Finite-volume formalism and the t-channel cut

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based on work with Andre Raposo

UK Research and Innovation

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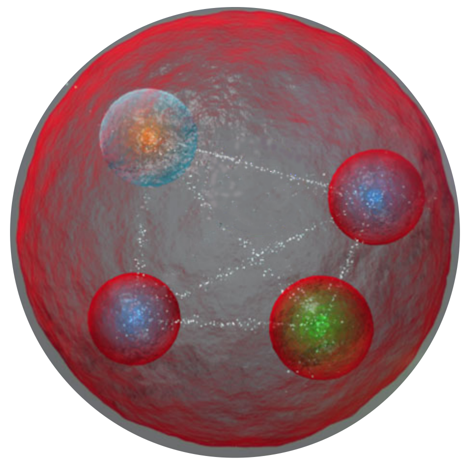
Multi-hadron observables

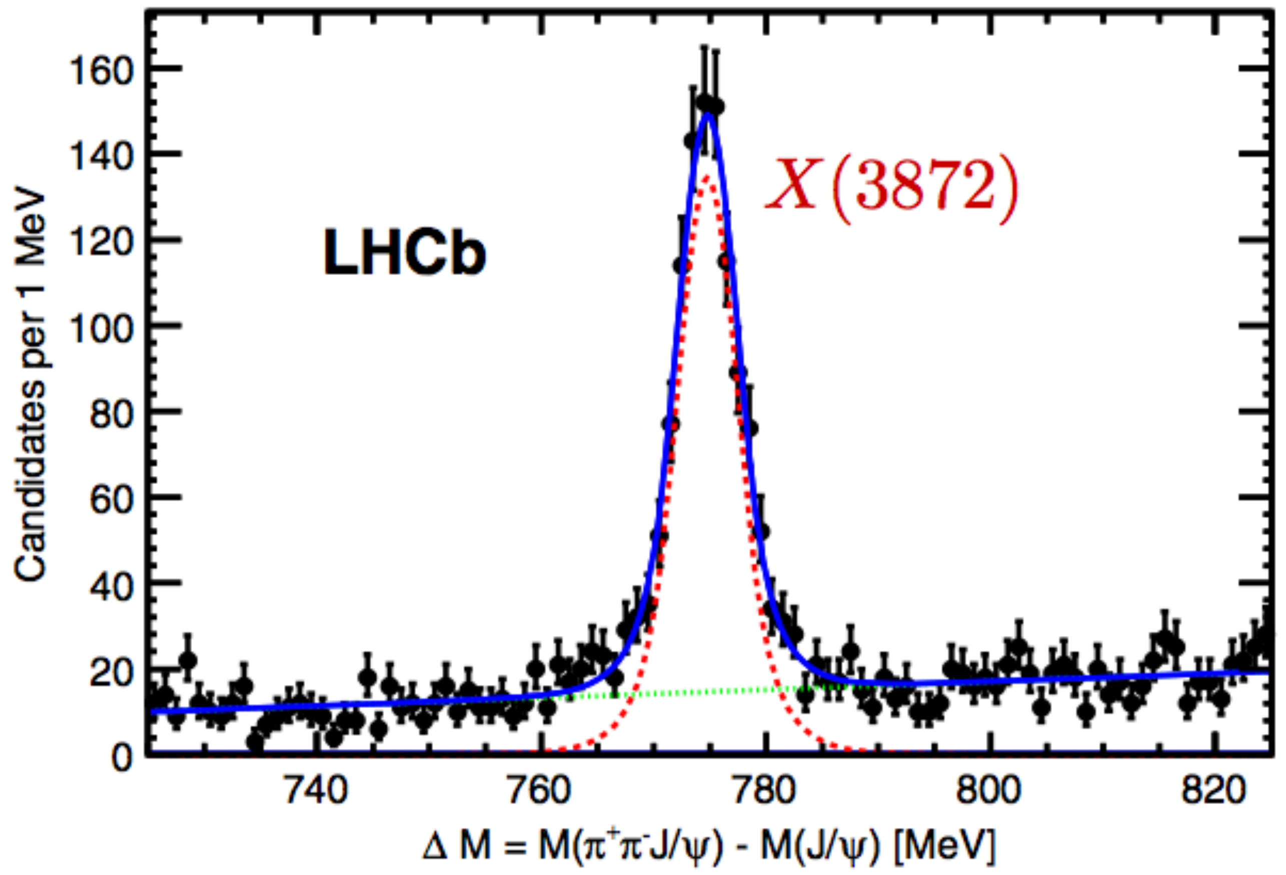
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

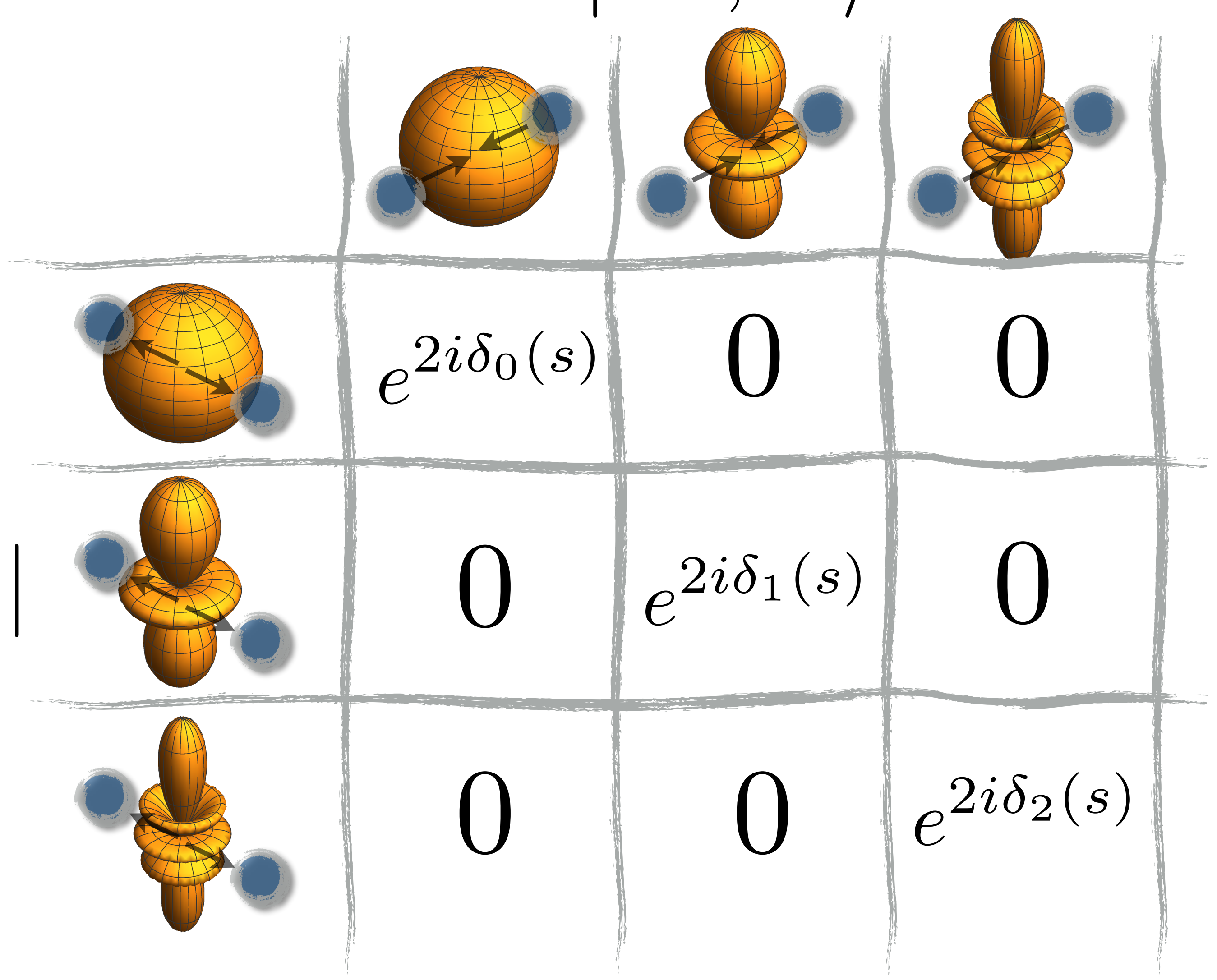
Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$ $|\pi\pi, \mathrm{in}\rangle$



depends on $s = E_{\mathrm{cm}}^2$
and angular variables

diagonal in angular momentum

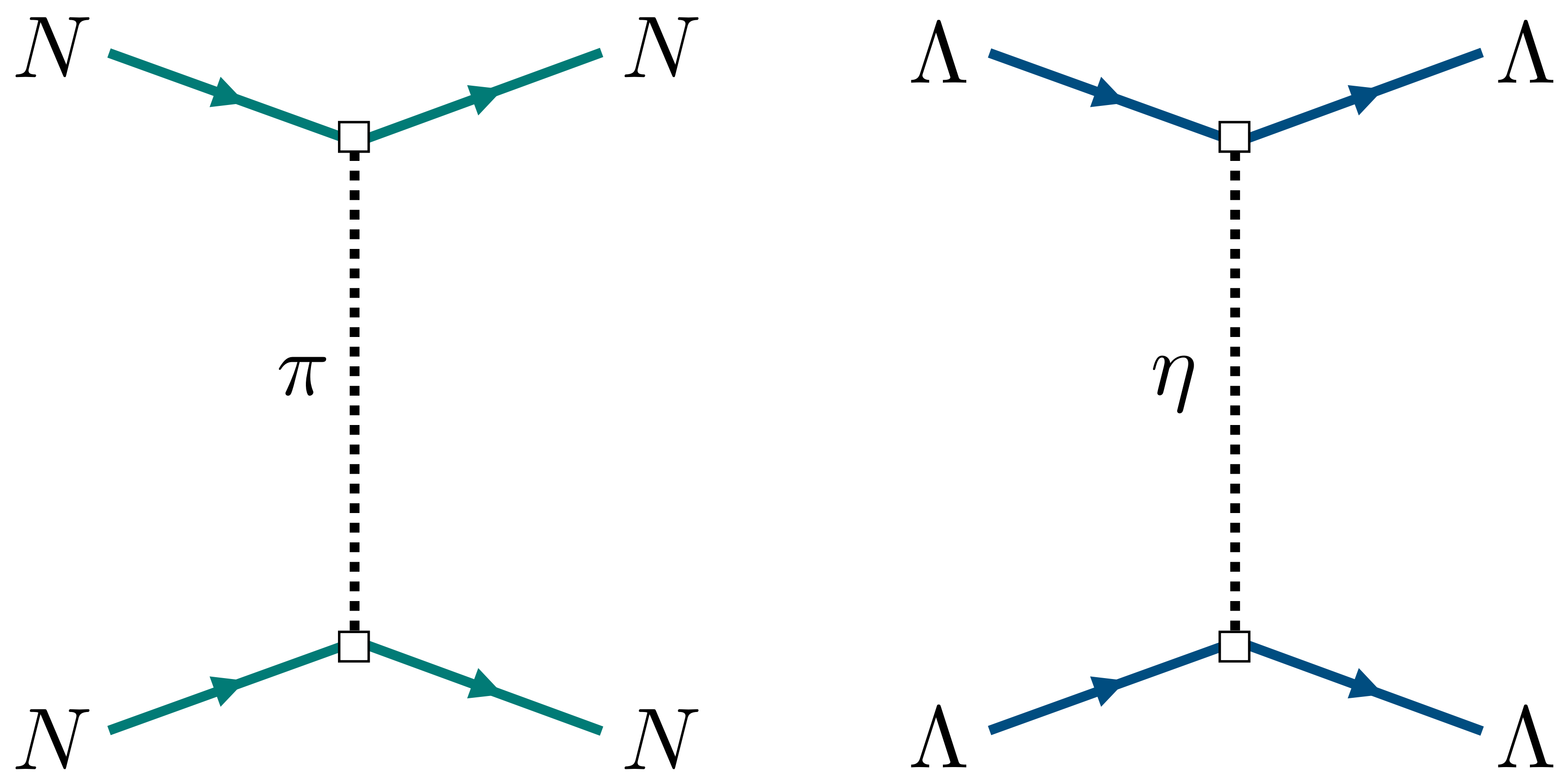
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

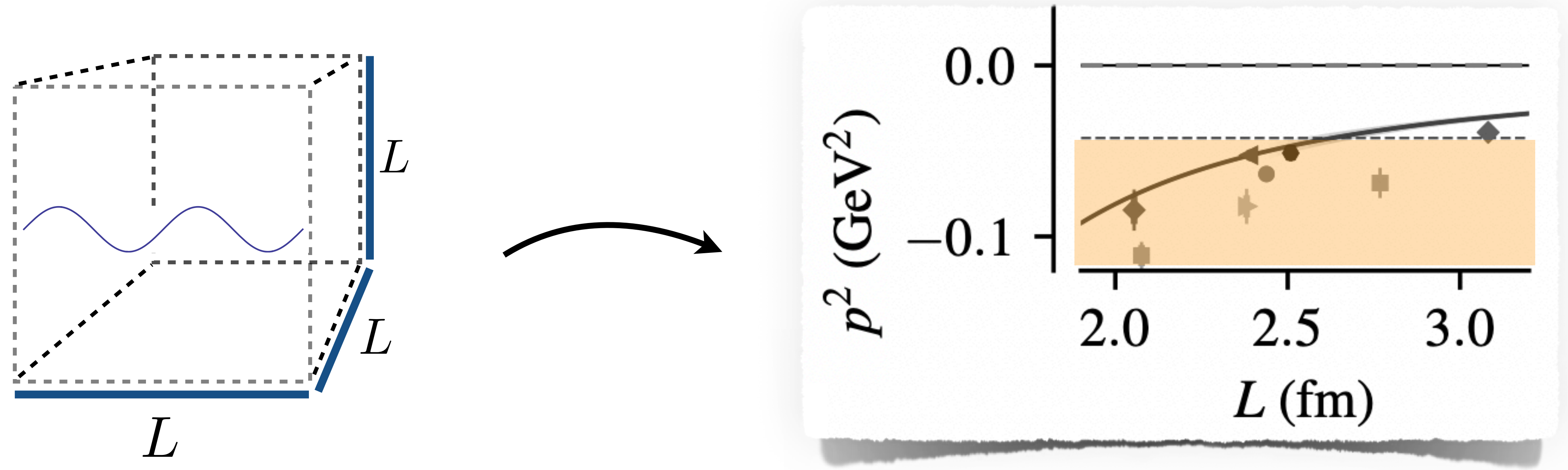
angular momentum also plays an important role in resonant analysis

- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

Motivation in this work…

- Lattice QCD has seen significant recent progress in multi-baryon scattering calculations (NPLQCD, CalLatt, Mainz)
- The extracted amplitudes have sub-threshold t-channel cuts (left-hand cuts) due to light meson exchanges
- Many calculations are extracting finite-volume energies on these cuts





Weakly bound H dibaryon from SU(3)-flavor-symmetric QCD
Green, Hanlon, Junnarkar, Wittig, **PRL** (2021)

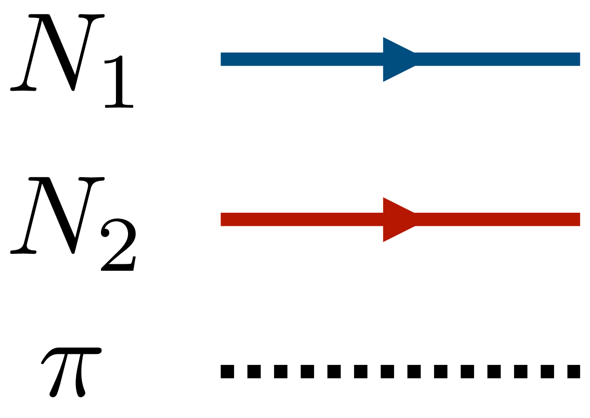
- The Lüscher scattering formalism (+extensions) is not applicable for these energies

$$p\cot\delta(p)_{(\text{true})}\Big|_{\text{cut}} \in \mathbb{C} \qquad\qquad p\cot\delta(p)_{(\text{f.v. extracted})}\Big|_{\text{cut}} \in \mathbb{R}$$

We aim to provide a modification (extension) that resolves this issue

t-channel reminder

- For concreteness, start with a theory of three scalars…

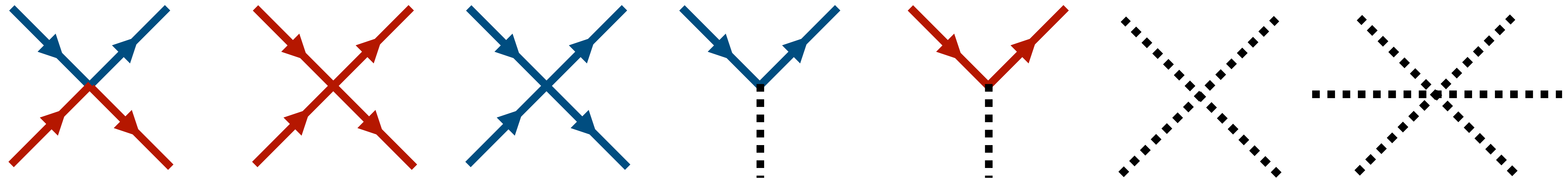


masses

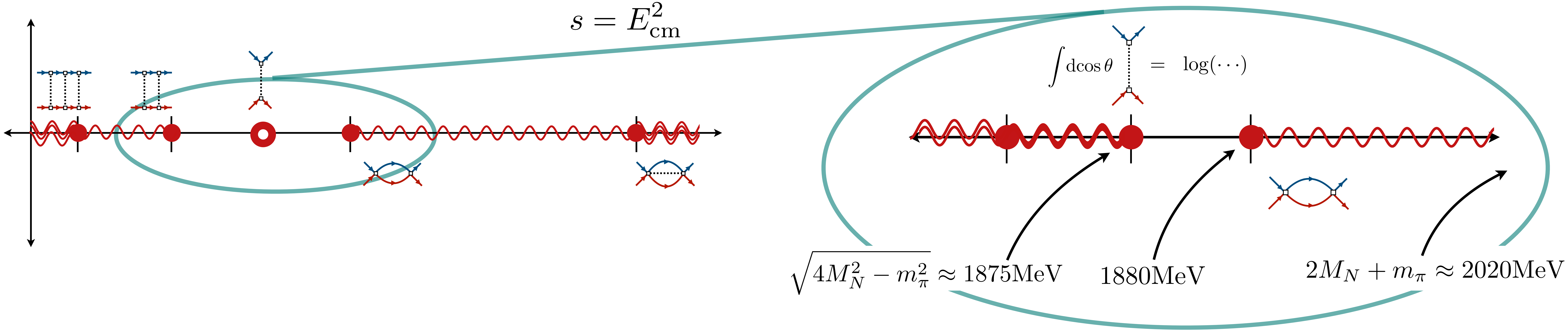
N_1, N_2: M_N

π: $m_\pi \ (< M_N)$

one-ness & two-ness are conserved



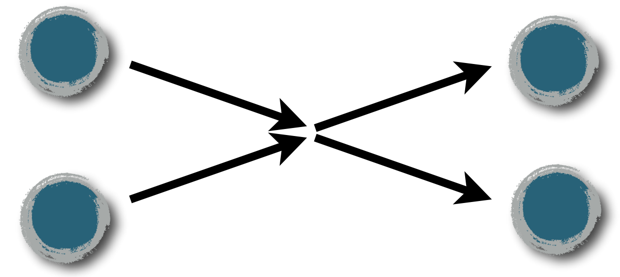
- Analytic structure of $\mathscr{M}(s)$, the $N_1 N_2 \to N_1 N_2$ amplitude



fixed t (forward limit $\rightarrow$ $u = 0,\ t = 4M_N^2 - s$)

angular momentum projected

s-channel cut (optical theorem)

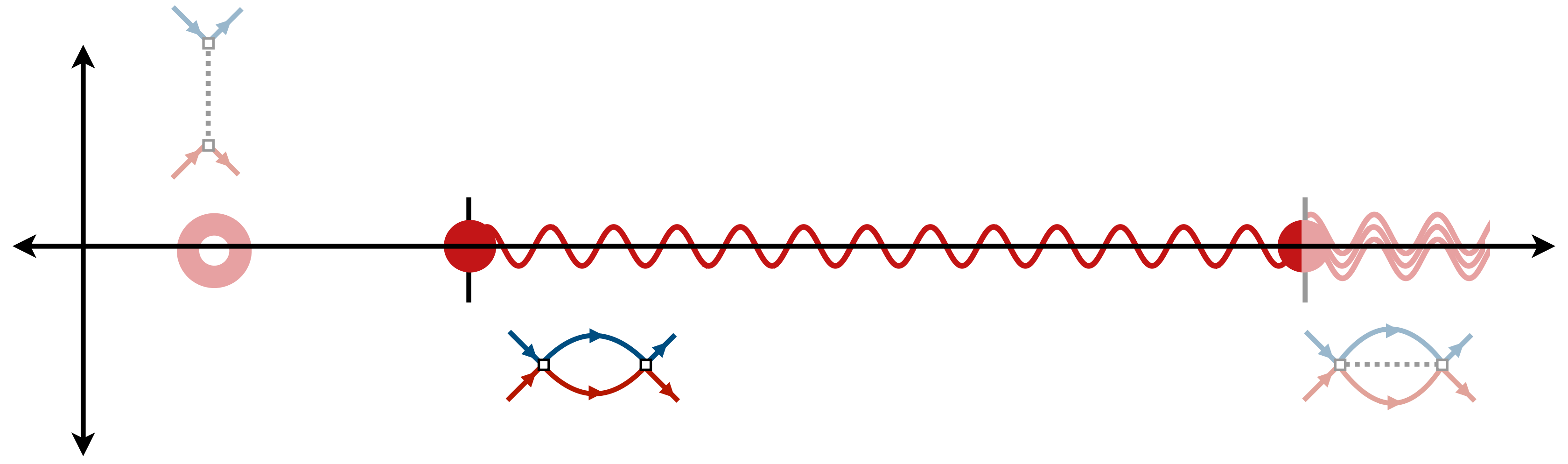
- For two-particle scattering energies, how do we know the analytic structure of $\mathcal{M}(s) =$  ?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \frac{\sqrt{1-4M_N^2/s}}{32\pi}$ is the two-particle phase space

$s = E_{\mathrm{cm}}^2$



- Unique solution is…

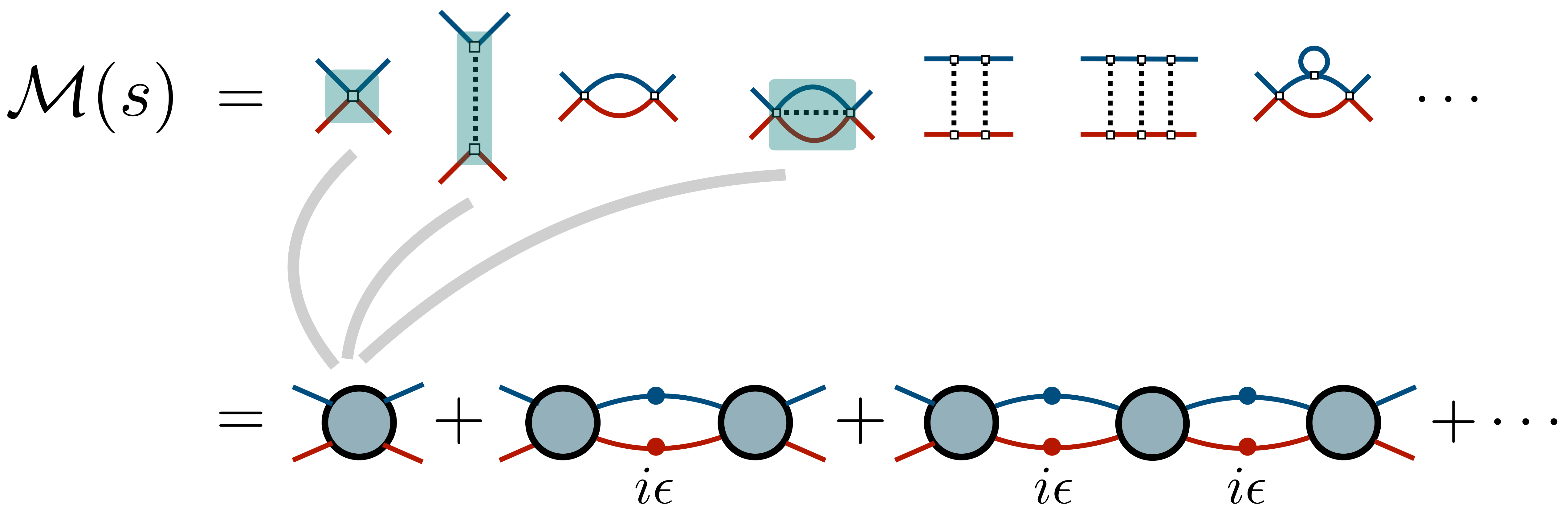
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

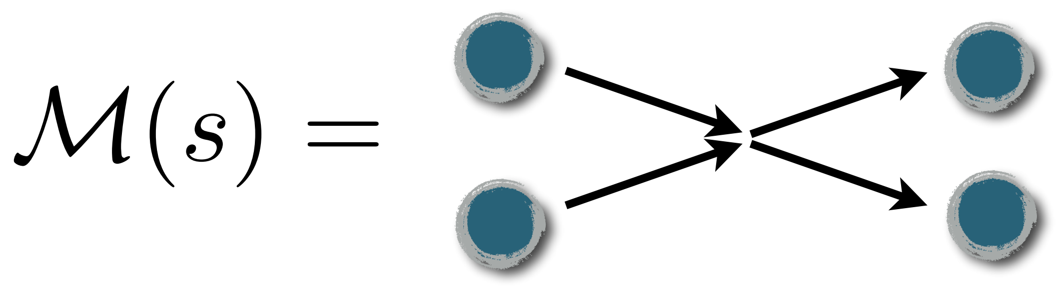
K matrix (short distance)

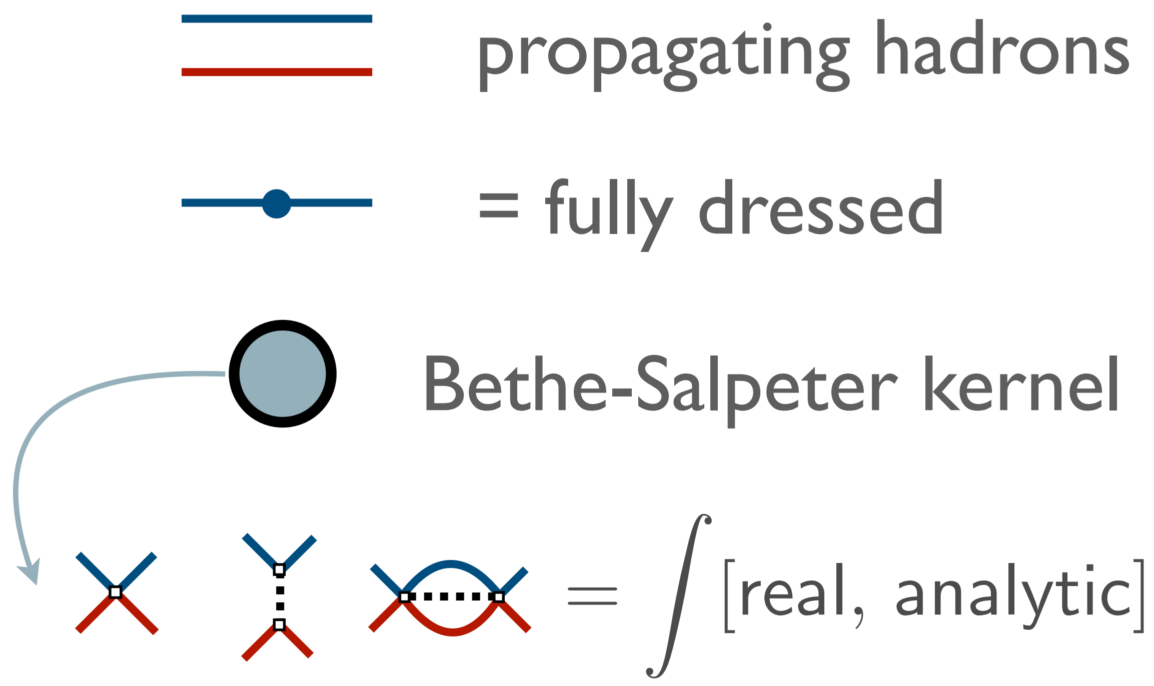
phase-space cut (long distance)

Key message: *s-channel square-root branch cut from the optical theorem*

s-channel cut (diagrammatic)



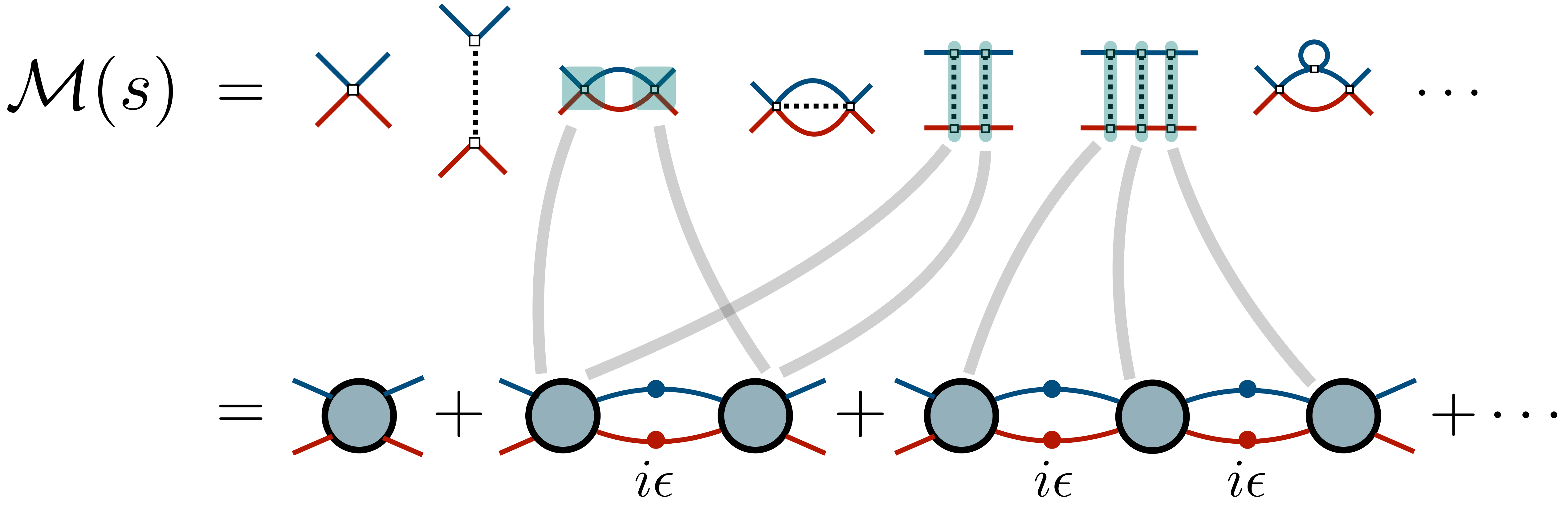


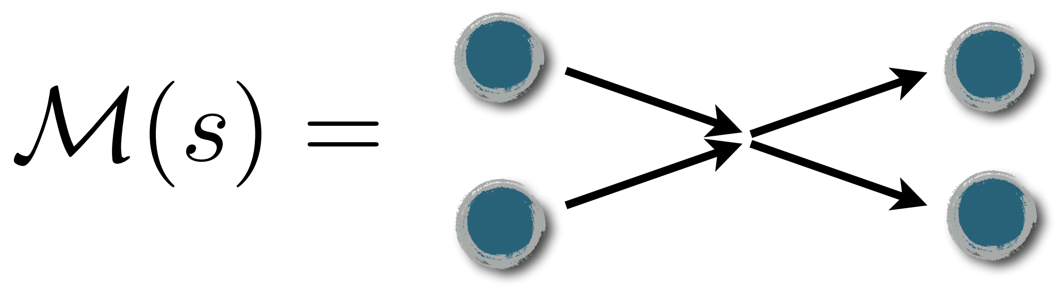


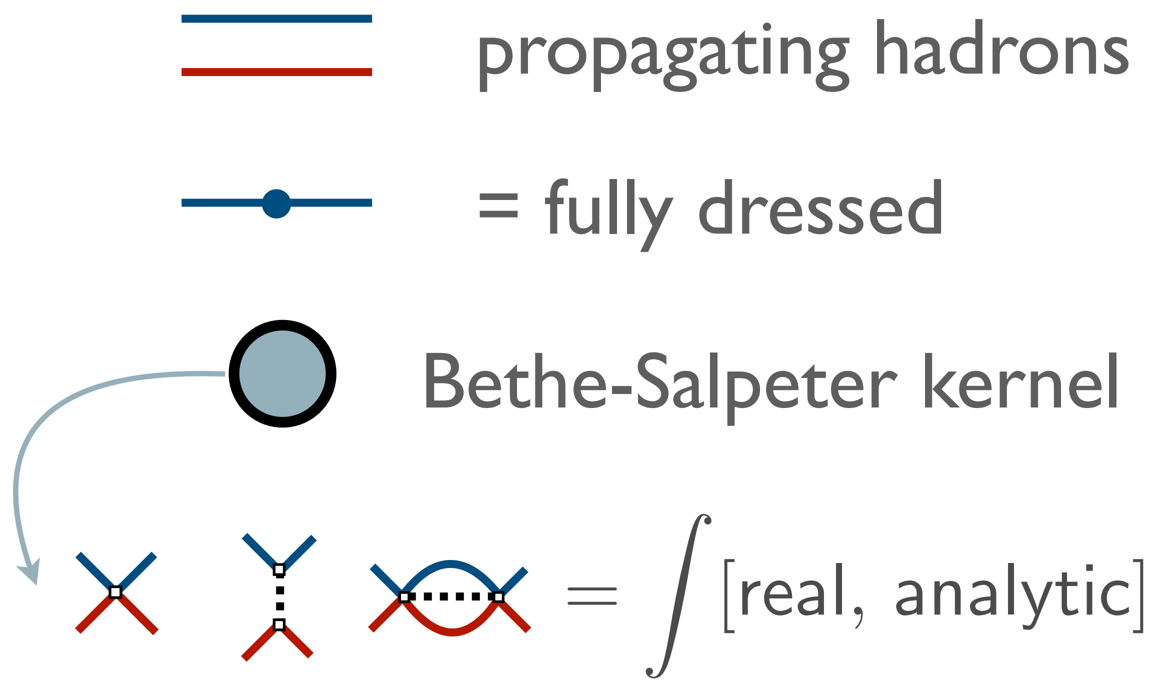
for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

s-channel cut (diagrammatic)





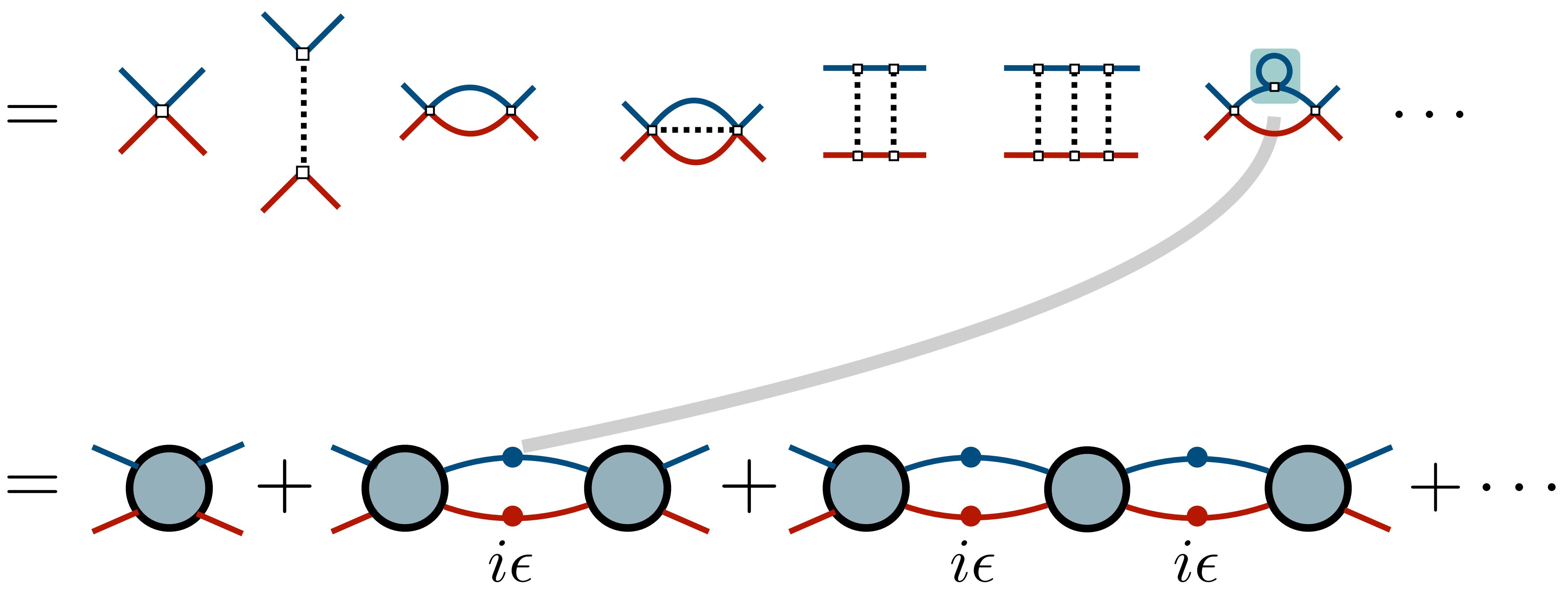


for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

s-channel cut (diagrammatic)

$\mathcal{M}(s) =$



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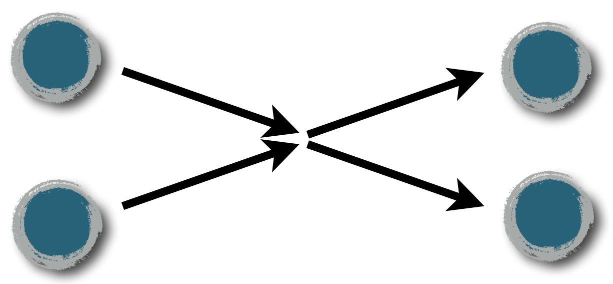

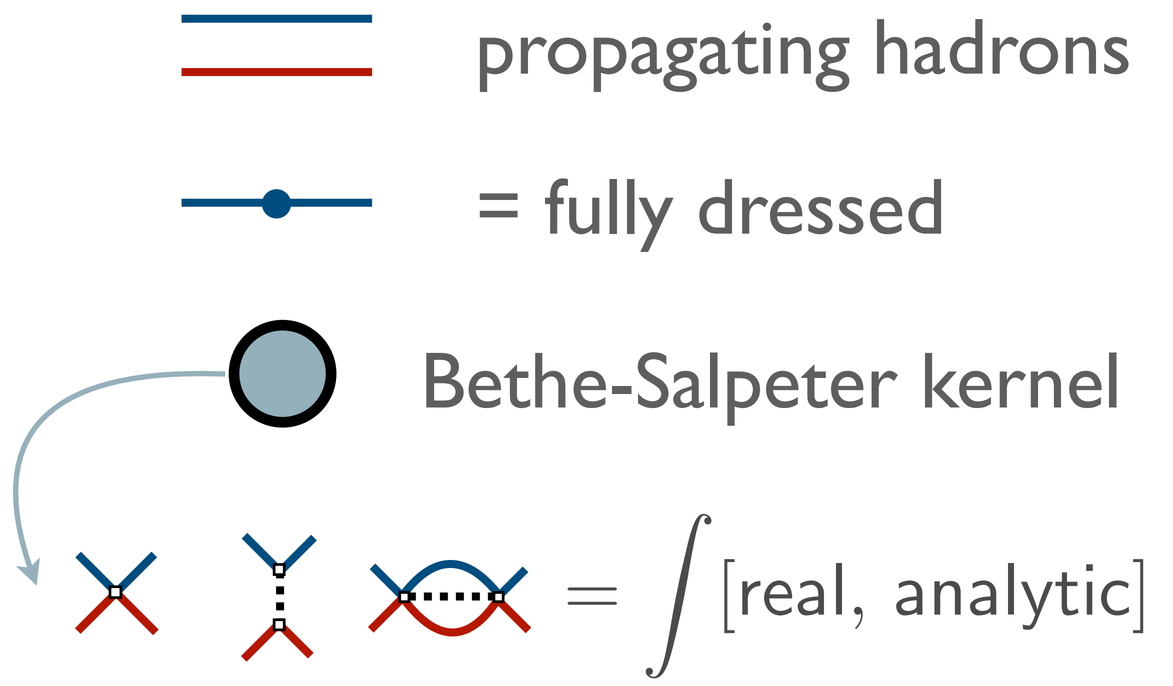

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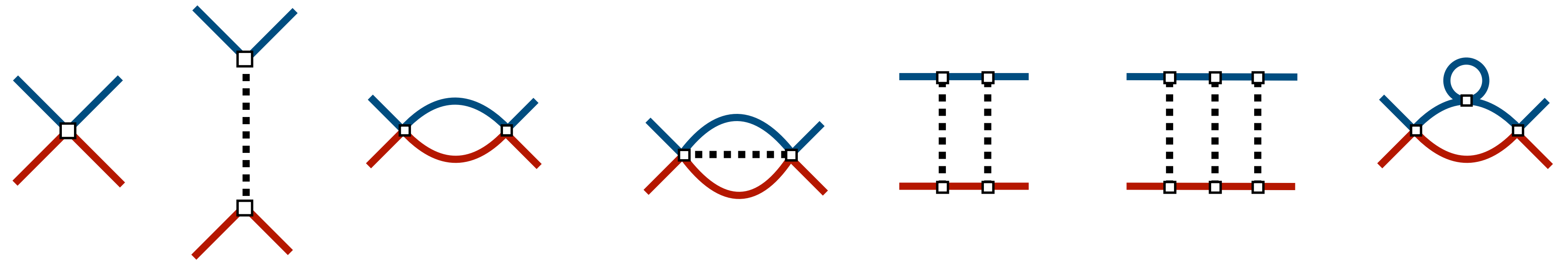
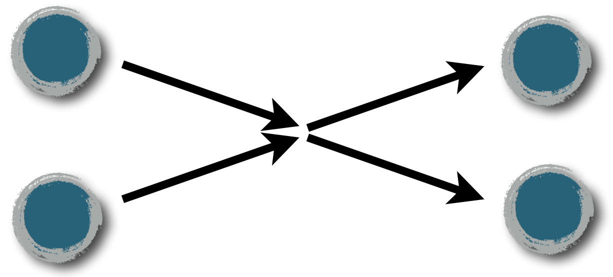
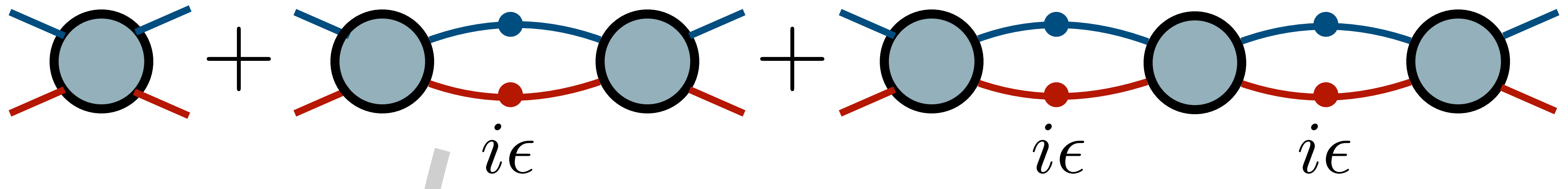
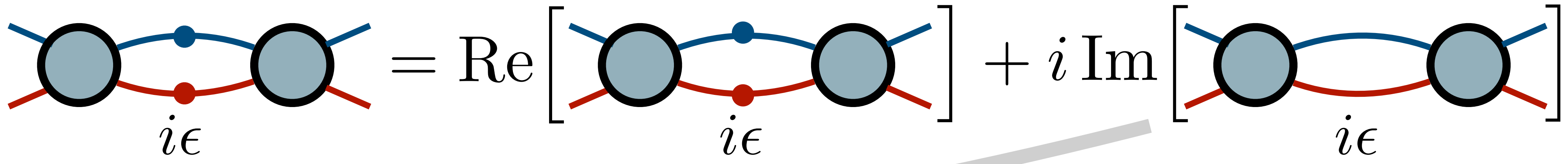
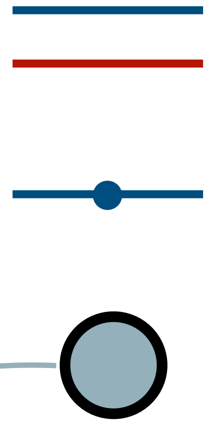
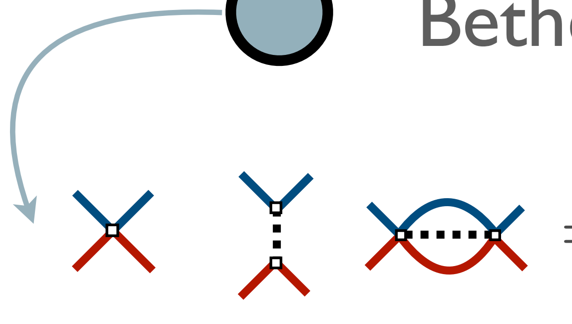
$\mathcal{M}(s) =$  $\cdots$

$\mathcal{M}(s) =$ 

$=$ 

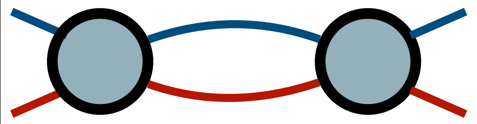




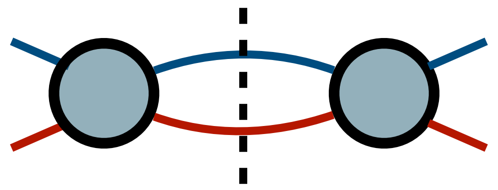
 $= \int$ [real, analytic]

for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

$i\,\mathrm{Im}\Big[$  $\Big]$ $= -i\mathrm{Im}\int_{\boldsymbol{k}} \frac{B(\boldsymbol{k})^2}{(2\omega_{\boldsymbol{k}})[(P-k)^2 - M_N^2 + i\epsilon]} = i\pi \int_{\boldsymbol{k}} \frac{\delta\big((P-k)^2 - m^2\big)}{2\omega_{\boldsymbol{k}}} B_{\ell m}(|\boldsymbol{k}_{\mathsf{cm}}|)^2$

$$\boldsymbol{k}_{\mathsf{cm}}^2 \overset{!}{=} E_{\mathsf{cm}}^2/4 - M_N^2 = p(E_{\mathsf{cm}})^2$$

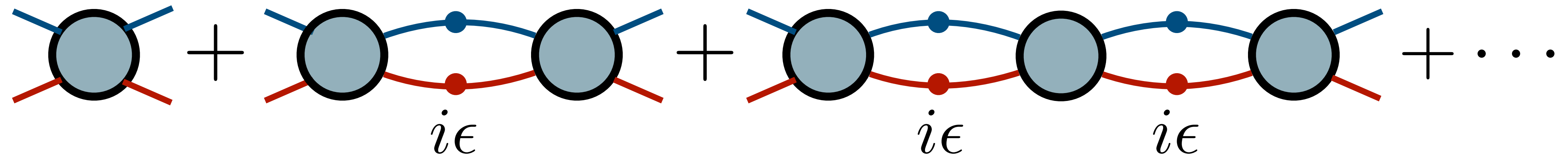
$= B_{\ell m}(s)\ i\rho(s)\ B_{\ell m}(s) =$ 

imaginary part is set exactly on-shell

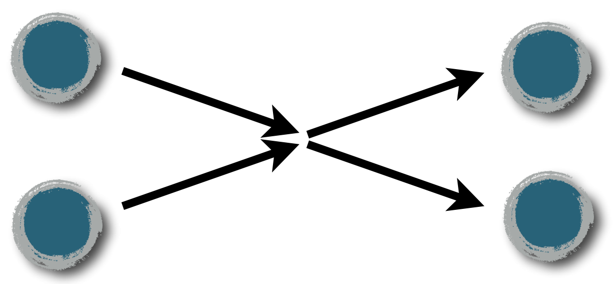
cut is trivial in angular momentum

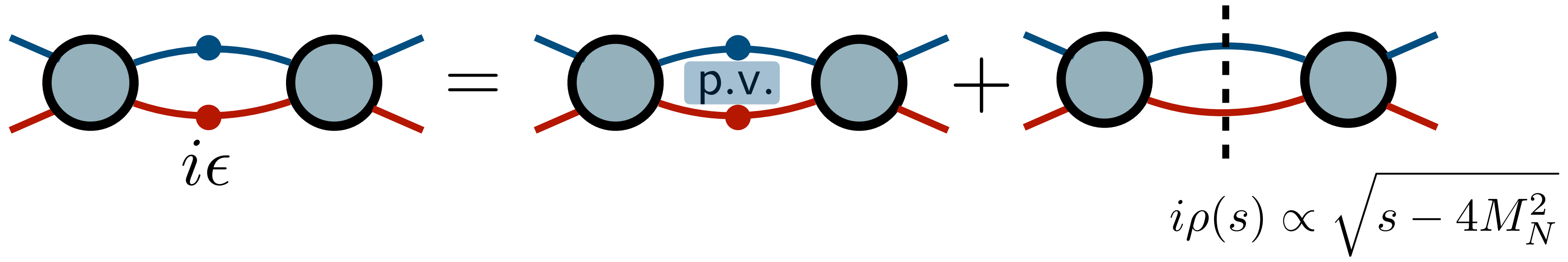
s-channel cut (diagrammatic)

$\mathcal{M}(s) =$

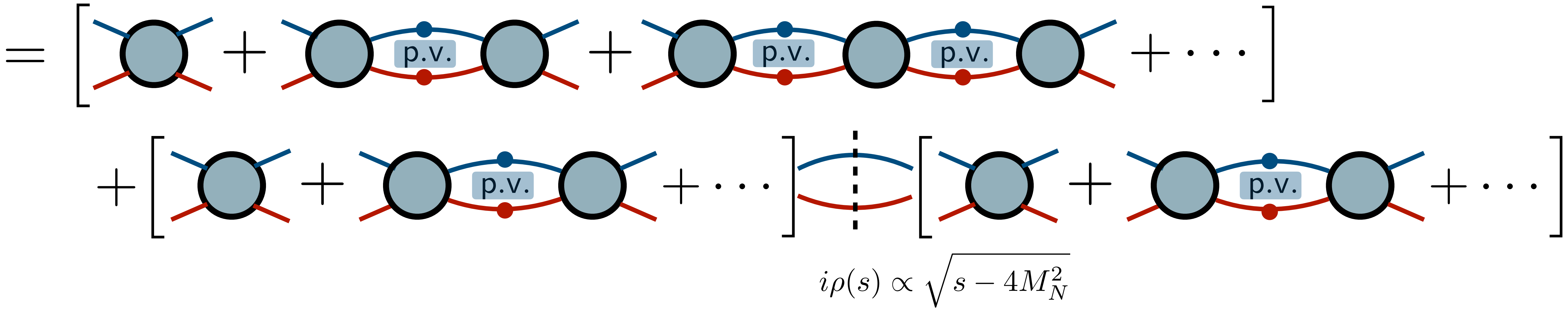


$\mathcal{M}(s) =$

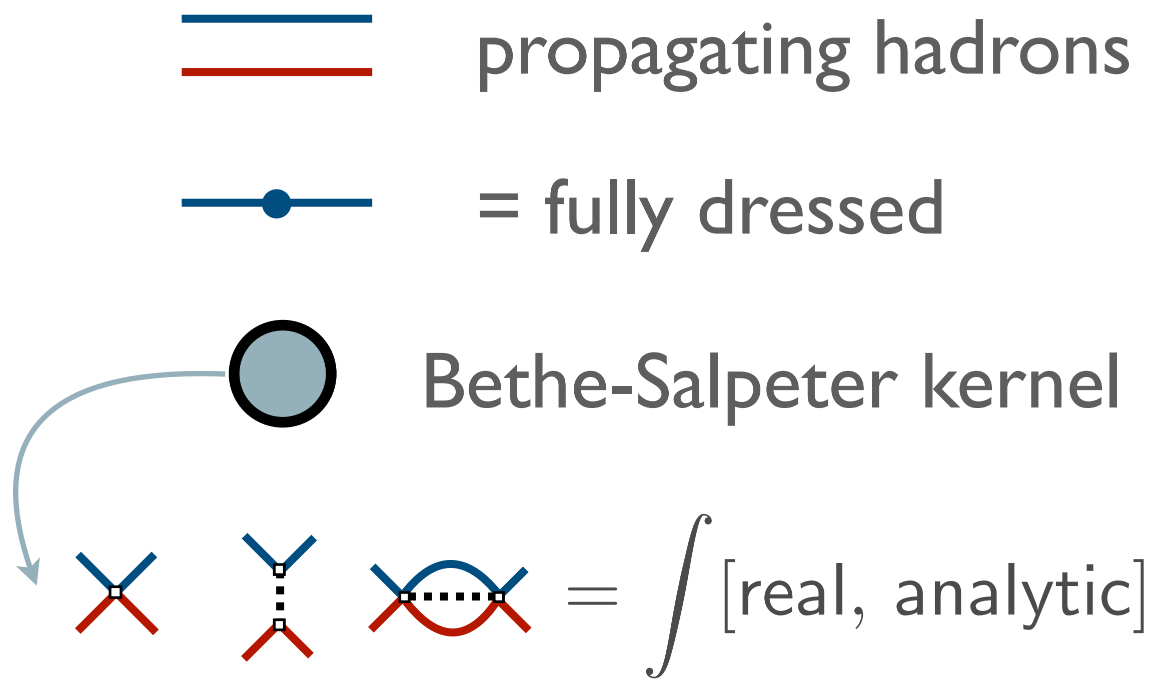




defines the K matrix $\mathcal{K}(s)$



$$= \mathcal{K}(s) + \mathcal{K}(s) i\rho(s) \mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$



for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

$$\boldsymbol{k}_{\text{cm}}^2 \stackrel{!}{=} E_{\text{cm}}^2/4 - M_N^2 = p(E_{\text{cm}})^2$$

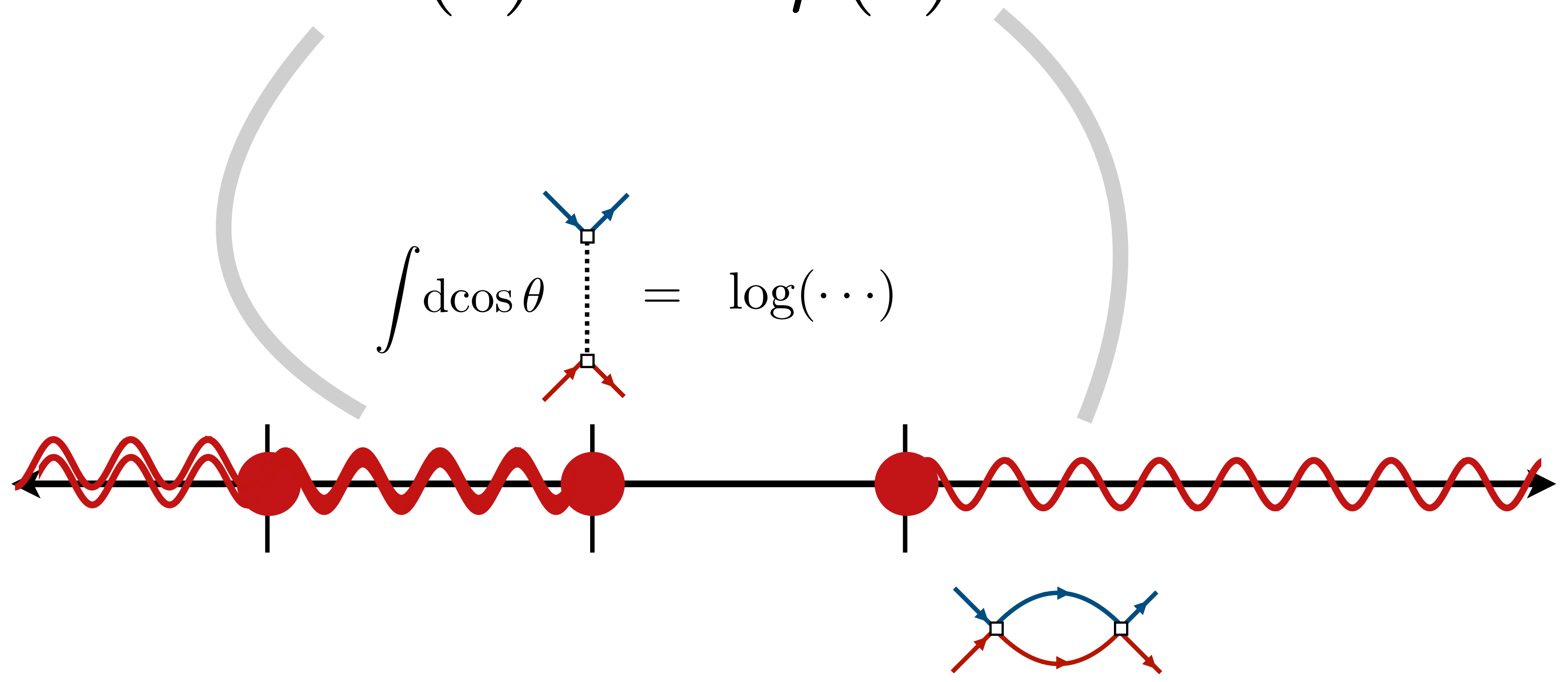
imaginary part is set exactly on-shell

cut is trivial in angular momentum

The importance of on-shell-ness

- s-channel cut arises from the imaginary part of two-particle loops
  - continuum of two-particle scattering states
  - loops can go on-shell, carry physical energy
  - pole is hit, need $i\epsilon$
  - Dirac delta enforces physical energy (on-shell)
- Above threshold, K-matrix is set identically on-shell
- Sub-threshold continuation follows from…
  - Riemann sheet specification: $i\rho(s) \to \pm\,|\,\rho(s)\,|$
  - near-threshold analyticity of $K(s)$

$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

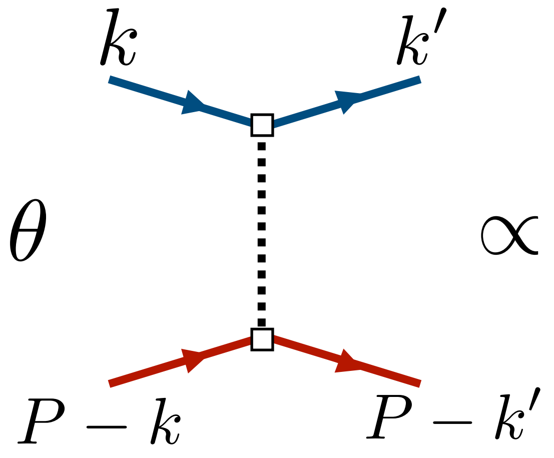


- t-channel appears further down but is intrinsically tied to the continuation of an on-shell diagram

Show me the cut!

- Evaluating the simple exchange diagram gives

$$\int \mathrm{d}\cos\theta$$



$$\propto \int \mathrm{d}\cos\theta \left[\frac{1}{-(\boldsymbol{k}_{\text{cm}} - \boldsymbol{k}'_{\text{cm}})^2 + m_\pi^2 - i\epsilon}\right] = \frac{1}{\boldsymbol{k}_{\text{cm}}^2} \log\left[\frac{4\boldsymbol{k}_{\text{cm}}^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

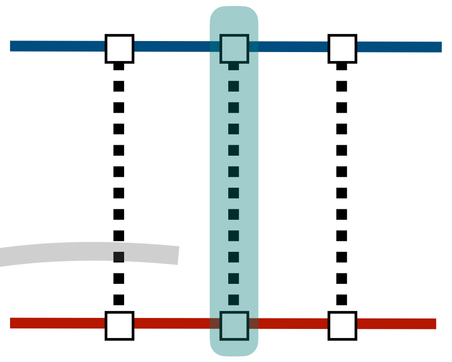
(analytic for $\boldsymbol{k}_{\text{cm}}^2 > -m_\pi^2/4$)

$\xrightarrow{\text{on-shell kinematics}}$

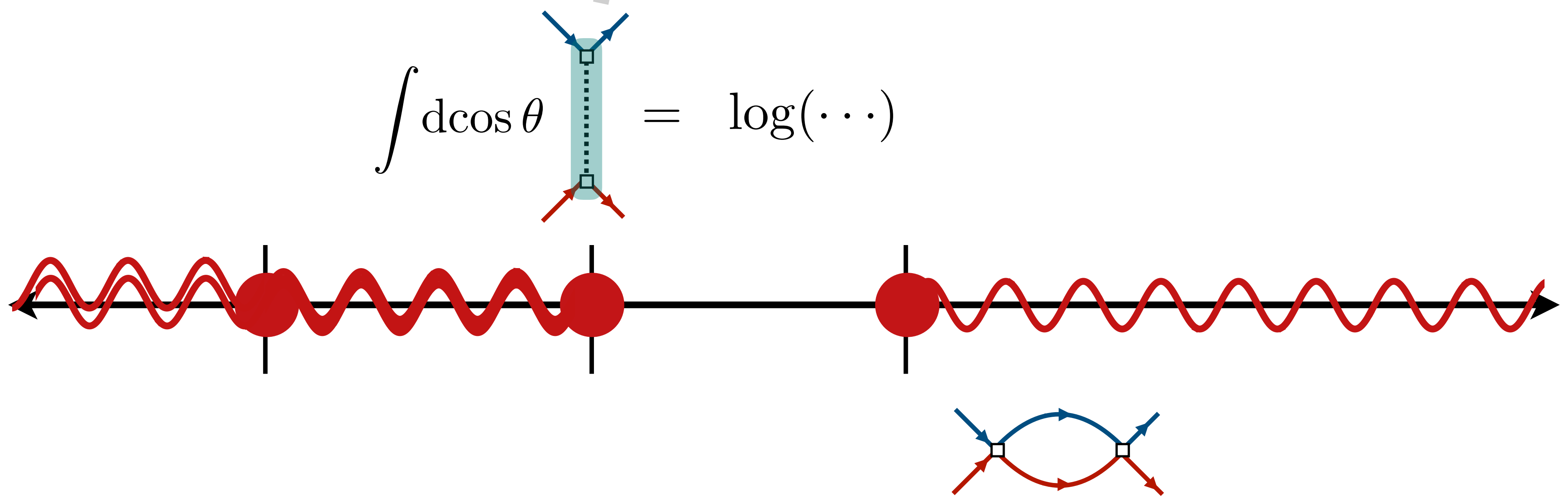
$$\frac{1}{s/4 - M_N^2} \log\left[\frac{s - 4M_N^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

(analytic for $s > 4M_N^2 - m_\pi^2$)

- What about diagrams like



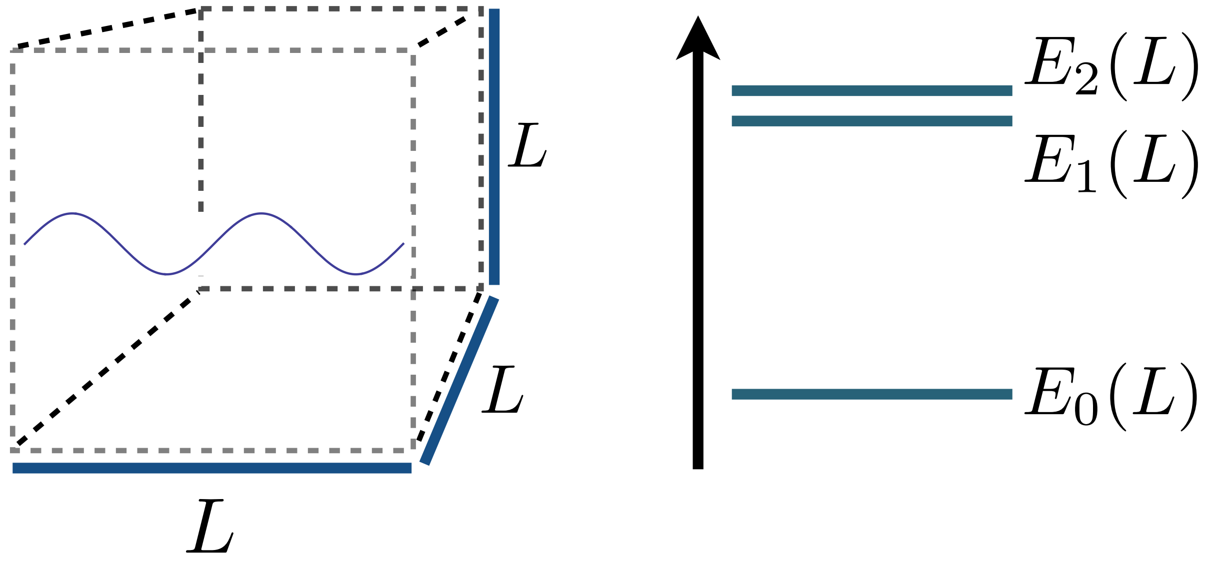
?



Naive statement: Internal loops are summed over real $\boldsymbol{k}$ → never get cuts from middle exchange

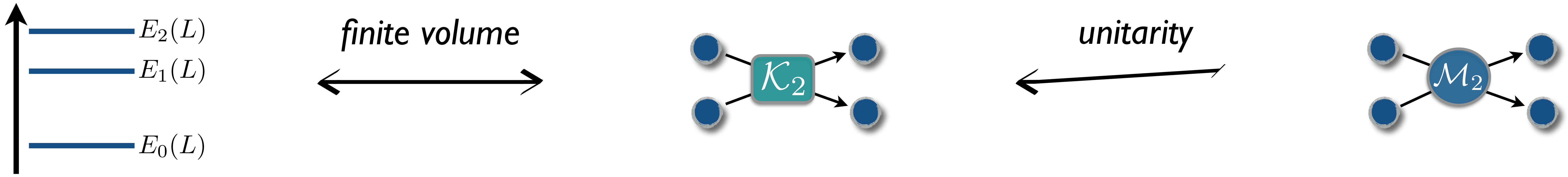
True statement: In the $\rho(s)$ decomposition, the middle exchange is set on-shell
→ contributes to the leading K-matrix cut

The finite-volume



- Finite, cubic, periodic spatial volume (extent L)
- Discrete momenta and energies
- L is large enough to neglect $e^{-m_\pi L}$
- Finite T and non-zero lattice spacing assumed negligible

Quantisation condition relates energies to K-matrices, and thereby amplitudes



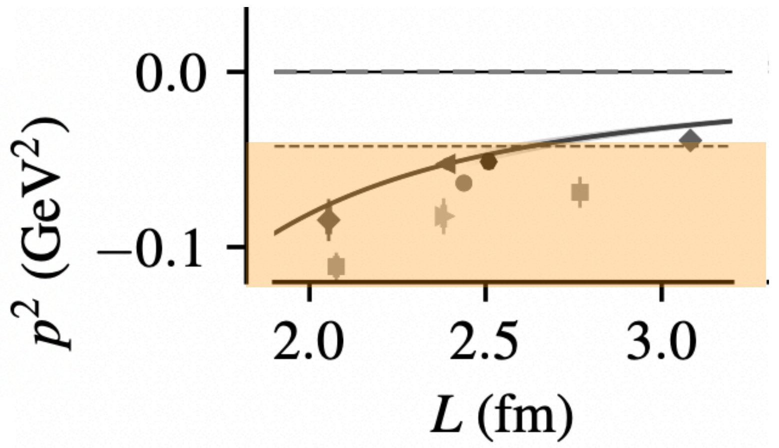
$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions

• Lüscher (1989) • *many others* •

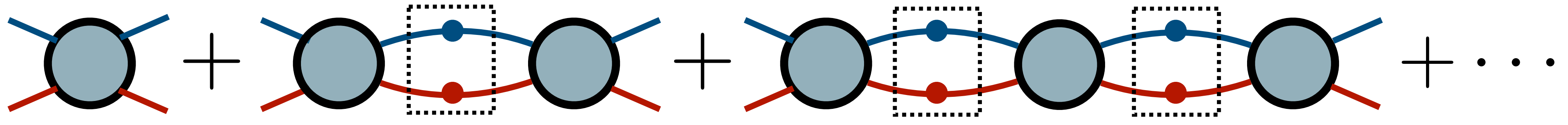
$F(P,L)$ is real below threshold

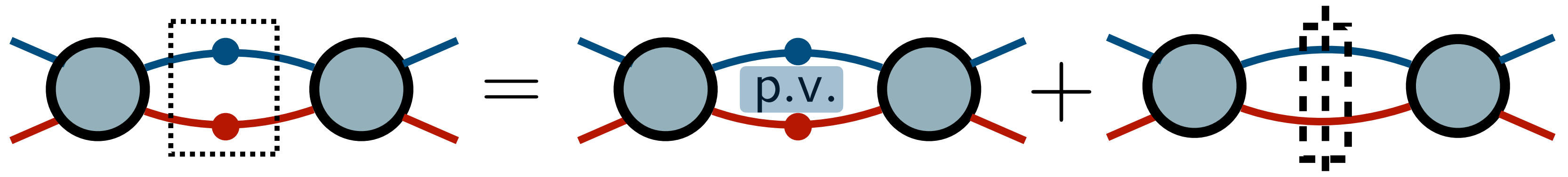
→ only solvable if $K(s)$ is real below threshold



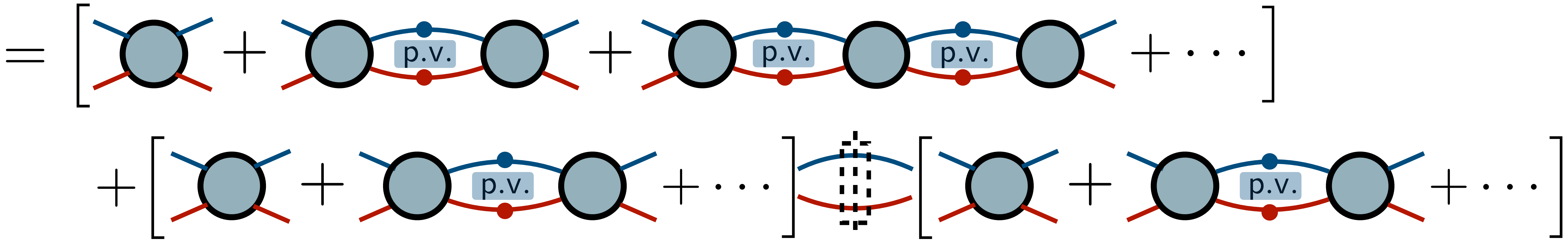
$$\frac{1}{s/4 - M_N^2} \log\left[\frac{s - 4M_N^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

Back to the derivation

$\mathcal{M}_L(P) =$ 



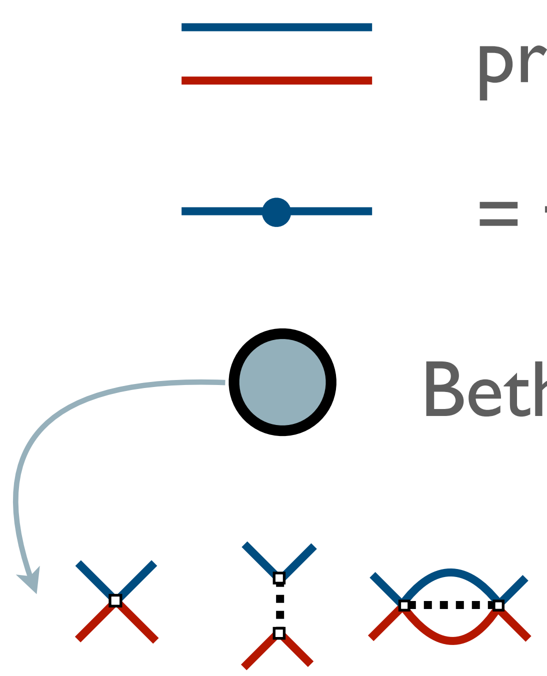
defines the K matrix $\mathcal{K}(s)$



$$= \mathcal{K}(s) - \mathcal{K}(s)F(P,L)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

$\mathcal{M}_L(P) =$ finite-volume correlator

poles are finite-volume energies



propagating hadrons

= fully dressed

Bethe-Salpeter kernel

$= \sum_{\boldsymbol{k}} [\text{real, analytic}]$

$= \int_{\boldsymbol{k}} [\text{real, analytic}] + e^{-\mu L}$

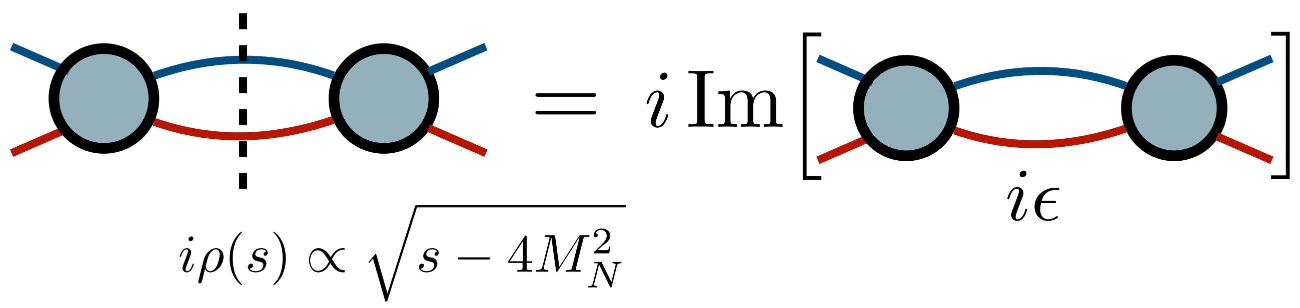
for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

Key details:

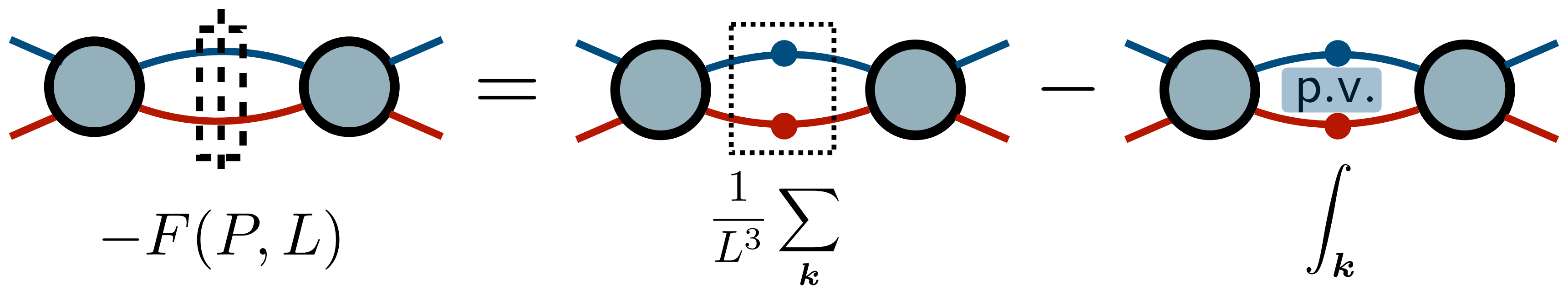
- matrices on angular momentum $\otimes$ channels
- $K(s)$ populated with physical (on-shell) partial waves

Origins / issues of on-shell projection



Exactly set on the mass shell by a Dirac delta function

… a result that leads to the correct sub-threshold analytic continuation



Set on the mass shell by the relation:

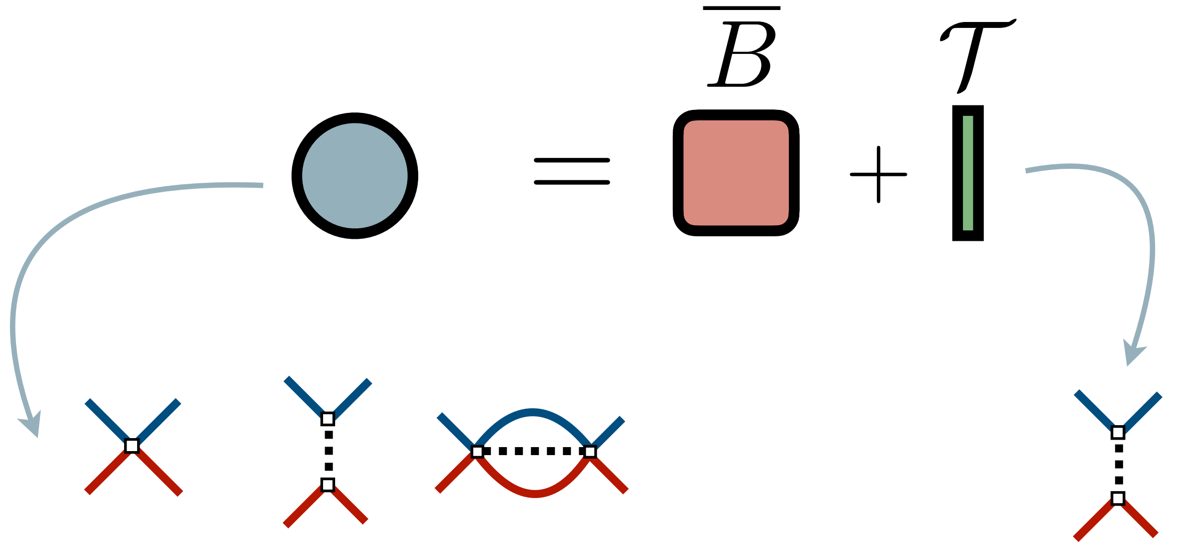
$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}\right] \frac{1}{2\omega_{\boldsymbol{k}}} \frac{\mathcal{S}(\boldsymbol{k}_{\text{cm}}^2) - \mathcal{S}(p(E_{\text{cm}})^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2} = e^{-\mu L}$$

… a result that breaks on the cut

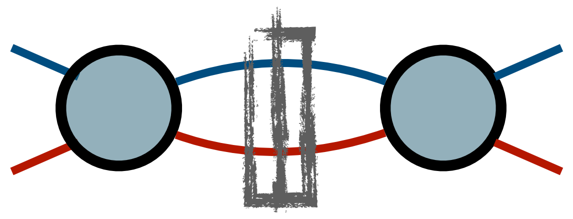
$$\frac{1}{\boldsymbol{k}_{\text{cm}}^2} \log\left[\frac{4\boldsymbol{k}_{\text{cm}}^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right] \qquad \frac{1}{s/4 - M_N^2} \log\left[\frac{s - 4M_N^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

Resolution:

separate the t-channel exchange from the Bethe Salpeter kernel



change the details of the cut to keep T off-shell



modify index space to reach a new quantisation condition

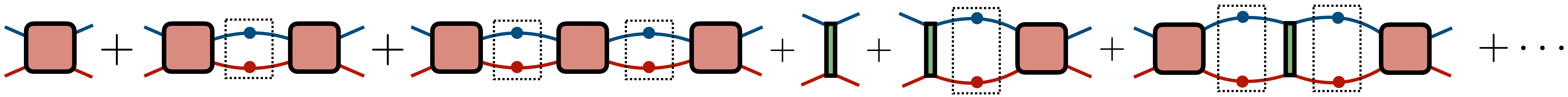
$$\ell, m \longrightarrow |\boldsymbol{k}_{\text{cm}}|, \ell, m$$

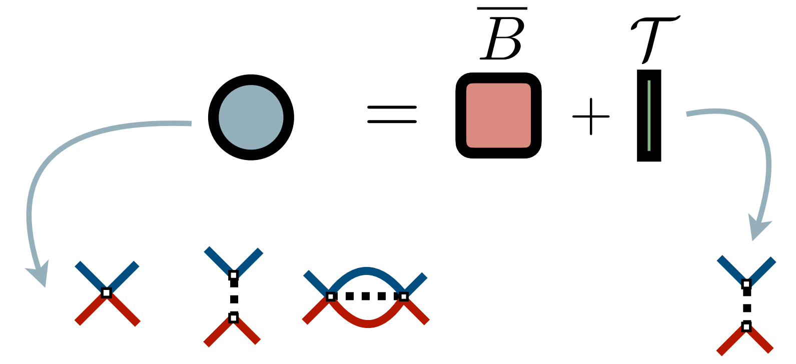
$E_n(L)$ can now constrain modified K matrix + the cut

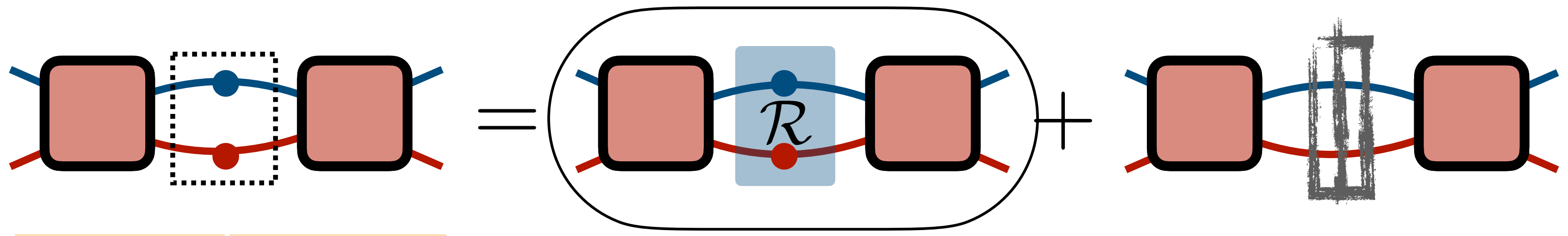
$$\overline{\mathcal{K}}(s) \qquad g^2 \log[\cdots]$$

Known integral equations relate this to the standard K (or the amplitude)

Some cutting details

$\mathcal{M}_L(P) =$ 





$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} \frac{\mathcal{A}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{B}(\boldsymbol{k}_{\text{cm}}^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2}$$

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} \frac{\mathcal{A}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{B}(p(E_{\text{cm}})^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2} H(\boldsymbol{k}_{\text{cm}})$$

smooth cutoff H

abbreviating slightly we write…

$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{off}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}$$

$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}(1-H) + \sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{off}} - \mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}$$

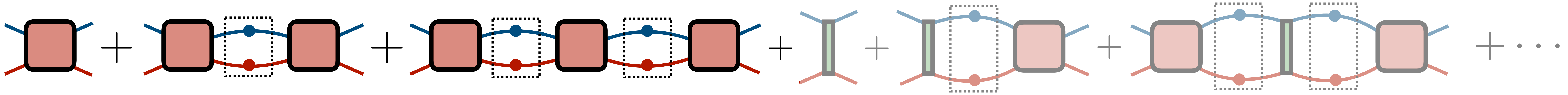
$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2} H$$

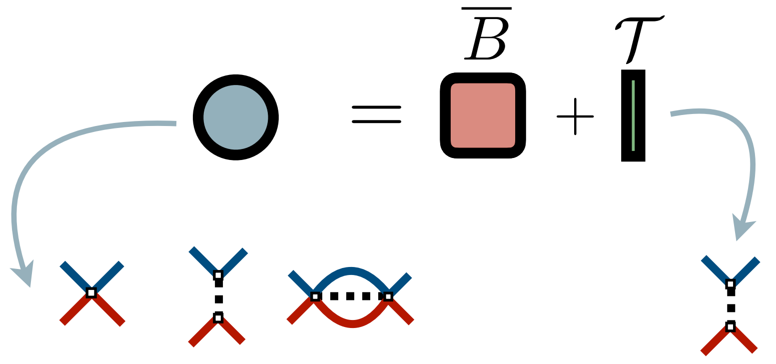
- ❑ We are still angular-momentum projecting everywhere *(not really the issue)*
- ❑ Sum is promoted to an additional index… can always be collapsed later
- ❑ Only the known log function is evaluated off-shell

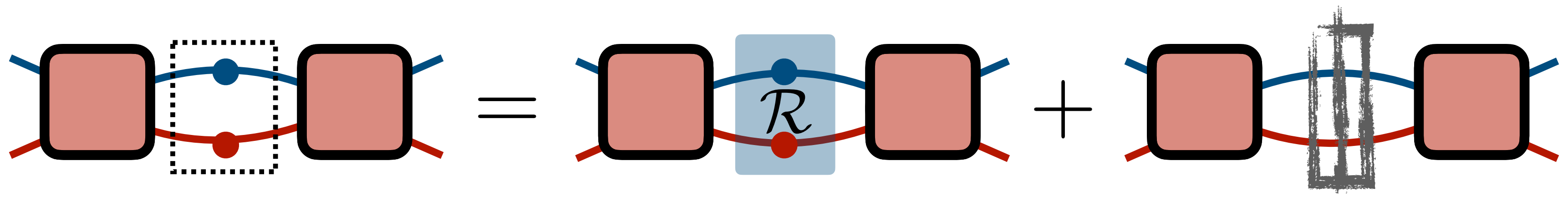
$$-S(P,L)_{|\boldsymbol{k}'_{\text{cm}}|,\ell',m'\;,\;|\boldsymbol{k}_{\text{cm}}|,\ell,m}$$

First sum the safe blobs

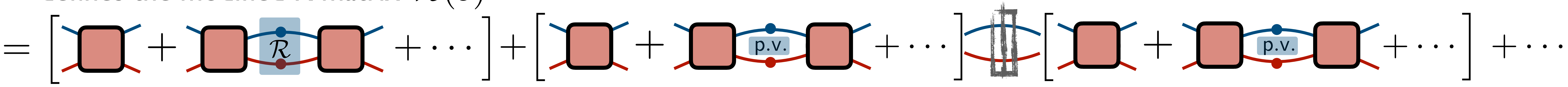
$\mathcal{M}_L(P) =$







defines the modified K matrix $\overline{\mathcal{K}}(s)$



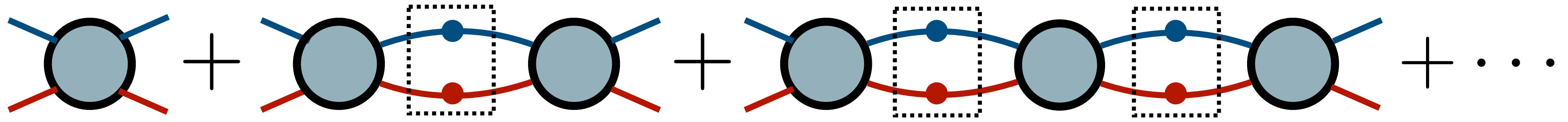
$-S(P,L)_{|\boldsymbol{k}'_{\text{cm}}|,\ell',m'\,,\;|\boldsymbol{k}_{\text{cm}}|,\ell,m}$

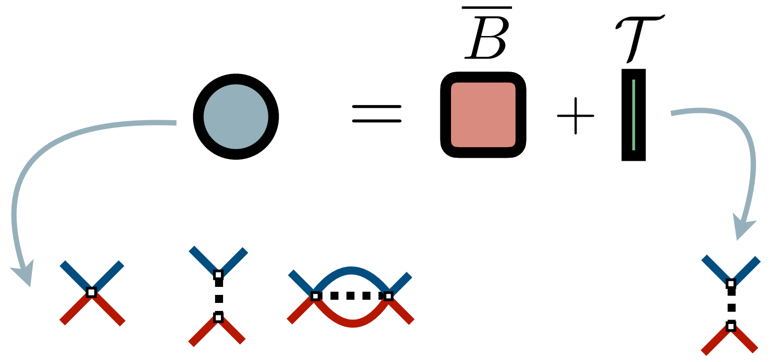
$$= \overline{\mathcal{K}}(s) - \overline{\mathcal{K}}(s) S(P,L) \overline{\mathcal{K}}(s) + \cdots = \frac{1}{\overline{\mathcal{K}}(s)^{-1} + S(P,L)}$$

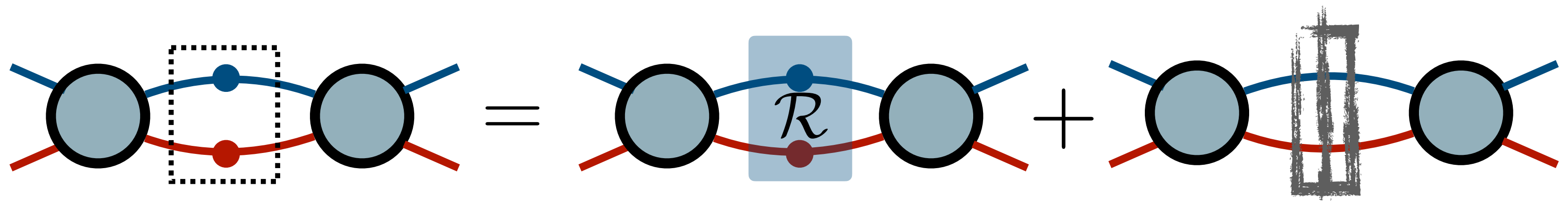
Key details:

- matrices on angular momentum $\otimes\ |\boldsymbol{k}_{\text{cm}}|$
- $\overline{K}(s)$ on shell but missing parts of diagrams
- but now have significant redefinition freedom

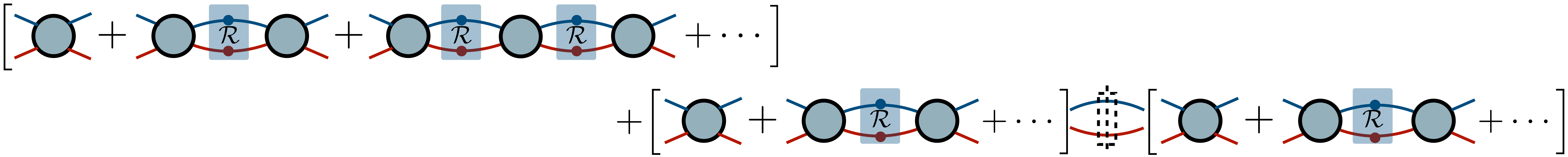
Completing the story…

$\mathcal{M}_L(P) =$ 



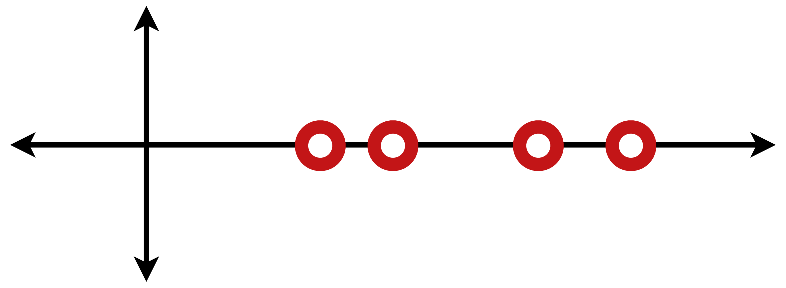


$-S(P,L)_{|\boldsymbol{k}'_{\rm cm}|,\ell',m'\,,\ |\boldsymbol{k}_{\rm cm}|,\ell,m}$

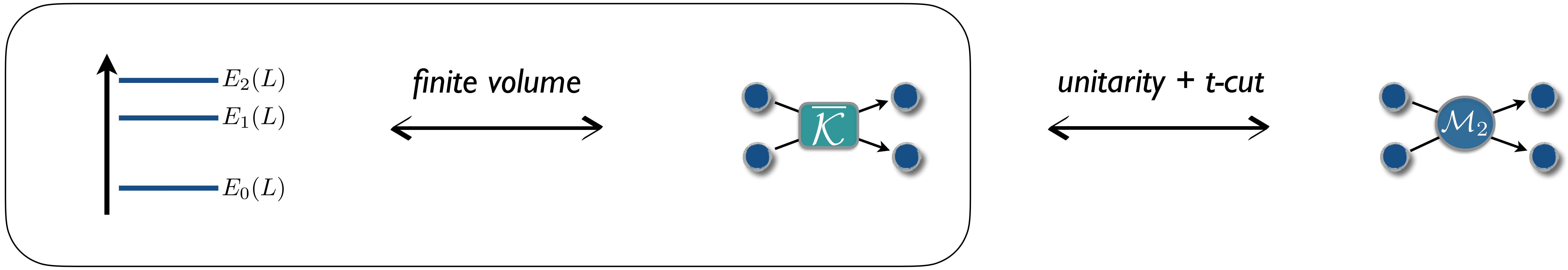
$=$ 

$-S(P,L)_{|\boldsymbol{k}'_{\rm cm}|,\ell',m'\,,\ |\boldsymbol{k}_{\rm cm}|,\ell,m}$

$$= [\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,] - [\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,]S(P,L)[\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,] + \cdots = \frac{1}{[\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,]^{-1} + S(P,L)}$$



Quantisation condition



$$\det_{|\boldsymbol{k}_{\rm cm}|\ell m}\left[[\overline{\mathcal{K}}(s)+g^2\mathcal{T}\,]^{-1}+S(P,L)\right]=0$$

$S(P,L) =$ Matrix of known geometric functions

$\mathcal{T} =$ Matrix of known off-shell logs

$$\mathcal{T}_{|\boldsymbol{k}_{\rm cm}|00,|\boldsymbol{k}'_{\rm cm}|00} = -\frac{1}{|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}|}\log\left[\frac{2\omega_{\boldsymbol{k}_{\rm cm}}\omega_{\boldsymbol{k}'_{\rm cm}} - 2|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}| - 2M_N^2 + m_\pi^2 - i\epsilon}{2\omega_{\boldsymbol{k}_{\rm cm}}\omega_{\boldsymbol{k}'_{\rm cm}} + 2|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}| - 2M_N^2 + m_\pi^2 - i\epsilon}\right]$$

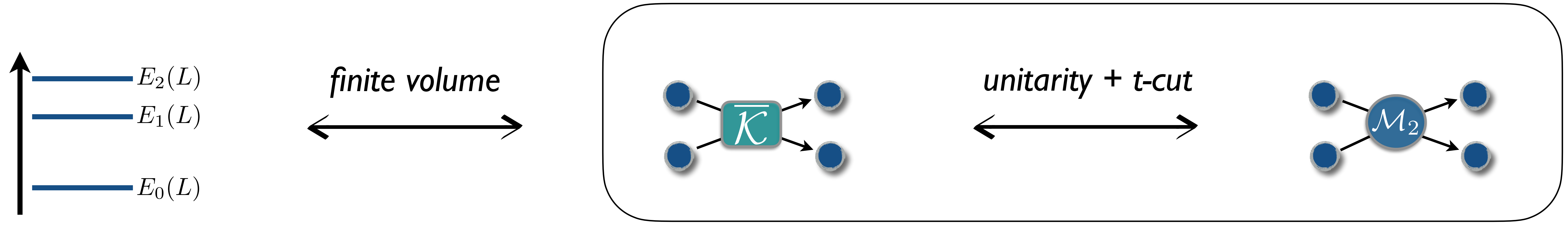
New matrix space, truncated by cutoff function

Holds up to $e^{-m_\pi L}$

Inspired in part three-particle work…

Blanton, Briceño, Döring, Draper, Mai, Meißner, Müller, Hammer, MTH, Pang, Romero-López, Rusetsky, Sharpe

Integral equations



- ❑ Exactly in the spirit of the three-particle approach
  - MTH, Sharpe (2015)
  - Agadjanov *et.al,* (2016) (Optical potential)
- ❑ Define a finite-volume amplitude with the correct limit:

$$\mathcal{M}_L(P) = \frac{1}{[\overline{\mathcal{K}}(s) + g^2 \mathcal{T}\,]^{-1} + S(P,L)}$$

- ❑ Formally take an infinite-volume limit to derive an integral equation

$$\mathcal{M}(s,t) = \lim_{\text{os}} \lim_{\epsilon \to 0} \lim_{L \to \infty} \mathcal{M}_L(P + i\epsilon)_{|\boldsymbol{k}'_{\text{cm}}|\ell m, |\boldsymbol{k}_{\text{cm}}|\ell m}$$

Note: *derivation strategy = logically separate from evaluation strategy*

Summary and outlook

- t-channel cut is an *identically* on-shell + angular-momentum projection + infinite-volume effect

- Using on-shell + angular-momentum projected + infinite-volume Bethe-Salpeter kernels gives incorrect finite-L description of the correlator

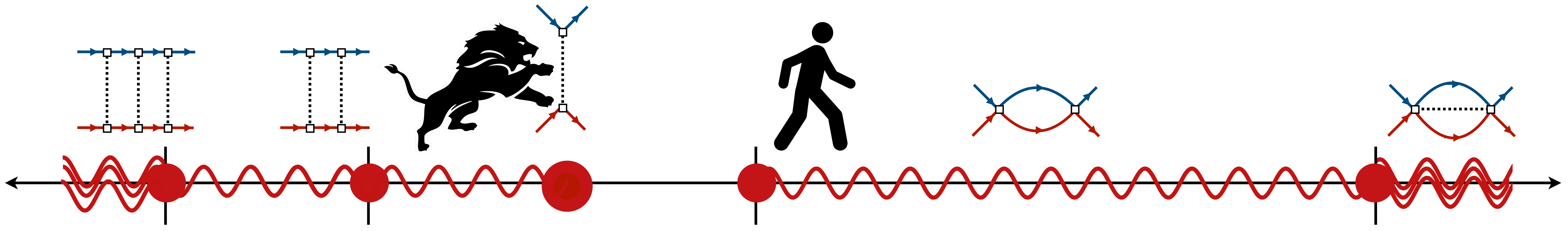
- A modified derivation solves this at the expense of a new quantisation condition (and a new K-matrix)

- Integral equations can be used to relate the latter to the standard quantity, with the cut included

- Publication forthcoming

- Much to explore… finite-L effects near (but not on) the cut, realistic systems, dispersive applications

Thanks to Jeremy Green, other Mainzers, CalLatt, Raúl Briceño for discussions!



Thanks for listening! Questions?….