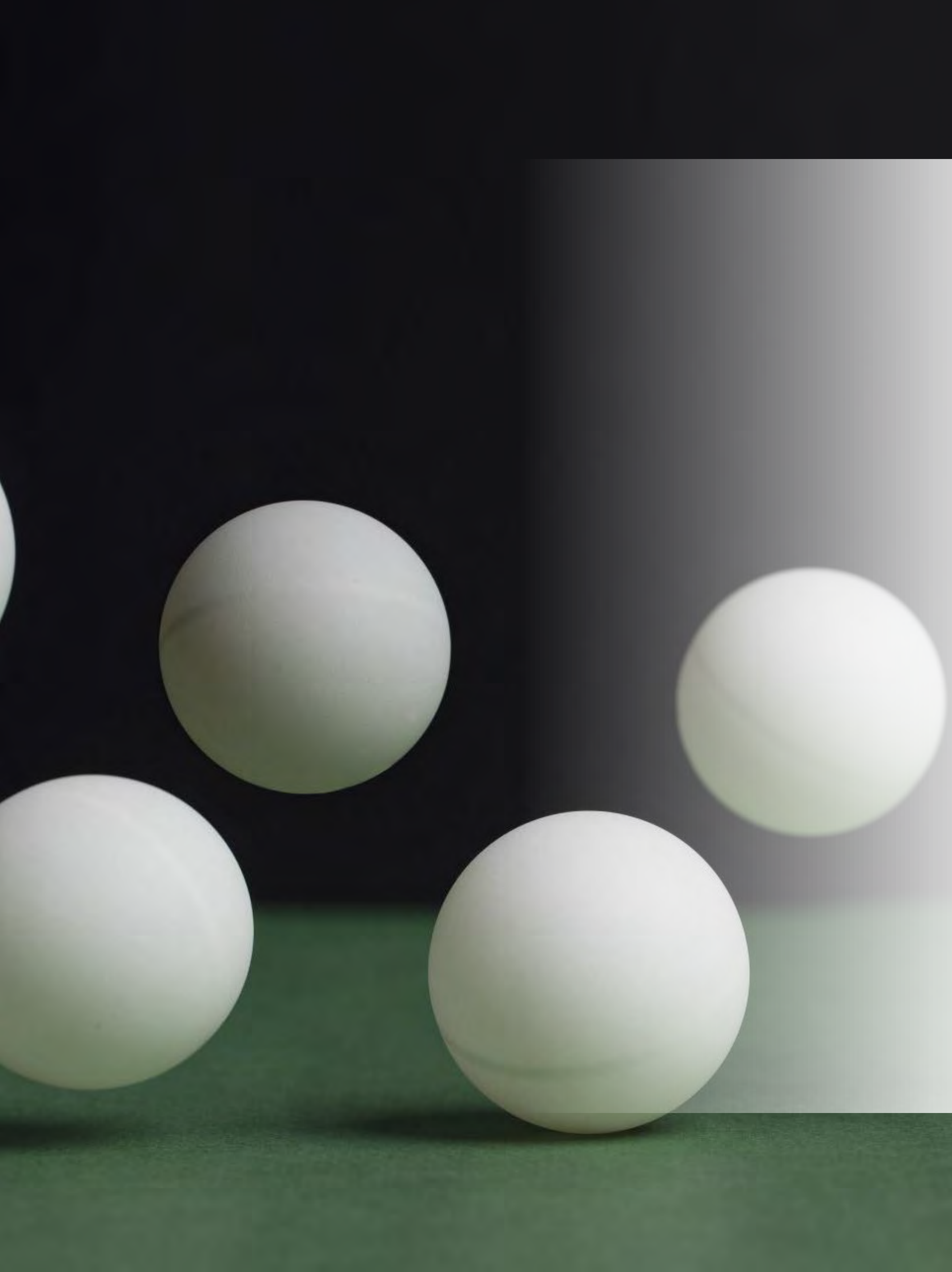


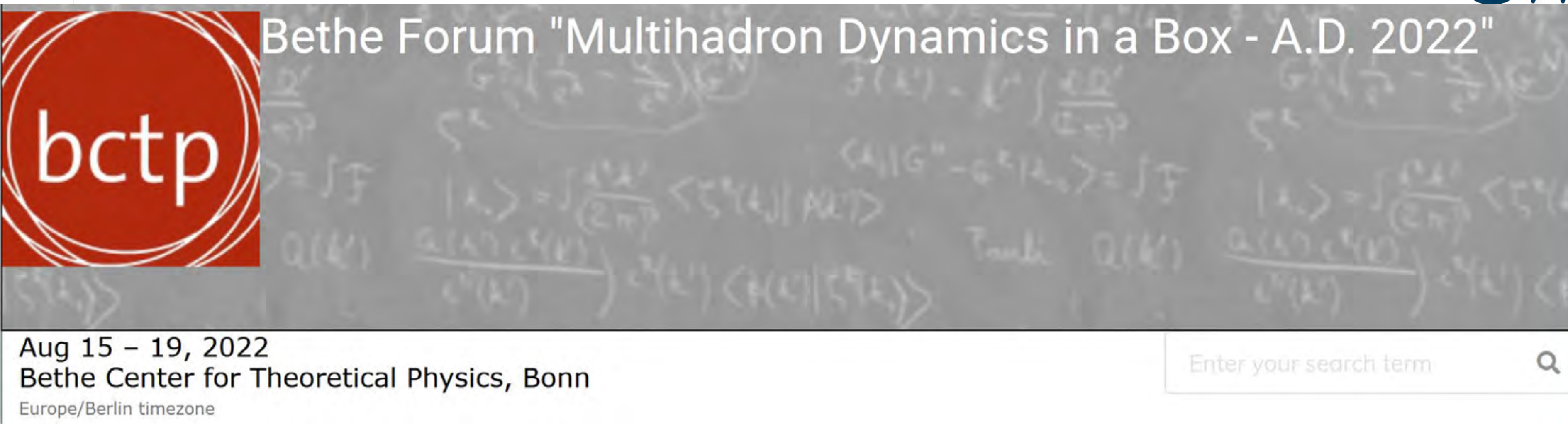
Simple three-pion and kaon systems

from lattice QCD

Michael Doering

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC

Jefferson Lab
Thomas Jefferson National Accelerator Facility



Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]

Intros to scattering can be found in these summer school lectures:

- The 2021 School on the Physics of Baryons (Baryons-21_School)
- Lectures on Scattering Resonances (2021)

Work supported by: National Science Foundation
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Introduction

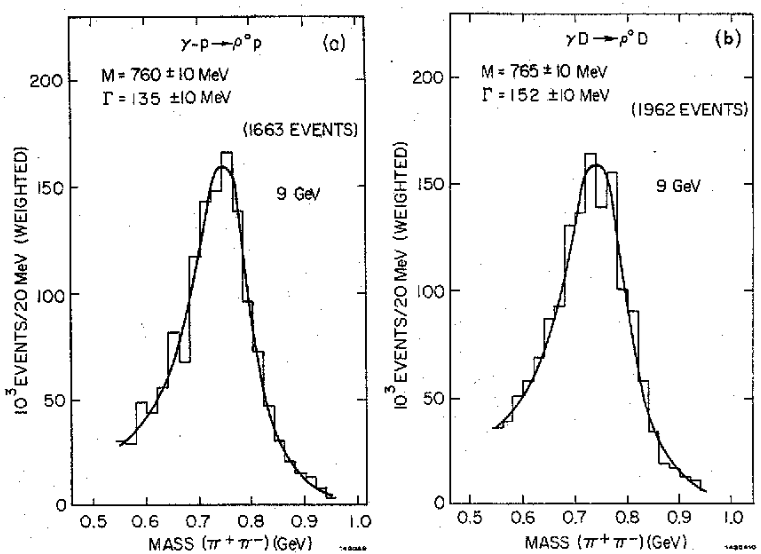
- Resonances and the role of three-body channels
- Resonances as poles in the complex plane: analytic continuation
- Example: Chiral trajectory of the $f_0(500)$ $''\sigma''$

Typical Breit-Wigner resonances

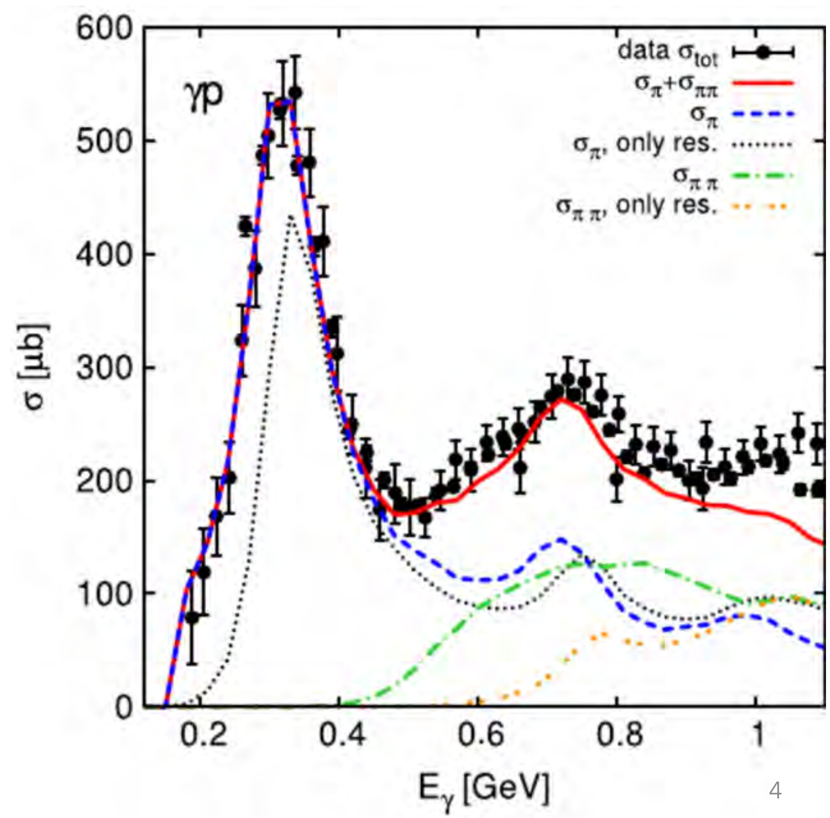
ρ-meson photoproduction

$$\left.\begin{array}{l} M = 760 \pm 10 \text{ MeV} \\ \Gamma = 135 \pm 10 \text{ MeV} \end{array}\right\} \quad \text{Hydrogen, 9 BeV.}$$

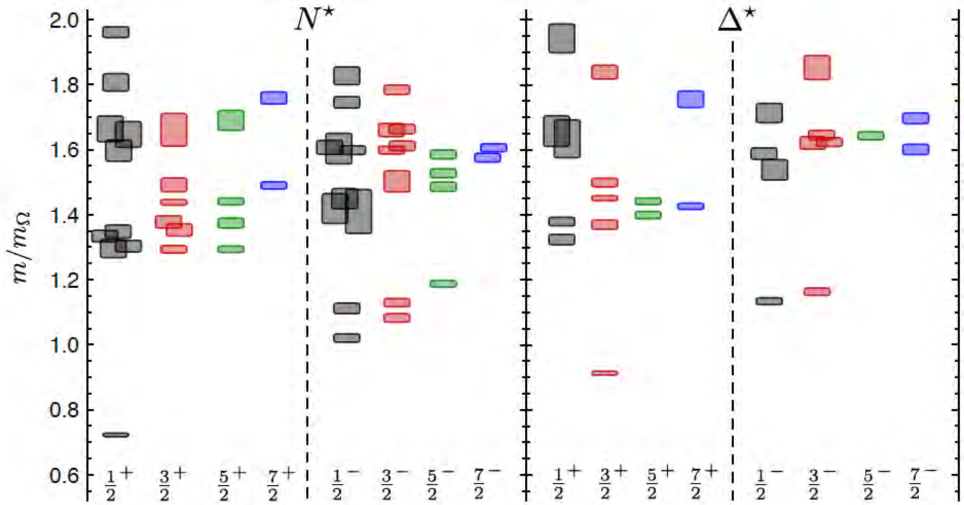
$$\left.\begin{array}{l} M = 765 \pm 10 \\ \Gamma = 152 \pm 10 \end{array}\right\} \quad \text{Deuterium, 9 BeV.}$$



Δ-baryon photoproduction

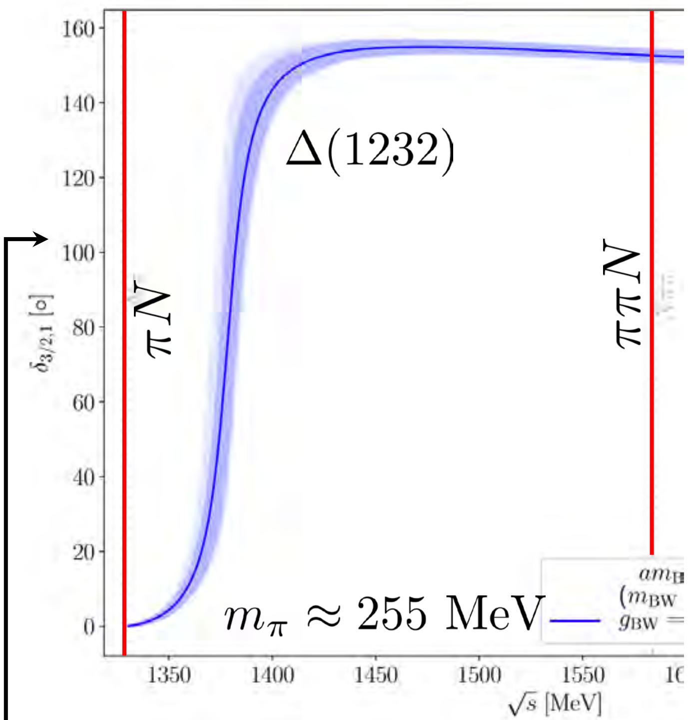


Lattice QCD for excited baryons



$m_\pi = 396$ MeV [Edwards *et al.*, Phys.Rev. D84 (2011)]

- Pioneering spectroscopic calculations
- Information on existence, width & properties of resonances requires
  - Meson-baryon interpolating operators
  - Detailed finite-volume analysis



[G. Silvi et. al., arXiv: 2101.00689]

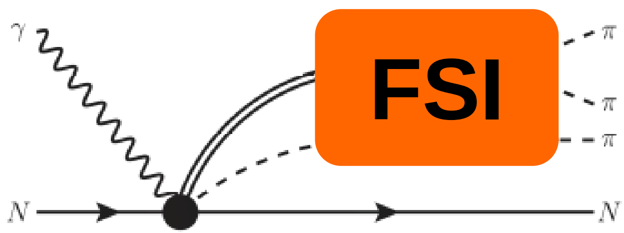
See also: Bulava et al., [2208.03867]

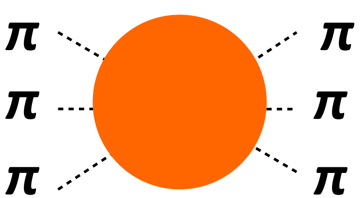
How about $\pi\pi N$? Roper resonance?

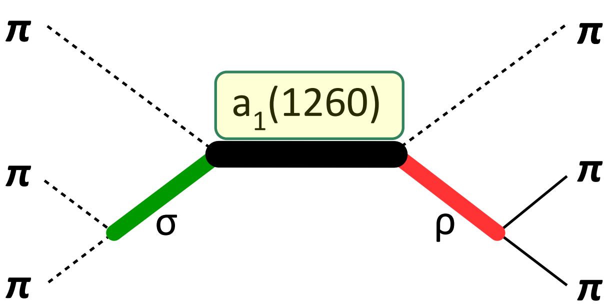
Three-body aspects: $\pi\pi N$ vs. $\pi\pi\pi$

Light mesons

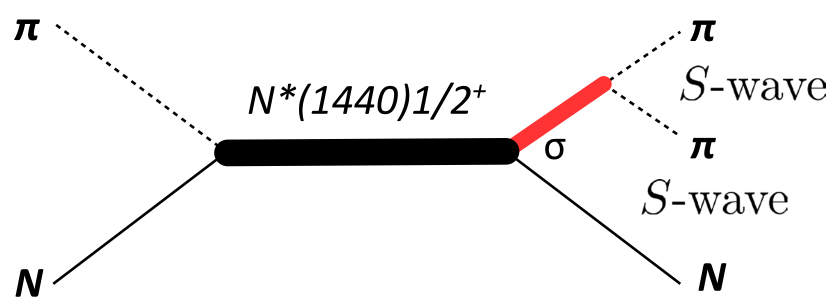
- COMPASS @ CERN: $\pi_1(1600)$ discovery
- GlueX @ Jlab in search of hybrids and exotics,
- Finite volume spectrum from lattice QCD:
  Lang (2014), Woss [HadronSpectrum] (2018) Hörz (2019), Culver (2020), Fischer (2020), Hansen (2020),...







Light baryons

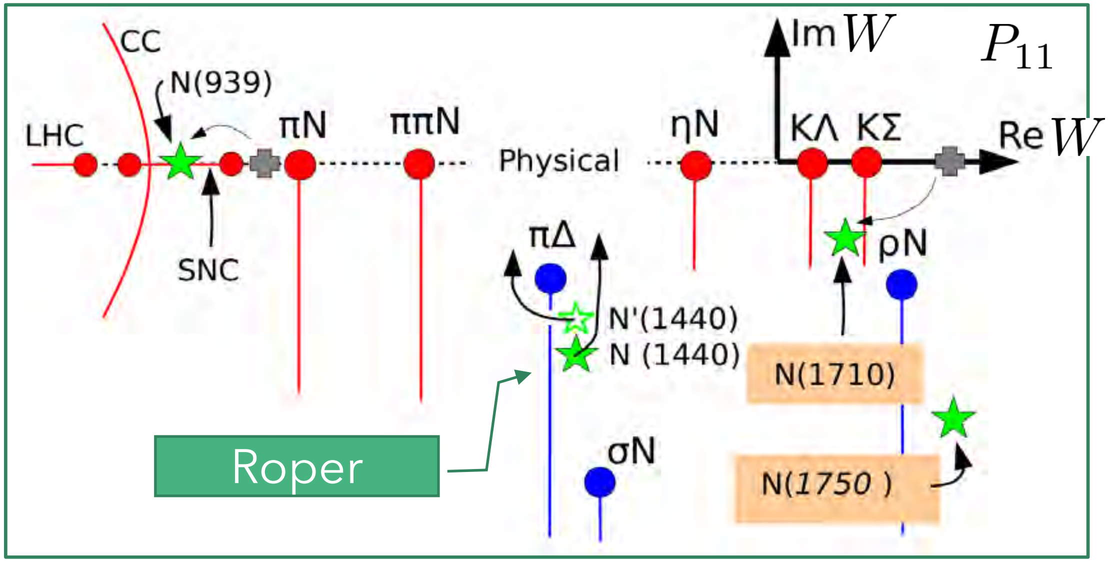


- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

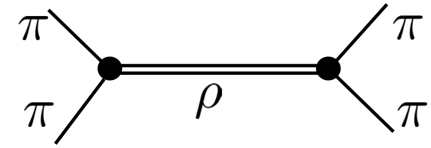
Hadronic resonances as poles

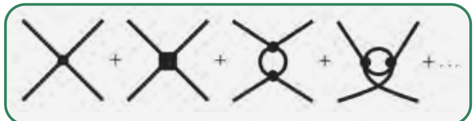
- Defining resonances as poles in amplitudes at complex energies resolves all mentioned problems
  - Real part of pole position ⟺ Mass
  - 2x Imaginary part of pole position ⟺ Width
  - Pole residue ⟺ Branching ratio into different channels because amplitudes factorize at poles [Doring 2009]

  - Next goal: What is this?
    - Red: Real thresholds
    - Blue: resonanat sub-channel thres.
    - Double pole Roper
    - Note partial-wave cuts (CC, SNC) that disappear in plane-wave amplitude



Analytic continuation (2B)





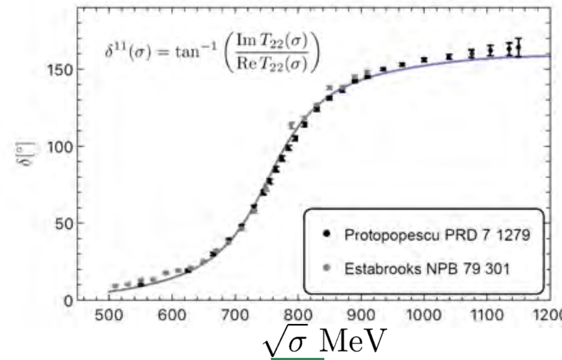
Inverse Amplitude Method [Dobado]

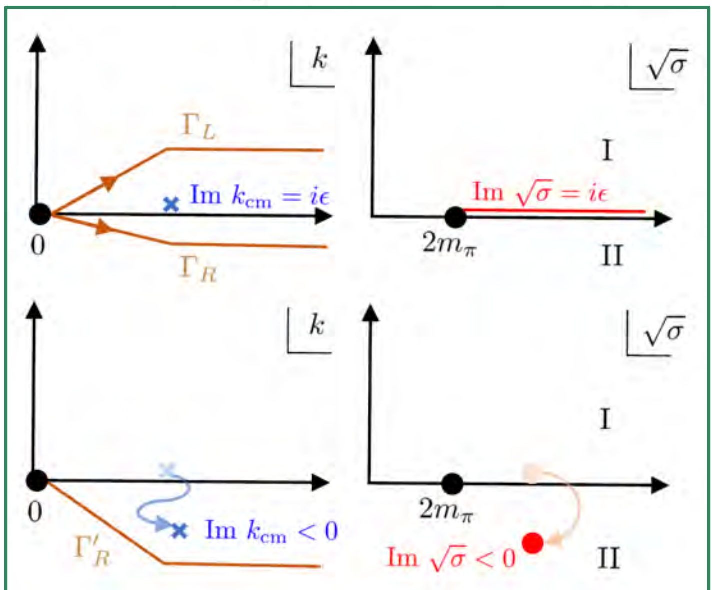
$$T_{22}(\sigma) = \tilde{v}(k_{\rm cm})\tau(\sigma)\tilde{v}^*(k_{\rm cm}),\;\; k_{\rm cm} = \sqrt{\frac{\sigma}{4} - m_\pi^2}\,,$$

$$\tau^{-1}(\sigma) = \boxed{K^{-1}} - \Sigma\,,$$

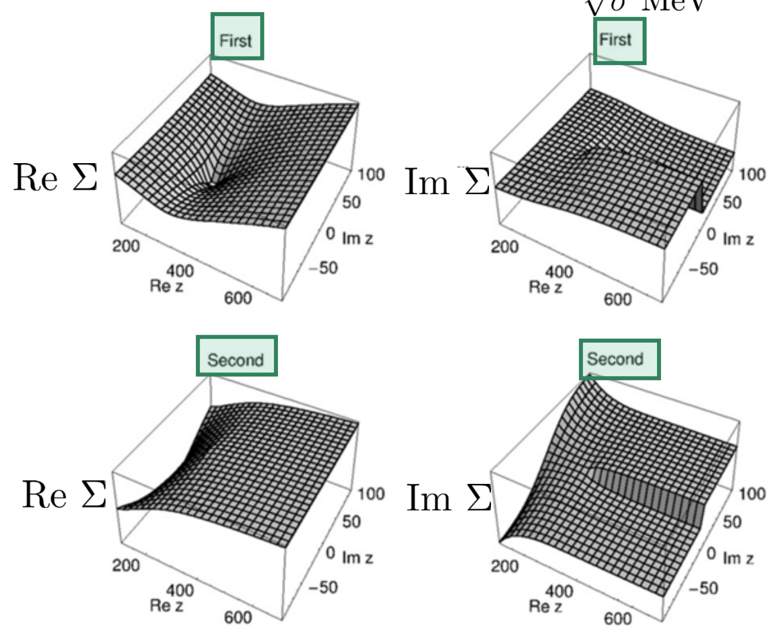
$$\Sigma = \int_0^\infty \frac{{\rm d}k\,k^2}{(2\pi)^3}\,\frac{1}{2E_k}\,\frac{\sigma^2}{\sigma'^2}\,\frac{\tilde{v}(k)^*\tilde{v}(k)}{\boxed{\sigma - 4E_k^2 + i\epsilon}}$$

$$E_k = \sqrt{k^2 + m_\pi^2}$$



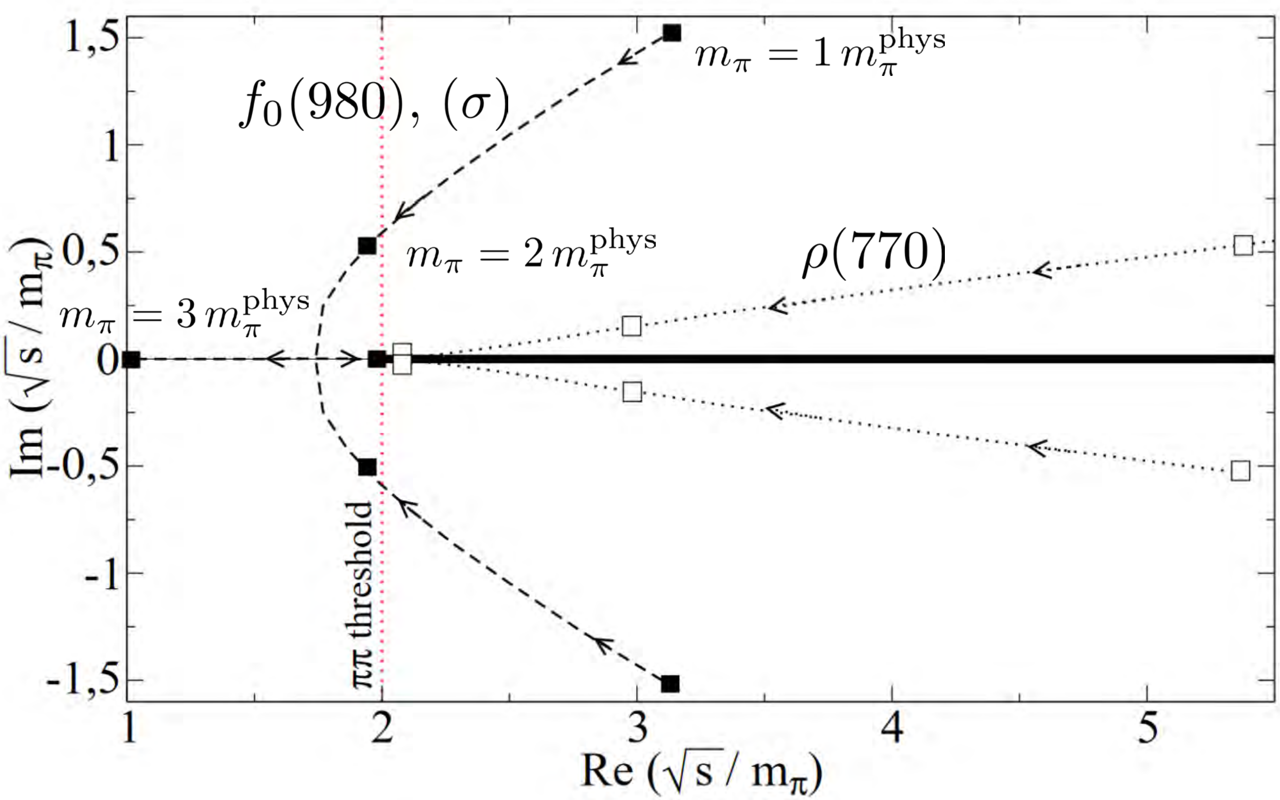


"Adiabatic" contour deformation
(remember for 3-body case later!)



Chiral trajectories of light mesons

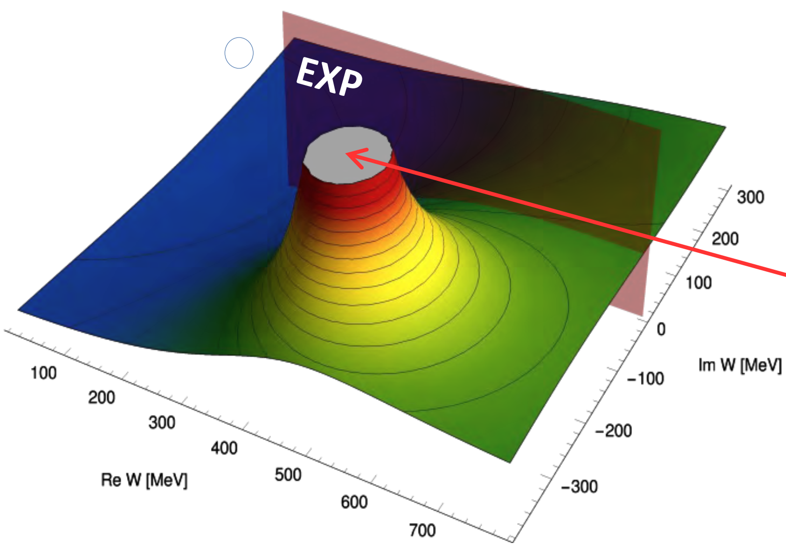
- Quark-mass dependence as predicted from "Inverse amplitude method" with one-loop ChPT [Hanhart et al., 0801.2871]

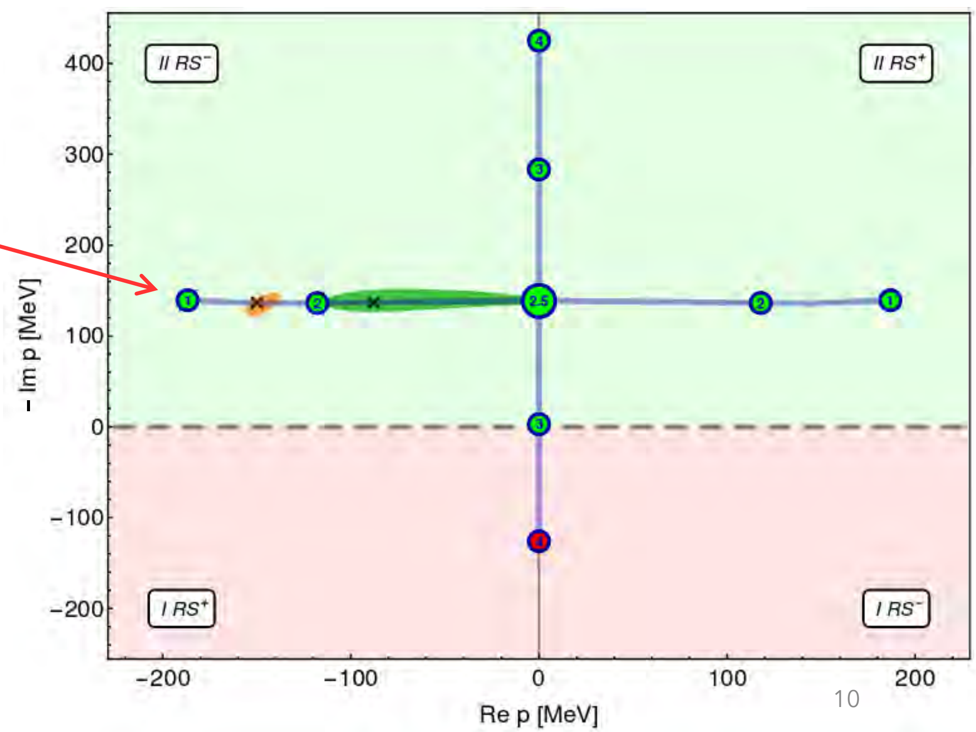


- Axes: $\sqrt{s}$ vs. k
- Resonances → Virtual state → bound state
- But rho-resonance: rather featureless conversion to bound state
- Wide scalar mesons are not at all conventional Breit-Wigner resonances
- Prominent molecular component [Morgan/Pennington] [Baru] [Guo]

Chiral trajectories in lattice QCD: $f_0(500)$ "σ"

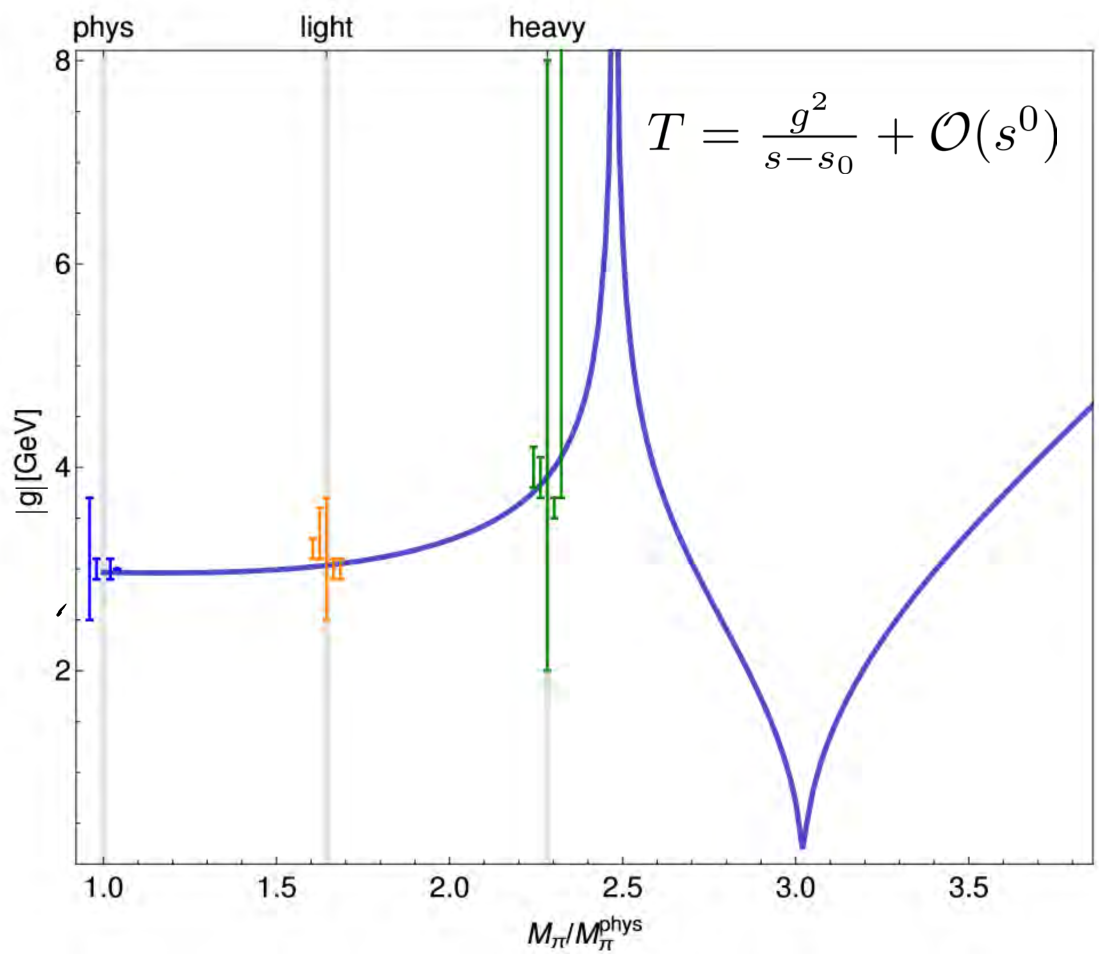
- A lattice calculation at M_π=227 MeV and 315 MeV [GWUQCD, 1803.02897]
- σ becomes a virtual state @ M_π = (345) 415 MeV





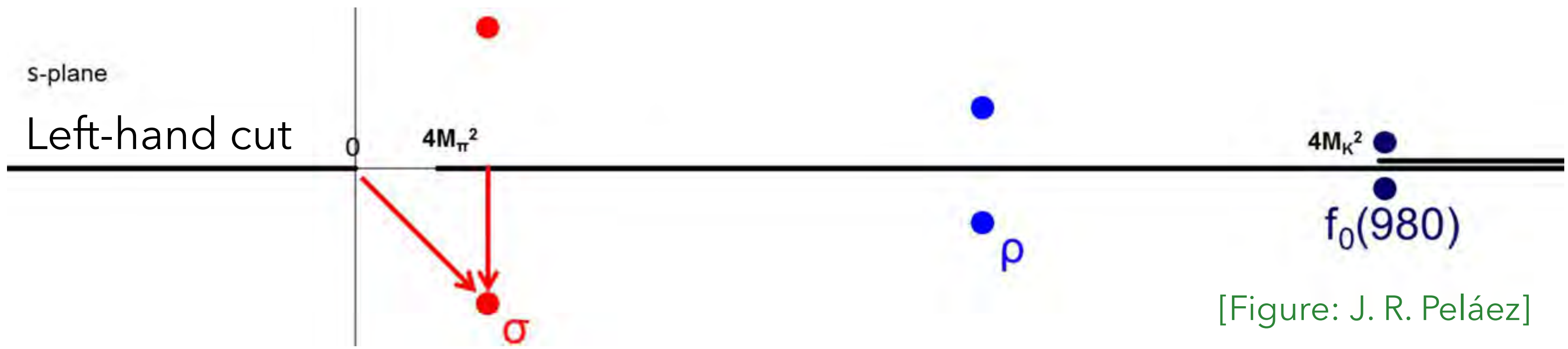
Resonance coupling *g* for $f_0(500) \to \pi\pi$

- The $f_0(500)$ remains a resonance at available masses
- Onset of virtual state domain visible
- More, heavier pion masses for the virtual state and bound state regimes needed



Challenge: left-hand cuts

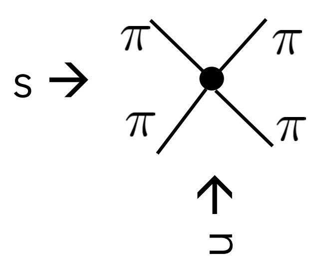
- Pole positions of wide resonances might be distorted if "left-hand cut" is not taken properly into account (and: analyticity in s, not $\sqrt{s}$)



[Figure: J. R. Peláez]

- Build in crossing symmetry manifestly through Roy-(like) equations

[Peláez]



Advantage: $\pi\pi$ scattering in u-channel is still $\pi\pi$
Problem: need amplitude for all s
πN: [Hoferichter]

Three-body amplitudes in infinite Vol.

- Three-body unitarity
- The $a_1(1260)$ resonance in a coupled-channel, three-body unitary framework: lineshape & Dalitz plots
- Analytic continuation for three-body amplitudes: contour deformation and the $a_1(1260)$ pole

Three-body unitarity with isobars * [Mai 2017]

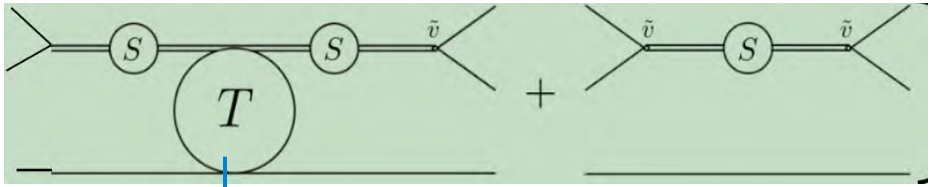
$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$
$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

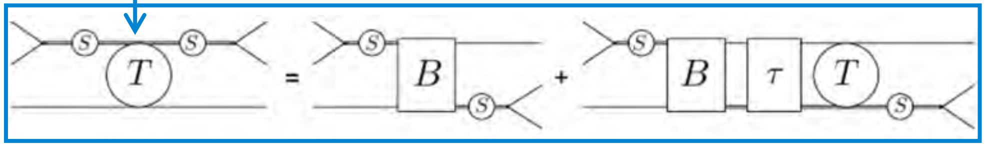
delta function sets all intermediate particles on-shell; no anti-particles!

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrization of full 2-body amplitude [Bedaque] [Hammer]

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

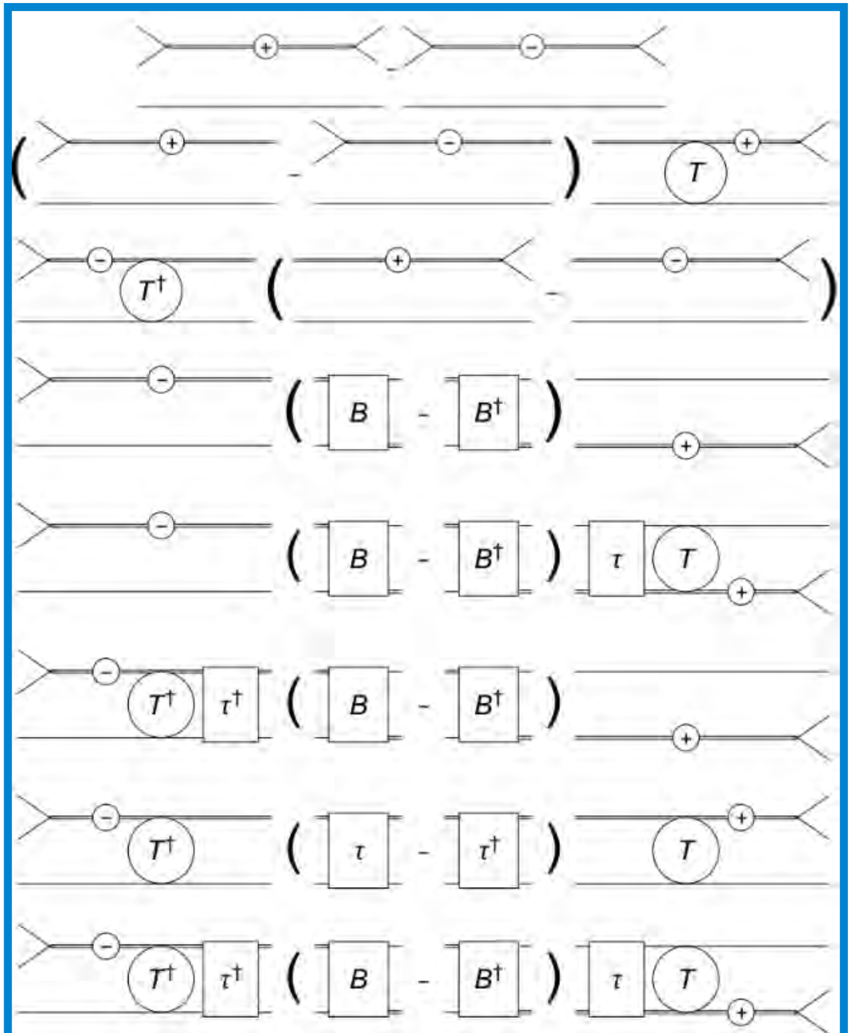




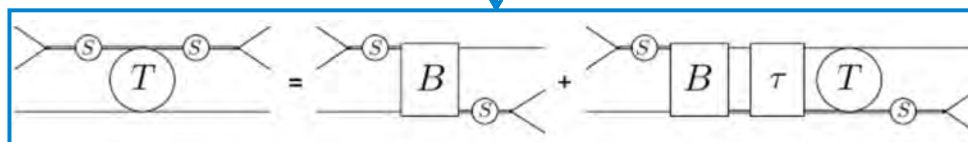
General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



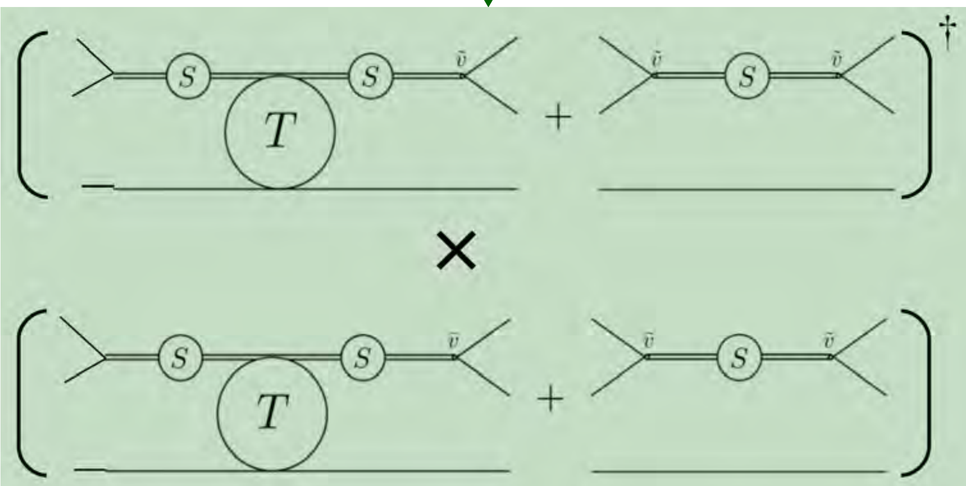
Bethe-Salpeter equation



$$\begin{aligned}\hat{T}-\hat{T}^\dagger &= v(S-S^\dagger)v + vSTSv - vS^\dagger T^\dagger S^\dagger v\\ &= v(S-S^\dagger)v + (vS - vS^\dagger)\,TSv + vS^\dagger T^\dagger\,(Sv - S^\dagger v) + vS^\dagger(T-T^\dagger)Sv\end{aligned}$$

Three-body unitarity

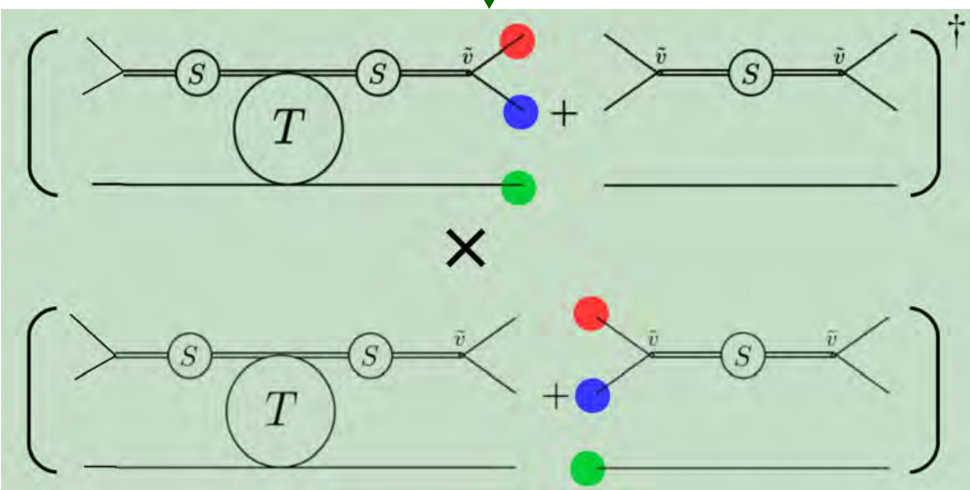
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

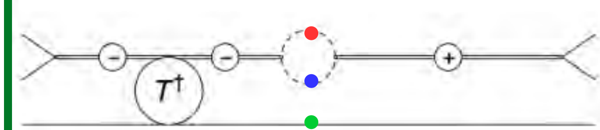


General connected-disconnected structure

Three-body unitarity

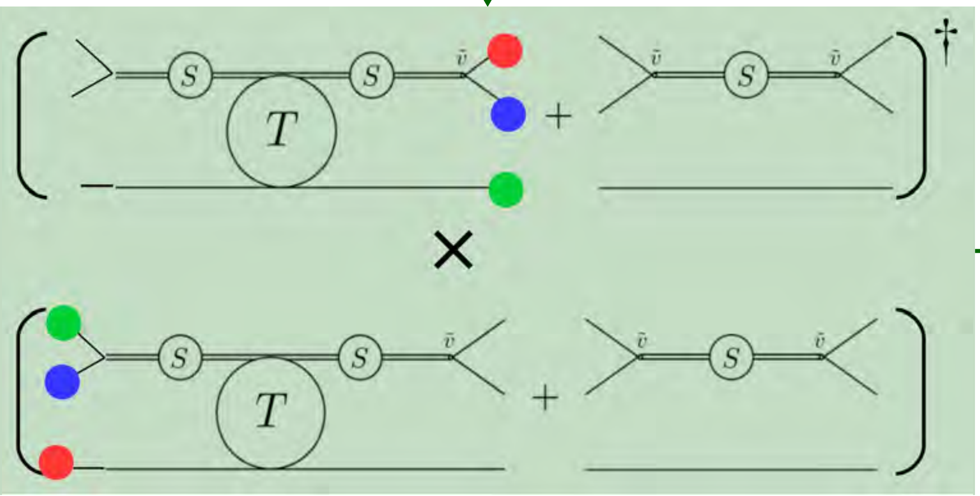
$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i \int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

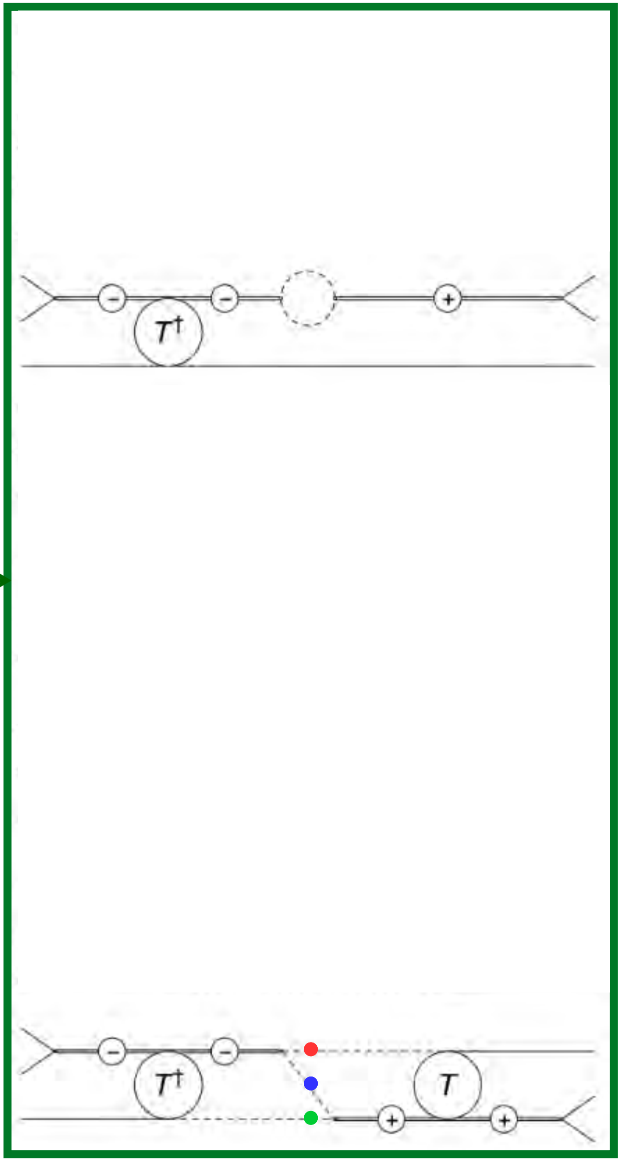




Three-body unitarity

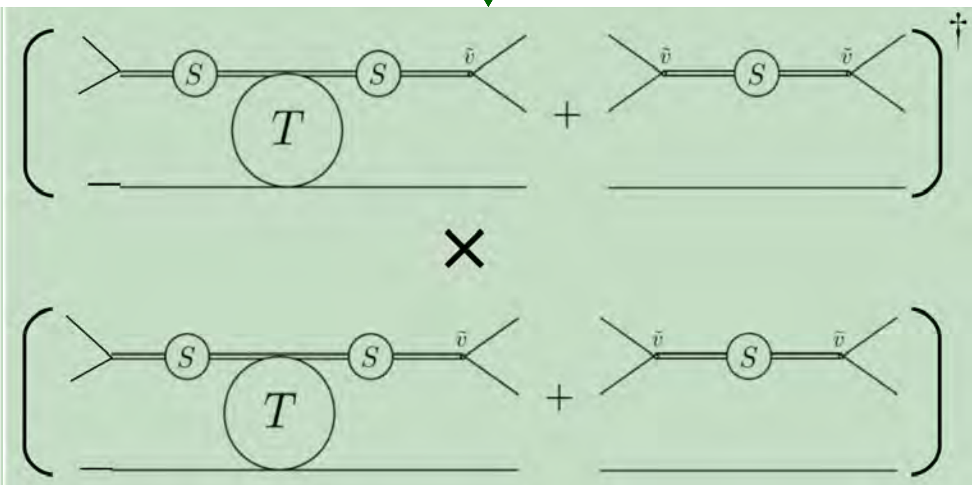
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



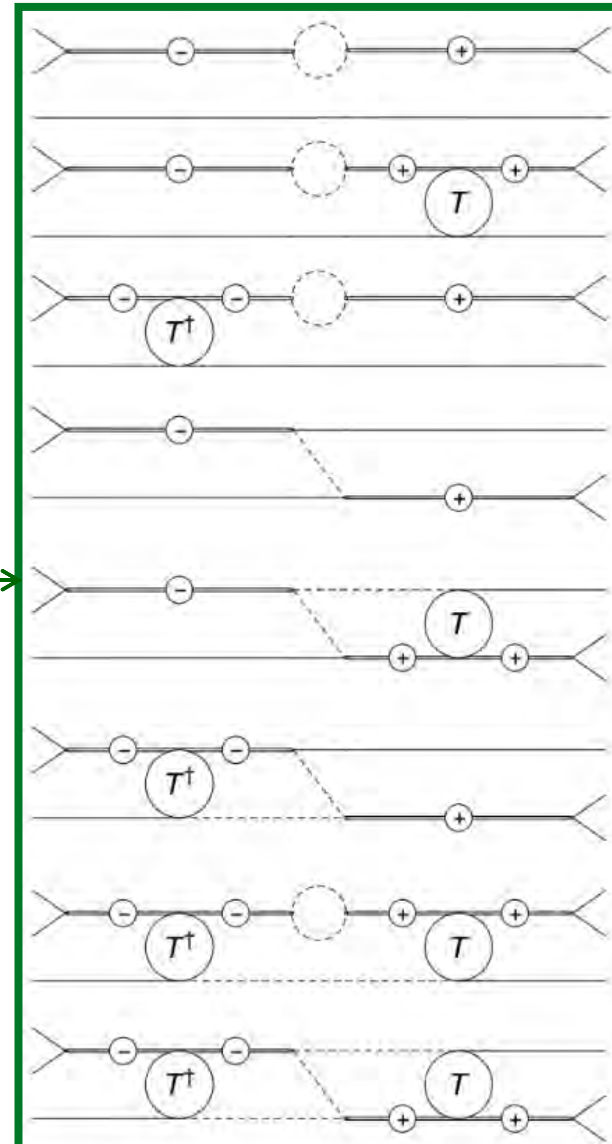


Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \quad = \quad i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

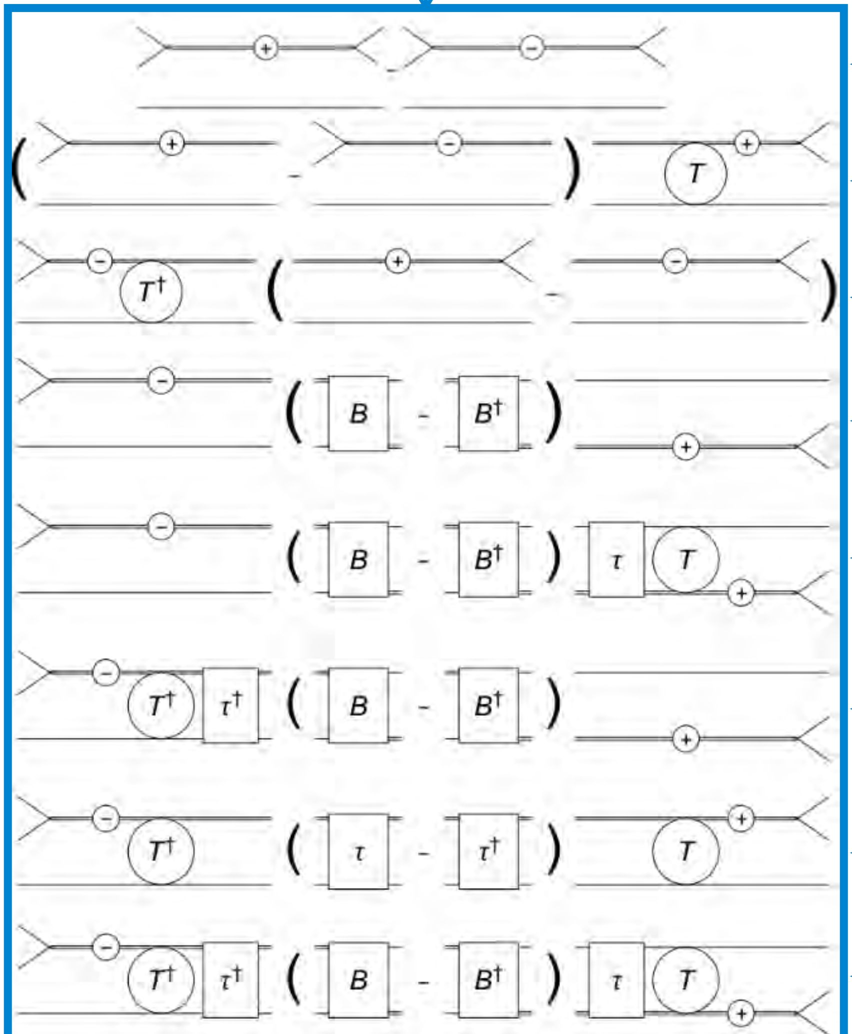


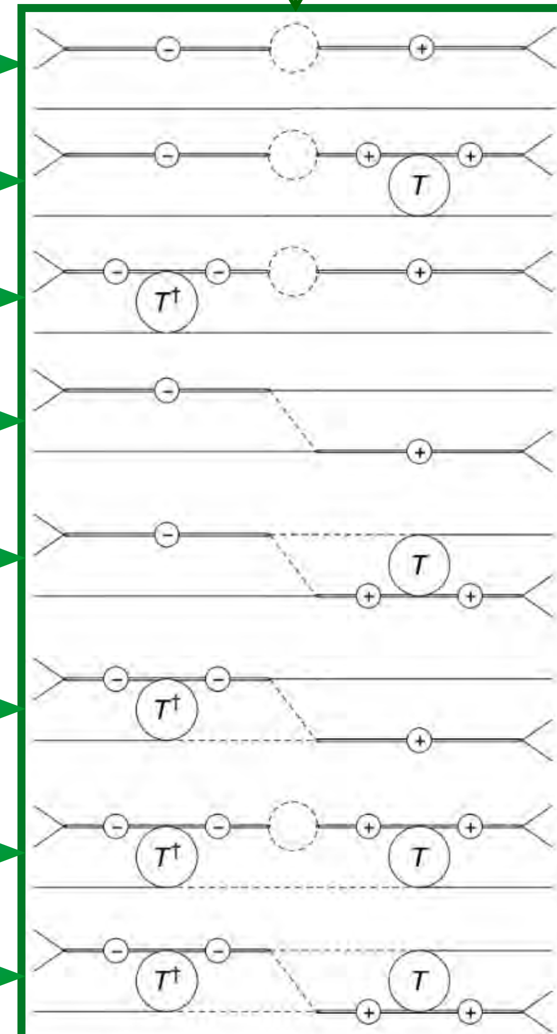
8 topologies



Three-body unitarity

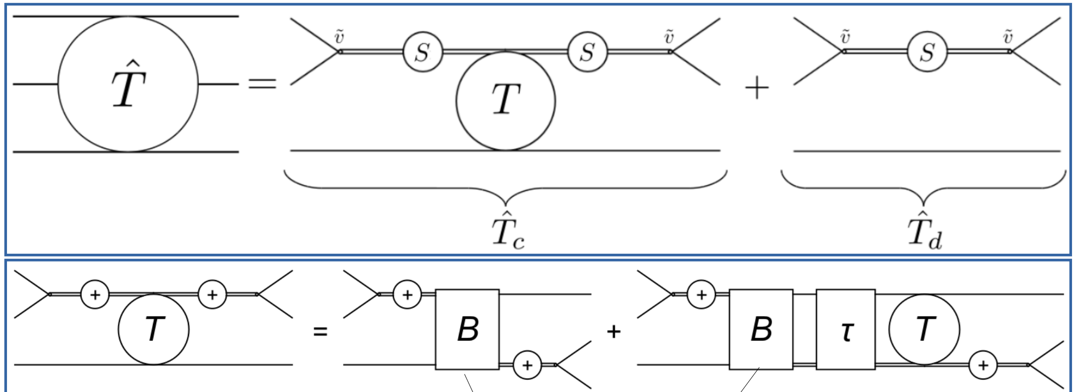
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$





Scattering amplitude (1)

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation

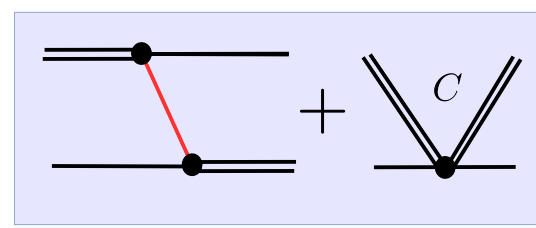


- Imaginary parts of ***B, S*** are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$



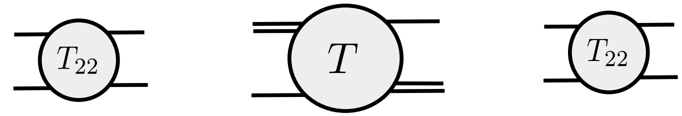
- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

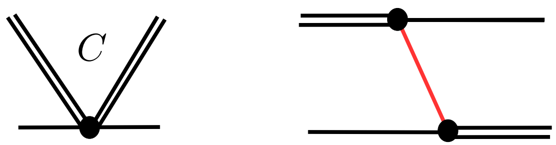
- One can map to field theory but does not have to. Result is a-priori dispersive.

Scattering amplitude (2)

Here: Version in which isobar rewritten in on-shell 2 → 2 scattering amplitude T_{22}

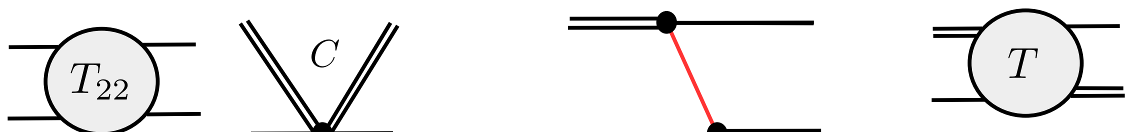
$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle = \frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$



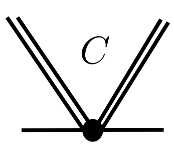


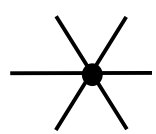
$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2 - (P-p-q)^2 - i\epsilon}$$

$$- \int \frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2E_\ell} T_{22}(\sigma(\ell))\left(\langle p|C(s)|\ell\rangle + \frac{1}{m^2 - (P-p-\ell)^2 - i\epsilon}\right)\langle \ell|T(s)|p\rangle$$



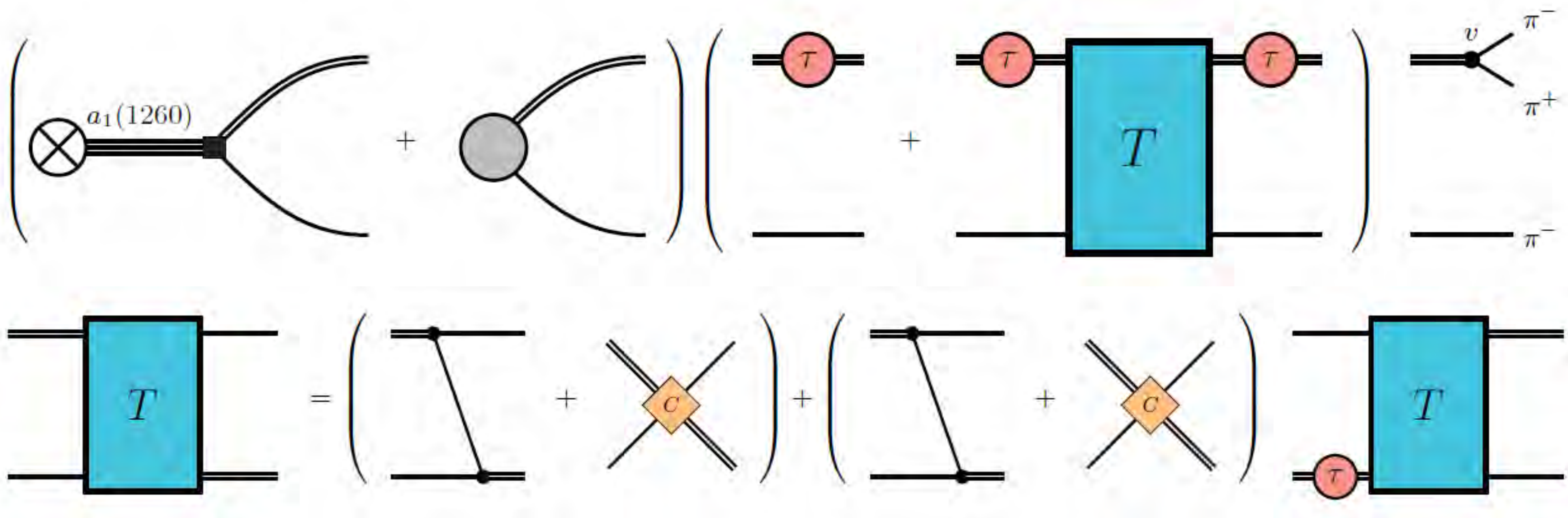
(S-wave)

Technical Detail: 

vs  Scheme-dependent 3-body force requires a mapping [Brett (2021)]

The a_1(1260) and its Dalitz plots

- Disconnected and connected decays for three-body untarity



- New complication: the rho has spin:

$$T_{\lambda'\lambda}(p,q_1) = (B_{\lambda'\lambda}(p,q_1) + C) + \sum_{\lambda''}\int \frac{d^3l}{(2\pi)^3 2E_l}\,(B_{\lambda'\lambda''}(p,l) + C)\,\tau(\sigma(l))\,T_{\lambda''\lambda}(l,q_1)$$

Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) + \sum_{\lambda''}\int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi\int_{-1}^{+1} \mathrm{d}x\, d^J_{\lambda\lambda'}(x) B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda' L'}$$

- JLS basis:

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

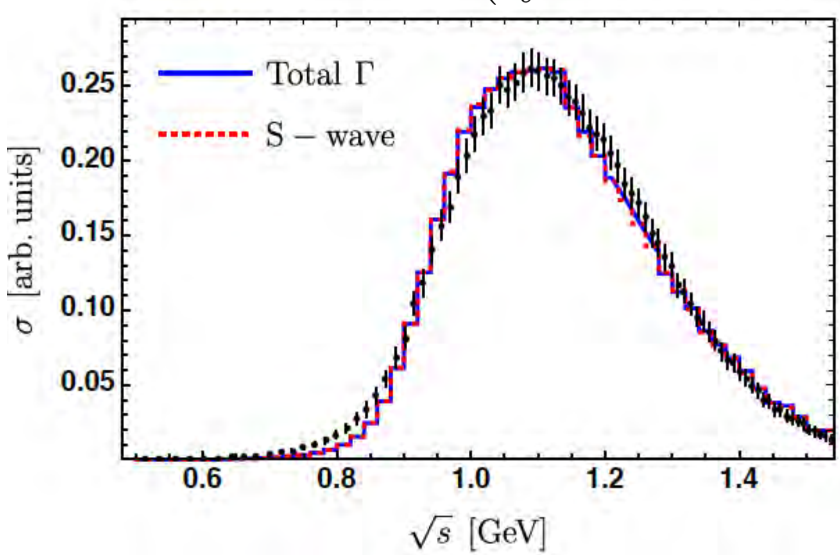
Fitting the lineshape & predicting Dalitz plots

[Sadasivan 2018]

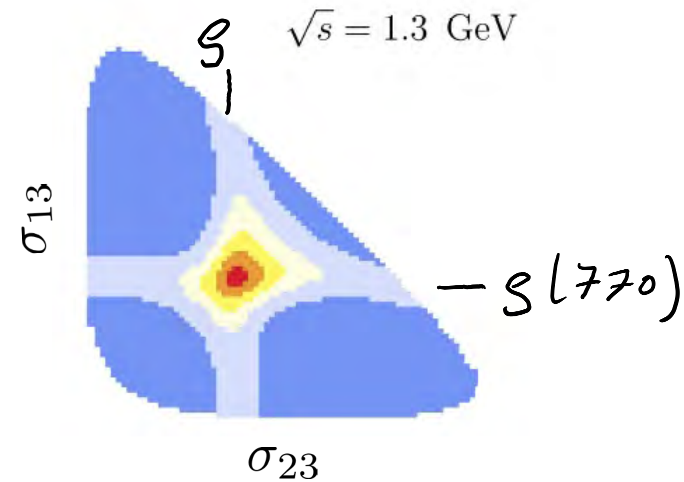
- One can have $\pi\rho$ in S- and D-wave coupled channels
- "Line shape": integrate all three final-state momenta,

$$\mathcal{L}(\sqrt{s}) = \frac{1}{\sqrt{s}} \int \frac{d^3\boldsymbol{q}_1}{(2\pi)^3} \frac{d^3\boldsymbol{q}_2}{(2\pi)^3} \frac{d^3\boldsymbol{q}_3}{(2\pi)^3} \frac{1}{2E_{q_1} 2E_{q_2} 2E_{q_3}} \times (2\pi)^4 \delta^4(P_3 - q_1 - q_2 - q_3) \overline{|\Gamma(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)|}^2 . \quad (18)$$

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize π^-'s!)



predict



Where is the resonance pole in s?

Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]

SMC

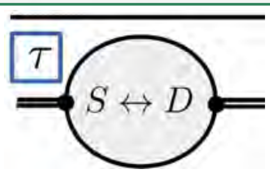
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p)+C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\boxed{dl}\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l)+C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

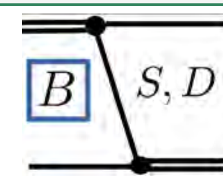
$$\Sigma = \int_0^\infty \frac{\boxed{dk}\, k^2}{(2\pi)^3}\frac{1}{2E_k}\frac{\sigma^2}{\sigma'^2}\frac{\tilde v(k)^*\tilde v(k)}{\boxed{\sigma-4E_k^2+i\epsilon}}$$

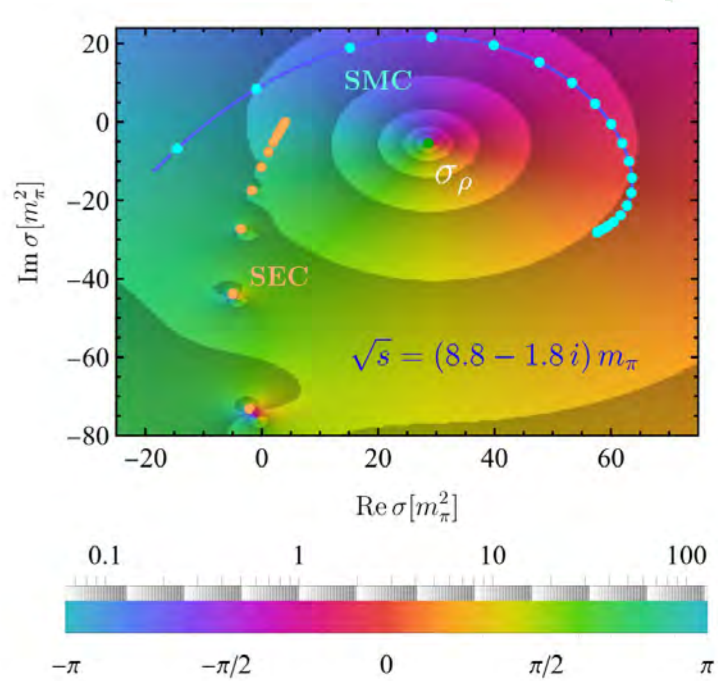
$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\,v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\boxed{(\sqrt{s}-E_p-E_{p'}-E_{p'+p}+i\epsilon)}}$$

SEC



Singularities



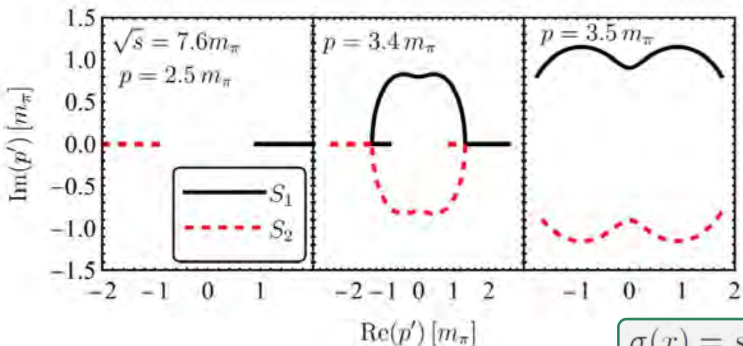


- Two contours (SMC and SEC)
- Deform both "adiabatically" to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

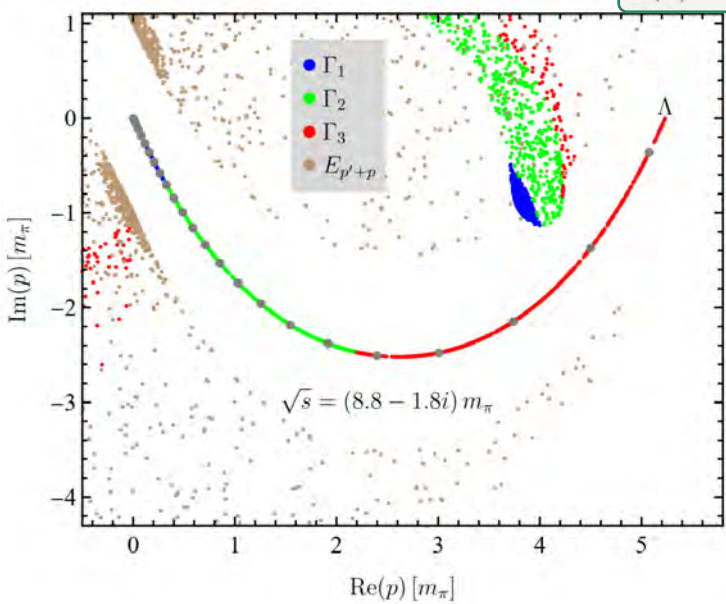
Analytic continuation 3-body (contd.)

- Three-body cuts

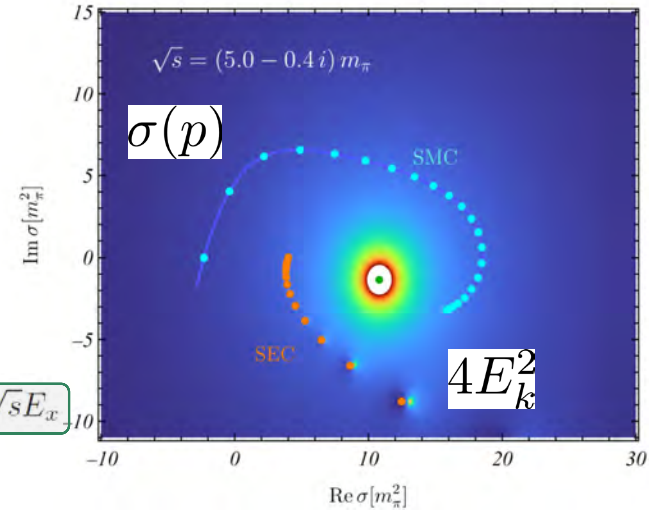
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$



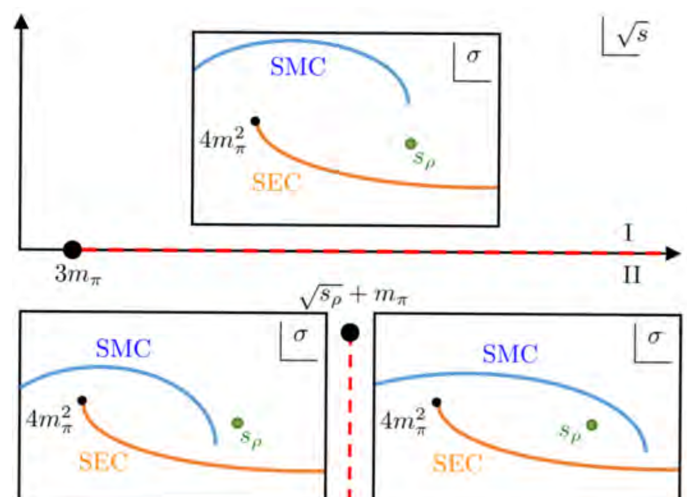
$$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$$



- Complex branch points

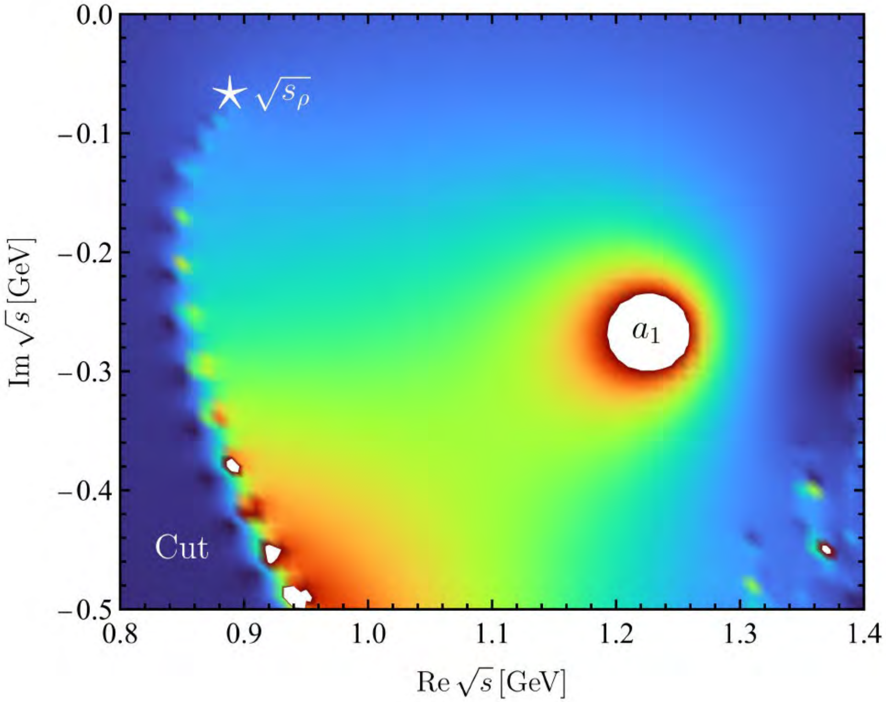


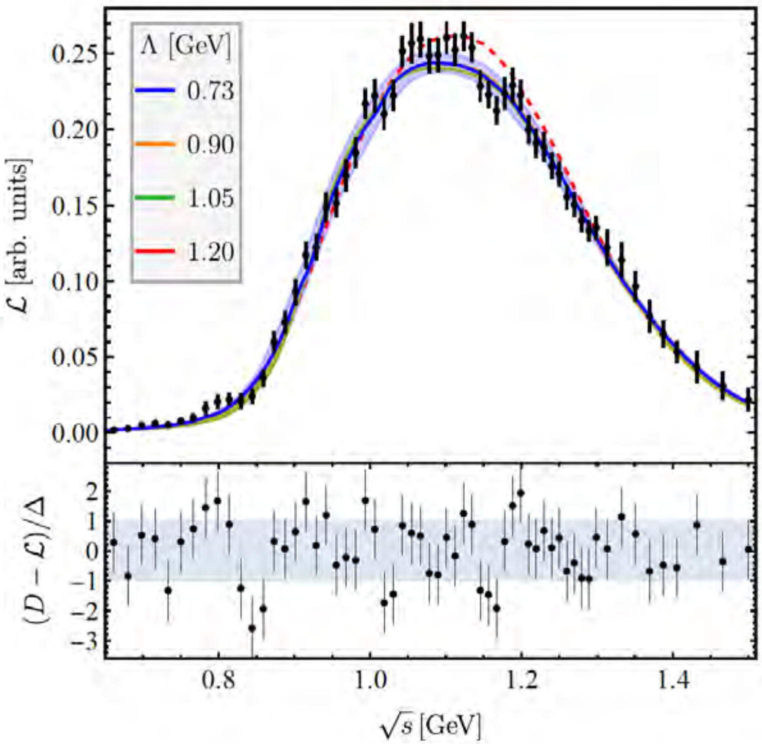
Integration limits at poles induce branch points



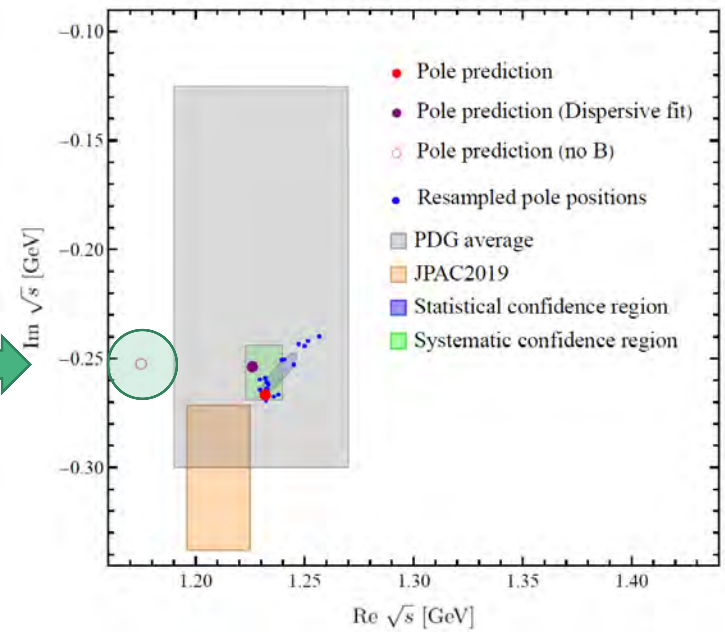
Result: Pole position

$|T_{00}|$





If the B-term is neglected + refit (unitarity violated)



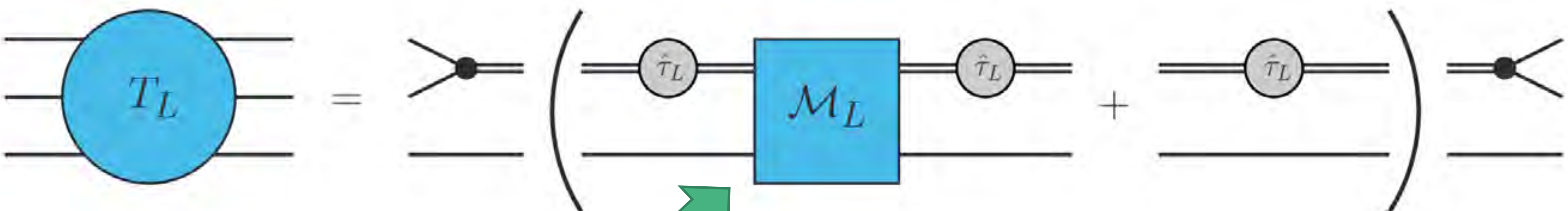
Finite volume quantization condition

- General procedure:
  - Formulate an amplitude in infinite volume
  - Discretize the momenta
  - Instead of partial-wave projection, as in infinite volume, project to irreps according to the symmetries of the problem
    - Three-body system at rest
    - Three-body system with finite momenta P=(1,0,0),…

Finite-volume quantization

[Doering 2018]

- Connectedness structure



$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{\mathbf{p}}) f_{\ell m}(p)$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{\mathbf{p}}) f(\mathbf{p})$$

(spinless isobar)

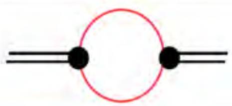
$$f^s(\hat{\mathbf{p}}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_u f_u^{\Gamma\alpha s} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) ,$$

$$f_u^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{\mathbf{p}}_j) \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) \quad \text{for} \quad \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}) \in U_s ,$$

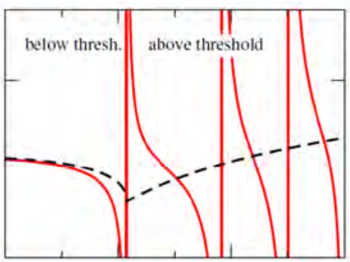
$\chi_u^{\Gamma\alpha s}$ are orthonormal basis functions for irrep Γ
Shell index s, row α

Three-body quantization condition

Two-body unitarity → **On-shell condition**



→ **Imaginary parts**



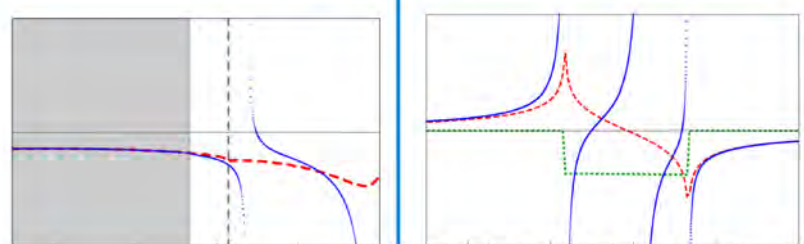
Infinite → Fin. Vol

→ **Power-law fin-vol. effects** → **Lüscher**

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)$$

Three-body unitarity → **On-shell condition**

→ **Imaginary parts**



→ **Power-law fin-vol. effects** → **Quantization Condition**

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W}^2) + \frac{\mathbf{2E_s\, L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W}^2)^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$

[Review Mai et al.]

Applications and extensions

- Three pions at maximal isospin (selection)
- Three kaons at maximal isospin
- Isobars with spin: The $a_1(1260)$ in coupled channels

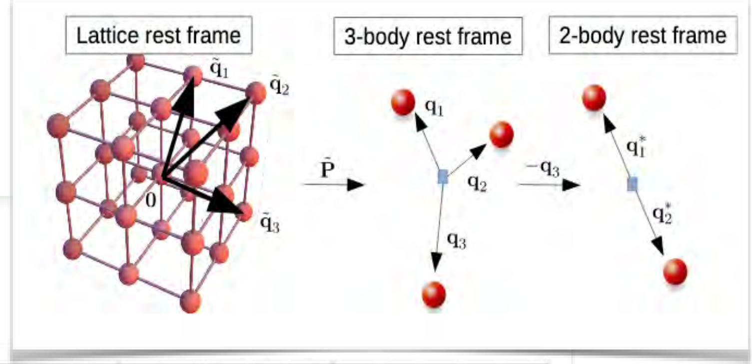
Three pions at maximal isospin

- Highlight qualitative insights
- Determination of scheme-dependent three-body force is possible and might be compared to LO CHPT (not discussed)

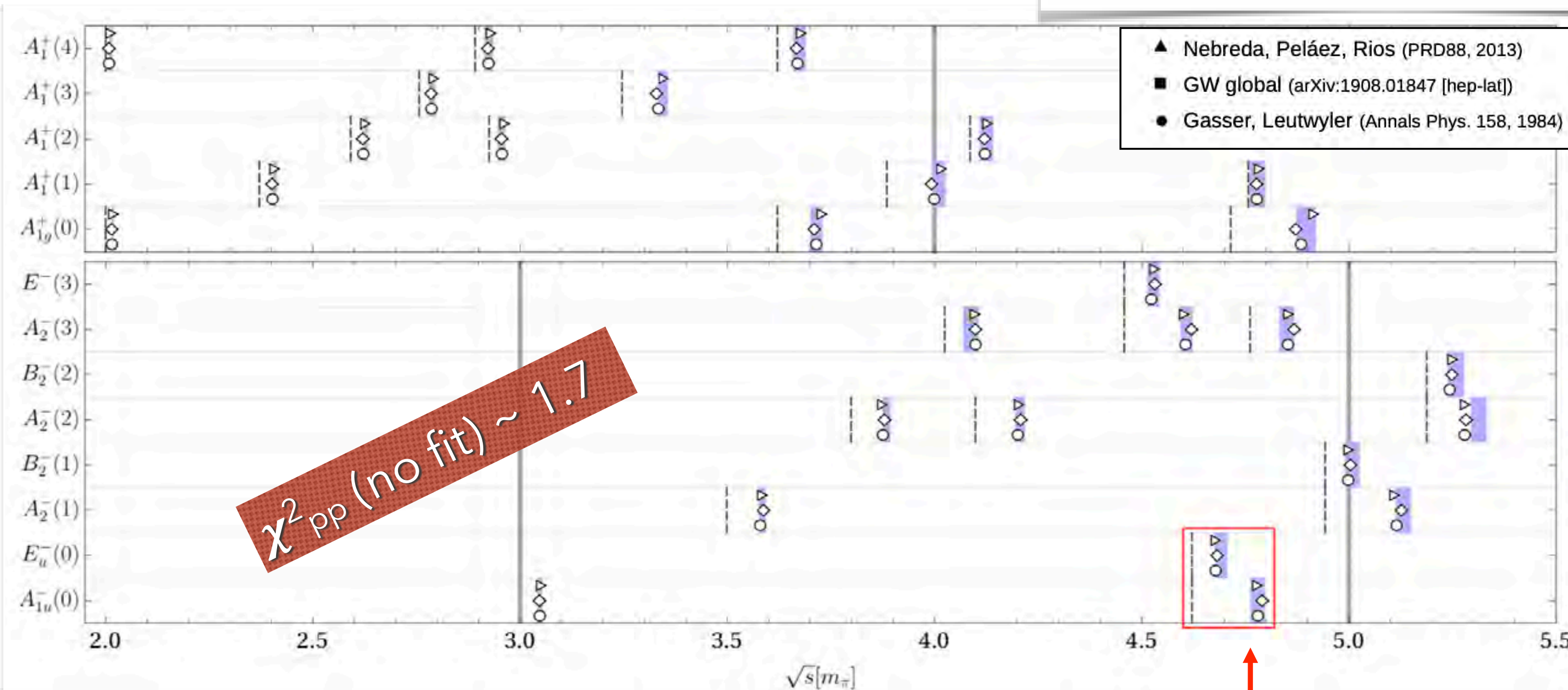
[Blanton PRL 2019, Brett 2021, Fischer 2021]

Hoerz/Hanlon data

[Hoerz 2019][see also Culver 2019]

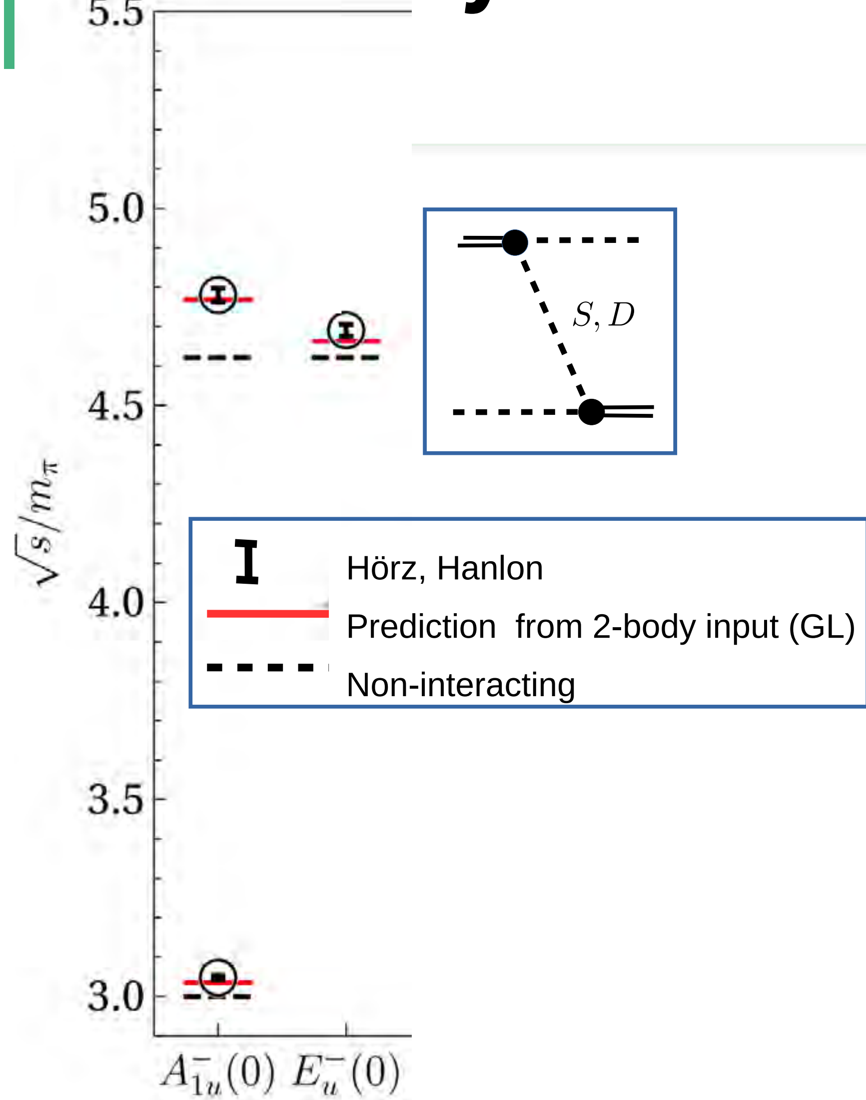


- Energy eigenvalues



- *Uncertainties dominated by the 2-body input. C~0*
- *Rel. strength between S/D waves fixed dominantly by 3b unitarity*

Unitarity shows in FV spectrum



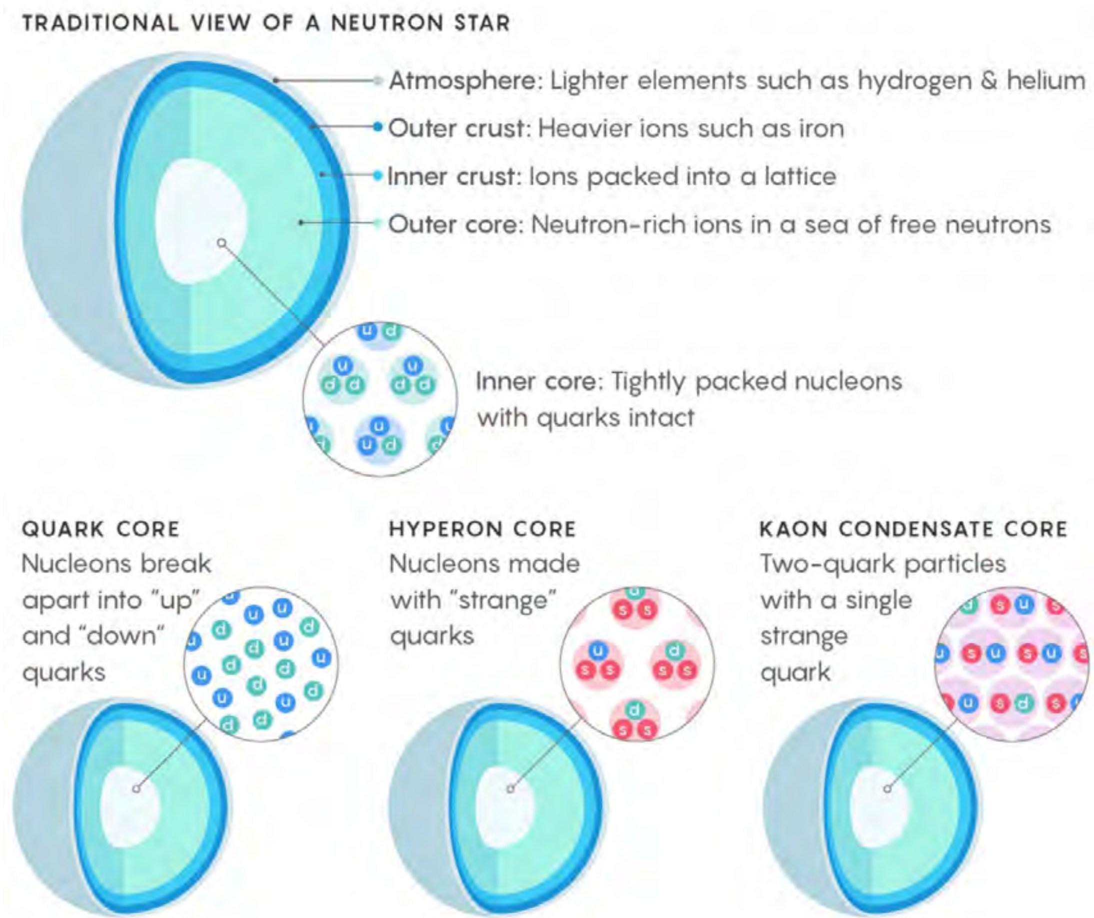
S **D** (lowest participating wave)

- S-wave prediction good at threshold (like for NPLQCD data)
- S-wave prediction good at high energies
  → **Energy dependence matched**
- 3-body "force" set to zero
- D-wave prediction qualitatively good
  → Relative/absolute strength between S- and D-wave matched
  → Consequence that 3-body interaction dominated by exchange
  → Consequence of 3-body Unitarity

- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD

Multi-kaon systems

- Relevant for heavy-ion collisions (ALICE)
- Matter with strangeness in neutron stars and equation of state:
  - Kaon condensate can soften the equation of state



[quantamagazine.org]

Three kaons at maximal isospin

[Alexandru 2020]

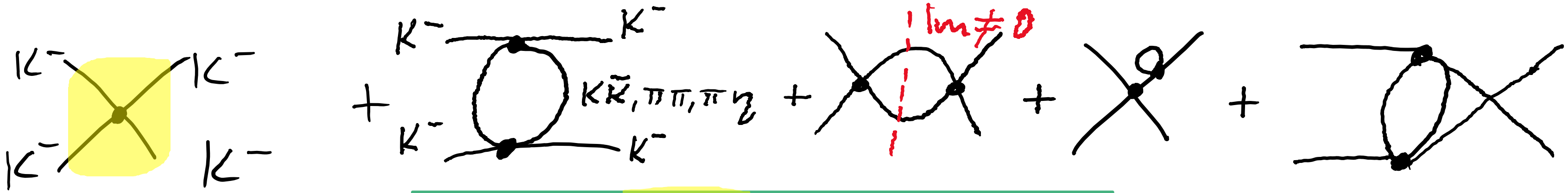
- First study of three kaons from lattice QCD with chiral amplitudes
- Other groups have improved on this in the meantime:
  - Max. isospin, non-identical masses ($\pi^+\pi^+K^+$, $\pi^+K^+K^+$)

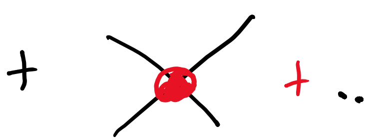
    [Blanton 2021]
  - Pions and kaons at maximal isospin with unprecedented accuracy and no. of levels ($\pi^+\pi^+\pi^+$, $K^+K^+K^+$) [Blanton 2021]

- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter w_0

Two kaons

- Crossing symmetry allows to get the amplitude $K^- K^- \to K^- K^-$ from $K^+ K^- \to K^+ K^-$
- SU(3) CHPT unitarized with inverse amplitude method

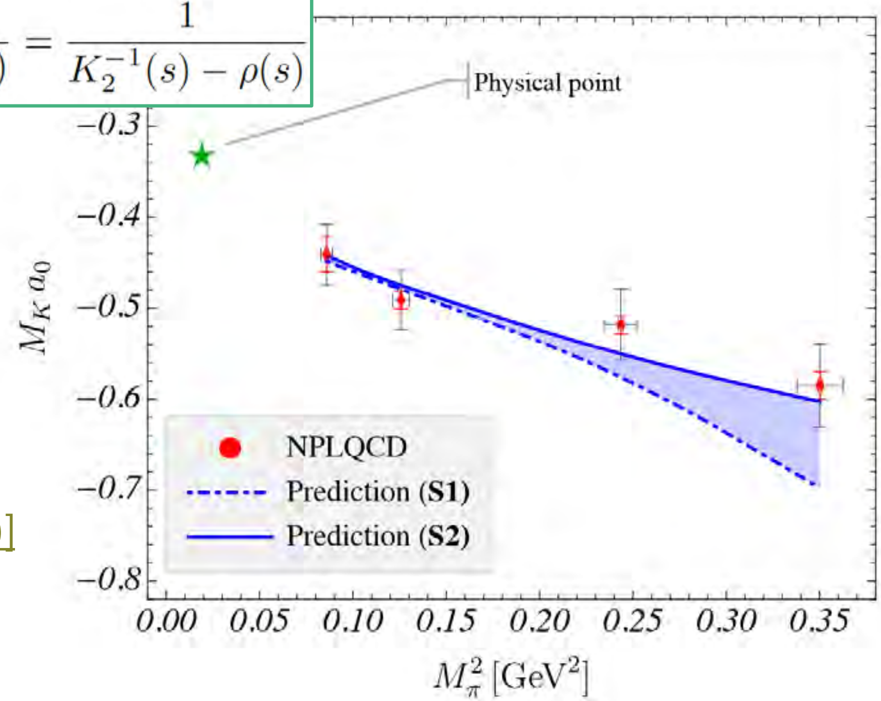




$$T_2(s) = \frac{(T_{\mathrm{LO}}(s))^2}{T_{\mathrm{LO}}(s) - T_{\mathrm{NLO}}(s)} = \frac{1}{K_2^{-1}(s) - \rho(s)}$$

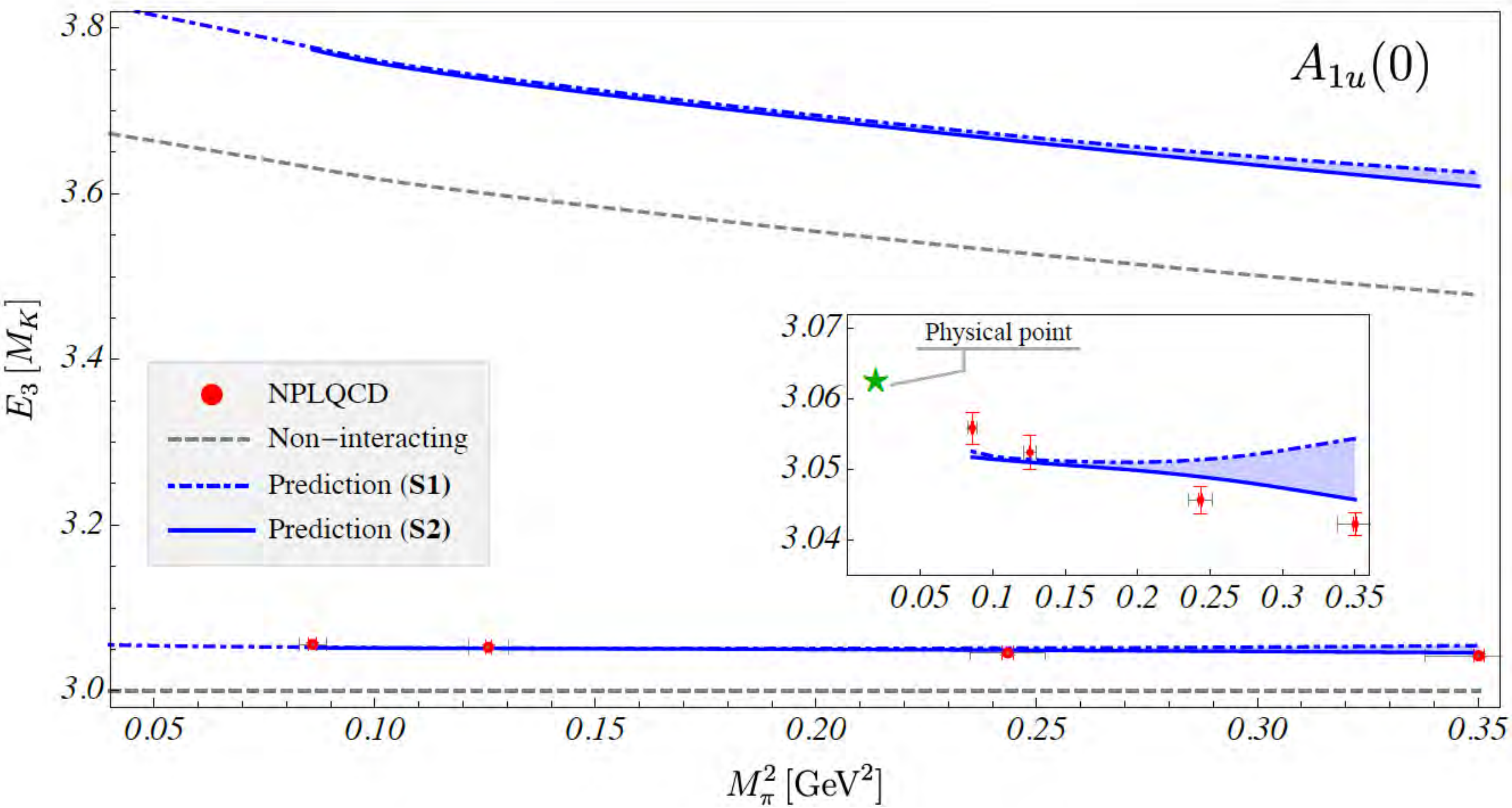
- Prediction of the I=1 KK scattering length: Data: [NPLQCD (2007)]

- LECs taken from most recent Global fit to lattice QCD [Molina (2020)]



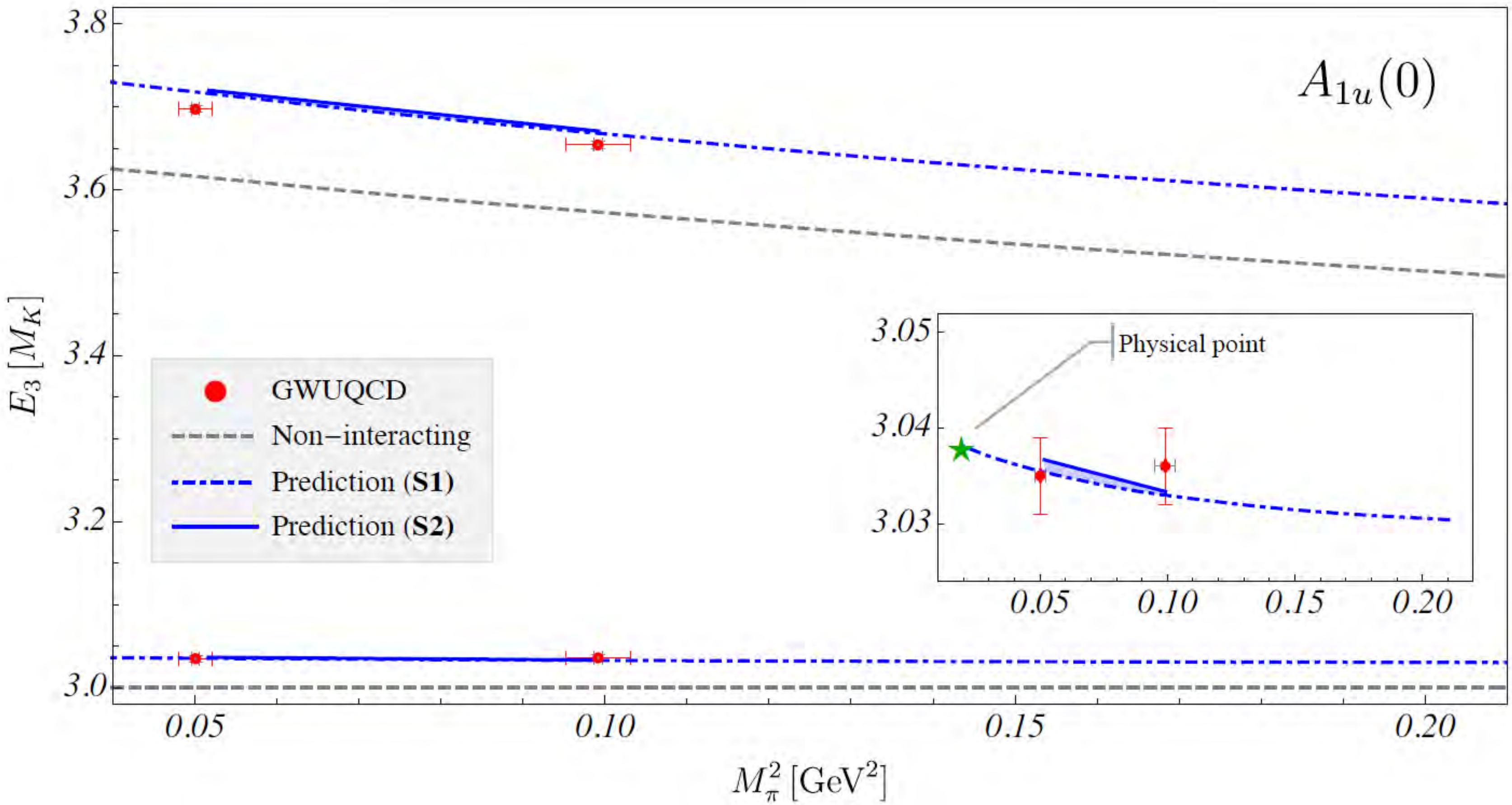
Three kaons – predictions vs lattice

- NPLQCD data – three-body term set to zero



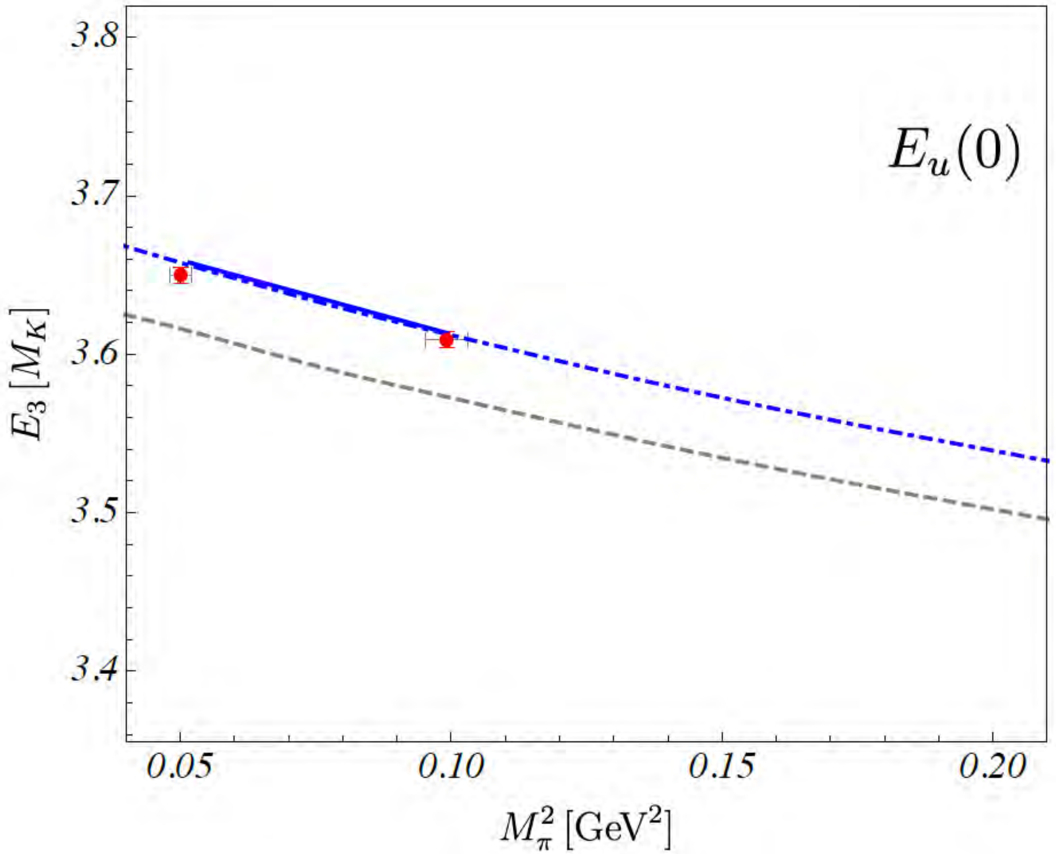
Three-kaons – predictions vs. lattice

- GWUQCD data – three-body term set to zero



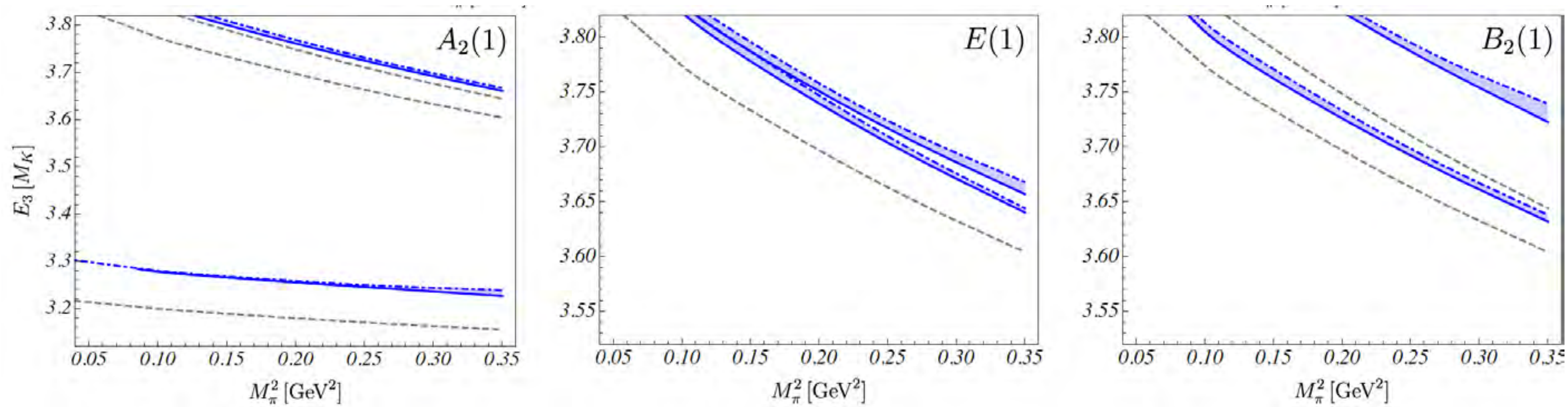
Predictions vs. lattice contd. (P=0)

- GWUQCD data – three-body term set to zero



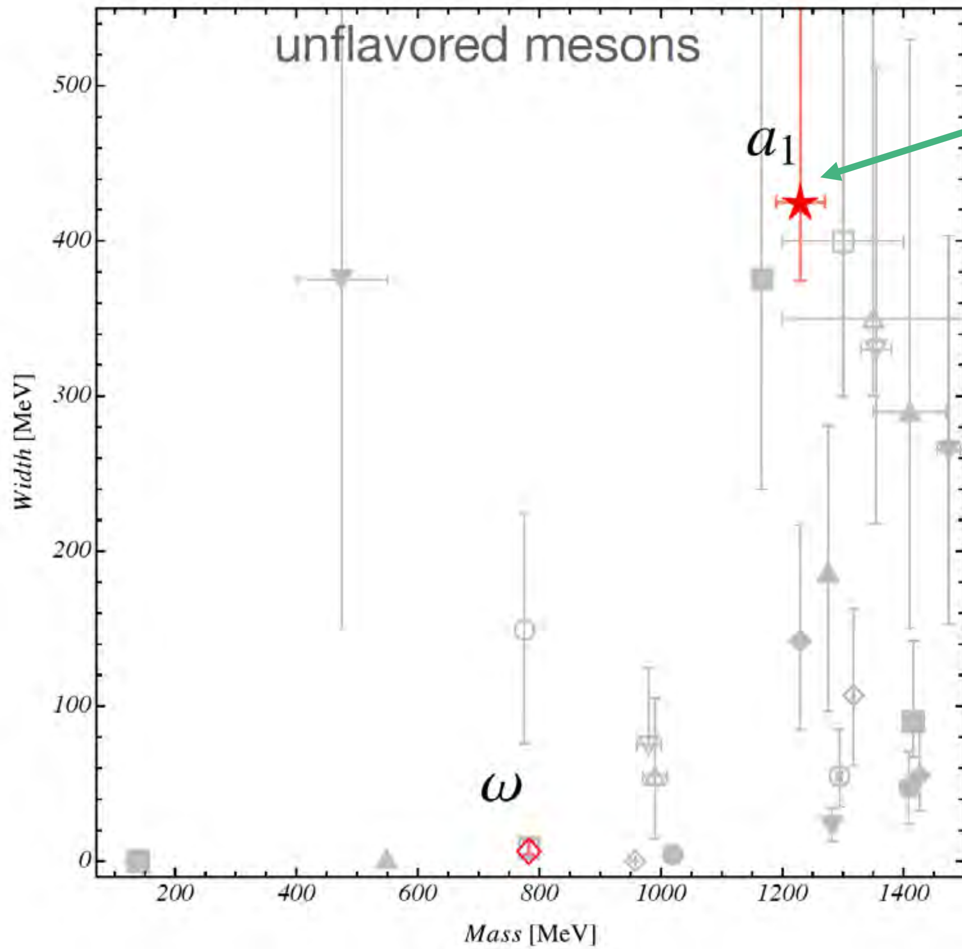
Predictions for 3-body system with momentum P=(1,0,0)

- NPLQCD lattice setup – three-body term set to zero



- Summary:
  - First lattice QCD & chiral calculation of three kaons at maximal isospin
  - At different masses, different irreps & different total momenta
  - Relativistic 3B quantization condition for strangeness sector,
  - IAM to NLO along different trajectories predicts data from two different lattice QCD calculations well (NPLQCD and GWUQCD)

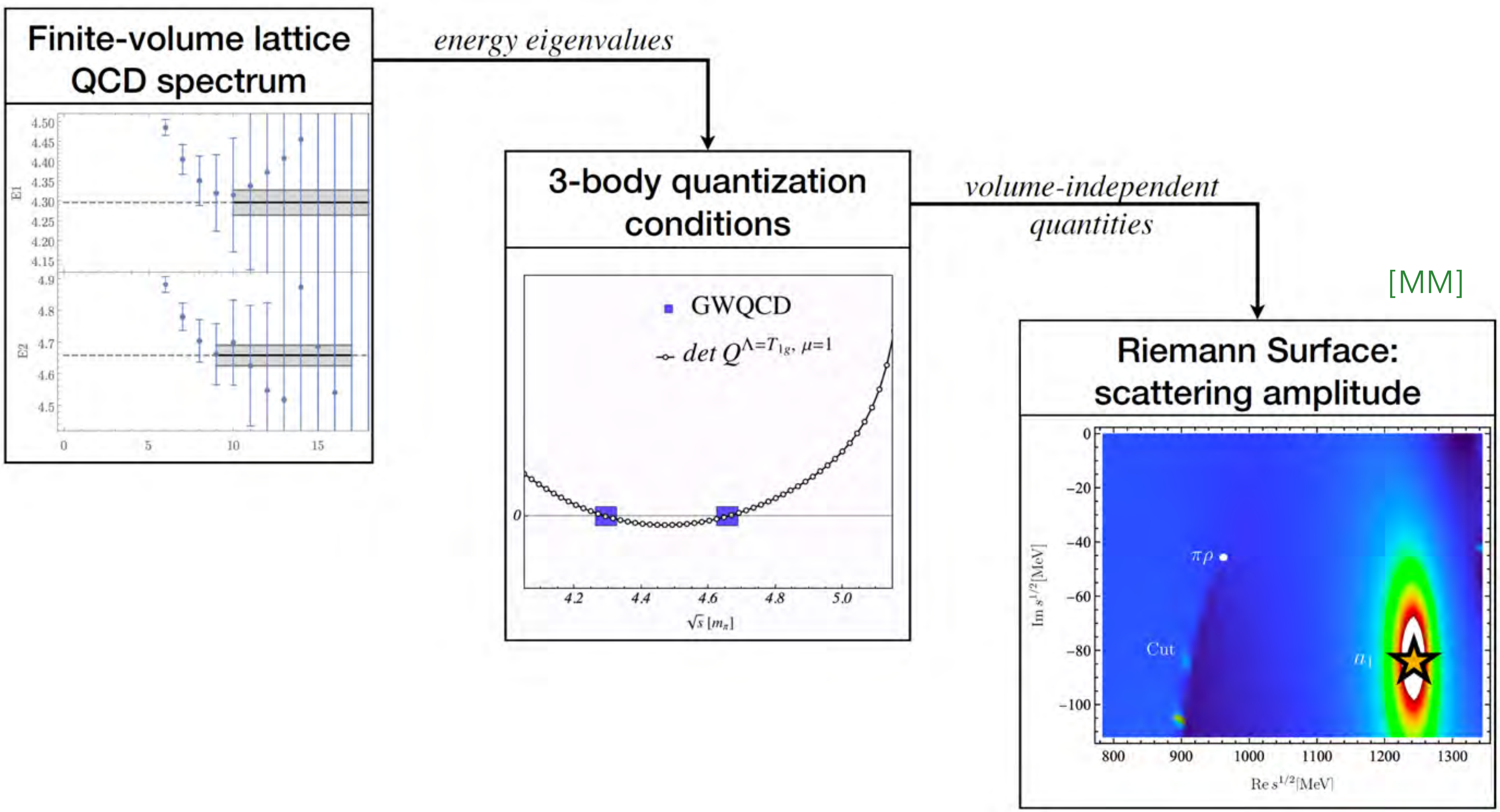
Light unflavored mesons: the a_1



We concentrate on this resonance!
(because 3-body)

Extraction of $a_1(1260)$ from lQCD [Mai/GWQCD]

- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



Three-particle propagation with helicities

- 2-body unitarity fixes only part of the interaction;

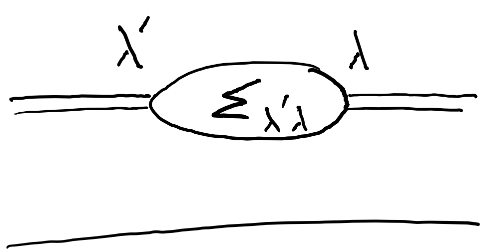
$$\tau^{-1}_{\lambda'\lambda}(\sigma_p) = \delta_{\lambda'\lambda}\tilde{K}^{-1}_n(s,\boldsymbol{p}) - \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p})\,,$$

$$\tilde{K}^{-1}_n(s,\boldsymbol{p}) = \sum_{i=0}^{n-1} a_i\sigma_p^i \quad \text{and} \quad \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p}) =$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{\sigma_p^n}{(4E_k^2)^n}\frac{\hat{v}^*_{\lambda'}(P-p-k,k)\hat{v}_\lambda(P-p-k,k)}{2E_k(\sigma_p-4E_k^2+i\epsilon)}$$

n-times subtracted for convergence (n=2)

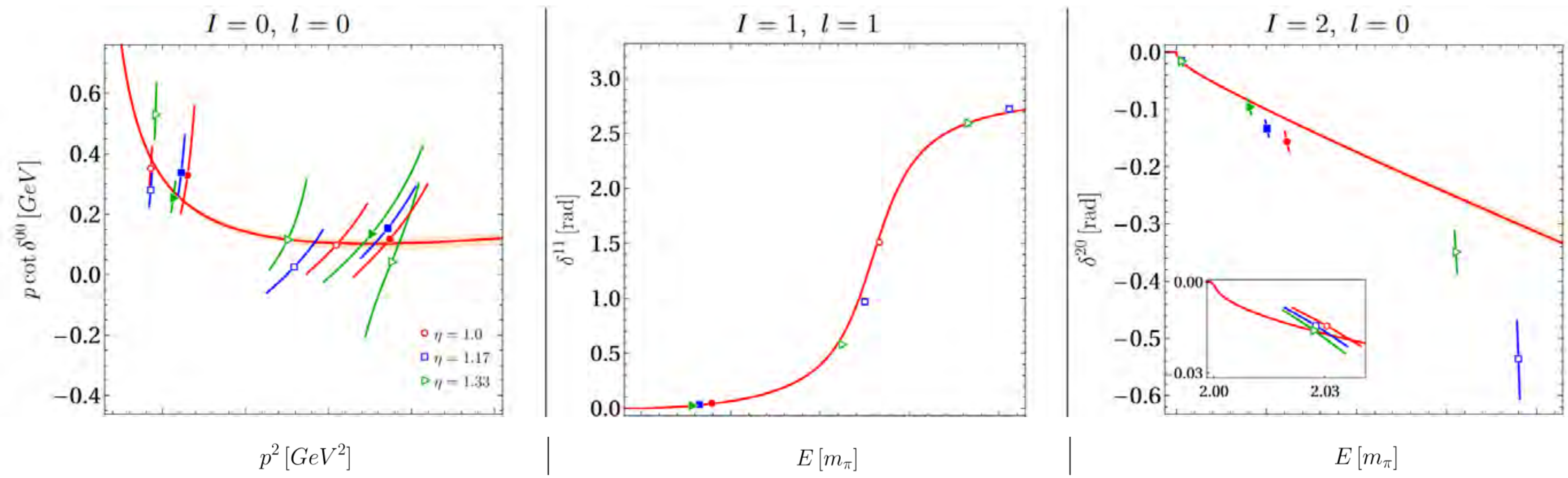
- The helicity of the ρ in flight can change!



$\Sigma_{\lambda'\lambda} \neq 0$ for $\lambda' \neq \lambda$

2-body input

- Global fit of 2-body sector across different isospins, including correlations across isospin [Mai 2019]



- Match to twice-subtracted dispersive expression to be used in three-body system

4 different fits to 2 energy eigenvalues

- Fitted isobar-spectator interaction (case 1, 2) for $|\boldsymbol{p}| \leq 2\pi/L|(1,1,0)| \approx 2.69\ m_\pi$

$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{i=-1}^{\infty} c^{(i)}_{\ell'\ell}(\boldsymbol{p}',\boldsymbol{p})(s-m^2_{a_1})^i$$

- Case 2: a_1 is generated as pole even though no built-in singularity

Non-zero coefficients	No of fit parameters	χ^2
c_{00}^0 (no built-in pole, m_{a1}=0)	1	9
c_{00}^0, c_{00}^1 (no built-in pole, m_{a1}=0)	2	0.15
g_0, g_2, m_{a1}	3	3.2
g_0, g_2, m_{a1}, c	4	10^{-7}

- Case 3, 4:

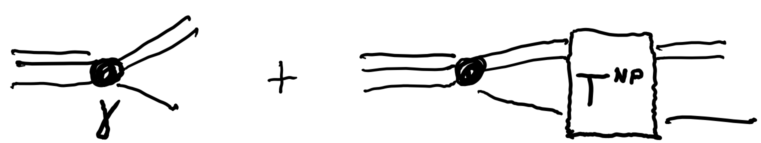
$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = g_{\ell'}\left(\frac{|\boldsymbol{p}'|}{m_\pi}\right)^{\ell'}\frac{m_\pi^2}{s-m^2_{a_1}}g_\ell\left(\frac{|\boldsymbol{p}|}{m_\pi}\right)^{\ell} + c\,\delta_{\ell'0}\delta_{\ell 0}$$

- In these cases, there is a built-in singularity, leading to resonance poles

Properties of 4-parameter fit

- 4 parameters for 2 energy eigenvalues produce remarkably stable results with resonance poles in a well-defined region and not all over the place.
- The D-wave coupling and a_1-mass are strongly correlated
  - Makes sense because D-wave a_1 self energy is mostly real

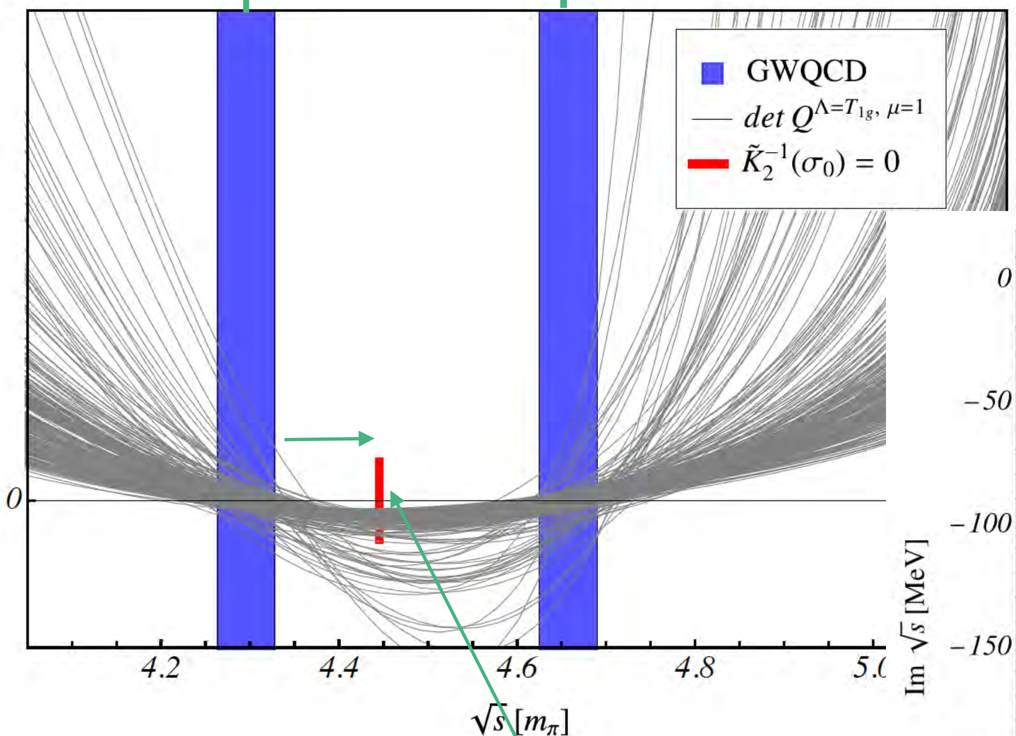
$$T^{Pole} = \frac{\Gamma\,\Gamma^{\dagger}}{s - m_{a_1}^2 - \Sigma_S - \Sigma_D}$$

$\Gamma =$ 

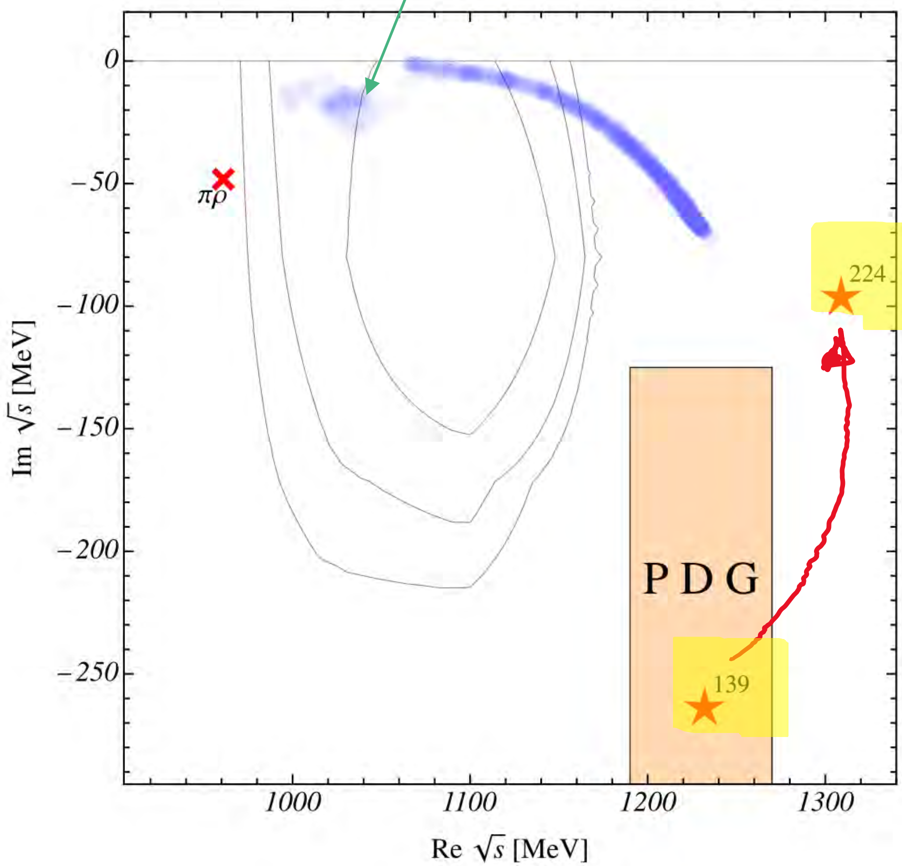
"Two-potential formalism"

Results – overview (4 parms)

Cluster: Large c and g_s compensate; disappears with c=0



$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{(\lambda'\lambda)(p'p)}$$



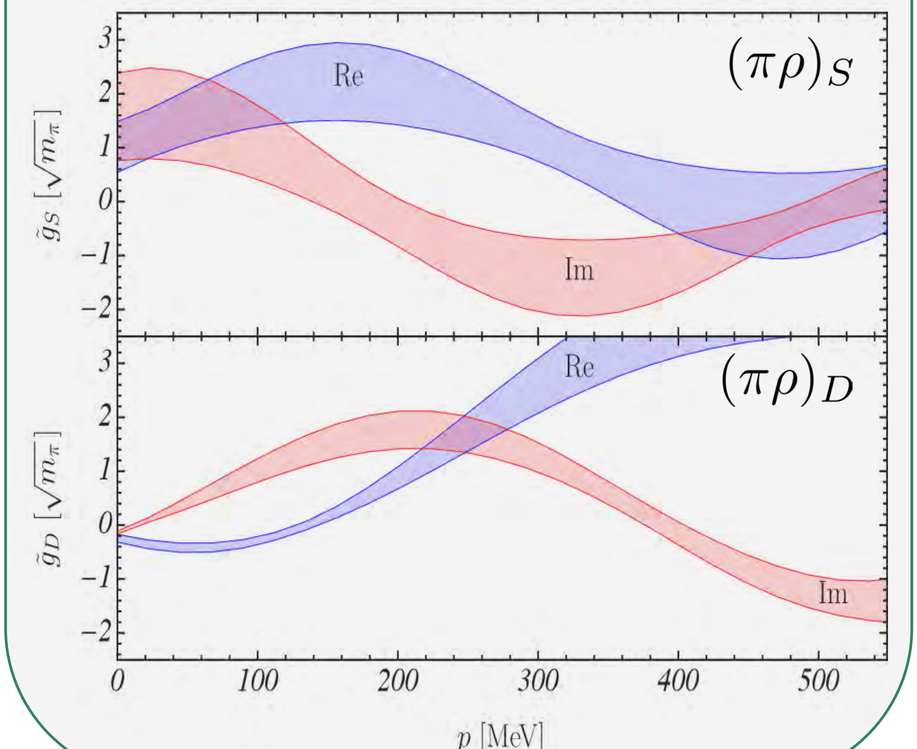
Branching ratios

- Calculate the residue at the pole:

$$\mathrm{Res}(T^c_{\ell'\ell}(\sqrt{s})) = \tilde{g}_{\ell'}\tilde{g}_\ell$$

- This result is not as reliable as pole position/existence of a_1
- More energy eigenvalues needed to better pin down the decay channels
- Other isospins needed, e.g., $(\pi\sigma)_P$ [Molina 2021]

"**Branching ratios**" in 3B decays are momentum -dependent, complex pole residues



Summary

- Lattice QCD progress in determining the explicit dynamics of three-body systems:
  - Three pions and kaons at maximal isospin
  - First determination of existence and properties of a three-body resonance - $a_1(1260)$ - in coupled channels using three-body unitarity
- **Outlook:** More (isospin) channels and more energy eigenvalues to assess uncertainties and put limits on decay properties.
- **Outlook:** Roper resonance as an interesting case of a baryon which has strong 3-body dynamics due to absence of centrifugal barriers (P_{11} has this special property).