Scattering from generalised ϕ^4

Marco Garofalo[a)], Maxim Mai[a)b)], Fernando Romero-López[c)], Akaki Rusetsky[a)d)] ,Carsten Urbach[a)]

a)HISKP (Theory), Rheinische Friedrich-Wilhelms-Universität Bonn, Germany
b) The George Washington University, Washington DC, USA
c)Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, USA
d)Tbilisi State University, Tbilisi, Georgia

Bethe Forum Multihadron Dynamics in a Box - August 15 - 19, 2022

- Many hadrons observed in nature are resonances
- The resonances are not asymptotic states but their existence are manifested through the behavior of the scattering amplitudes
- E.g. ρ decaying via $\rho \to \pi\pi$ or $a_1(1260) \to \pi\pi\pi$
- Lattice QCD (LQCD) offers a systematic first-principles approach to compute hadrons properties from QCD
- Using LQCD to calculate resonances properties is more challenging than for single-particle properties
  - The finite volume does not allow states with well-separated decay products
  - Real-time processes such as scattering are inaccessible due to the practical difficulty of the analytic continuation to Minkowski correlators.
- To study resonance with LQCD need to use an indirect approach

Three-particle quantization

- Three approaches can be used in in lattice QCD to study scattering process of three particle
  - Relativistic fields theory (RFT) [M. T. Hansen , S. R. Sharpe (2014)]
  - Non-relativistic effective field theory (NREFT) [H. W. Hammer , J. Y. Pang , A. Rusetsky (2017)] is algebraically equivalent to FVU. A separate check of NREFT is thus not needed.
  - Finite-volume unitarity (FVU) [M. Mai, M. Döring (2017)]
- We are testing the RFT and FVU approach.

- There have been studies of systems with no resonances e.g.:
  [T. D. Blanton et al.(2021)] [T. D. Blanton et al.(2019)]
  [Hadron Spectrum Collaboration M. T. Hansen et al. (2020)]
  [M. Fischer at al. (2020)]
  [A. Alexandru et al.(2020)]
  [C. Culver et al. (2019)]
  [B. Hörz and A. Hanlon (2019)]
- Our aim is to study whether the formalism is usable for resonances in a toy model
  [GWQCD Collaboration, M. Mai et al. (2021)]

model 3-particle resonance

- We use complex field
- masses $3m_0 < m_1 < 5m_0$

$$S = \int dx \left[\sum_{i=0,1} \left(\frac{1}{2}\partial_\mu \phi_i^\dagger \partial_\mu \phi_i + \frac{1}{2} m_i \phi_i^\dagger \phi_i + \lambda_i (\phi_i^\dagger \phi_i)^2 \right) + \frac{g}{2} \phi_1^\dagger \phi_0^3 + h.c. \right]$$

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- At $g = 0$
  - $U(1) \times U(1)$ global symmetry $\phi_0 \to e^{i\theta}\phi_0 \quad \phi_1 \to e^{i\theta'}\phi_1$
  - ϕ_1 is a stable particle

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- With $g > 0$
  - residual global symmetry is $\phi_0 \to e^{i\theta}\phi_0$ together with $\phi_1 \to e^{i3\theta}\phi_1$
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- The residual symmetry avoid mixing $\phi_0 \leftrightarrow \phi_0^3$
- The model is most likely trivial. However, with small but finite lattice spacing the model effectively describes an interacting continuum field theory.

- Lattice discretization with $\partial_\mu \varphi(x) = \varphi(x+\mu) - \varphi(x)$, $m_i = \frac{1-2\lambda_i}{\kappa_i} - 8$, $\hat{\lambda}_i = \frac{\lambda_i}{4\kappa_i^2}$, $\hat{g} = \frac{1}{4}\sqrt{\frac{1}{\kappa_0 \kappa_1^3}}$ and $\phi_i = \sqrt{2\kappa_i}\varphi_i$

$$S = \sum_x \sum_{i=0,1} \kappa_i \left[\sum_\mu \varphi_i^\dagger(x)\varphi_i(x+\mu) + \hat{\lambda}_i(\varphi_i^\dagger(x)\varphi_0(x) - 1) + \varphi_i^\dagger \varphi_i(x) + h.c. \right] + \frac{\hat{g}}{2}\varphi_1^\dagger \varphi_0^3 + h.c.$$

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- The limit $\hat{\lambda} \to \infty$ restrict the filed to satisfy

$$\varphi_i^\dagger(x)\varphi_i(x) = 1$$

and thus the fields can be described with a phase $\varphi_i = e^{i\theta_i}$

- The action becomes

$$S = \sum_x \sum_{i=0,1} \left[\kappa_i \sum_\mu \varphi_i^\dagger(x)\varphi_i(x+\mu) + h.c. \right] + \frac{\hat{g}}{2}\varphi_1^\dagger \varphi_0^3 + h.c.$$

Monte Carlo simulation

- Metropolis-Hastings algorithm to generate ensembles
  - We did not implement more advanced algorithm such Cluster Algorithm [U. Wolff (1989)] since we are simulating at values of m_0 not too small and it is difficult to parallelize.
- Implementation with Kokkos to have a performance portable implementation [H. C. Edwards, C. R. Trott, D. Sunderland (2014)]

Spectrum $g = 0$

- Single particle energy level

$$\tilde{\varphi}_i^\dagger(t)\tilde{\varphi}_i(0) \qquad |A_{\varphi_i \to 0}| \; e^{-E_1 t} - e^{-E_1(T-t)}$$

Spectrum $g = 0$

- Single particle energy level

$$\tilde{\varphi}_i^\dagger(t)\tilde{\varphi}_i(0) \qquad |A_{\varphi_i \to 0}| \left(e^{-E_1 t} - e^{-E_1(T-t)} \right)$$

- Two particle energy level

$$\tilde{\varphi}_0^\dagger(t)^2\tilde{\varphi}_0(0)^2 \qquad |A_{2\varphi_0 \to 0}| \left(e^{-E_2 t} - e^{-E_2(T-t)} \right) + |A_{\varphi_0 \to \varphi_0}| e^{-E_1 T}$$

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- Three particle energy level

$$\tilde{\varphi}_0^\dagger(t)^3\tilde{\varphi}_0(0)^3 \qquad |A_{3\varphi_0 \to 0}| \left(e^{-E_3 t} - e^{-E_3(T-t)} \right)$$
$$+ |A_{2\varphi_0 \to \varphi_0}| e^{-E_1 T} \left(e^{-t(E_2 - E_1) + e^{-(T-t)(E_2-E_1)}} \right)$$

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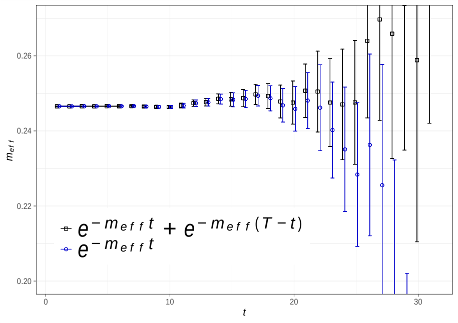
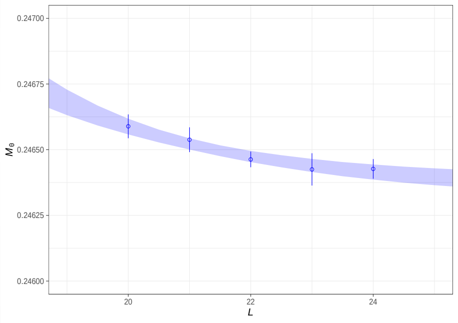
- We simulate at T large enough to neglect finite volume pollution (T=64)

- Simulation parameters: $\kappa_0 = 0.148522$, $\kappa_1 = 0.134228$, $g = 0$
- $T = 64$, $L = 20 - 24$
- Fit Exponential volume effects in the light mass $M_0 = M_\infty + P_0 \frac{e^{-M_0 L}}{\sqrt{M_0 L}}$

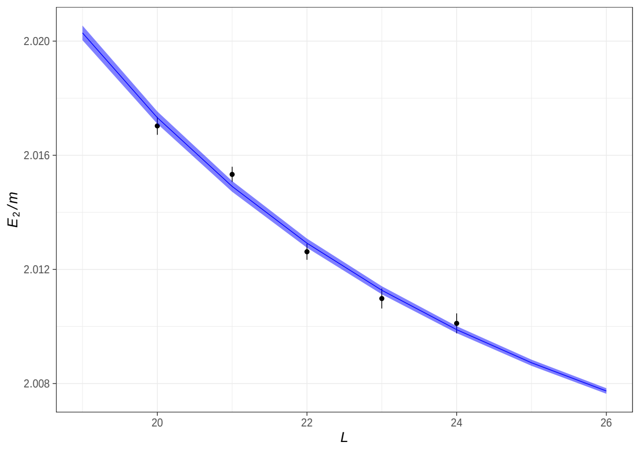




- $M_0 L \quad [4.8, 5.76]$
- Exponential finite volume effects are neglected in the quantization condition

- Two particle energy level fit

$$\cot\delta = \frac{Z_{00}(1,q^{\,2})}{\pi^{3/2}\gamma q} \qquad \text{with} \qquad \frac{q}{M_0}\cot\delta = \frac{1}{a_0 M_0}$$



$$\chi^2/d.o.f. = 1.4$$

$$a_0 M_0 = -0.1514(17)$$

- FVU

$$\det\left[B + C - 2L^3 E_{\mathbf{p}}\left(\tilde{K}_2^{-1} - \Sigma_2^L\right)\right] = 0$$

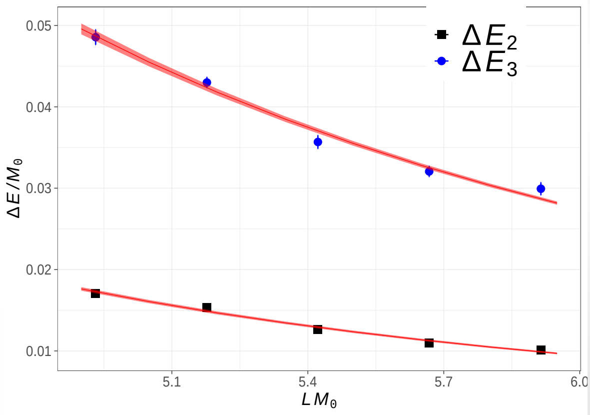
- RFT

$$\det\left[K_{df,3} + L^3\left(\tilde{F}/3 - \tilde{F}(K_2^{-1} + \tilde{F} + \tilde{G})^{-1}\tilde{F}\right)^{-1}\right] = 0$$

- Both depend on the spectator moment
- B, Σ_2^L, $\tilde{F}$ and $\tilde{G}$ can be computed from the finite volume spectrum
- $\tilde{K}_2$ and K_2 related to the two-body scattering amplitude
- $C(s)$ and $K_{df,3}$ the three body forces, infinite-volume quantity related to the scattering amplitude
- Isotropic approximation:
  - s-wave dominance
  - Three body force independent from the spectator momentum
- truncation of the momentum space.

Two-particle phase shift $\qquad \frac{q^*}{M_0}\cot\delta = \frac{1}{a_0 M_0}$

Three-particle force $\qquad K_3 = P_0$

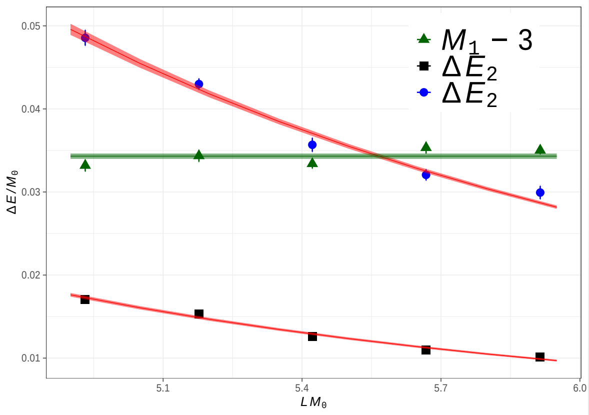


- fit:

$$\chi^2/d.o.f. = 1.8$$
$$P_0 = 1351(500)$$
$$a_0 M_0 = -0.1514(20)$$

- $a_0 M_0$ compatible with the only two-particle fit

Two-particle phase shift	$\frac{q^*}{M_0} \cot\delta = \frac{1}{a_0 M_0}$
Three-particle force	$K_3 = P_0$
One heavy-particle φ_1 mas	M_1=const



- fit:

$$\chi^2/d.o.f. = 1.8$$
$$P_0 = 1351(500)$$
$$a_0 M_0 = -0.1514(20)$$
$$M_1/M_0 = 3.0343(3)$$

- $a_0 M_0$ compatible with the only two-particle fit

Spectrum $g > 0$

- Single particle energy level

$$\tilde{\varphi}_0^\dagger(t)\tilde{\varphi}_0(0) \qquad |A_{\varphi_0 \to 0}|e^{-M_0 t}$$

- Two particle energy level

$$\tilde{\varphi}_0^\dagger(t)^2\tilde{\varphi}_0(0)^2 \qquad |A_{2\varphi_0 \to 0}|e^{-E_2 t}$$

- Three particle operator with the same quantum number $\tilde{\varphi}_0^3$ and $\tilde{\varphi}_1$ thus we measure the matrix

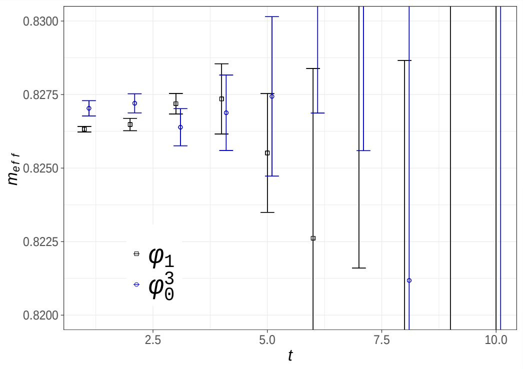
$$c(t) = \begin{bmatrix} \tilde{\varphi}_0^\dagger(t)^3\tilde{\varphi}_0(0)^3 & \tilde{\varphi}_0^\dagger(t)^3\tilde{\varphi}_1(0) \\ \tilde{\varphi}_1^\dagger(t)\tilde{\varphi}_0(0)^3 & \tilde{\varphi}_1^\dagger(t)\tilde{\varphi}_1(0) \end{bmatrix}$$

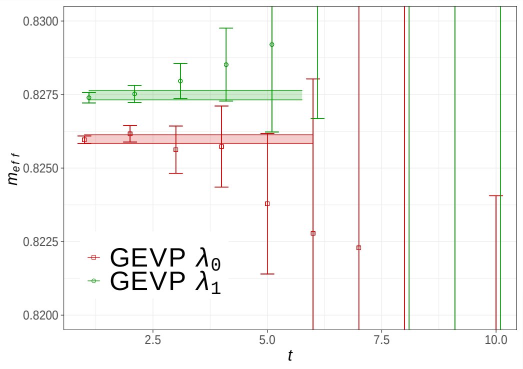
and we solve the GEVP [B. Blossier, M. Della Morte, G von Hippel, T. Mendes, R. Sommer (2009)]

$$C(t)v_n = \lambda(t)C(t_0)v_n$$

- The eigenvalues are $\lambda_n \qquad e^{-(t-t_0)E_3^n}$

- Simulation parameters: $\kappa_0 = 0.14771$, $\kappa_1 = 0.131062$, $g = 10$
- $T = 64$, $L = 23$





- We try two Ansatz for the three-body force K_3
  - Resonance

$$K_3 = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$$

  - No-resonance $c_0 = 0$ and φ_1 a stable particle

$$K_3 = c$$

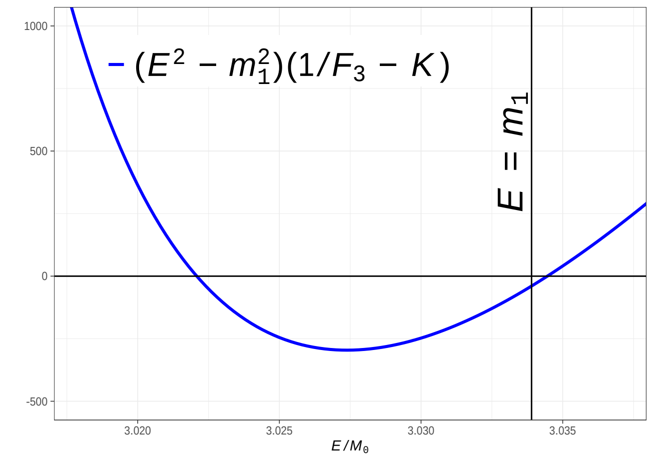
- In order to change smoothly between the two Ansatz we multiply the quantization condition by the pole of K_3, e.g. in RFT

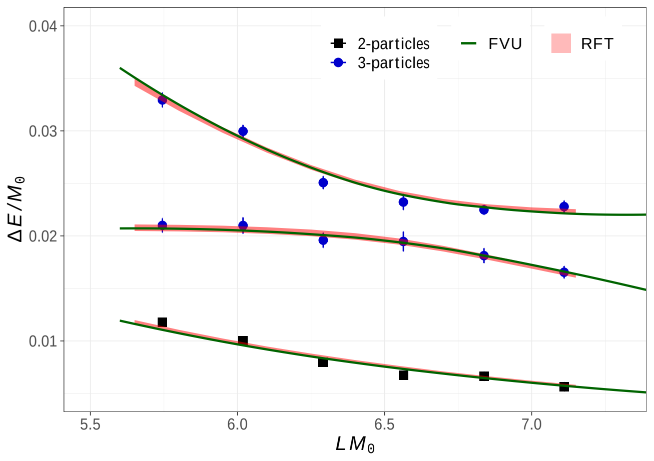
$$(E^2 - m_1^2)\left[1/F_3^{iso} + K_{df,3}^{iso}\right] = 0$$

so

$$\begin{cases} c_0 > 0 \implies & E = m_1 \text{ is not a solution} \\ c_0 = 0 \implies & E = m_1 \text{ is a solution} \end{cases}$$

- Solutions of the quantization condition



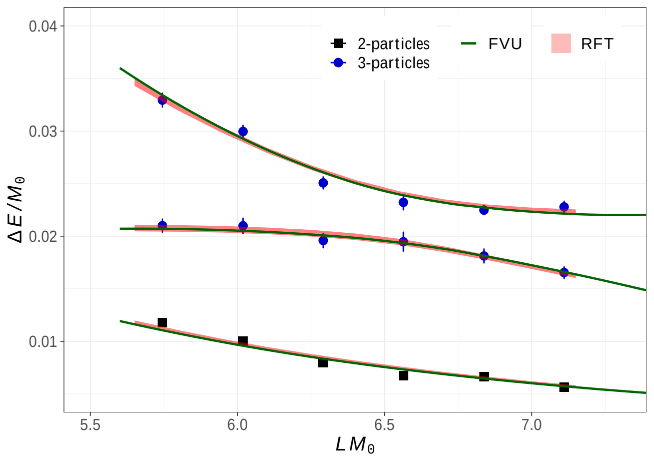


Two-particle	$\frac{q^*}{M_0}\cot\delta = \frac{1}{a_0 M_0}$
Three-particle	$K^{iso}_{df,3} = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$

- Fit:

$$\chi^2/d.o.f. \quad 1.45$$

- Very similar prediction of the energy levels between FVU and RFT
- Fit without c_1: $\chi^2/d.o.f. \quad 1.63$



Two-particle	$\frac{q^*}{M_0} \cot\delta = \frac{1}{a_0 M_0}$
Three-particle	$K^{iso}_{df,3} = \frac{c_0 M_0^2}{E^2 - m_1^2} + c_1$

- Two-particle only fit: $a_0 M_0 = -0.1562(29)$
- Two- and three-particle fit:
  - $a_0 M_0 = -0.1563(27)$
  - with $c_1 = 0$ $a_0 M_0 = -0.1531(13)$

We also try explicitly the t with $c_0 = 0$, no resonance and a stab
particle $_1$ ^{2}=d:o:f 3:1

$c_0 > 0$ reproduce better the data
^{2}=d:o:f: 1:5

We study the g dependence generating various ensembles

Simulation parameters:

0	1	g	L
0.148522	0.134228	0	20-24
0.147957	0.131234	5	20-25
0.147710	0.131062	10	21-26
0.147145	0.131062	20	21-27

$T = 64$, $L = 20 \quad 27$

$M_0 L = 5{:}4 \quad 7{:}2$

Spectrum for $g = 5; 10; 20$

^{2}=d:o:f: $2\ [1; 1{:}5]$

Avoid level crossing more visible with larger g

Compute the scattering amplitude $\mathcal{M}$ from the three body force ($g = 20$)

FVU

$$T_3 = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + C)}{2E_\ell} \frac{1}{\tilde{K}_2^{-1} - \Sigma_n} T_3$$

RFT

$$\mathcal{M}_{3s}(E) = \frac{\sum_i L(p_i)R(p_i)}{1/K_3 + F^{-1}}$$

Fit F^{-1} as a polynomial in the energy

Find the pole as $1/K_3 + F^{-1} = 0$

Fit F^1 as a polynomial in the energy in a region close to the pole

Near resonance we expect the amplitude

$$M_3 = \frac{c}{E \quad M_R + i \quad =2}$$

We nd a very small width $=M_0 \quad 10^{7}$

$$= \frac{1}{2M_R}\frac{1}{s!}\int dQ_3 |M_{1! \quad 3}|^2$$

$$\int dQ_3 \quad 10^{7}$$

We also have a set of data at $g = 5$ with smaller $M_0 L$ ($_0 = 0{:}147973$, $_1 = 0{:}131234$)

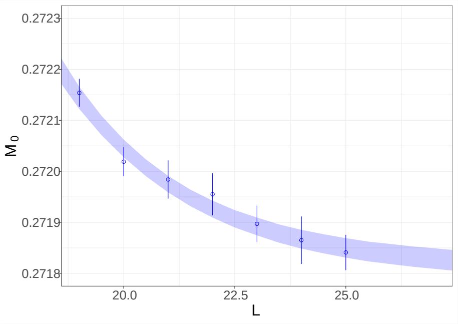
$g = 5$ (small M_0) $g = 10$

^{2}=d:o:f: = 2:4 ^{2}=d:o:f: = 1:5

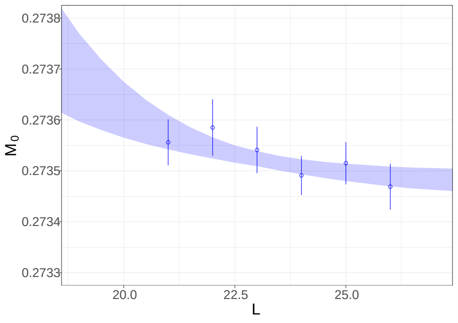
Large ^{2} at $g = 5$ could be due to exponential nite volume e ect in M_0

- Two- and Three-particle quantization conditions do not take in to account exponential effects

- $g = 5$ (small M_0)



- $g = 10$



- $M_0 L \quad [5.1, 6.8]$
- $M_0 L \quad [5.7, 7.1]$

Conclusion and Outlook

- We have evidence of a resonance and the avoided level crossing for $g > 0$
- The three-particle quantization condition describe with good χ^2 the energy level
- We extract the pole position in the scattering amplitude M_3
- Further comparison of M_3 extraction from FVU and RFT away from the pole

Thank you for your attention