Non-perturbative range corrections and spurious poles in finite volume

Martin Ebert

Institut für Kernphysik, Technische Universität Darmstadt

in collaboration with:

H.-W. Hammer, F. Müller, J.-Y. Pang, A. Rusetsky, J.-J. Wu

Bethe Forum: Multihadron Dynamics in a Box - A.D. 2022

Objective and Set-Up

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Objective

- Include range corrections in the quantization condition (**QC**)
- Non-perturbative approach $\rightarrow$ spurious pole (**SP**)
- Develop a non-perturbative method to avoid issues caused by **SP**

$\rightarrow$ Improve accuracy of **QC**

- Three identical bosons in non-relativistic effective field theory (**NREFT**)
- Pionless effective field theory

$$\mathbf{m}_{high} = m_\pi \approx 140 \text{ MeV}$$

- Shallow bound two-body state
- Unnaturally large two-body **scattering length** a and naturally sized **effective range** r

$$\frac{1}{a} \sim k \sim \mathbf{m}_{low}; \qquad \frac{1}{r} \sim \mathbf{m}_{high}$$

- Use particle-dimer picture

Non-Perturbative Range Corrections and Spurious Poles in Finite Volume

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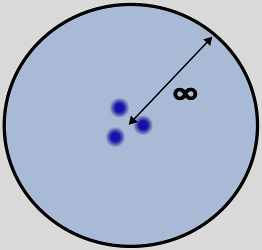
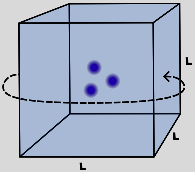
1. Infinite Volume



ME, Hammer, Rusetsky,
Eur. Phys. J. A **57**, 332 (2021)

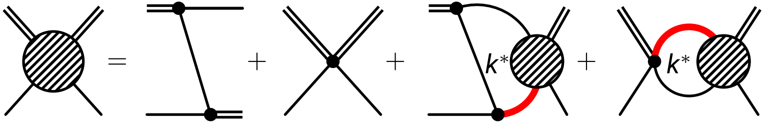
2. Finite Volume



Pang, ME, Hammer, Müller, Rusetsky, Wu,
J. High Energ. Phys. **2022**, 19 (2022)

Effective Range in the Particle-Dimer Picture and the Spurious pole

- Use an auxiliary-field (dimer) to simplify the three-particle problem (S-wave)



- Dimer-propagator with effective range expansion (**ERE**)

$$\tau(k^*) = \left(k^* \cot\delta(k^*) + k^*\right)^{-1} = \left(-\frac{1}{a} - \frac{r}{2}k^{*2} + \ldots + k^*\right)^{-1}$$

- Two poles for $r \ll a$

$$k_1^* \approx \frac{1}{a}\,; \qquad k_2^* \approx \frac{2}{r}$$

$$k^{*2} = 3/4\,k^2 - mE$$

Effective Range in the Particle-Dimer Picture and the Spurious pole

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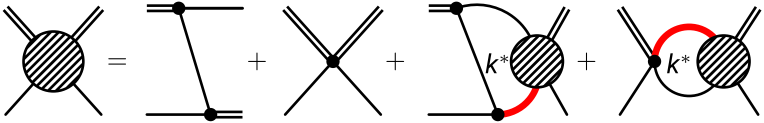

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- Two poles for $r \ll a$

$$k_1^* \approx \frac{1}{a}\,; \qquad k_2^* \approx \frac{2}{r}$$

$k^{*2} = 3/4\,k^2 - mE$

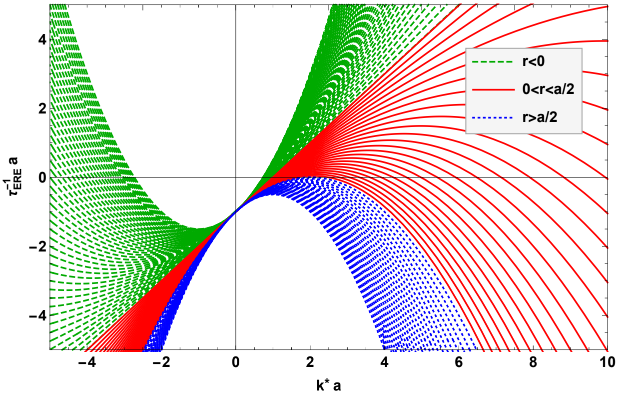
k_2^* has a negative residue:
$\rightarrow$ violates unitarity / causality
$\Rightarrow$ unphysical / spurious

Condition for the Spurious Pole

- Appears at **NLO** in the power-counting
- k_2^* causes problems if it lies on the path of integration

→ Condition for the issues:

$$0 < r < \frac{a}{2}$$



Established Methods

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- Choose UV cut-off $\Lambda < k_2^*$ $\rightarrow$ Spurious pole does not appear in integration path

$\rightarrow$ Limited accuracy

- Fully **perturbative method**
  Vanasse, Phys. Rev. C **88**, 044001 (2013)
  Ji, Phillips, Platter, Annals Phys. **327**, 1803 (2012)

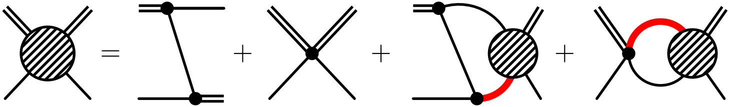
$$\tau_{pert}^{m}(k^*) = \sum_{n=0}^{m} \frac{n!(k^*)^{2n}}{(-1/a + k^*)^{n+1}} \left(\frac{r}{2}\right)^n$$

$\rightarrow$ Convergence issues in finite volume

$\Rightarrow$ Develop a **non-perturbative method**

The Non-Perturbative Method
The Effective Potential

- Original Faddeev equation



$$M(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + 8\pi \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} Z(\mathbf{p},\mathbf{k};E)\tau(k^*)M(\mathbf{k},\mathbf{q};E)$$

- Introduce an effective potential W

$$M(\mathbf{p},\mathbf{q};E) = W(\mathbf{p},\mathbf{q};E) + 8\pi \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} W(\mathbf{p},\mathbf{k};E)(\tau(k^*) - f(k^*))M(\mathbf{k},\mathbf{q};E)\,;$$

$$W(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + 8\pi \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} Z(\mathbf{p},\mathbf{k};E)f(k^*)W(\mathbf{k},\mathbf{q};E)$$

The Non-Perturbative Method
The Function f

- Separate the physical pole k_1^* and the spurious pole k_2^*

$$\tau(k^*) = \frac{2(k_1^* + k_2^*)/r}{(k_2^* - k_1^*)(k^* + k_2^*)(k^* - k_1^*)} - \frac{4k_2^*/r}{(k_2^* - k_1^*)(k^{*2} - k_2^{*2})}$$

- Subtract a function $f(k^*)$, such that

$$\tau_i(k^*) = \tau(k^*) - f(k^*) = \frac{2(k_1^* + k_2^*)/r}{(k_2^* - k_1^*)(k^* + k_2^*)(k^* - k_1^*)} + \frac{4k^{*2}/r}{(k_2^* - k_1^*)k^{*2}_2}\left\{1 + \frac{k^{*2}}{k_2^{*2}} + \frac{k^{*4}}{k_2^{*4}} + \dots\right\}$$

- We have shown analytically that

$$W(\mathbf{p}, \mathbf{q}; E) = Z(\mathbf{p}, \mathbf{q}; E) + \# + \#' p^2 + \dots = Z'(\mathbf{p}, \mathbf{q}; E)$$

- Absorb the polynomial in the three-body forces, drop un-physical imaginary part
  $\rightarrow$ Change renormalization prescription

The Non-Perturbative Method
The Function f

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- Separate the phys

τ

- Subtract a functio

> Solve new Faddeev equation:
>
> $$M(\mathbf{p},\mathbf{q};E) = Z'(\mathbf{p},\mathbf{q};E) + 8\pi\int^{\Lambda}\frac{d^3\mathbf{k}}{(2\pi)^3}\,Z'(\mathbf{p},\mathbf{k};E)\tau_i(k^*)M(\mathbf{k},\mathbf{q};E)$$

$$\tau_i(k^*) = \tau(k^*) - f(k^*) = \frac{2(k_1^* + k_2^*)/r}{(k_2^* - k_1^*)(k^* + k_2^*)(k^* - k_1^*)} + \frac{4k^{*2}/r}{(k_2^* - k_1^*)k_2^{*2}}\left\{1 + \frac{k^{*2}}{k_2^{*2}} + \frac{k^{*4}}{k_2^{*4}} + \dots\right\}$$

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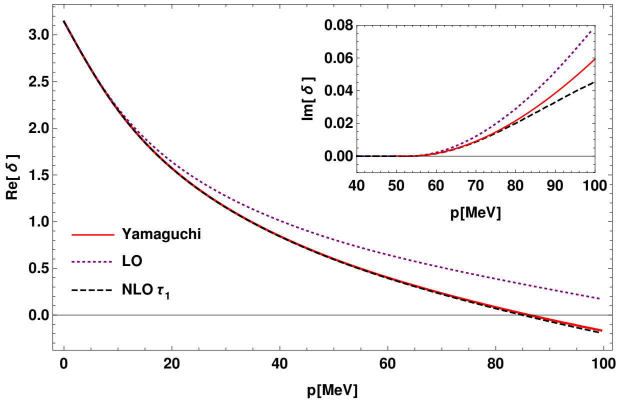
Numerical Test
Particle-Dimer Phase Shift

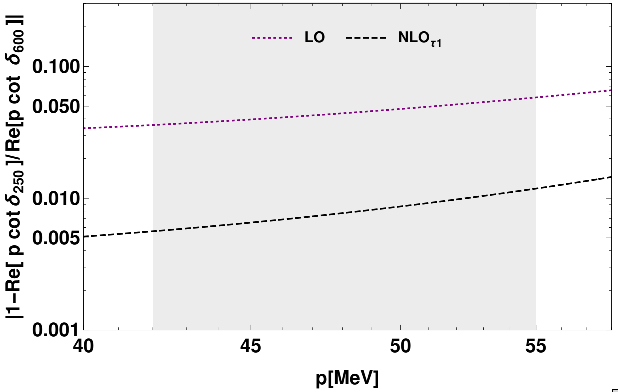
- Compare to Yamaguchi model (S-wave)
- $a = 5.42$ fm, $r = 1.76$ fm

→ Method can describe the model

→ Clear improvement compared to LO



Numerical Test
Consistency Assessment



- Investigate the internal consistency of our method

  Grießhammer, Eur. Phys. J. A **56**, 118 (2020)

$$\frac{p\cot\delta_{\Lambda_2} - p\cot\delta_{\Lambda_1}}{p\cot\delta_{\Lambda_2}} = c\cdot\left(\frac{p}{\Lambda_b}\right)^{n+1-\eta} + \ldots$$

- Double logarithmic plot $\rightarrow$ linear function
- Slope $\Leftrightarrow$ order of the **EFT**

Λ_b: Breakdown scale; Λ_1, Λ_2: UV cutoff

Numerical Test
Consistency Assessment

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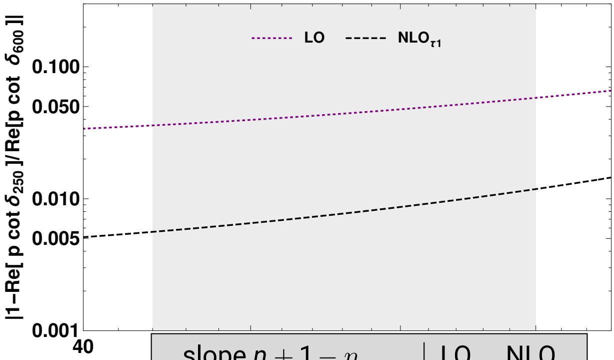

slope $n + 1 - \eta$	LO	NLO
fit here	1.8	2.8
fit by Grießhammer	1.9	2.9

- Investigate the internal consistency of our method

  Grießhammer, Eur. Phys. J. A **56**, 118 (2020)

$$\frac{p\cot\delta_{\Lambda_2} - p\cot\delta_{\Lambda_1}}{p\cot\delta_{\Lambda_2}} = c \cdot \left(\frac{p}{\Lambda_b}\right)^{n+1-\eta} + \dots$$

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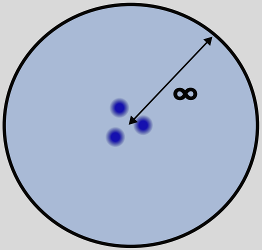
A Non-Perturbative Method for Spurious Poles in Infinite and Finite Volume

Infinite Volume



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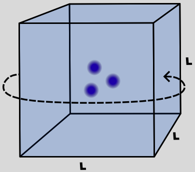
Finite Volume



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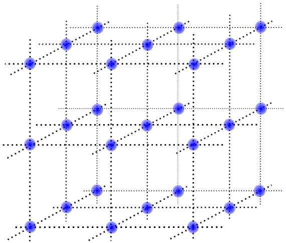
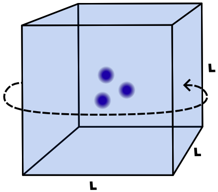




- Discretized position-space, a **lattice**:
  $\rightarrow$ Countably infinite number of degrees of freedom

- Finite volume:
  $\rightarrow$ Finite number of degrees of freedom
- Apply periodic boundary conditions

$\rightarrow$ Discrete momentum space

$\rightarrow$ Reduced symmetry of the cubic group

Quantization Condition

- Three-particle quantization condition (**QC**) in **NREFT**
  Hammer, Pang, Rusetsky, JHEP **1709**, 109 (2017); JHEP **1710** (2017) 115

$$\det\left(\tau_L^{-1}(k^*) - Z(\mathbf{p},\mathbf{k},E)\right) = 0$$

with the dimer propagator in finite volume

$$\tau_L(k^*) = \frac{1}{k^* \cot\delta(k^*) + S(\mathbf{k},E)} \quad \text{and} \quad S(\mathbf{k},E) = -\frac{4\pi}{L^3}\sum_{\mathbf{p}} \frac{1}{\mathbf{p}^2 + \mathbf{p}\mathbf{k} + \mathbf{k}^2 - mE}$$

- **QC** in irreducible representation Γ of the cubic group
  Döring, Hammer, Mai, Pang, Rusetsky, Wu, Phys. Rev. D **97**, 114508 (2018)

$$\det\left(\tau_L^{-1}(s)\delta_{r,s}\delta_{\lambda,\rho} - \frac{\theta(s)}{GL^3} Z^{\Gamma}_{\lambda,\rho}(r,s)\right) = 0$$

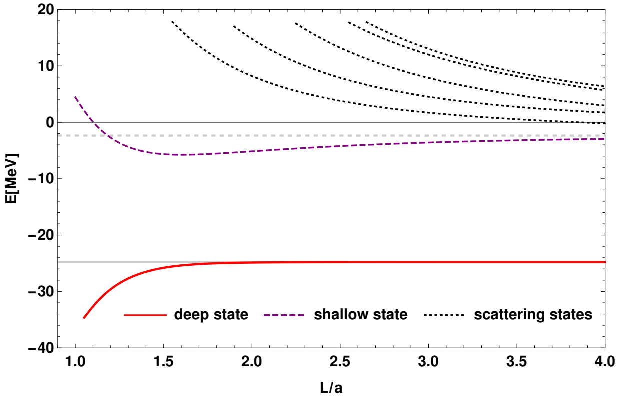
Spectrum of the Yamaguchi Model

- Compare to Yamaguchi model (S-wave)
- $a = 5.42$ fm, $r = 1.76$ fm

→ Two bound states

→ Three-body scattering

→ Particle-dimer scattering



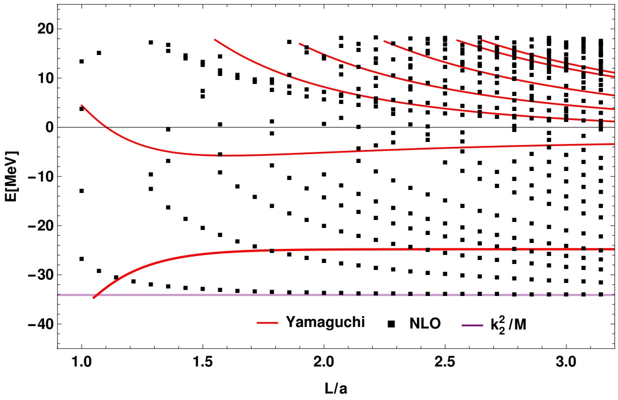
The Spurious Pole in Finite Volume

- Ignore the pole ⇔ Use **QC** with unmodified dimer propagator

$$\tau_L^{-1}(k^*) = -\frac{1}{a} - \frac{r}{2}(k^*)^2 + S(k,E)$$

→ Incorrect spectrum

→ Spurious scattering states



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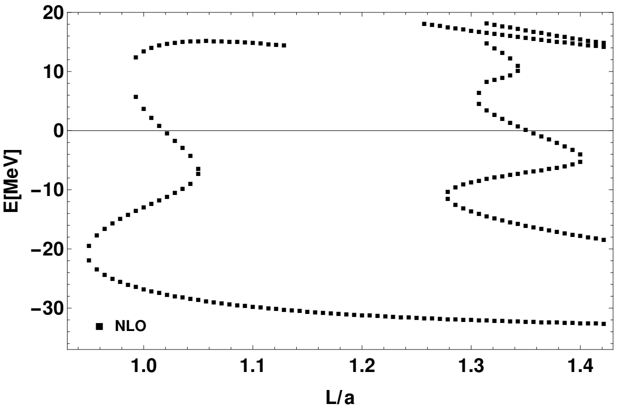
$\rightarrow$ Incorrect spectrum

$\rightarrow$ Spurious scattering states

$\rightarrow$ Un-physical merging of states (expected due to the negative residue)



The Non-Perturbative Method in Finite Volume

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- Same idea as in infinite volume:
  (a) Introduce an effective potential
  (b) New Faddeev equation with $\tau_L(k^*) \to \tau_L(k^*) - f_L(k^*)$
  (c) The difference between the effective potential and the original potential is a low-energy polynomial
  (d) Absorb the difference in the three-body forces
- The analytical argument for (c) in infinite volume relies on perturbation theory
  Additional poles of the exchange diagram above two-body threshold
  $\rightarrow$ Cannot be used!

$\rightarrow$ Use modified function $f_L(k^*)$

$$f_L(k^*) = \begin{cases} f(k^*) & (k^*)^2 \geq 0\,, \\ 0 & \text{else} \end{cases}$$

Numerical Test
Energy Spectrum in Finite Volume

- Use **QC** with modified dimer propagator

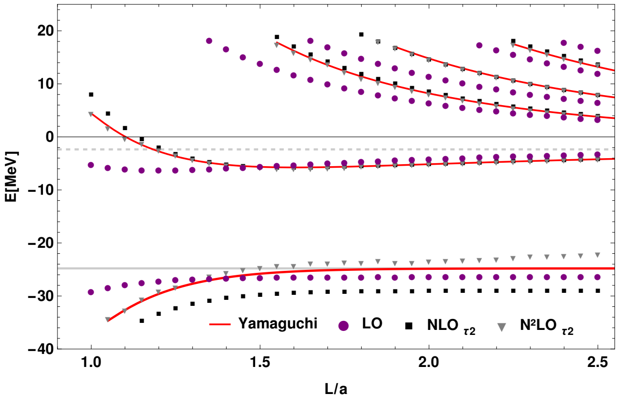
$$\tau_L(k^*) \to \tau_L(k^*) - f_L(k^*)$$

and changed renormalization prescription

→ Correct spectrum

→ Clear improvement to **LO**

→ Systematic improvement order by order



Summary and Outlook

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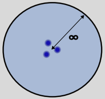
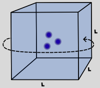
Infinite Volume



- Range corrections → Spurious pole k_2^*
- Developed non-perturbative method
- Describes models and is consistent with power-counting

→ Higher orders of the **EFT/ERE**

Finite Volume



- Method can be adopted to finite volume
- **QC** for **NREFT** with the method describes the energy spectrum

→ Extend **QC** to nucleons

→ Include range corrections

Summary and Outlook

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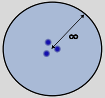
Infinite Volume



- Range corrections $\rightarrow$ Spurious pole k_2^*
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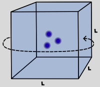
- Describes models and is consistent with power-counting

$\rightarrow$ Higher orders of the **EFT/ERE**

Thank you for your attention!

Finite Volume



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- **QC** for **NREFT** with the method describes the energy spectrum

$\rightarrow$ Extend **QC** to nucleons

$\rightarrow$ Include range corrections

Backup
The Propagator with Range Corrections

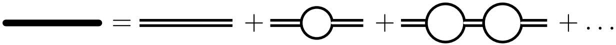
- Lagrangian of the particle-dimer picture

$$\mathcal{L}_d = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + \sigma d^\dagger \left({\color{red} i\partial_0 + \frac{\nabla^2}{4m}} + \Delta \right) d + \frac{f_0}{2} \left(d^\dagger \psi^2 + \text{h.c.} \right)$$
$$+ h_0 d^\dagger d \psi^\dagger \psi + h_2 d^\dagger d (\psi^\dagger \nabla^2 \psi + (\nabla^2 \psi^\dagger)\psi) + \cdots$$

- Bare propagator at LO (NLO)

$$S_D^{NLO}(p_0, \mathbf{p}) = -\frac{i}{\Delta + {\color{red} p_0 - \mathbf{p}^2/(4m)}}$$

- Dyson equation for the dressed propagator $\tau(k^*)$



Backup
Potential

- The potential: One particle exchange and three-body forces

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{pq}-mE} + \frac{H_0(\Lambda)}{\Lambda^2} + \frac{H_2(\Lambda)}{\Lambda^4}\frac{3}{8}\left(\mathbf{p}^2+\mathbf{q}^2\right) + \cdots$$

- Energy scheme

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{pq}-mE} + \frac{H_0}{\Lambda^2} + \frac{H_2}{\Lambda^4}(mE+\gamma^2) + \cdots$$

- S-wave projection

$$Z_S(p,q,E) = \frac{1}{2pq}\frac{p^2+q^2+pq-mE}{p^2+q^2-pq-mE} + \frac{H_0}{\Lambda^2} + \frac{H_2}{\Lambda^4}(mE-mE_d) + \cdots$$

Backup
Avoided avoided-level crossing

- Consider two states and a volume size L' such that $E_1(L') = E_2(L')$ for $g = 0$
- The energy is given by

$$\det \begin{pmatrix} E - E_1 & g\,H_{12} \\ g\,H_{12} & E - E_2 \end{pmatrix} = 0$$

$$\Rightarrow E_{\pm} = \frac{1}{2}(E_1 - E_2) \pm \sqrt{\frac{1}{4}(E_1 - E_2)^2 + g^2 H_{12}^2}$$

- Consistent theory: The root can not vanish $\rightarrow$ avoided level crossing
- Due to the negative residue it can $\rightarrow$ merging of states
- Also the root can be negative $\rightarrow$ disappearing states