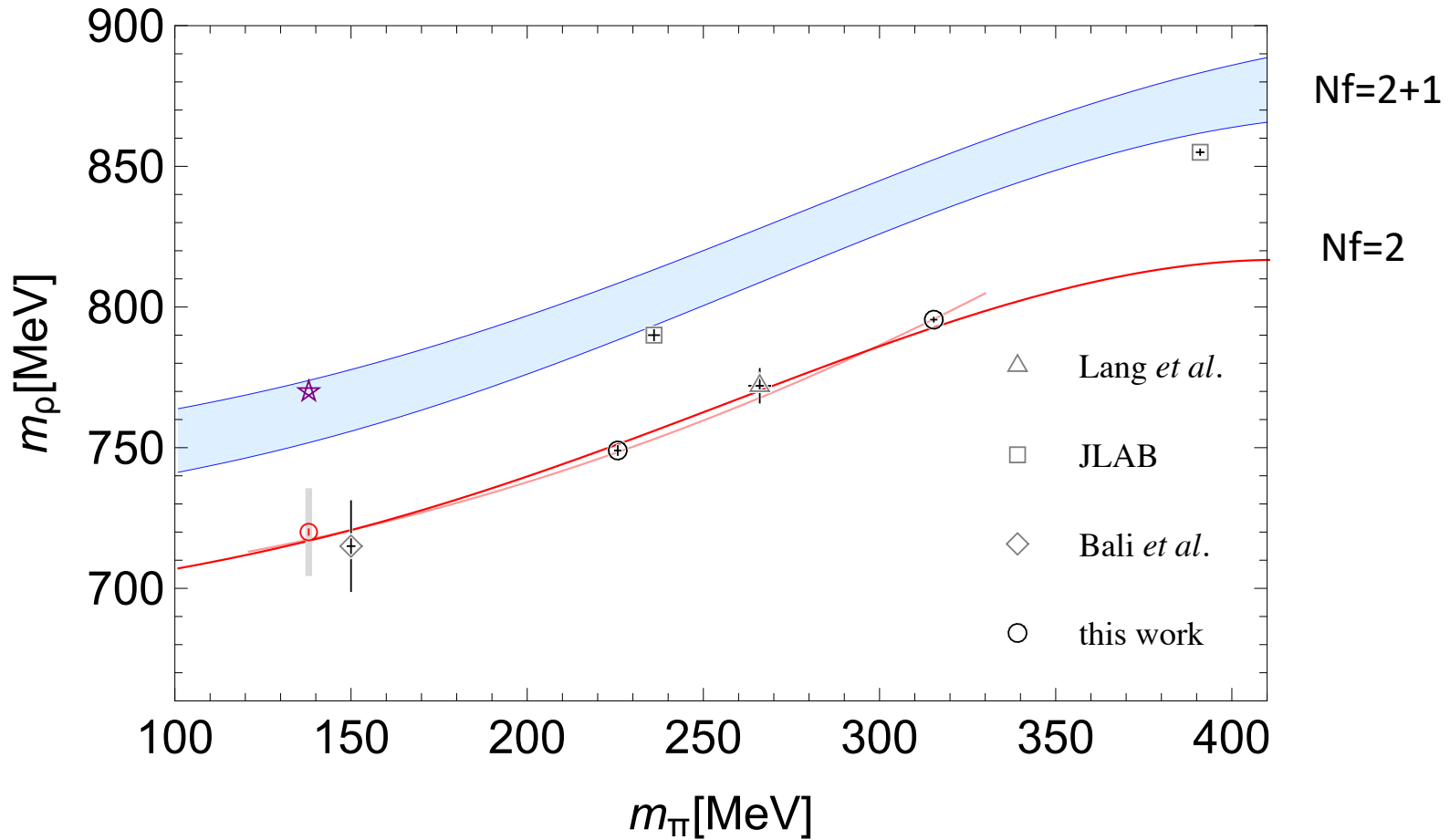
$I=1$ strange quark effects



Effects of **strange quarks** due to $\pi\pi \rightarrow K \bar{K}$ and $K \bar{K} \rightarrow K \bar{K}$ channels estimated from a $U\chi$ PT fit to our data at both pion masses.

D. Guo, AA, R. Molina, and M. Doering, Phys. Rev. D94 (2016), no. 3 034501, [arXiv:1605.03993].

ρ mass chiral extrapolation

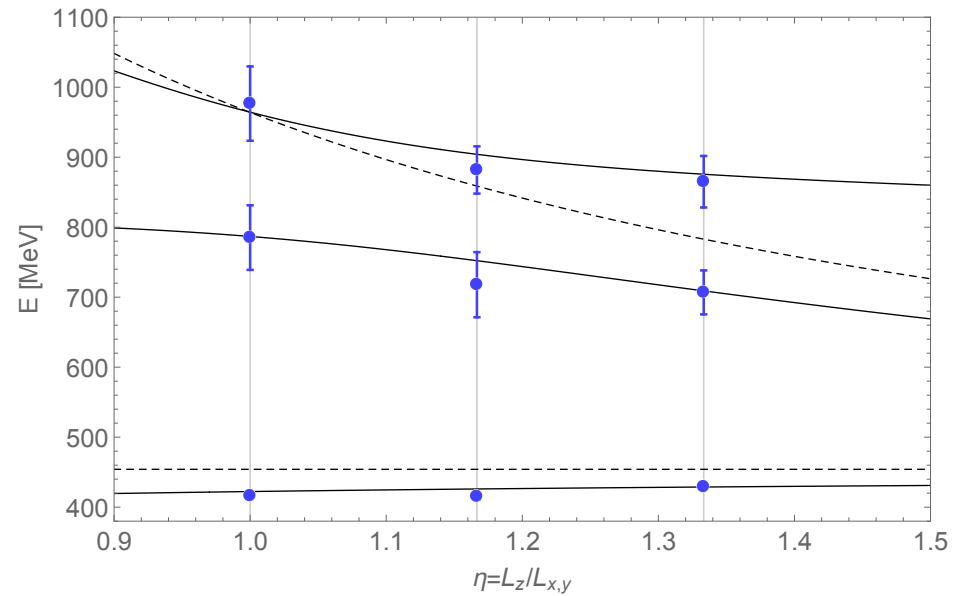


B. Hu, R. Molina, M. Doering, and AA, Phys. Rev. Lett. 117 (2016), no. 12 122001, [[arXiv:1605.04823](https://arxiv.org/abs/1605.04823)]

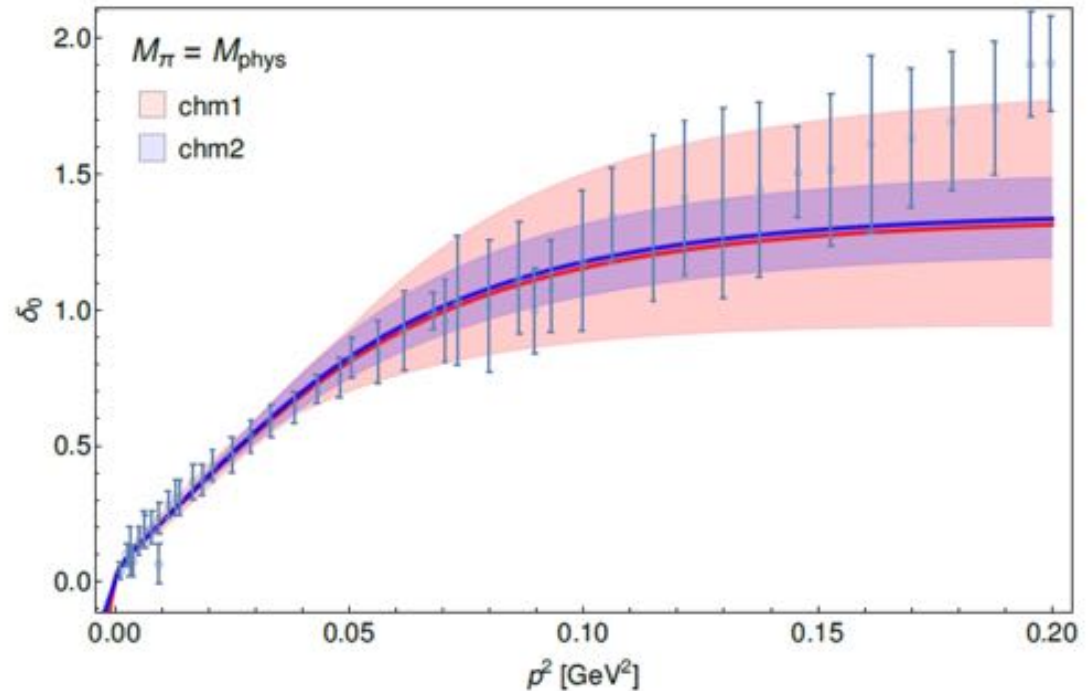
D. Guo, AA, R. Molina, and M. Doering, Phys. Rev. D94 (2016), no. 3 034501, [[arXiv:1605.03993](https://arxiv.org/abs/1605.03993)].

I=0

Vacuum subtracted energy levels
For $m_\pi=227$ MeV.

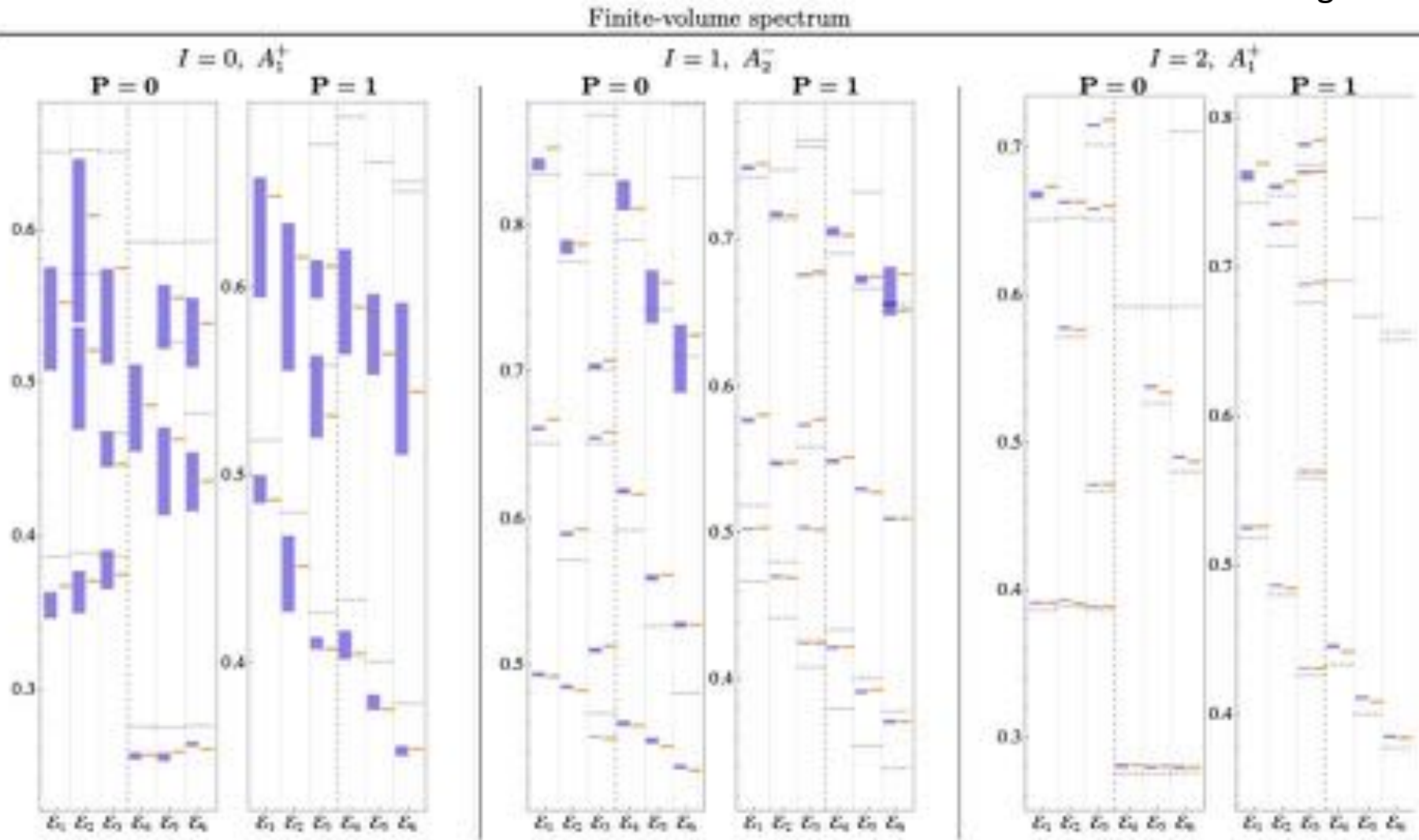


Phaseshifts extrapolated to physics point from two fits: σ (blue), $\sigma+\rho$ (red). Compared to experimental data (points).



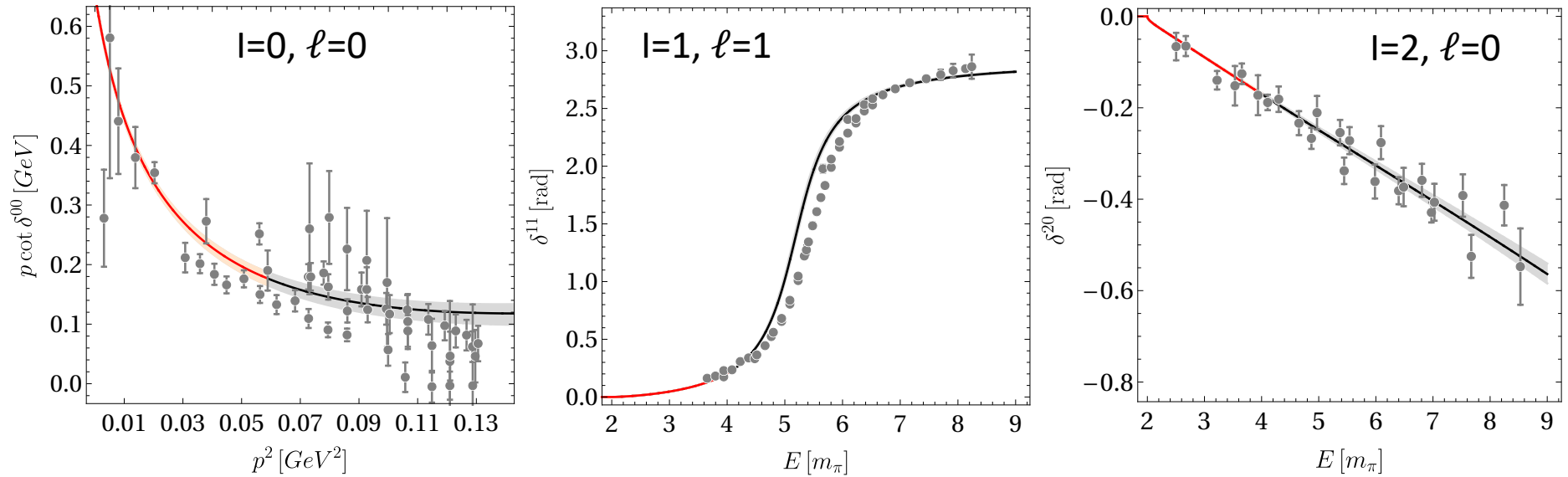
Global Fit: energy levels

Blue: lattice levels
Orange: fitted levels
Dash: noninteracting levels



M. Mai, C. Culver, AA, M. Doring, and F. X. Lee, A cross-channel study of pion scattering from lattice QCD, arXiv:1908.01847.

Global Fit: theory vs. experiment



$m_\pi [\text{MeV}]$	~ 315	~ 224	139	Phenomenology
$m_\pi a_0^{l=0}$	$+1.9008^{+0.0521}_{-0.0593}$	$+0.6985^{+0.0010}_{-0.0015}$	$+0.2132^{+0.0008}_{-0.0009}$	$+0.2198(46)_{\text{stat}}(16)_{\text{syst}}(64)_{\text{th}} [86]$
$m_\pi a_0^{l=2}$	$-0.1538^{+0.0021}_{-0.0018}$	$-0.0952^{+0.0010}_{-0.0009}$	$-0.0433^{+0.0002}_{-0.0002}$	$-0.0445(11)_{\text{stat}}(4)_{\text{syst}}(8)_{\text{th}} [86]$
$m_\sigma [\text{MeV}]$	$+591^{+6}_{-5} - i109^{+4}_{-4}$	$+502^{+4}_{-4} - i175^{+6}_{-5}$	$+443^{+3}_{-3} - i221^{+6}_{-6}$	$+449^{+22}_{-16} - i275^{+12}_{-12} [48]$
$g_{\sigma\pi\pi} [\text{MeV}]$	533^{+2}_{-2}	426^{+2}_{-2}	$397.8^{+0.6}_{-0.6}$	$345^{+25}_{-22} [48]$
$m_\rho [\text{MeV}]$	$+789^{+1}_{-1} - i20^{+0}_{-0}$	$+738^{+2}_{-1} - i43^{+1}_{-1}$	$+724^{+2}_{-4} - i67^{+1}_{-1}$	$+763.1^{+1.5}_{-1.5} - i73.3^{+1.4}_{-1.4} [87]$
$g_{\rho\pi\pi} [\text{MeV}]$	226^{+2}_{-2}	282^{+3}_{-2}	323^{+5}_{-3}	$349^{+4}_{-4} [87]$

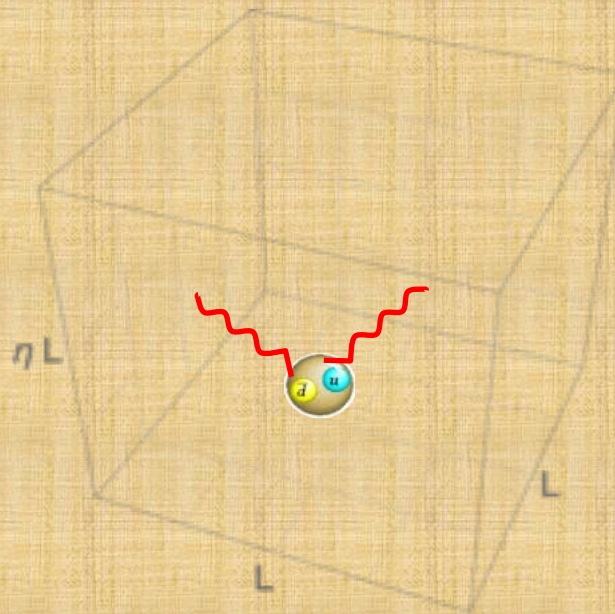
M. Mai, C. Culver, AA, M. Doring, and F. X. Lee, A cross-channel study of pion scattering from lattice QCD, arXiv:1908.01847.

Conclusion for multihadron scattering

- (Elongated box + LapH + GEVP + $U\chi$ PT) is an effective combination for studying multihadron systems.
 - $\pi\pi$ phaseshifts across $l=0,1,2$ channels
 - Three-body systems: $\pi\pi\pi$, KKK , $a_1(1260)$ resonance
 - Next target: $\Delta(1232)$ and $N(1440)$ in πN scattering

Part 3

Low-energy Compton scattering from four-point functions in lattice QCD

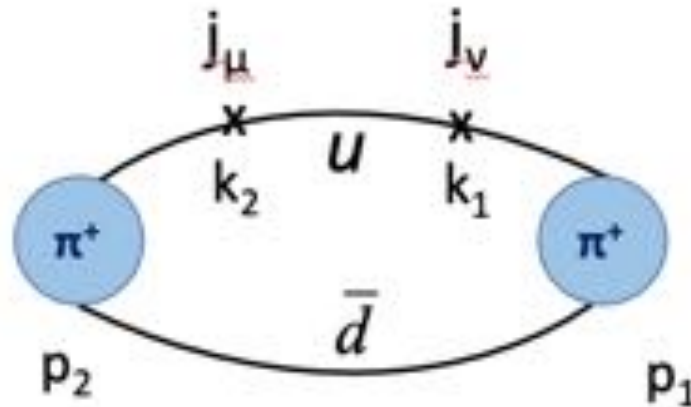


Scattering amplitude in continuum

$$\frac{d\sigma}{d\Omega} \propto |\mathcal{T}|^2 \quad \mathcal{T} = \alpha \epsilon_2^{\mu*} T_{\mu\nu} \epsilon_1^\nu$$

Four-point
function

$$T_{\mu\nu} = i \int d^4x e^{ik_2 \cdot x} (\pi(p_2) | T j_\mu(x) j_\nu(0) | \pi(p_1))$$



Compton tensor

- Lorentz invariant
- Gauge invariant
- Crossing symmetry

$$\sqrt{2E_1 2E_2} T_{\mu\nu} =$$

$$\begin{aligned} & - \frac{T_\mu(p_1 + k_1, p_1) T_\nu(p_2, p_2 + k_2)}{(p_1 + k_1)^2 - m^2} \\ & - \frac{T_\mu(p_2, p_2 - k_1) T_\nu(p_1 - k_2, p_1)}{(p_1 - k_2)^2 - m^2} + 2g_{\mu\nu} \\ & + A(k_1^2 g_{\mu\nu} - k_{1\mu} k_{1\nu} + k_2^2 g_{\mu\nu} - k_{2\mu} k_{2\nu}) \end{aligned}$$

Born
(or elastic)

$$\begin{aligned} & + B(k_1 \cdot k_2 g_{\mu\nu} - k_{2\mu} k_{1\nu}) \\ & + C(k_1 \cdot k_2 Q_\mu Q_\nu + Q \cdot k_1 Q \cdot k_2 g_{\mu\nu} \\ & \quad - Q \cdot k_2 Q_\mu k_{1\nu} - Q \cdot k_1 Q_\nu k_{2\mu}), \end{aligned}$$

Non-Born
(or inelastic)

$$Q = p_1 + p_2$$

form factor:

$$T_\mu(p', p) = (p'_\mu + p_\mu) F_\pi(q^2) + q_\mu \frac{p'^2 - p^2}{q^2} (1 - F_\pi(q^2))$$

$$q = p' - p$$

small q expansion:
$$F_\pi(q^2) = 1 + \frac{\langle r^2 \rangle}{6} q^2 + \frac{\langle r^4 \rangle}{120} q^4$$

Compton tensor (continued)

Comparing non-relativistic reduction tensor with low-energy form,

$$\alpha \epsilon_2^{\mu*} T_{\mu\nu} \epsilon_1^\nu =$$

$$\hat{\epsilon}_1 \cdot \hat{\epsilon}_2^* \left[-\frac{\alpha}{m} \left(1 + \frac{\langle r^2 \rangle}{6} (k_1^2 + k_2^2) \right) + \alpha_E \omega_1 \omega_2 \right]$$

$$+ \beta_M (\hat{\epsilon}_1 \times \vec{k}_1) \cdot (\hat{\epsilon}_2^* \times \vec{k}_2)$$

Lab frame:

Transversality condition (Coulomb gauge)

$$\epsilon_1 = (0, \hat{\epsilon}_1), \quad \epsilon_2 = (0, \hat{\epsilon}_2),$$

$$\hat{\epsilon}_1 \cdot \vec{k}_1 = 0 = \hat{\epsilon}_2 \cdot \vec{k}_2$$

$$k_1 = (\omega_1, \vec{k}_1),$$

$$k_2 = (\omega_2, \vec{k}_2),$$

$$p_1 = (m, \vec{0}),$$

$$p_2 = (m + \omega_1 - \omega_2, \vec{k}_1 - \vec{k}_2)$$

Constants of theory are related to low-energy parameters:

$$A = \frac{\langle r^2 \rangle}{3}, \quad B = \frac{2m\beta_M}{\alpha}, \quad C = -\frac{\alpha_E + \beta_M}{2m\alpha}$$

Compton tensor (continued)

Small momentum expansion in zero-momentum Breit frame.

Electric polarizability ($\mu=v=0$):

$$\begin{aligned} T_{00}(\vec{k}) &= \frac{4m_\pi}{\vec{k}^2} + \left(\frac{1}{m_\pi} - \frac{4}{3}m_\pi \langle r^2 \rangle \right) \\ &+ \left[-\frac{\langle r^2 \rangle}{3m_\pi} + \frac{1}{9}m_\pi \langle r^2 \rangle^2 + \frac{1}{15}m_\pi \langle r^4 \rangle + \frac{\alpha_E^\pi}{\alpha} \right] \vec{k}^2 \\ &\equiv T_{00}^{Born}(\vec{k}) + \frac{\alpha_E^\pi}{\alpha} \vec{k}^2, \end{aligned}$$

Magnetic polarizability ($\mu=v=1$):

$$T_{11}(\vec{k}) = -\frac{1}{m_\pi} + \vec{k}^2 \left(\frac{\langle r^2 \rangle}{3m_\pi} + \frac{\beta_M}{\alpha} \right) \equiv T_{11}^{Born}(\vec{k}) + \frac{\beta_M^\pi}{\alpha} \vec{k}^2,$$

Scattering amplitude on lattice

Consider a lattice of isotropic spacing a with N_s number of spatial sites.

$$T_{\mu\nu} = i N_s a \int dt \sum_{\vec{x}} e^{ik_2 \cdot x} \langle \pi(p_2) | T j_\mu^L(x) j_\nu^L(0) | \pi(p_1) \rangle$$

$$|n(p)\rangle \rightarrow V^{1/2} |n(p)\rangle,$$

$$j_\mu(x) \rightarrow \frac{Z_V}{a^3} j_\mu^L(x),$$

$$\int d^4x \rightarrow a^4 \int_{-\infty}^{\infty} dt \sum_{\vec{x}}$$

$$N_s = N_x \times N_y \times N_z$$

$$V = N_s a^3$$

We are still in Minkowski spacetime. We keep the time continuous but dimensionless for later convenience.

On the lattice, the contribution to this tensor when $p_1=p_2$, called Vacuum Expectation Value (VEV), must be subtracted out.

Normal ordering instead of time ordering:

$$: j_\mu^L(x) j_\nu^L(0) : = T j_\mu^L(x) j_\nu^L(0) - \langle 0 | T j_\mu^L(x) j_\nu^L(0) | 0 \rangle$$

Scattering amplitude on lattice (continued)

$$T_{\mu\nu} = i N_s a \int dt \sum_{\vec{x}} e^{ik_2 \cdot x} \langle \pi(p_2) | T j_\mu^L(x) j_\nu^L(0) | \pi(p_1) \rangle$$

Inserting a complete set of intermediate states,

$$\sum_{n, \vec{p}_n, s} |n(\vec{p}_n, s)\rangle \langle n(\vec{p}_n, s)| = 1$$

$$j_\mu^L(x) = e^{ip_\pi \cdot x} j_\mu^L(0) e^{-ip_n \cdot x} = e^{im_\pi t a} j_\mu^L(0) e^{-i(E_n a t - \vec{p}_n \cdot \vec{x})}$$

$$\sum_{\vec{x}} e^{i(-\vec{q} + \vec{p}_n) \cdot \vec{x}} = \delta_{\vec{q}, \vec{p}_n} N_s$$

$$\int_{-\infty}^{\infty} dt e^{ia(m_\pi - E_n)t} = \frac{2}{ia(E_n - m_\pi)}$$

Diagonal components:

$$T_{\mu\mu}(\vec{q}) = 2N_s^2 \sum_{n,s} \frac{|\langle \pi(\vec{0}) | j_\mu^L(0) | n(s, \vec{q}) \rangle|^2}{E_n - m_\pi} - 2N_s^2 \sum_{n,s} \frac{|\langle 0 | j_\mu^L(0) | n(s, \vec{q}) \rangle|^2}{E_n}$$


$T_{\mu\mu}^{elas}$ is the first term in the sum

Polarizability formulas

Charged Pion

$$\alpha_E^\pi = \alpha \left[\frac{\langle r^2 \rangle}{3m_\pi} + \frac{T_{00}(\vec{q}) - T_{00}^{elas}(\vec{q})}{\vec{q}^2} \right]$$

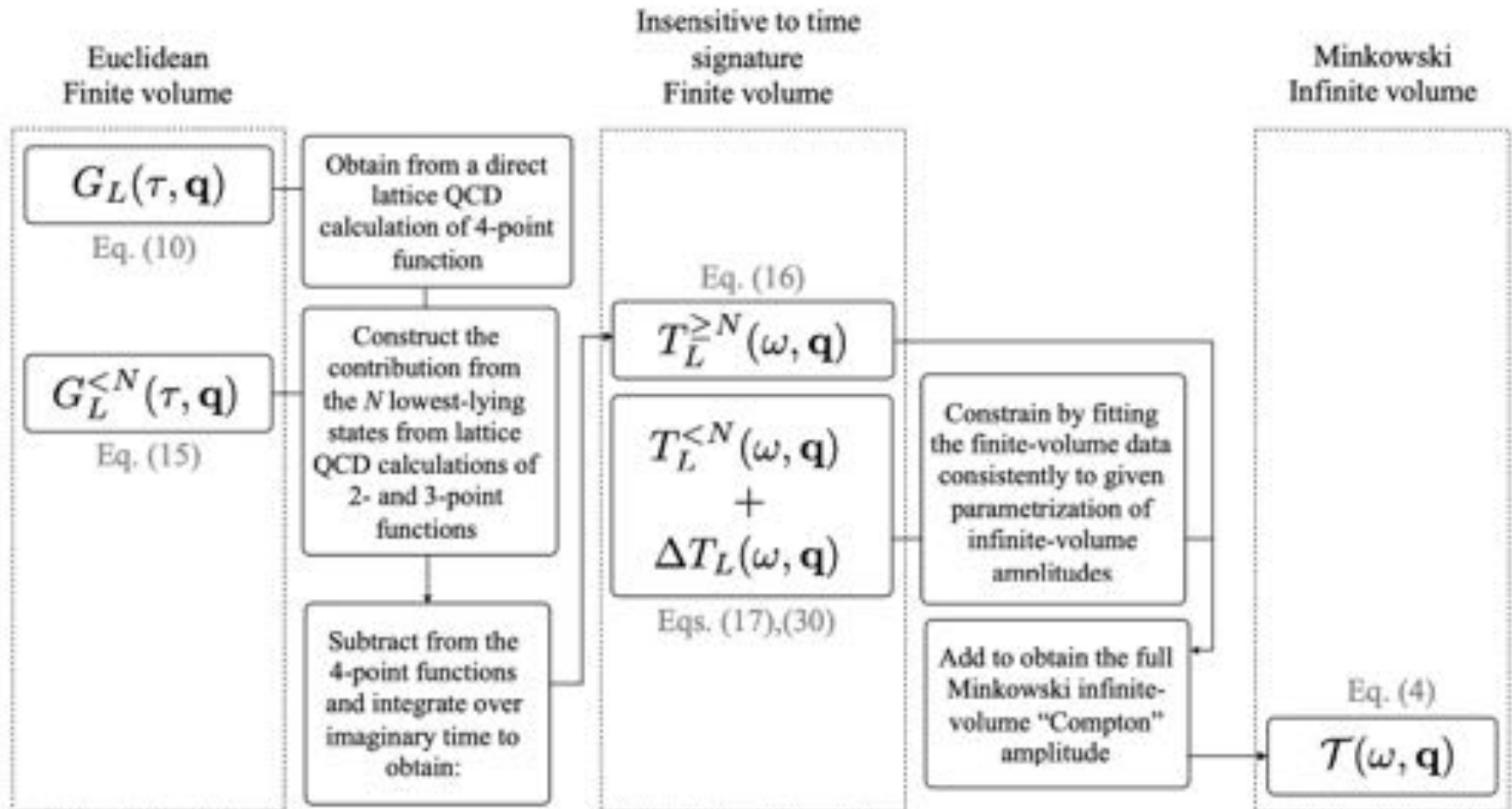
$$\beta_M^\pi = \alpha \left[-\frac{\langle r^2 \rangle}{3m_\pi} + \frac{T_{11}(\vec{q}) - T_{11}(0)}{\vec{q}^2} \right]$$

Proton

$$\alpha_E^p = \alpha \left[\frac{\langle r_E^2 \rangle}{3m_p} + \frac{1 + \kappa^2}{4m_p^3} + \frac{T_{00}(\vec{q}) - T_{00}^{elas}(\vec{q})}{\vec{q}^2} \right]$$

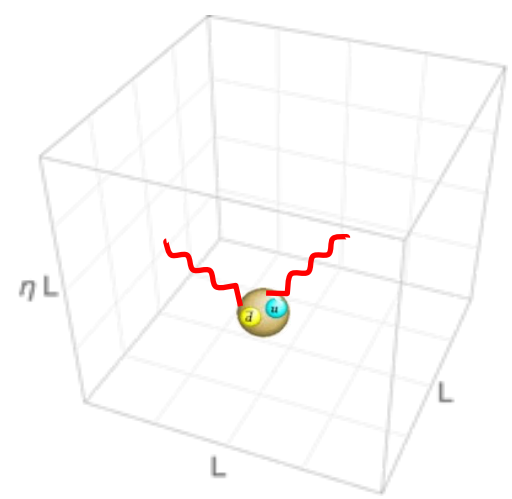
$$\beta_M^p = \alpha \left[-\frac{\langle r_E^2 \rangle}{3m_p} - \frac{1 + \kappa + \kappa^2}{2m_p^3} + \frac{T_{11}(\vec{q}) - T_{11}^{elas}(\vec{q}) - T_{11}(\vec{0})}{\vec{q}^2} \right]$$

How is it related to



“Long-range electroweak amplitudes of single hadrons from Euclidean finite-volume correlation functions”, Raúl A. Briceño, Zohreh Davoudi, Maxwell T. Hansen, Matthias R. Schindler, Alessandro Baroni, Phys. Rev. D 101, 014509 (2020).

Charged pion polarizability on the lattice



$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{q^2} \int_0^\infty dt \left[Q_{44}(\mathbf{q}, t) - Q_{44}^{elas}(\mathbf{q}, t) \right]$$

m_π can be extracted from two-point functions.

Charge radius can be extracted from elastic part of the same Q_{44} ,

$$Q_{44}^{elas}(\mathbf{q}, t) \xrightarrow{t \gg 1} \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\mathbf{q}^2) e^{-a(E_\pi(\mathbf{q}) - m_\pi)t}$$

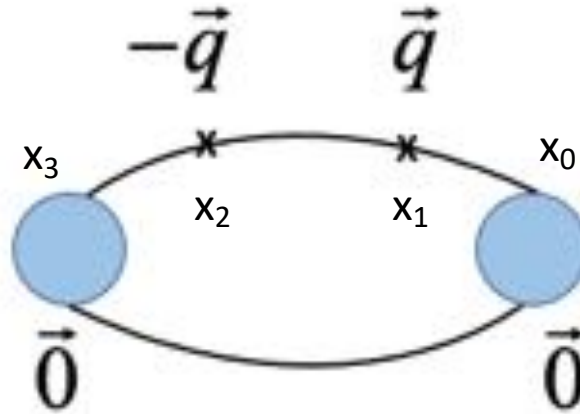
Four-point function

$$\frac{\sum_{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0} e^{-i\mathbf{q} \cdot \mathbf{x}_2} e^{i\mathbf{q} \cdot \mathbf{x}_1} \langle \Omega | \psi^\dagger(x_3) : j_\mu^L(x_2) j_\nu^L(x_1) : \psi(x_0) | \Omega \rangle}{\sum_{\mathbf{x}_3, \mathbf{x}_0} \langle \Omega | \psi^\dagger(x_3) \psi(x_0) | \Omega \rangle}$$

$$\equiv Q_{\mu\nu}(\mathbf{q}, t_3, t_2, t_1, t_0)$$

Kinematics

(zero-momentum Breit frame)



Euclidean
spacetime

Zero-momentum pions are approximated by [wall sources](#) without gauge-fixing.

At large time separations,

$$Q_{\mu\mu}(\vec{q}, t) = N_s^2 \sum_n |\langle \pi(\vec{0}) | j_\mu^L(0) | n(\vec{q}) \rangle|^2 e^{-a(E_n - m_\pi)t} - N_s^2 \sum_n |\langle 0 | j_\mu^L(0) | n(\vec{q}) \rangle|^2 e^{-aE_n t}.$$

[Elastic contribution](#) is from $n=1$ term in the first sum.

Local vs. Conserved current

$$\frac{\sum_{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0} e^{-i\mathbf{q} \cdot \mathbf{x}_2} e^{i\mathbf{q} \cdot \mathbf{x}_1} \langle \Omega | \psi^\dagger(\mathbf{x}_3) : j_\mu^L(\mathbf{x}_2) j_\nu^L(\mathbf{x}_1) : \psi(\mathbf{x}_0) | \Omega \rangle}{\sum_{\mathbf{x}_3, \mathbf{x}_0} \langle \Omega | \psi^\dagger(\mathbf{x}_3) \psi(\mathbf{x}_0) | \Omega \rangle}$$

$$\equiv Q_{\mu\nu}(\mathbf{q}, t_3, t_2, t_1, t_0)$$

Pion operator $\psi_{\pi^+}(x) = \bar{d}(x) \gamma_5 u(x)$

Local current $j_\mu^{(PC)} = Z_V (q_u \bar{u} \gamma_\mu u + q_d \bar{d} \gamma_\mu d)$

Requires renormalization, but simpler correlation functions:

20 diagrams if u and d distinct, 12 diagrams if isospin symmetry ($m_u=m_d$)

Conserved-current (point-split) for Wilson-like fermions,

$$j_\mu^{(PS)}(x) = q_u \kappa [-\bar{u}(x)(1 - \gamma_\mu) U_\mu(x) u(x + a\hat{\mu}) + \bar{u}(x + a\hat{\mu})(1 + \gamma_\mu) U_\mu^\dagger(x) u(x)] \\ + q_d \kappa [-\bar{d}(x)(1 - \gamma_\mu) U_\mu(x) d(x + a\hat{\mu}) + \bar{d}(x + a\hat{\mu})(1 + \gamma_\mu) U_\mu^\dagger(x) d(x)]$$

No renormalization ($Z_V=1$) but more complicated functions:

80 diagrams if u and d distinct, 48 diagrams if isospin symmetry.

Involves gauge links explicitly.

$Q_{44}(\mathbf{q}=0, t_1, t_2)$ = charge factor (independent of t_1 and t_2)

Wick contractions



connected insertion: different flavor



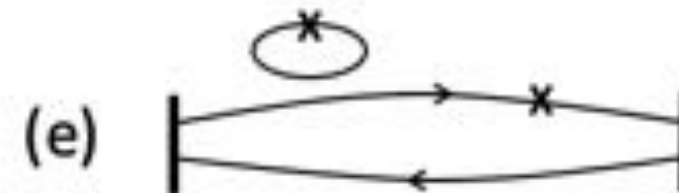
connected insertion: same flavor



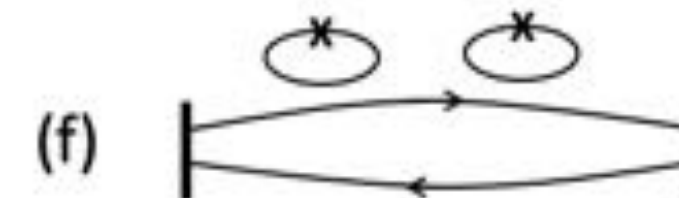
connected insertion: same flavor Z-graph



disconnected insertion: single loop, double current



disconnected insertion: single loop



disconnected insertion: double loop

Lattice setup

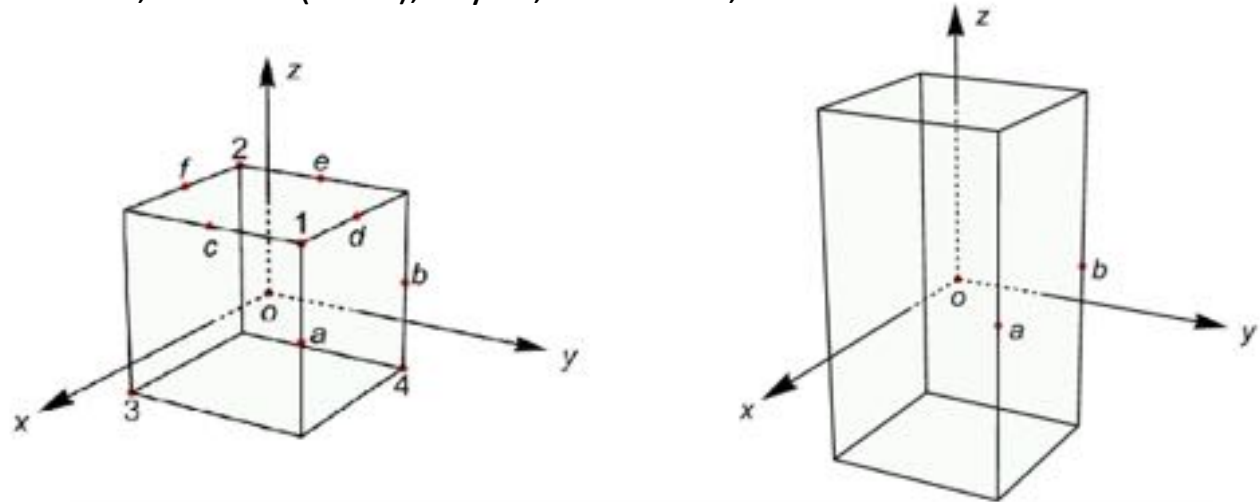
Proof-of-concept:

Quenched Wilson action at $\beta=6.0$ and $\kappa=0.152$ and 0.1555 , 24^4 lattice, Dirichlet bc, 100 configs.

Production runs:

Two-flavor nHYP clover fermion + Lüscher-Weisz gauge action.

Scale for this action set in PRD102, 094506 (2020), Niyazi, Alexandru, Lee



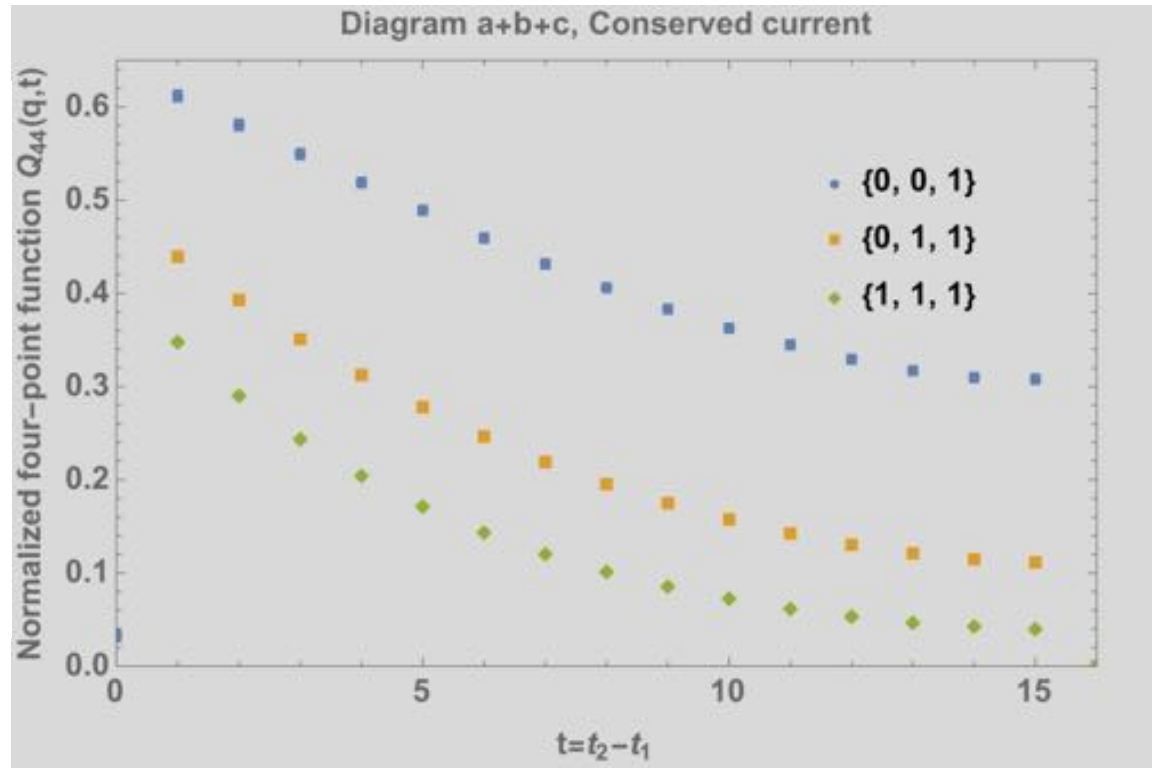
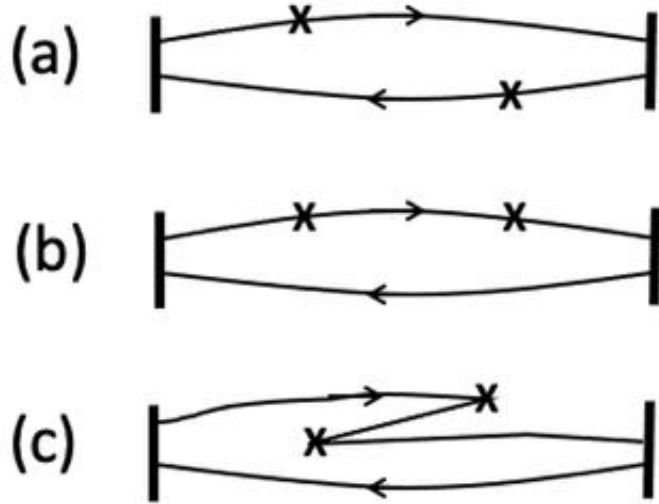
$m_\pi=315$ MeV

$m_\pi=227$ MeV

ensemble	$N_t \times N_{x,y}^2 \times N_z$	η	$a[\text{fm}]$	N_{cfg}	aM_π	am_N	$am_{u/d}^{\text{pcac}}$	af_π
$\mathcal{E}_1$	$48 \times 24^2 \times 24$	1.0	0.1210(2)(24)	300	0.1934(5)	0.644(6)	0.01237(9)	0.0648(8)
$\mathcal{E}_2$	$48 \times 24^2 \times 30$	1.25	—	—	—	—	—	—
$\mathcal{E}_3$	$48 \times 24^2 \times 48$	2.0	—	—	—	—	—	—
$\mathcal{E}_4$	$64 \times 24^2 \times 24$	1.0	0.1215(3)(24)	400	0.1390(5)	0.62(1)	0.00617(9)	0.060(1)
$\mathcal{E}_5$	$64 \times 24^2 \times 28$	1.17	—	—	—	—	—	—
$\mathcal{E}_6$	$64 \times 24^2 \times 32$	1.33	—	—	—	—	—	—

Elongated lattices to study volume dependence

Diagrams (a+b+c)

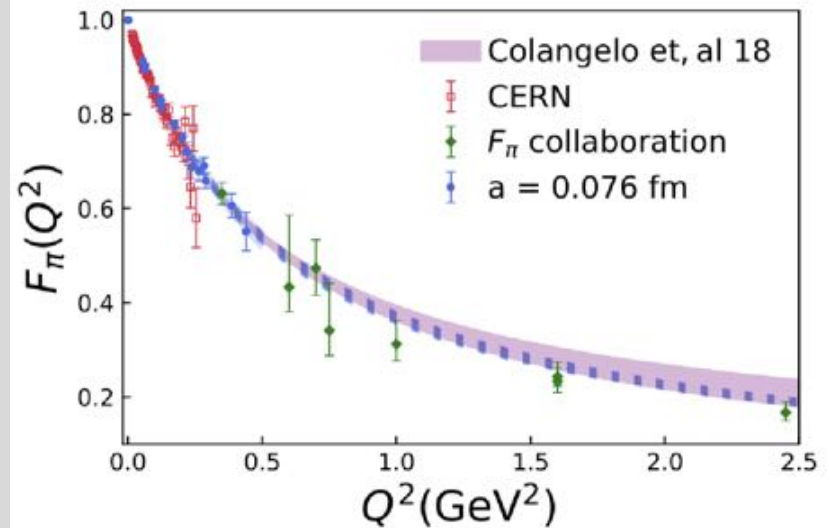
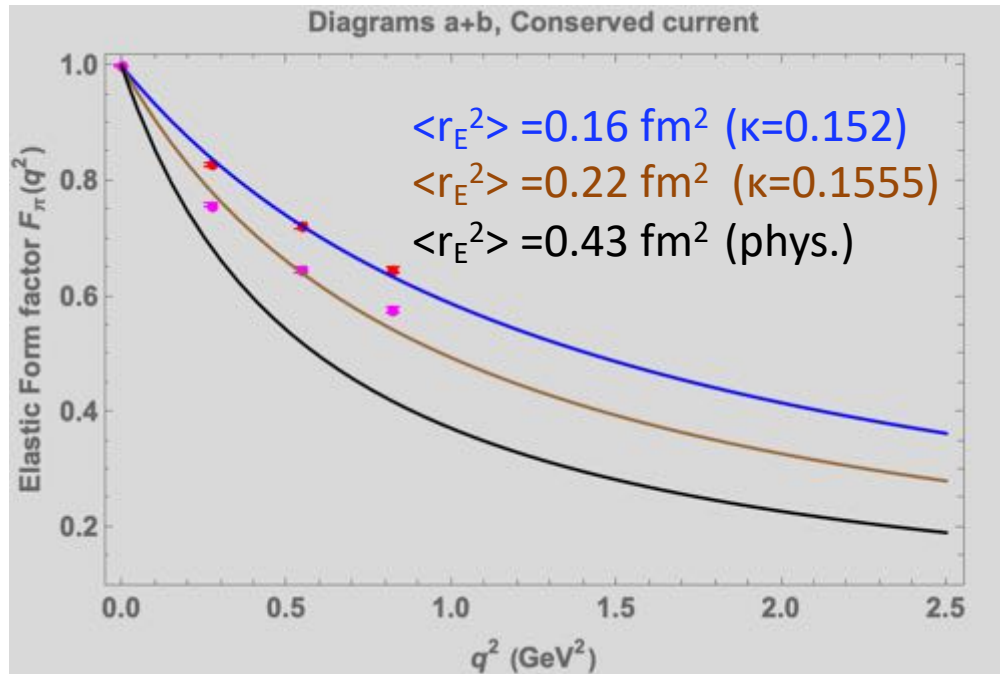


$$\alpha_{\pi E}^{\pi} = \frac{\alpha \langle r_E^2 \rangle}{3m_{\pi}} + \frac{2\alpha a}{q^2} \int_0^{\infty} dt \left[Q_{44}(\mathbf{q}, t) - Q_{44}^{elas}(\mathbf{q}, t) \right]$$

Pion form factor

$$Q_{44}^{elas}(\mathbf{q}, t) = \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\mathbf{q}^2) e^{-a(E_\pi - m_\pi)t}$$

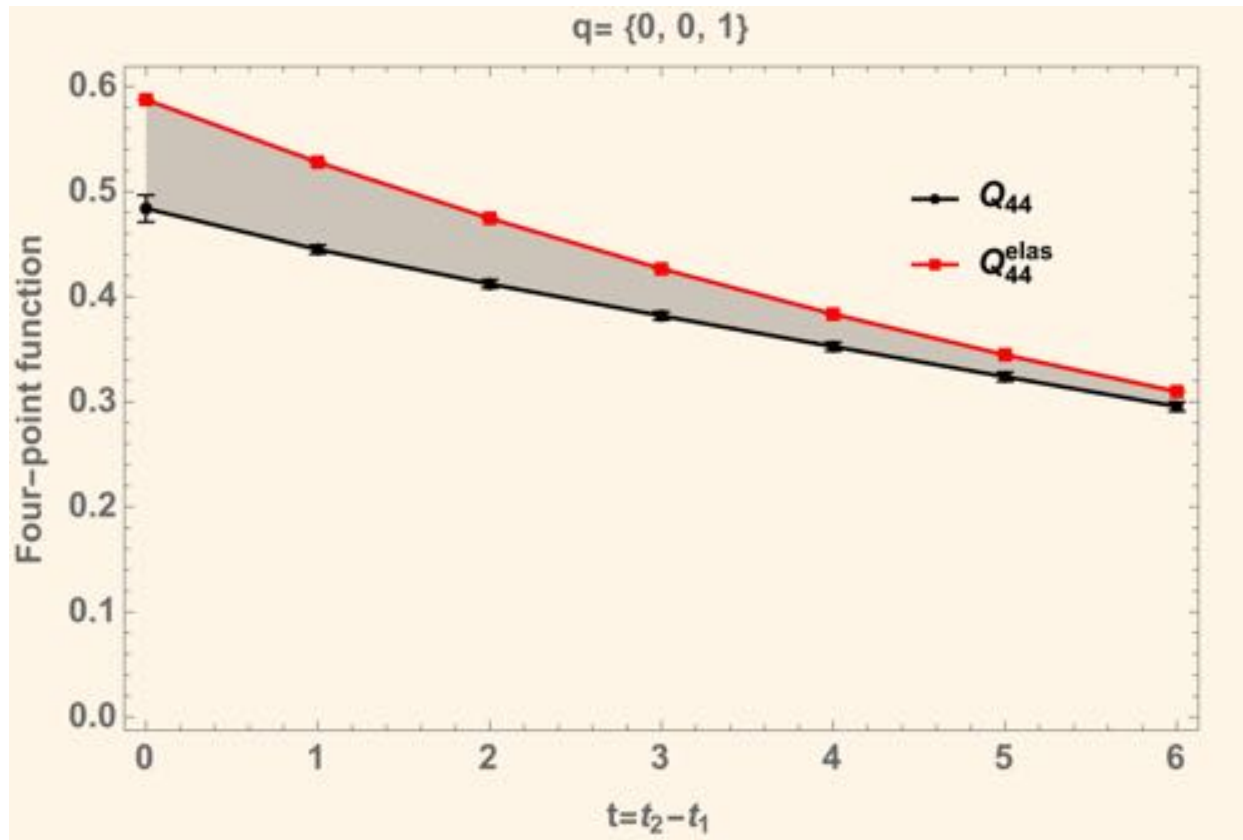
$$F_\pi(\mathbf{q}^2) = 1 / \left(1 + \frac{\mathbf{q}^2}{m^2}\right) \quad \langle r_E^2 \rangle = -6 \frac{dF_\pi(\mathbf{q}^2)}{d\mathbf{q}^2} \quad E_\pi = \sqrt{\mathbf{q}^2 + m_\pi^2}$$



(arXiv:2102.06047, Gao et al)

Q integral

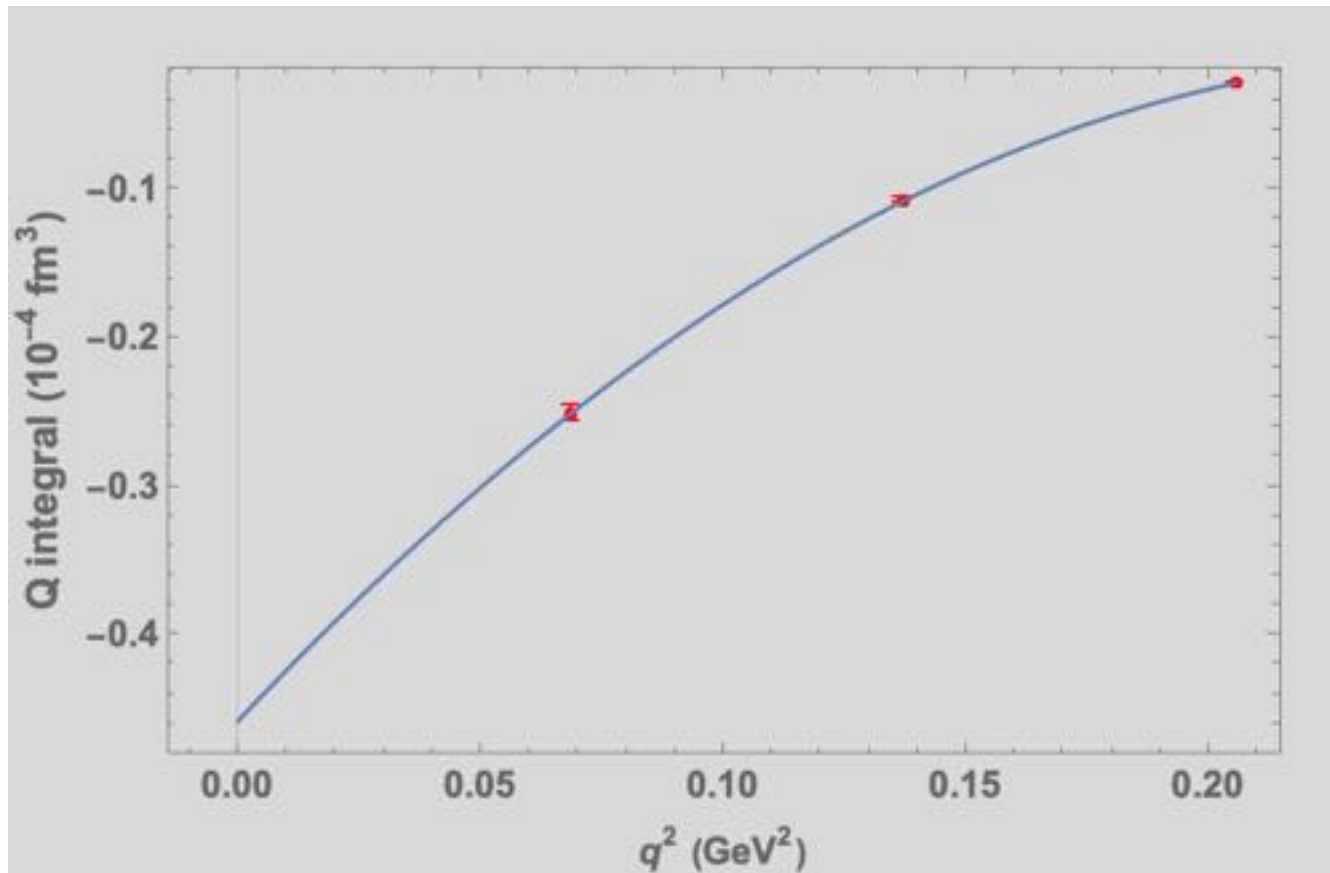
$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{q^2} \int_0^\infty dt [Q_{44}(\mathbf{q}, t) - Q_{44}^{elas}(\mathbf{q}, t)]$$



Negative of the shaded area

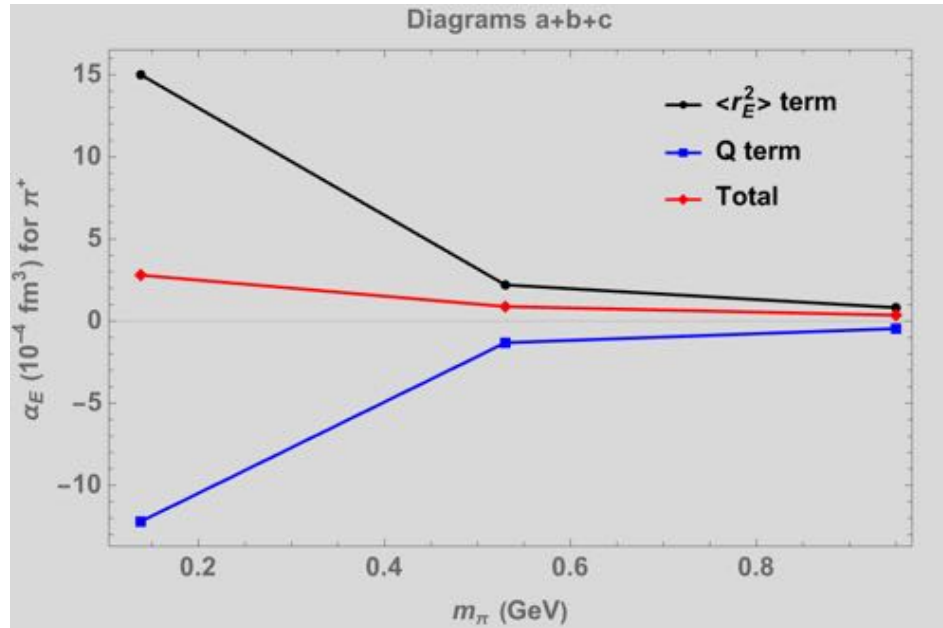
Extrapolation to $q^2=0$

$$\alpha_{\pi}^{\pi} = \frac{\alpha \langle r_E^2 \rangle}{3m_{\pi}} + \frac{2\alpha a}{q^2} \int_0^{\infty} dt [Q_{44}(\mathbf{q}, t) - Q_{44}^{elas}(\mathbf{q}, t)]$$



Pion mass dependence

$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{q^2} \int_0^\infty dt [Q_{44}(\mathbf{q}, t) - Q_{44}^{elas}(\mathbf{q}, t)]$$



α_E (10^{-4} fm^3)	$m_\pi=960 \text{ MeV}$	$m_\pi=530 \text{ MeV}$	$m_\pi=138 \text{ MeV}$
$\langle r_E^2 \rangle$ term	0.82	2.21	15.0
Q term	-0.46	-1.32	-12.2
Total	0.58	1.71	2.8

Strong m_π dependence of individual terms but opposite sign. Gentle dependence of the total.

Conclusion on Compton scattering

- Four-point functions offer good physics payout
 - form factors (charge radius)
 - polarizabilities
 - renormalization constant Z_V (local vs. conserved current)
- Alternative to background field method for charged hadrons that avoids the issues of
 - acceleration in electric field
 - Landau levels in magnetic field
- Proof-of-concept simulations for π^+ with quenched Wilson fermions on 24^4 lattices so far
 - validated the connected diagrams for conserved currents
 - yielded encouraging results on $\langle r_E^2 \rangle$ and α_E
 - Issues to study: $t=0$ and $q^2=0$ extrapolations, Dirichlet bc
- Outlook
 - Dynamical ensembles (two-flavor nhyp fermions, smaller m_{π} , infinite-volume extrapolation)
 - magnetic polarizability (β_M) from Q_{11} components
 - disconnected insertions
 - proton ($\langle r_E^2 \rangle$, α_E , μ_p , β_M)
 - Understand and utilize the general formalism developed by Raul and Co.
 - Generalized polarizabilities in virtual Compton scattering?