Bethe Forum: "Modular Flavor Symmetries"

Modular and eclectic flavor symmertries from string compactifications

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May 04, 2021

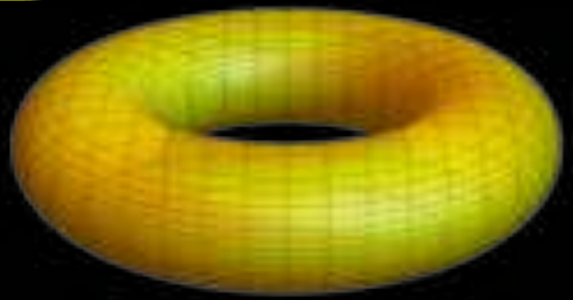
Outline

* Brief summary of modular symmetries
    - Symmetries of a 2D torus
    - Finite groups and modular forms

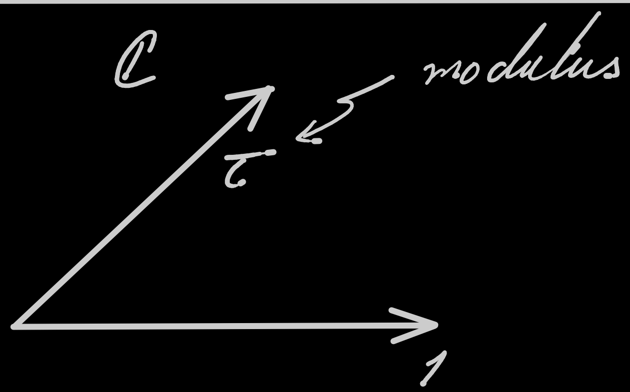
* Flavor from orbifold compactifications
    - Toroidal orbifold compactifications
    - Traditional flavor symmetries
    - Narain orbifolds as common source of all kinds of flavor and CP
    - Eclectic Flavor scheme from strings
    - Local Flavor Unification

* Flavor from compactifications on magnetized tori
    - Recall: metaplectic symmetries
    - Dirac eq. solutions on tori with fluxes
    - Couplings
    - Metaplectic flavor symmetries from tori

* Modular symmetries



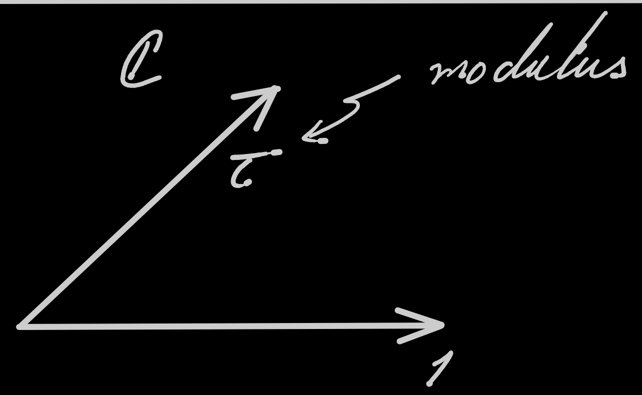
$\Rightarrow$



τ: describes the torus $\mathbb{T}^2$

* Modular symmetries

$\Rightarrow$



τ: describes the torus $\mathbb{T}^2$

Transforms under $SL(2,\mathbb{Z})$ as

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}\,, \quad \gamma \in SL(2,\mathbb{Z})$$

τ & τ' describe the <u>same</u> $\mathbb{T}^2$ $\Rightarrow$ $SL(2,\mathbb{Z})$ is a symmetry of $\mathbb{T}^2$

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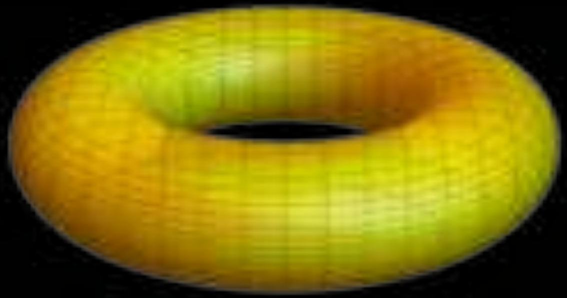

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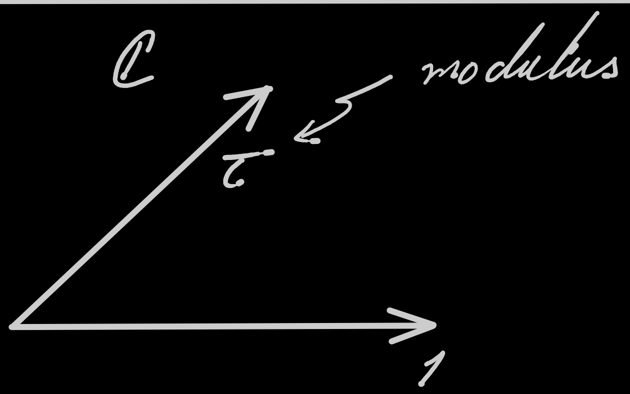

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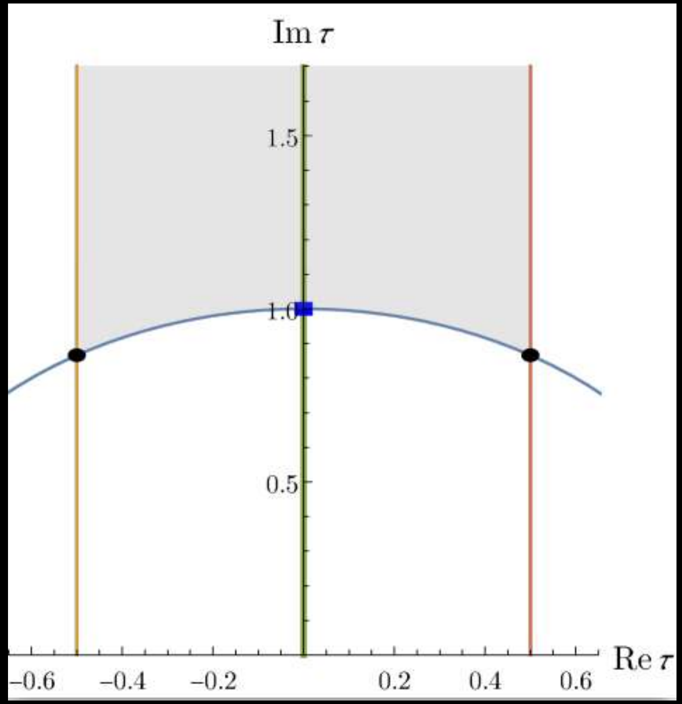
$$SL(2,\mathbb{Z}) = \langle S, T \mid S^4 = (ST)^3 = \mathbb{1},\ S^2T = TS^2 \rangle$$

generators represented by

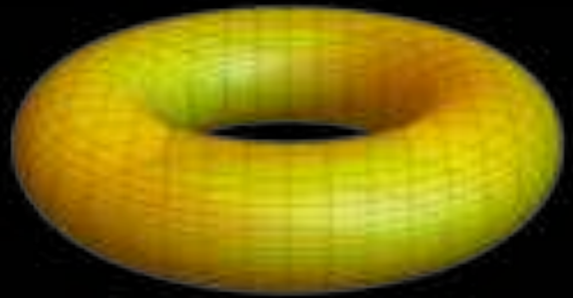
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det\gamma = 1, \quad a,b,c,d \in \mathbb{Z}$$



Moduli Space

* Modular symmetries



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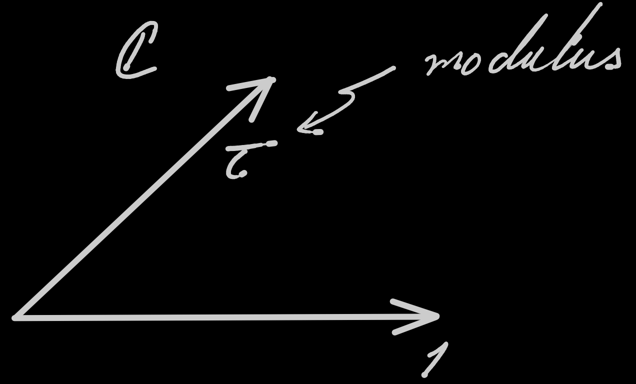

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– Modular forms of weight n and level N: modular weight

$$Y^{(n)}(\tau) \xrightarrow{\gamma} Y^{(n)}(\tau') = \underbrace{(c\tau+d)^n}_{\text{automorphy factor}}\ \underbrace{\rho(\gamma)}_{\text{representation of } \gamma \text{ in a finite modular group}}\, Y^{(n)}(\tau)$$

$\{Y^{(n)}(\tau)\}$ vector space

level N	Γ_N	GAP	Γ'_N	GAP
2	S_3	[6, 1]	S_3	[6, 1]
3	A_4	[12, 3]	T'	[24, 3]
4	S_4	[24, 12]	SL(2, 4)	[48, 30]
5	A_5	[60, 5]	SL(2, 5)	[120, 5]

Examples: $\Gamma'_N = \dfrac{SL(2,\mathbb{Z})}{\Gamma(N)} \Longleftrightarrow n \in \mathbb{N}$; $\quad \Gamma_N = \dfrac{PSL(2,\mathbb{Z})}{\Gamma(N)} \Longleftrightarrow n \in 2\mathbb{N}$

see e.g. Liu, Ding (2019)

see e.g de Adelhaart, Feruglio, Hagedorn (2011)

* Flavor from toroidal orbifold compactifications

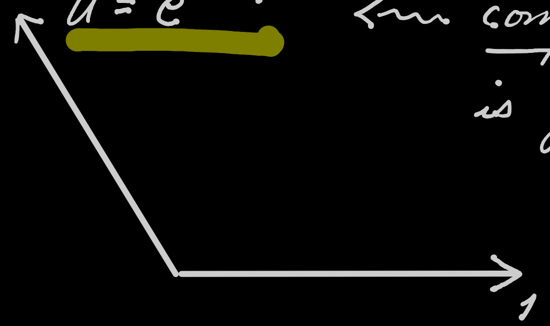
- Orbifold compactifications

we live here!

String theory is 10D $\longrightarrow$ 6D must be compactified: $\mathbb{M}^{10} = \mathbb{M}^4 \otimes X_6 \leftarrow$ this is compact & small

E.g. on a (factorizable) orbifold: $X_6 = \dfrac{\mathbb{T}^6}{\mathbb{Z}_N} = \dfrac{\mathbb{T}^2}{\mathbb{Z}_{N_1}} \otimes \dfrac{\mathbb{T}^2}{\mathbb{Z}_{N_2}} \otimes \dfrac{\mathbb{T}^2}{\mathbb{Z}_{N_3}}$

$\mathbb{T}^2/\mathbb{Z}_3$ $\qquad U = e^{2\pi i/3} \longleftarrow$ complex structure (analogous to τ) is geometrically fixed to make $\mathbb{T}^2$ compatible with $\mathbb{Z}_3$



In terms of the metric $U = \dfrac{1}{G_{11}}\left(G_{12} + i\sqrt{\det G}\right)$

$\mathbb{Z}_3 : \{\vartheta \mid \vartheta^3 = 1\}$ $\qquad \vartheta = e^{2\pi i/3}$ rotation (2x2 matrix)

$y \in \mathbb{T}^2/\mathbb{Z}_3 \Rightarrow y \sim \vartheta y + en$

e: 2D geometrical vielbein

n: 2D integer vector (winding)

Orbifold space group: $\{(\vartheta, en)\}$

* Flavor from toroidal orbifold compactifications

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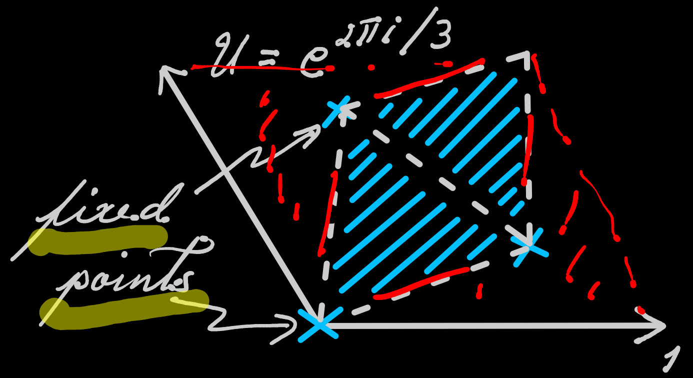
$\mathbb{T}^2/\mathbb{Z}_3$



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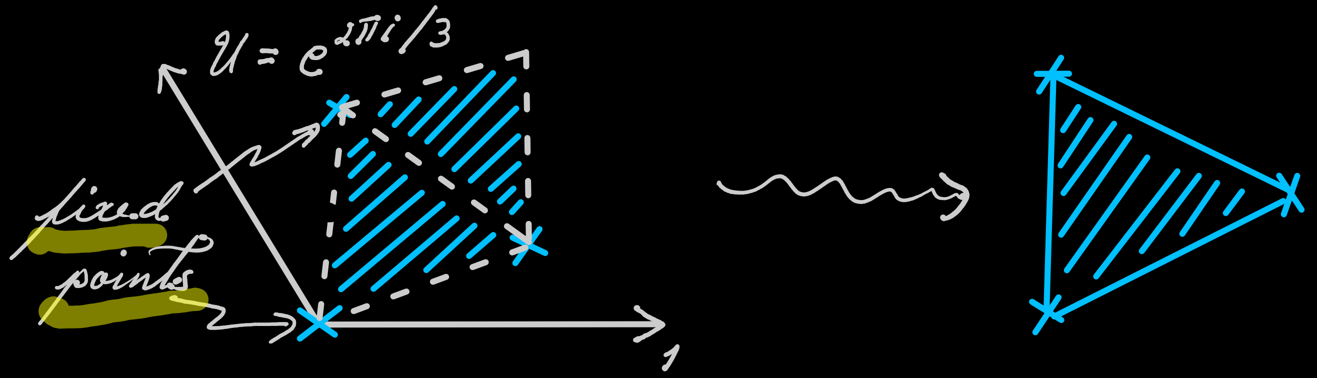

orbifold $\updownarrow$ triangular "pillow"

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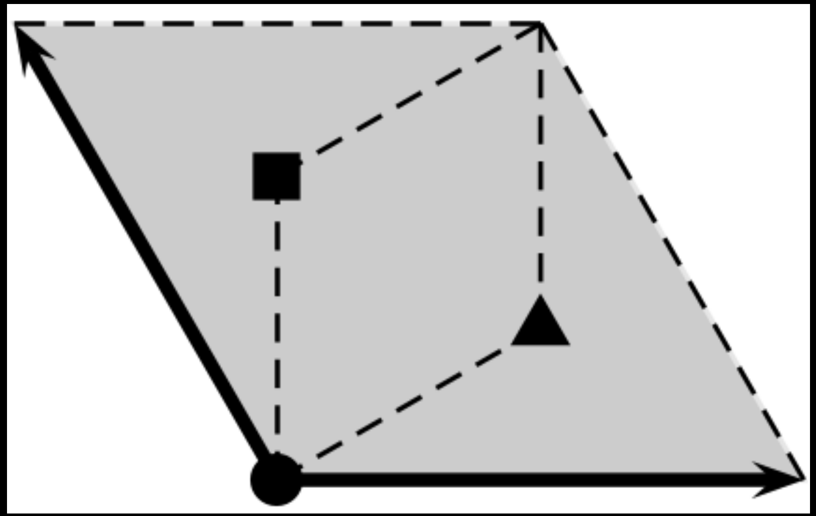

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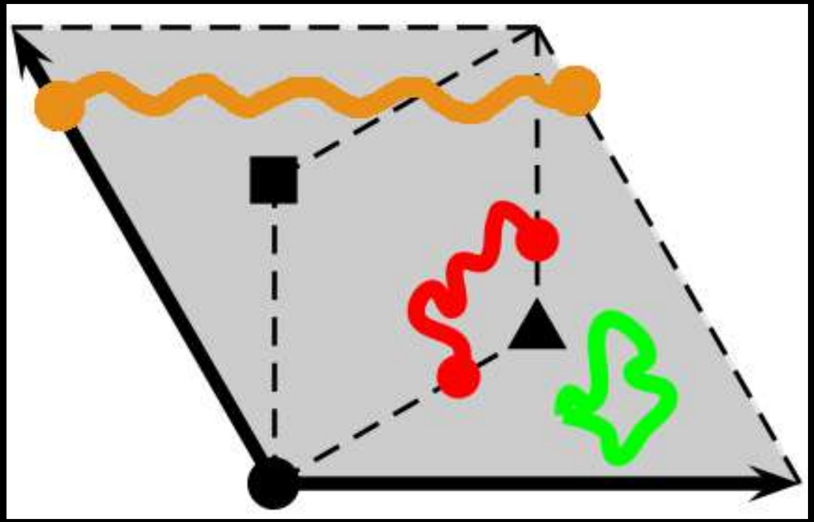

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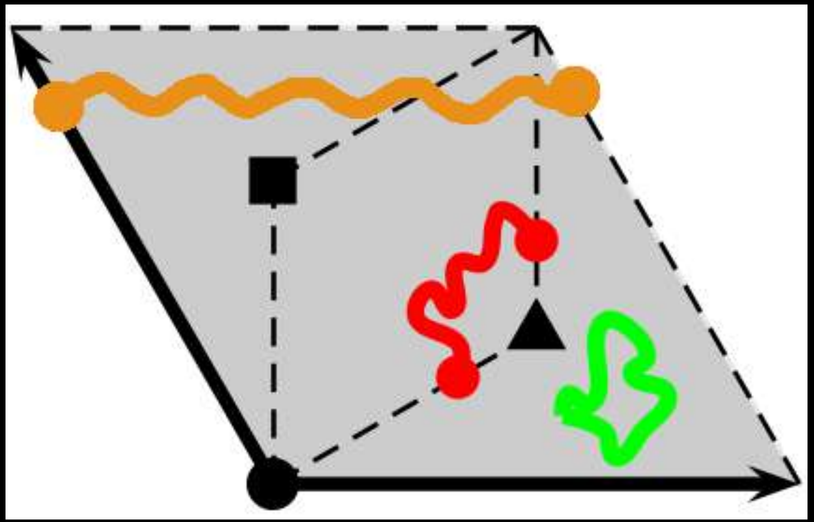
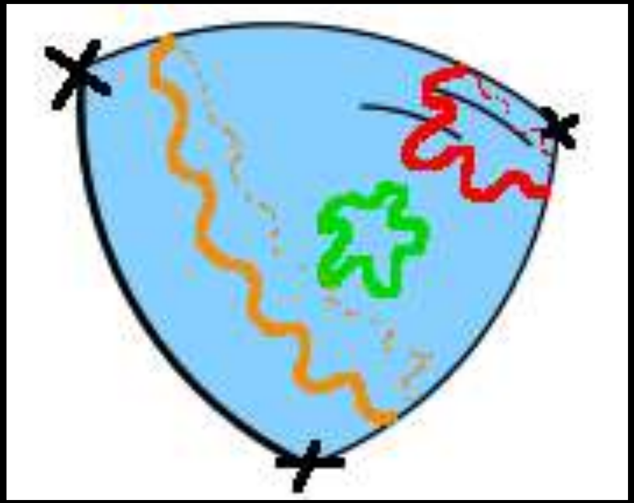


Strings on the orbifold $\rightsquigarrow$ 4D Matter states



Twisted (red) strings are attached to fixed points

Identical strings appear at each fixed point $\leftsquigarrow$ important for flavor

In orbifolds with strings: additional modulus $T = B_{12} + i\sqrt{\det G}$

anti-symmetric B-field

"volume" of $\mathbb{T}^2$

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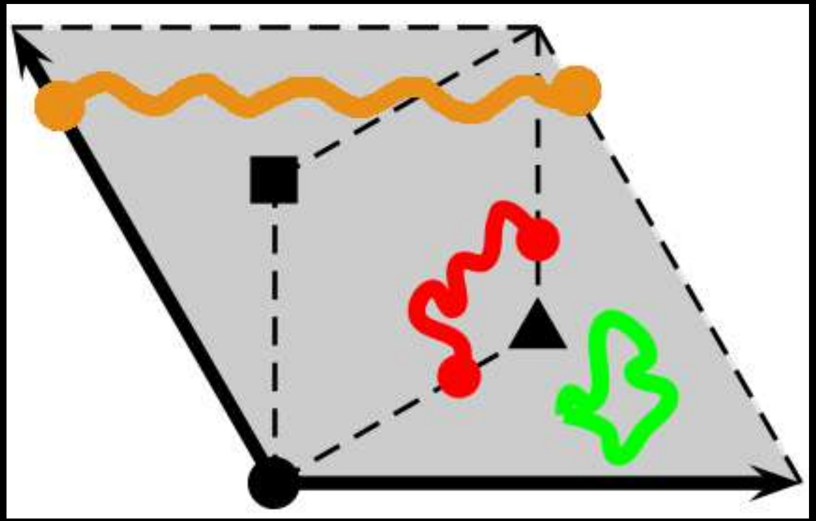
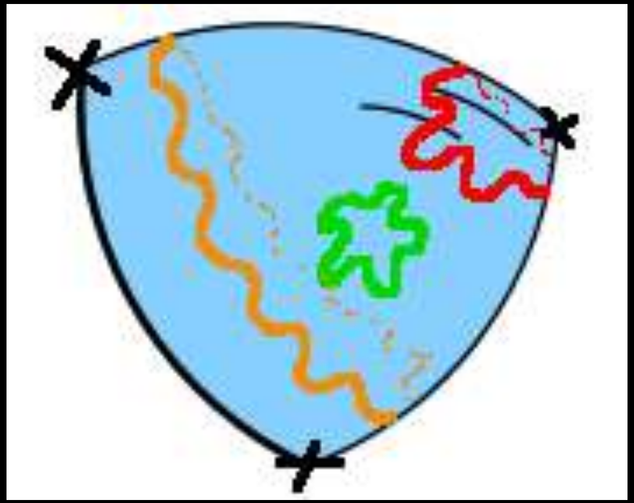
$\mathbb{T}^2/\mathbb{Z}_3$



Strings on the orbifold ⇝ 4D Matter states



Twisted (red) strings are attached to fixed points

Identical strings appear at each fixed point ⇜ important for flavor

In orbifolds with strings: additional modulus $T = B_{12} + i\sqrt{\det G}$

This Kähler modulus is free: admits $SL(2,\mathbb{Z})_T$ transformations

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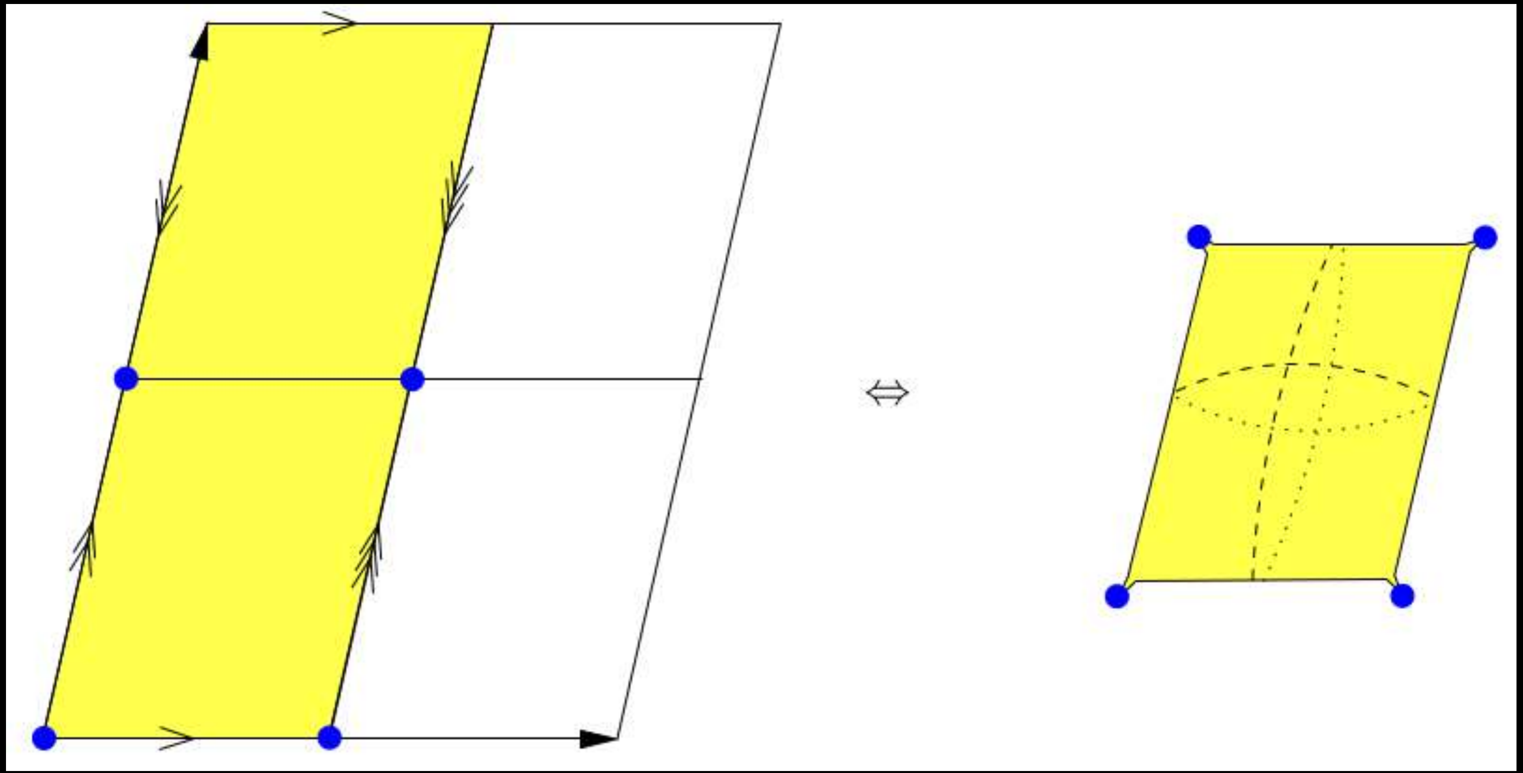
$\mathbb{T}^2/\mathbb{Z}_2$



* 4 fixed points
* U & T are free moduli
* Invariant under mirror transformations

$$U \xleftrightarrow{M} T$$

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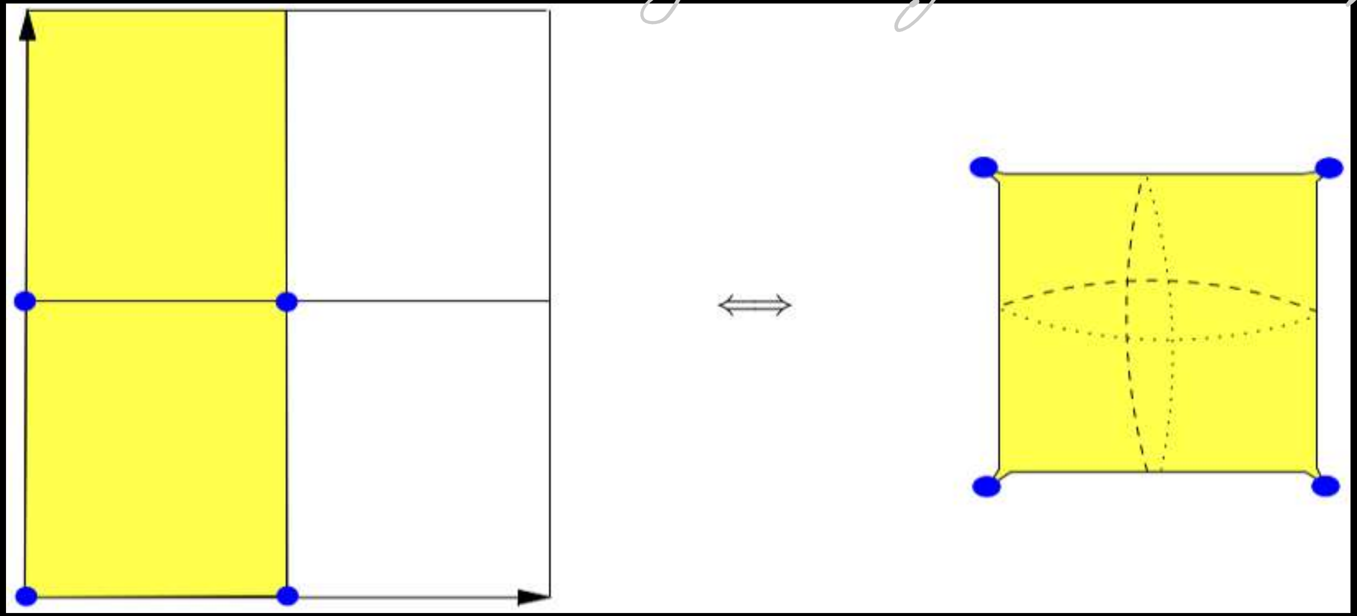
$\mathbb{T}^2/\mathbb{Z}_2$

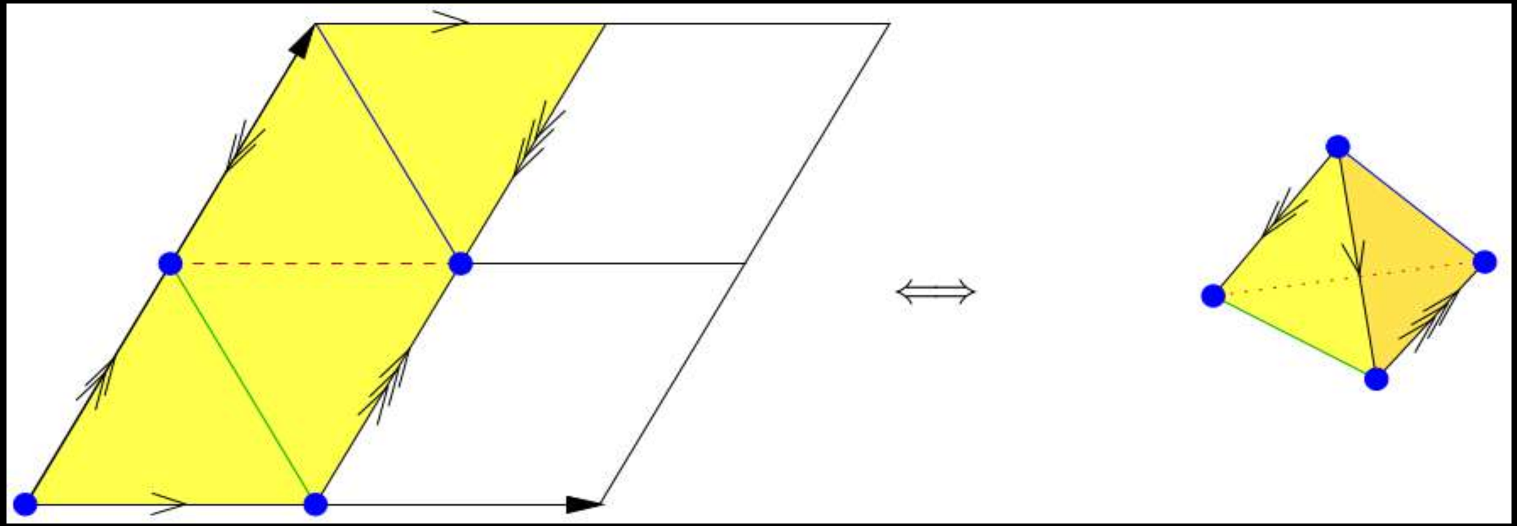
* Fixing U "by hand", we find two special cases:



- "Raviolo"

$U = i$

(T remains free)



- "Tetrahedron"

$U = e^{\pi i/3}$

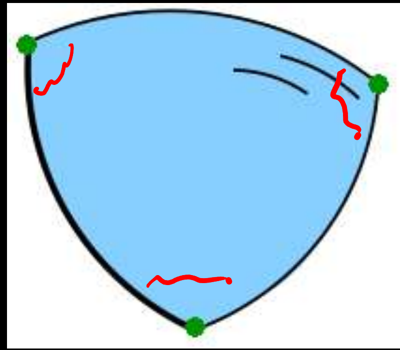
(T is free)

* Flavor from toroidal orbifold compactifications

- Traditional flavor symmetries

Degeneration of states at the fixed points $\longrightarrow$ permutation symmetries

$T^2/\mathbb{Z}_3$



S_3

String selection rules: given their localization in the orbifold, which strings are allowed to interact?

only 3, 6, 9, ... twisted strings $\longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_3 \longleftarrow$ only if located at the same point or at all 3 points

Traditional flavor symmetry: multiplicative closure of permutations & selection rules

$S_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \Delta(54) \cong [54, 8]$ moduli independent!

GAP ID

* Flavor from toroidal orbifold compactifications

- Traditional flavor symmetries

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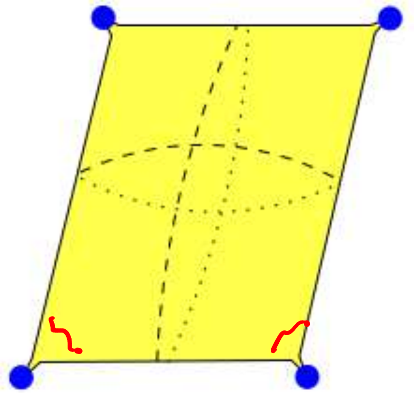

Permutations

Stringy Selection rules

$$(S_2 \times S_2) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \frac{D_8 \times D_8}{\mathbb{Z}_2} \cong \frac{Q_8 \times Q_8}{\mathbb{Z}_2} \cong [32, 49]$$

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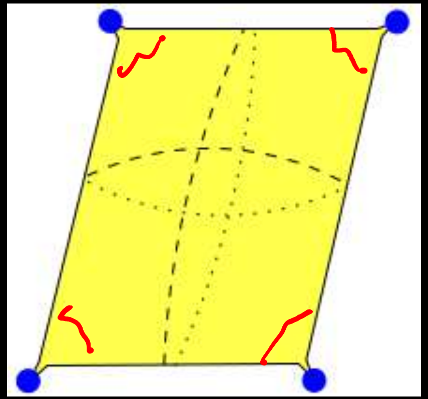
$\mathbb{T}^2/\mathbb{Z}_2$ 

Permutations, Stringy Selection rules

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Massless strings build only restricted representations:

Nilles, Ratz, Trautner, Vaudrevange (2018)

$\mathbb{T}^2/\mathbb{Z}_3$: $\underline{1}, \underline{1}'$ (untwisted) $\underline{3}_1, \underline{3}_2$ (twisted) of $\Delta(54)$ — all doublets are massive winding strings

$\mathbb{T}^2/\mathbb{Z}_2$: $\underline{1}_0$ (untwisted) $\underline{4}$ (twisted) of $(D_8 \times D_8)/\mathbb{Z}_2{}^*$

Baur, Kade, Nilles, Ramos-Sanchez, Vaudrevange (2021)

* combined with a $\mathbb{Z}_4^R$

* Flavor from toroidal orbifold compactifications

- Traditional flavor symmetries

multiplicative closure

Kobayashi, Nilles, Ploeger, Raby, Ratz (2006)

This can be generalized to other orbifolds: $G_{\text{traditional}} = \text{Permutations} \cup \text{Stringy sel. rules}$

Ramos-Sánchez (2017); Olguín-Trejo, Pérez-Martínez, Ramos-Sánchez (2018)

It has been used to classify traditional flavor symmetries in Heterotic Orbifolds

Carballo-Pérez, Peinado, Ramos-Sánchez (2016)

It can lead to interesting neutrino phenomenology (a generic $\mathbb{T}^2/\mathbb{Z}_3$ with $\Delta(54)$ & MSSM-like)

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BUT these are only one kind of the available symmetries

Lauer, Mas, Nilles (1989,1990)

- Modular transformations can mix different twisted states too!

$\Rightarrow$ Modular flavor symmetries? Can we determine systematically all flavor sym.?

* Flavor from toroidal orbifold compactifications

- Narain Tori and Narain Orbifolds

A string has right-moving & left-moving independent modes.

Decompose the string coordinates as $y = \begin{pmatrix} y_R \\ y_L \end{pmatrix}$

Narain (1986); Narain, Samadi, Witten (1987); Groot-Nibbelink, Vaudrevange (2017)

Narain proposal: compactify y_R & y_L as independent degrees of freedom

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We can translate the geometric orbifold $\mathbb{T}^2/\mathbb{Z}_2$ to the Narain toroidal orbifold

twist of $\mathbb{T}^2/\mathbb{Z}_N$ $\qquad y \sim \Theta y + E\hat{N}$

Narain twist $\rightarrow \Theta = \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta \end{pmatrix}$, $\hat{N} = \begin{pmatrix} n \\ m \end{pmatrix}$ $\begin{matrix} \leftarrow \text{winding} \\ \leftarrow \text{Kaluza-Klein} \end{matrix}$, $E = E(T, U)$: Narain vielbein

$\Rightarrow$ 4x4 metric $\mathcal{H}(T,U) = E^T E \longrightarrow$ Narain toroidal lattice Γ of signature $(2,2)$

Combine Θ & Γ in the Narain space group $S_{\text{Narain}} = \{(\Theta, E\hat{N})\}$

* Flavor from toroidal orbifold compactifications

  - Flavor symmetries from outer automorphisms of the Narain space group

"Simetrías" of S_{Narain}

orbifold rotations

$\{(\Theta, E\hat{N})\}$

orbifold translations

$\{(\Sigma, 0)\}$

rotational automorphisms of S_{Narain}

if $\Sigma \sim \Theta \Rightarrow$ inner $\rightarrow$ in S_{Narain}

if $\Sigma \neq \Theta \Rightarrow$ outer $\rightarrow$ NEW

Σ affects $\mathcal{M}(T, U)$:

modular symmetry

Baur, Nilles, Trautner, Vaudrevange (2019)

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Σ affects $\mathcal{H}(T,U)$:

modular symmetry

$\{(\mathbb{1}, E\hat{T})\}$

translational automorphisms of S_{Narain}

If $\hat{T} \in \mathbb{Z}^4 \Rightarrow$ inner $\to$ in S_{Narain}

If $\hat{T} \notin \mathbb{Z}^4 \Rightarrow$ outer $\to$ NEW

$\hat{T}$ doesn't affect $\mathcal{H}(T,U)$, but relates states at fixed points:

traditional symmetry

Baur, Nilles, Trautner, Vaudrevange (2019)

* Flavor from toroidal orbifold compactifications

- Flavor symmetries from outer automorphisms of the Narain space group

- Traditional flavor symmetries: $\{(\mathbb{1}, E\hat{T}) \mid \hat{T} \notin \mathbb{Z}^4\}$ that leave S_{Narain} invariant

Extra: $(-\mathbb{1}, 0)$ is moduli universal, too!

$\mathbb{T}^2/\mathbb{Z}_3$: $\{(\mathbb{1}, E\hat{T}_A), (\mathbb{1}, E\hat{T}_B), (-\mathbb{1}, 0)\}$ e.g. $\hat{T}_A = (\frac{1}{3}, \frac{2}{3}, 0, 0)^T$

$\mathbb{T}^2/\mathbb{Z}_2$: $\{(\mathbb{1}, E\hat{T}_k) \mid k=1,\ldots,4\}$ e.g. $\hat{T}_1 = (\frac{1}{2}, 0, 0, 0)^T$

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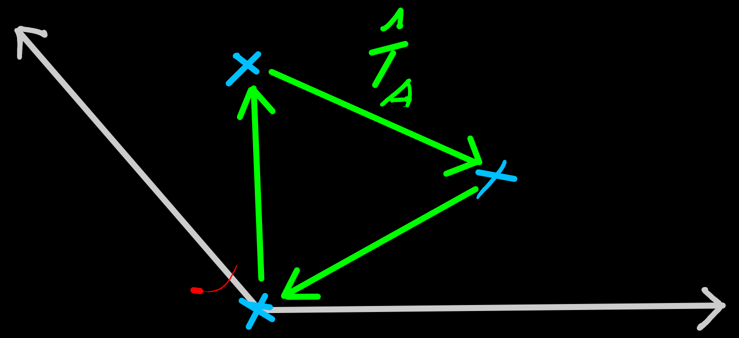
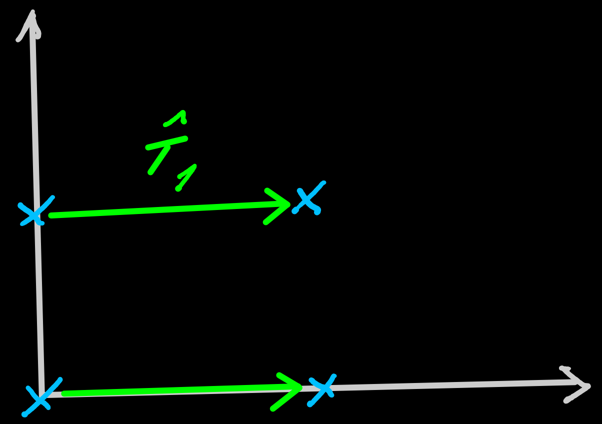
  - Inspect its action on fixed points

E.g.





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- Inspect its action on fixed points

Results: $\mathbb{T}^2/\mathbb{Z}_3$ $\mathbb{T}^2/\mathbb{Z}_2$

$\Delta(54) \cong [54, 8]$ $\frac{D_8 \times D_8}{\mathbb{Z}_2} \cong [32, 49]$

* Flavor from toroidal orbifold compactifications

  - Flavor symmetries from outer automorphisms of the Narain space group

Baur, Nilles, Trautner, Vaudrevange (2019)

- Modular symmetries : subgroup of $\Sigma \in \{ \underbrace{K_S, K_T}_{\text{Kähler } T}, \underbrace{C_S, C_T}_{U}, M, \overset{CP}{\Sigma_*} \}$

K_S, K_T : $T \xrightarrow{K_S} -\frac{1}{T}$, $T \xrightarrow{K_T} T+1$, $U \longrightarrow U$ $\Rightarrow$ $SL(2,\mathbb{Z})_T$

C_S, C_T : $U \xrightarrow{C_S} -\frac{1}{U}$, $U \xrightarrow{C_T} U+1$, $T \to T$ $\Rightarrow$ $SL(2,\mathbb{Z})_U$

M : $U \overset{M}{\longleftrightarrow} T$, $K_S \longleftrightarrow C_S$, $K_T \longleftrightarrow C_T$ $\Rightarrow$ $\mathbb{Z}_2^M$

Σ_*: $U \xrightarrow{\Sigma_*} -\bar{U}$, $T \xrightarrow{\Sigma_*} -\bar{T}$ $\Rightarrow$ $\mathbb{Z}_2^{CP}$ CP-like trafo.

compare with bottom-up with Novichkov, Penedo, Petcov, Titov (2019)

* Flavor from toroidal orbifold compactifications

  - Flavor symmetries from outer automorphisms of the Narain space group

Baur, Nilles, Trautner, Vaudrevange (2019)

– Modular symmetries : subgroup of $\Sigma \in \{K_S, K_T, C_S, C_T, M, \Sigma_*\}$

K_S, K_T : $T \xrightarrow{K_S} -\frac{1}{T}$, $T \xrightarrow{K_T} T+1$, $U \longrightarrow U$ $\Longrightarrow$ $SL(2,\mathbb{Z})_T$

C_S, C_T : $U \xrightarrow{C_S} -\frac{1}{U}$, $U \xrightarrow{C_T} U+1$, $T \rightarrow T$ $\Longrightarrow$ $SL(2,\mathbb{Z})_U$

M : $U \xleftrightarrow{M} T$, $K_S \longleftrightarrow C_S$, $K_T \longleftrightarrow C_T$ $\Longrightarrow$ $\mathbb{Z}_2^M$

Σ_*: $U \xrightarrow{\Sigma_*} -\bar{U}$, $T \rightarrow -\bar{T}$ $\Longrightarrow$ $\mathbb{Z}_2^{CP}$ CP-like trafo

compare with bottom-up with Novichkov, Penedo, Petcov, Titov (2019)

– Breakdown arises from geometrical fixing of U

Recall in $\mathbb{T}^2/\mathbb{Z}_3$: $U = e^{2\pi i/3}$. Since $C_S C_T \circ e^{2\pi i/3} = e^{2\pi i/3}$ $\Longrightarrow$ $\mathbb{Z}_3^U$ unbroken

In $\mathbb{T}^2/\mathbb{Z}_2$: ALL these symmetries are unbroken at generic point in moduli space!

* Flavor from toroidal orbifold compactifications

  - Flavor symmetries from outer automorphisms of the Narain space group

    - Modular symmetries: Finite modular groups are obtained by inspecting 4D matter trafos.

      * Modular weights n are determined by compactification

        4D massless superfield $\rightarrow \Phi_n \qquad n = \pm \frac{\text{integer}}{N} \quad \text{in} \quad \mathbb{T}^2/\mathbb{Z}_N$

        Ibáñez, Lüst (1992)

      * In CFT formalism, we can compute the modular trafos. of untwisted strings

      * OPE of twisted strings yield untwisted strings

        Lauer, Mas, Nilles (1989, 1990)

      * We can read off modular trafos. of twisted strings

        $$\Phi_n \xrightarrow{\gamma} (c\tau + d)^n \rho(\gamma)\, \Phi_n \,, \qquad \rho(\gamma) \text{ fixed by theory}$$

      * Determine the finite modular group generated by $\{\rho(\gamma)\}\ \forall$ unbroken γ

        use e.g. GAP 4.11.1

* Flavor from toroidal orbifold compactifications

- Flavor symmetries from outer automorphisms of the Narain space group

- Modular symmetries : Finite modular groups

$T^2/\mathbb{Z}_3$ $\quad \gamma \in \{K_S, K_T, C_S\Sigma_*, C_S C_T\}$ $\quad$ Recall $\quad \omega = e^{2\pi i/3}$

$$\langle \rho(K_S), \rho(K_T)\rangle \cong T' \cong [24,3] \qquad \text{note} \quad T' \cong \Gamma_3' = \frac{SL(2,\mathbb{Z})}{\Gamma(3)}$$

$$\langle \rho(C_S\Sigma_*)\rangle \cong \mathbb{Z}_2^{CP} : \Phi_n \longrightarrow \Phi_n^*$$

modular repr. of $\mathbb{Z}_3$ twist

Lorentz remnant

$$C_S C_T : \Phi_n \longrightarrow (-e^{2\pi i/3} - 1)^{n_u} \rho(C_S C_T)\, \Phi_n = e^{2\pi R/9}\, \Phi_n \qquad \mathbb{Z}_9^R$$

R-charge (moduli universal)

4D matter Φ_n		mod. weight n	T' repr.	R-charge
bulk	{	0	$\underline{1}$	0
		-1	$\underline{1}$	3
twisted	{	-2/3	$\underline{1} \oplus \underline{2}'$	1
		-5/3	$\underline{1} \oplus \underline{2}'$	-2

Nilles, Ramos-Sánchez, Vaudrevange (2020)

* Flavor from toroidal orbifold compactifications

- Flavor symmetries from outer automorphisms of the Narain space group

- Modular forms. Since the finite modular group is $\Gamma_3' \cong T'$

$\Rightarrow$ couplings are modular forms* (up to a $j(T)$ function)

$Y^{(n_Y)}(T)$ with $n_Y \geq 1$ & $n_Y \in \mathbb{N}$

$n_Y = 1 \longrightarrow Y^{(1)}(T)$ builds a $\underline{2}''$ of T' (as usual)

& $Y^{(1)}(T) \xrightarrow{\gamma} (cT+d)^{1}\rho(\gamma)\, Y^{(1)}(T)$

Higher weights arise from products of these modular forms

---

* Note: one can verify this by computing them explicitly via CFT techniques

Erler, Jungnickel, Spalinski, Stieberger (1993)

* Flavor from toroidal orbifold compactifications

- Flavor symmetries from outer automorphisms of the Narain space group

- Modular symmetries : Finite modular groups

$T^2/\mathbb{Z}_2$ $\gamma \in \{K_S, K_T, C_S, C_T, M, \Sigma_*\}$

Recall U, T = free

$$\langle \rho(K_S), \rho(K_T), \rho(C_S), \rho(C_T), \rho(M) \rangle \cong [144, 115] \cong (S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^M$$

two mod. weights $\longleftrightarrow SL(2,\mathbb{Z})^2$

$$\langle \rho(\Sigma_*) \rangle \cong \mathbb{Z}_2^{CP} : \Phi_{(n_T,n_U)} \longrightarrow \Phi^*_{(n_T,n_U)}$$

modular repr. of $\mathbb{Z}_2$ twist

Lorentz remnant

$$C_S^2 : \Phi_{(n_T,n_U)} \longrightarrow (-1)^{n_U} \rho(C_S^2)\, \Phi_{(n_T,n_U)} = e^{2\pi R/4}\, \Phi_{(n_T,n_U)} \qquad \mathbb{Z}_4^R$$

R-charge (moduli universal)

	$\Phi_{(n_T,n_U)}$	n_T	n_U	[144,115]	R-charge
bulk		0	0	$\mathbf{1}_0$	0
		-1	-1	$\mathbf{1}_0$	2
twisted		$-1/2$	$-1/2$	$\mathbf{4}_1$	3
		$-3/2$	$1/2$	$\mathbf{4}_1 \oplus \mathbf{4}_1$	1
		$1/2$	$-3/2$		1

Baur, Kade, Nilles, Ramos-Sánchez, Vaudrevange (2020,2021)

* Flavor from toroidal orbifold compactifications

- Flavor symmetries from outer automorphisms of the Narain space group

The finite modular group $[144, 115] \cong (S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^M$ acts on moduli T, U as

$$[72, 40] \cong (S_3^T \times S_3^U) \rtimes \mathbb{Z}_2^M \cong (\Gamma_2^T \times \Gamma_2^U) \rtimes \mathbb{Z}_2^M \subset \Gamma_{2,2}$$

$\Gamma_{2,2}$: symplectic finite group of genus $g=2$ and level $n=2$

Ding, Feruglio, Liu (2020)

$$\Gamma_{2,2} = \frac{Sp(4,\mathbb{Z})}{\Gamma_2(2)}$$

Siegel modular group

It breaks to $[72, 40]$ for the modulus choice $\Omega = \begin{pmatrix} U & Z \\ Z & T \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & T \end{pmatrix}$

where Z can be identified with a Wilson line modulus in the context of string orbifolds

Baur, Kade, Nilles, Ramos-Sánchez, Vaudrevange (2020)

* Flavor from toroidal orbifold compactifications

  - Flavor symmetries from outer automorphisms of the Narain space group

$\sim$ Modular forms. Coupling among states are Siegel modular forms

$Y^{(n_Y)}(T,U)$ with "parallel" weight $n_Y \geq 2$ (equal for both $SL(2,\mathbb{Z})$)

$n_Y = 2 \longrightarrow Y^{(2)}(T,U)$ builds a $\underline{4}_3$ of $[144,115]$

$\& \quad Y^{(2)}(T,U) \xrightarrow{\gamma} (c_T T + d_T)^2 (c_U U + d_U)^2 \rho(\gamma)\, Y^{(2)}(T,U)$

(ignoring CP-like trafos.)

* Flavor from toroidal orbifold compactifications

Nilles, Ramos-Sánchez, Vaudrevange (2020)

- Eclectic Flavor Scheme

Eclectic flavor symmetries : nontrivial combination of a finite modular group & a traditional flavor symmetry.

universal trafos. in moduli space

Trafos. that affect the moduli

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

multiplicative closure

- $G_{\text{modular}} \subset \text{Out}(G_{\text{traditional}})$
- $G_{\text{traditional}} \cap G_{\text{modular}} \supset \{\mathbb{1}\}$ It contains more!

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme

Nilles, Ramos-Sánchez, Vaudrevange (2020)

Eclectic flavor symmetries: nontrivial combination of a finite modular group & a traditional flavor symmetry.

universal trafos. in moduli space

Trafos. that affect the moduli

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

E.g. $\mathbb{T}^2/\mathbb{Z}_3$

nature of symmetry		outer automorphism of Narain space group	flavor groups				
eclectic	modular	rotation S $\in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_4$	T'			$\Omega(2)$
		rotation T $\in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_3$				
	traditional flavor	translation A	$\mathbb{Z}_3$	$\Delta(27)$	$\Delta(54)$	$\Delta'(54,2,1)$	
		translation B	$\mathbb{Z}_3$				
		rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$	$\mathbb{Z}_2^R$				
		rotation $\hat{\mathrm{R}}_1 \Leftrightarrow \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$	$\mathbb{Z}_9^R$				

Excluding CP

$\Delta'(54,2,1) \cap T' = \{\pm 1\}$

$\Delta'(54,2,1) \cong [162, 44]$

$\Omega(2) \cong [1944, 3448]$

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme

The power of eclectic flavor symmetries in the effective 4D theory.

$\mathbb{T}_2/\mathbb{Z}_3$

sector	fields Φ_n	osc.	modular T' subgroup: irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	traditional $\Delta(54)$ subgroup: irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$
bulk	Φ_0	no	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$
	Φ_{-1}	no	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1
θ	$\Phi_{-2/3}$	no	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$
	$\Phi_{-5/3}$	yes	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$

~ Kähler potential for Φ_n with $n=-\frac{2}{3}$

$\underline{3}_2 : (X, Y, Z)^T$ in $\Delta(54)$

$\underline{2}'\oplus\underline{1} : \left(\frac{1}{\sqrt{2}}(Y+Z), -X\right)^T \oplus \frac{1}{\sqrt{2}}(Y-Z)$ in T'

Most general structure

$$K \supset \sum_{\Phi_n}\sum_{n_Y\geq 0}(-\mathrm{i}T+\mathrm{i}\bar{T})^{n+n_Y}\sum_a \kappa_a^{(n_Y)}\left[\hat{Y}_{\boldsymbol{s}}^{(n_Y)}(T)\otimes\Phi_n\otimes\left(\hat{Y}_{\boldsymbol{s}}^{(n_Y)}(T)\right)^*\otimes\bar{\Phi}_n\right]_{\mathbf{1}}$$

Chen, Ramos-Sánchez, Ratz (2019)

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme

The power of eclectic flavor symmetries in the effective 4D theory.

$\mathbb{T}_2/\mathbb{Z}_3$

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bulk	Φ_0	no	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$
	Φ_{-1}	no	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1
θ	$\Phi_{-2/3}$	no	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$
	$\Phi_{-5/3}$	yes	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$

~ Kähler potential for Φ_n with $n=-\frac{2}{3}$

$\underline{3}_2 : (X, Y, Z)^T$ in $\Delta(54)$

$\underline{2}'\oplus\underline{1} : \left(\frac{1}{\sqrt{2}}(Y+Z), -X\right)^T \oplus \frac{1}{\sqrt{2}}(Y-Z)$ in T'

If only T'

$$
\begin{aligned}
K \supset \sum_{n_Y\geq 0}(-\mathrm{i}T+\mathrm{i}\bar{T})^{n+n_Y}\Big[&A_1^{(n_Y)}(T,\bar{T})|X|^2 + A_2^{(n_Y)}(T,\bar{T})|Y|^2 + A_3^{(n_Y)}(T,\bar{T})|Z|^2\\
&+A_4^{(n_Y)}(T,\bar{T})\left(X\bar{Y}+\bar{X}Y\right) + A_5^{(n_Y)}(T,\bar{T})\left(X\bar{Z}+\bar{X}Z\right)\\
&+A_6^{(n_Y)}(T,\bar{T})\left(Y\bar{Z}+\bar{Y}Z\right)\Big]
\end{aligned}
$$

Predictability issues!

!!

Chen, Ramos-Sanchez, Ratz (2019)

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme

The power of eclectic flavor symmetries in the effective 4D theory.

$\mathbb{T}_2/\mathbb{Z}_3$

sector	fields	osc.	modular T' subgroup				traditional $\Delta(54)$ subgroup			
	Φ_n		irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$
bulk	Φ_0	no	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$
	Φ_{-1}	no	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1
θ	$\Phi_{-2/3}$	no	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$
	$\Phi_{-5/3}$	yes	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$

~ Kähler potential for Φ_n with $n=-\frac{2}{3}$

$\underline{3}_2 : (X, Y, Z)^T$ in $\Delta(54)$

$\underline{2}'\oplus\underline{1} : \left(\frac{1}{\sqrt{2}}(Y+Z), -X\right)^T \oplus \frac{1}{\sqrt{2}}(Y-Z)$ in T'

With $\Delta(54)$:

$$K \supset \sum_{n_Y\geq 0} (-\mathrm{i}T+\mathrm{i}\bar{T})^{n+n_Y} A_1^{(n_Y)}(T,\bar{T}) \left(|X|^2+|Y|^2+|Z|^2\right)$$

canonical ! ☺

Generic : traditional component of $\mathcal{G}_{\text{eclectic}}$ fixes the Kähler

Nilles, Ramos-Sánchez, Vaudrevange (2020)

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme

The power of eclectic flavor symmetries in the effective 4D theory.

~ Also the superpotential is quite restricted

$$\mathcal{W}(T, X_i, Y_i, Z_i) \supset c^{(1)}\Big[\hat{Y}_2(T)\big(X_1\,X_2\,X_3 + Y_1\,Y_2\,Y_3 + Z_1\,Z_2\,Z_3\big) - \frac{\hat{Y}_1(T)}{\sqrt{2}}\big(X_1\,Y_2\,Z_3 + X_1\,Y_3\,Z_2 + X_2\,Y_1\,Z_3 + X_3\,Y_1\,Z_2 + X_2\,Y_3\,Z_1 + X_3\,Y_2\,Z_1\big)\Big]$$

$\Rightarrow$ Moduli dependent mixing patterns assuming e.g. $X_1 = Q$, $X_2 = \bar{u}$, $X_3 = \varphi$ flavon necessary (breaks T' & $\Delta(54)$)

$\Downarrow$

Open question: Pheno ??

$c^{(1)}$: number & $\langle H_u \rangle$

Baur, Nilles, Ramos-Sánchez, Trautner, Vaudrevange (to appear in 2021)

* Flavor from toroidal orbifold compactifications

  - Eclectic Flavor Scheme: Local Flavor Unification

At different fixed points or loci in moduli space (i.e. $\langle T\rangle$ for $\mathbb{T}^2/\mathbb{Z}_3$, $(\langle T\rangle, \langle U\rangle)$ for $\mathbb{T}^2/\mathbb{Z}_2$)

"Stabilizer" $\gamma_h \in H_{(\langle T\rangle, \langle U\rangle)}$ $\therefore$ $\langle T\rangle \xrightarrow{\gamma_h} \langle T\rangle$, $\langle U\rangle \xrightarrow{\gamma_h} \langle U\rangle$

New enhanced traditional flavor symmetry = unified flavor symmetry

$$G_{UFS} = G_{\text{traditional}} \cup H_{\langle T\rangle} \subset G_{\text{eclectic}}$$

all elements are linearly realized:

$$\text{e.g.} \quad \Phi_n \xrightarrow{\gamma_h} \underbrace{(c\langle T\rangle + d)^n}_{\text{phase}} \rho(\gamma_h)\, \Phi_n =: \underbrace{\rho'(\gamma_h)\, \Phi_n}_{\text{universal in moduli space}}$$

universal in moduli space
↳ traditional sym.

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme: Local Flavor Unification

At different fixed points or loci in moduli space (i.e. $\langle T\rangle$ for $\mathbb{T}^2/\mathbb{Z}_3$, $(\langle T\rangle,\langle U\rangle)$ for $\mathbb{T}^2/\mathbb{Z}_2$)

"Stabilizer" $\gamma_h \in H_{(\langle T\rangle,\langle U\rangle)}$ $\therefore$ $\langle T\rangle \xrightarrow{\gamma_h} \langle T\rangle$, $\langle U\rangle \xrightarrow{\gamma_h} \langle U\rangle$

New enhanced traditional flavor symmetry = unified flavor symmetry

$$G_{UFS} = G_{\text{traditional}} \cup H_{\langle T\rangle} \subset G_{\text{eclectic}}$$

all elements are linearly realized:

$$\text{e.g.}\quad \Phi_n \xrightarrow{\gamma_h} (c\langle T\rangle + d)^n \rho(\gamma_h)\,\Phi_n =: \rho'(\gamma_h)\,\Phi_n$$

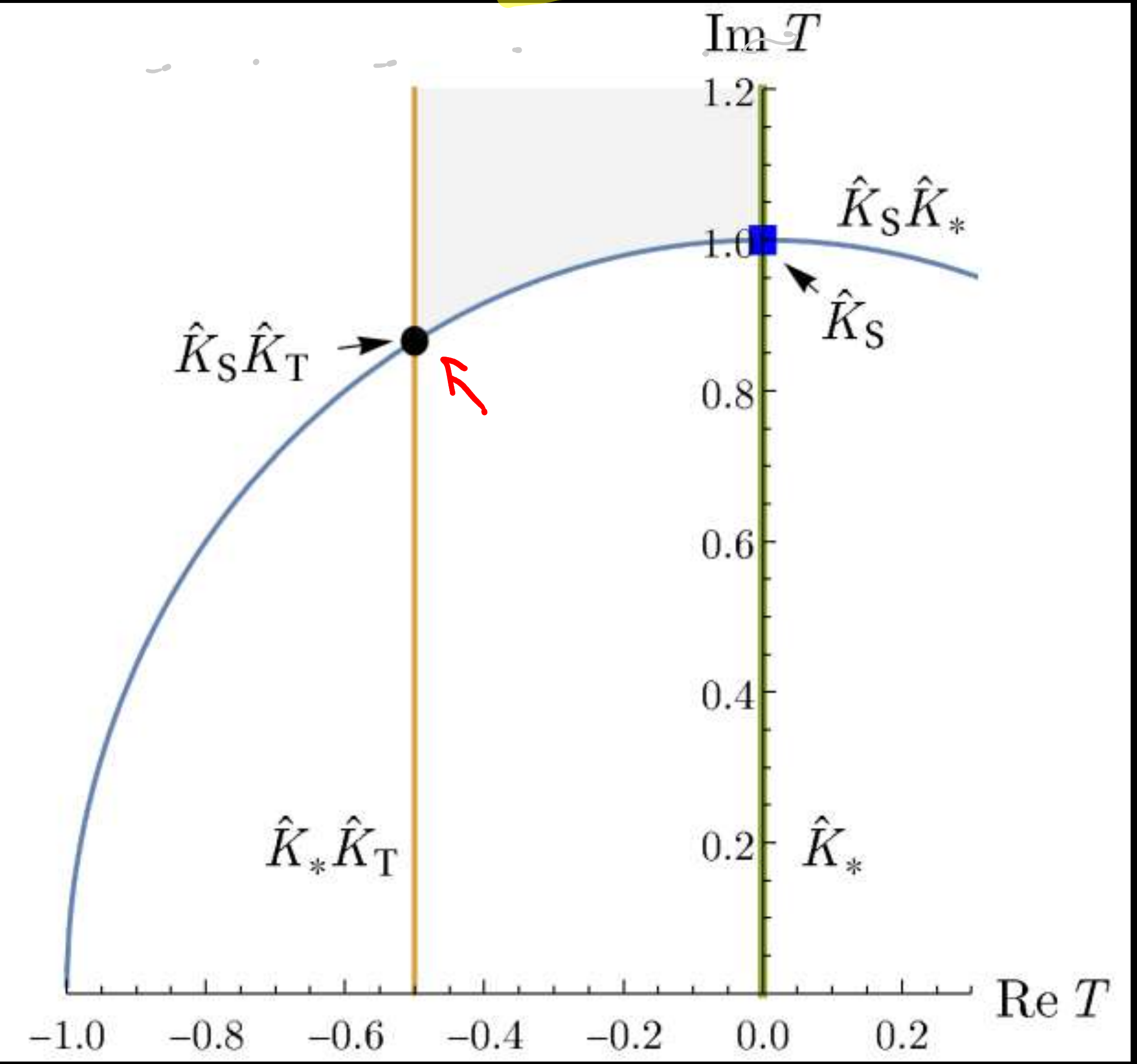
"alignement" of $Y^{(n_Y)}$

$$\& \quad Y^{(n_Y)}(\langle T\rangle) \xrightarrow{\gamma_h} Y^{(n_Y)}(\langle T\rangle) = (c\langle T\rangle + d)^{n_Y} \rho_Y(\gamma_h)\, Y^{(n_Y)}(\langle T\rangle) \Leftrightarrow \rho_Y(\langle T\rangle)\, Y^{(n_Y)} \overset{!}{=} (c\langle T\rangle + d)^{-n_Y}\, Y^{(n_Y)}$$

Baur, Nilles, Trautner, Vaudrevange (2019), Nilles, Ramos-Sánchez, Vaudrevange (2020)

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme: Local Flavor Unification



Various G_{UFS} at the different fixed points and fixed loci in moduli space of $\mathbb{T}^2/\mathbb{Z}_3$

$\langle T\rangle$	stabilizer generators		unified flavor symmetries	
	non $\mathcal{CP}$-like	$\mathcal{CP}$-like	without $\mathcal{CP}$	with $\mathcal{CP}$
i	$\hat{K}_\mathrm{S}$	$\hat{K}_*$	$\Xi(2,2)\cong[324,111]$	$[648,548]$
ω	$\hat{K}_\mathrm{S}\hat{K}_\mathrm{T}$	$\hat{K}_*\hat{K}_\mathrm{T}$	$H(3,2,1)\cong[486,125]$	$[972,469]$
$\mathrm{Re}\langle T\rangle=0$		$\hat{K}_*$	$\Delta'(54,2,1)\cong[162,44]$	$[324,125]$
$\mathrm{Re}\langle T\rangle=-1/2$		$\hat{K}_*\hat{K}_\mathrm{T}$	$\Delta'(54,2,1)\cong[162,44]$	$[324,125]$
$\lvert\langle T\rangle\rvert=1$		$\hat{K}_\mathrm{S}\hat{K}_*$	$\Delta'(54,2,1)\cong[162,44]$	$[324,125]$

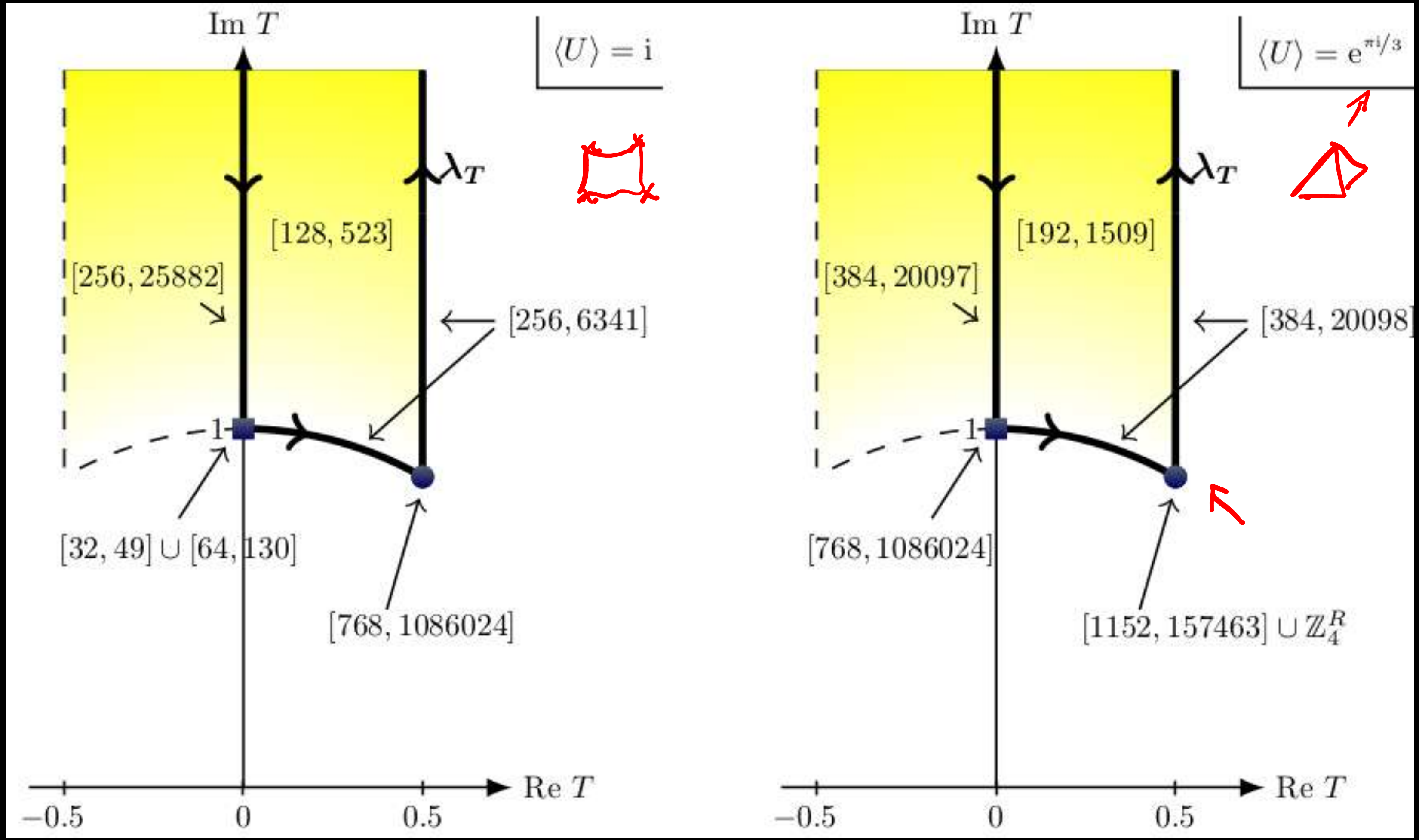
Nilles, Ramos-Sánchez, Vaudrevange (2020)

$K_* = C_S\,\Sigma_*$ CP-like trafo.

* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme: Local Flavor Unification

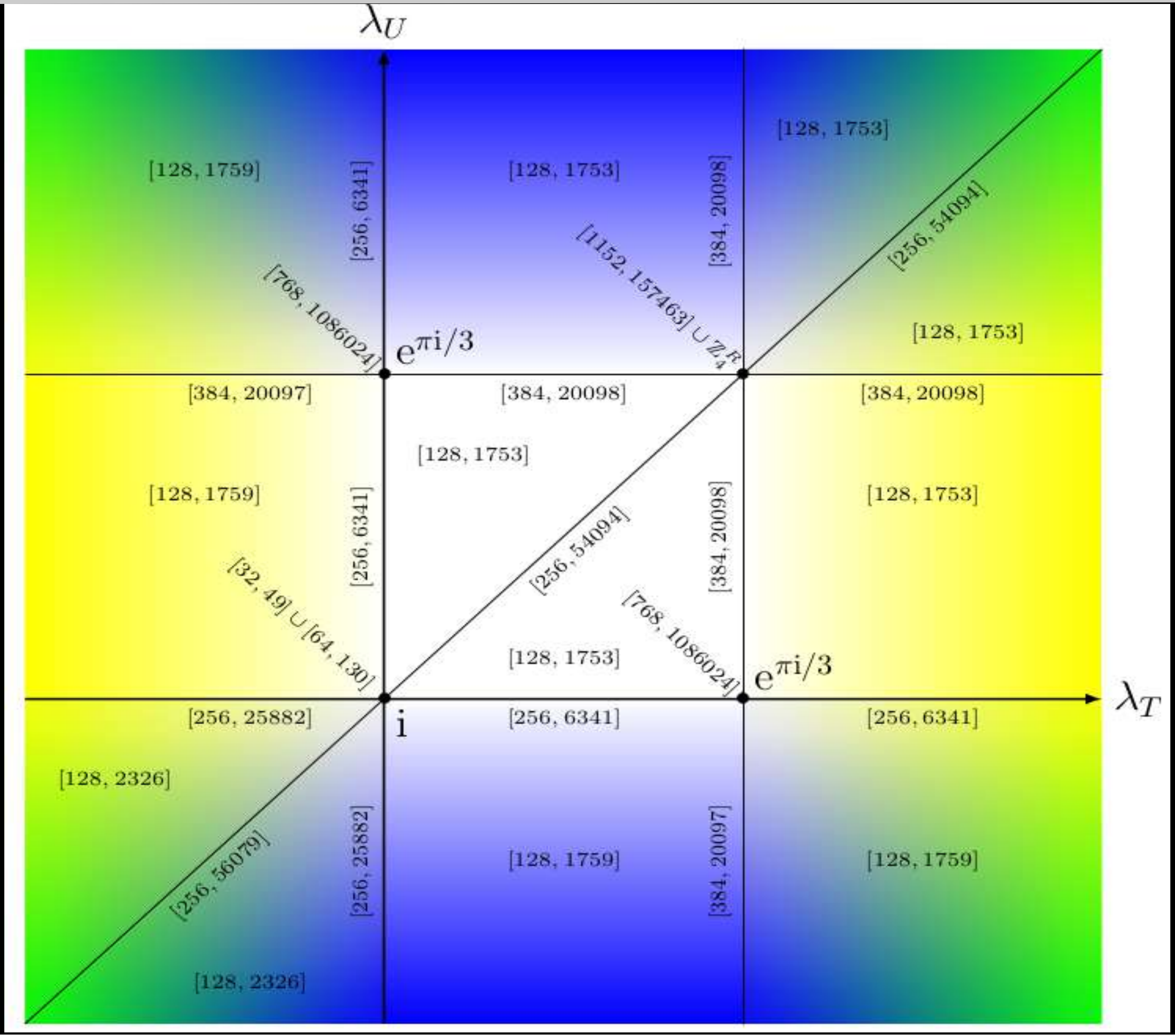
in $\mathbb{T}^2/\mathbb{Z}_2$



* Flavor from toroidal orbifold compactifications

- Eclectic Flavor Scheme: Local Flavor Unification

in $\mathbb{T}^2/\mathbb{Z}_2$



Flavor symmetries from magnetized tori

Cremades, Ibáñez, Marchesano (2004)
Kikuchi, Kobayashi, Takada, Tatsuishi, Uchida (2020)
Ishiguro, Kobayashi, Otsuka (2020)
...
Tatsuta (2021)
Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla (2021)

* Flavor from compactifications on magnetized tori

- Metaplectic modular groups

For half-integral mod. weights, consider $Mp(2,\mathbb{Z}) = \tilde{\Gamma}$ instead of $SL(2,\mathbb{Z})$

$$Mp(2,\mathbb{Z}) = \{[\gamma, \epsilon] \,/\, \gamma \in SL(2,\mathbb{Z}),\ \epsilon = \pm 1\}$$

multiplier

double covering of

generated by $\tilde{S} := [S, -1]$, $\tilde{T} := [T, +1]$

$\tilde{S}^2 \tilde{T} = \tilde{T} \tilde{S}^2$, $\tilde{S}^8 = (\tilde{S}\tilde{T})^3 = [\mathbb{1}, +1]$ analogous to relations of $SL(2,\mathbb{Z})$

Modular forms with $n = 1/2$ of level $4N$

Liu, Yao, Qu, Ding (2020)

$$Y^{(n)}(\tau) \xrightarrow{\tilde{\gamma}} \underbrace{\epsilon\,(c\tau + d)^{1/2}}\ \rho(\tilde{\gamma})\, Y^{(n)}(\tau)$$

modified automorphy factor

unitary representation of $\tilde{\gamma}$ in $\tilde{\Gamma}_{4N}$

* Flavor from compactifications on magnetized tori

  - Metaplectic modular groups

Finite metaplectic groups

for finiteness

$$\tilde{\Gamma}_{4N} := \frac{Mp(2,\mathbb{Z})}{\tilde{\Gamma}(4N)} = \langle \tilde{S}^8 = (\tilde{S}\tilde{T})^3 = \tilde{T}^{4N} = [\mathbb{1},+1],\ \tilde{S}^2\tilde{T} = \tilde{T}\tilde{S}^2,\ \text{extra relations} \rangle$$

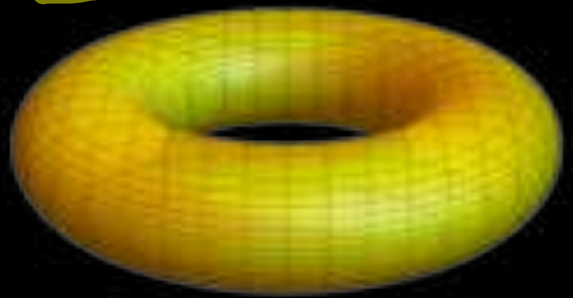
metaplectic congruence subgroups

$\tilde{\Gamma}_4 \cong [96,67]$, $\tilde{\Gamma}_8 \cong [768,1085324]$, $|\tilde{\Gamma}_{12}| = 2304$, ... huge orders!

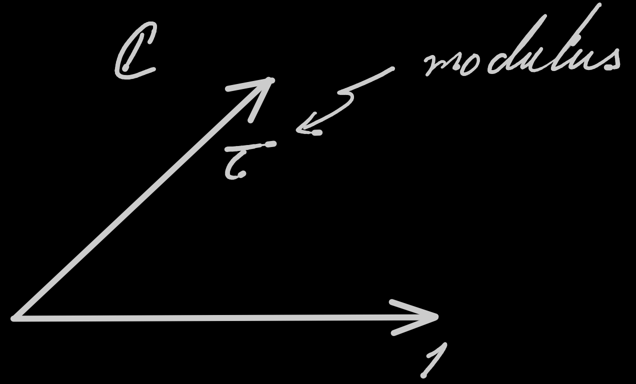
(Double cover of $\Gamma'_4, \Gamma'_8, \Gamma'_{12}, \ldots$)

* Flavor from compactifications on magnetized tori

- Magnetized tori



$\Rightarrow$



$\mathbb{T}^2$ boundary conditions

$z \sim z+1$

$z \sim z+\tau$ ⟵ geometric complex coordinate

Usual transformations $(z,\tau) \xrightarrow{\gamma} \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$

~ Add a $U(1)$ magnetic flux F quantized as $\frac{1}{2\pi}\int_{\mathbb{T}^2} F = M \in \mathbb{Z}$ ⟵ flux quanta

F induced by $A(z) = \frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}(\bar{z}\,dz)$

with

$$\left.\begin{aligned} A(z+1) &= A(z) + d\left(\frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im} z\right) = A(z) + d\chi_1 \\ A(z+\tau) &= A(z) + d\left(\frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}(\bar{\tau} z)\right) = A(z) + d\chi_2 \end{aligned}\right\} U(1)$$

* Flavor from compactifications on magnetized tori

- Magnetized tori

Consider compact components of 6D fields can be associated with spinorial wavefunctions satisfying

- 2D Dirac eq. $i\not{D}\,\psi_{T^2}(z,\tau)=0$ 0-modes ⟵ massless states

- boundary conds. $\psi_{T^2}(z+1,\tau) = e^{i\chi_1}\,\psi_{T^2}(z,\tau)$ (use $q=1$)

$$\psi_{T^2}(z+\tau,\tau) = e^{i\chi_2}\,\psi_{T^2}(z,\tau)$$

with $\psi_{T^2}=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$, ψ_+ takes positive fluxes, $M>0$

Sol. $\psi^{j}_{T^2}(z,\tau) = \mathcal{N} e^{\pi i M z \frac{\operatorname{Im} z}{\operatorname{Im}\tau}}\,\vartheta\begin{bmatrix} j/M \\ 0\end{bmatrix}(Mz, M\tau)$, $\underbrace{0\le j\le M-1}_{M\ \text{0-modes}}$, $\mathcal{N}=(2M\operatorname{Im}\tau)^{1/4}$

Jacobi theta function

normalization

M 0-modes ⟿ M flavors

- It turns out that $\psi^j_{T^2}$ has modular weight $k=\frac{1}{2}$ (computed from Kähler, too)

$$\Rightarrow \quad \Psi_{T^2}(z,\tau) := \begin{pmatrix} \psi^0_{T^2}(z,\tau) \\ \vdots \\ \psi^{M-1}_{T^2}(z,\tau) \end{pmatrix} \xrightarrow{\tilde\gamma} \epsilon\,(c\tau+d)^{1/2}\rho(\tilde\gamma)\,\Psi_{T^2}(z,\tau)$$

(up to irrelevant phase & z-shift)

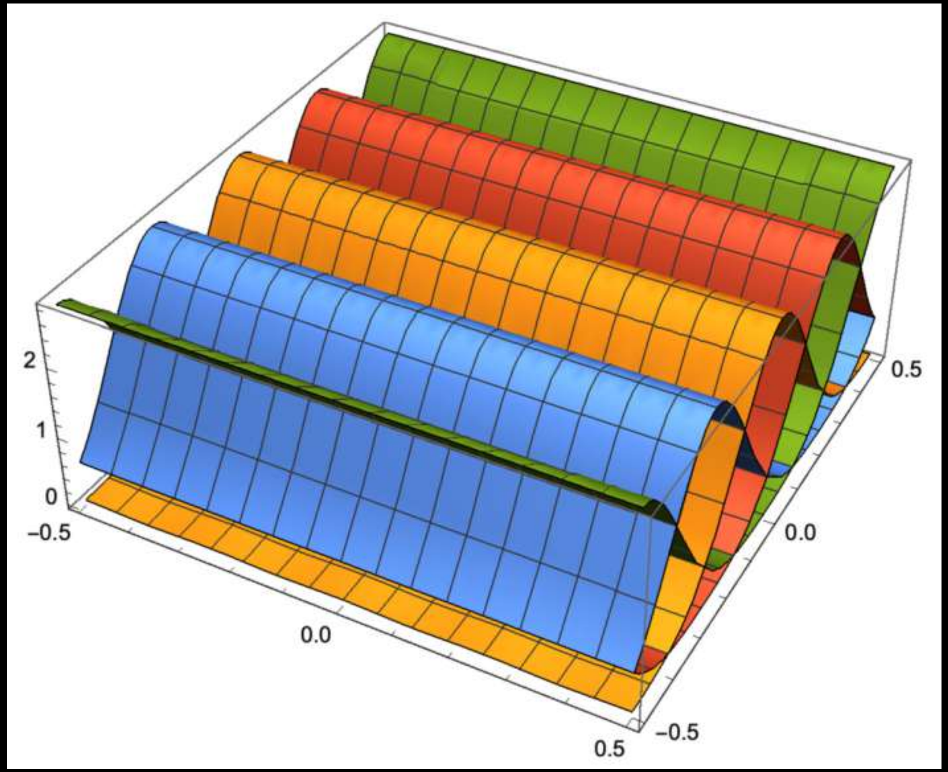
- Possible to compute explicitly the S and T trafos of $\vartheta[\ ](\ ,\ )$

$$\Rightarrow \quad \rho(\tilde S)_{j\ell} = -\frac{e^{\pi i/4}}{\sqrt{M}}\, e^{2\pi i \frac{j\ell}{M}}, \quad \rho(\tilde T)_{j\ell} = e^{\pi i j\left(\frac{j}{M}+1\right)}\,\delta_{j,\ell} \qquad 0 \le j \le M-1, \text{ any } M$$

- The M solutions $\psi^j_{T^2}(z,\tau)$ satisfy their boundary conditions even after a modular transformation

Tatsuta (2021); Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla (2021)

- Couplings among Ψ are given by the overlaps encoded in integrals



$$Y_{jkl}(\tau_i) \sim \int \Psi^j \Psi^k \bar{\Psi}^l \propto \vartheta[\;](\;)$$

Jacobi

- Y_{jkl} build vector spaces and can be accomodated in vectors $Y_{\hat\alpha}$ transforming as

$$Y_{\hat\alpha} \xrightarrow{\tilde\gamma} \pm(c\tau+d)^{1/2}\rho(\tilde\gamma)_{\hat\alpha\hat\beta}\, Y_{\hat\beta}$$

with $\rho(\tilde{S})_{\hat\alpha\hat\beta} = -\dfrac{e^{\pi i/4}}{\sqrt{\lambda}}\, e^{\frac{2\pi i\hat\alpha\hat\beta}{\lambda}}$, $\rho(\tilde{T})_{\hat\alpha\hat\beta} = e^{\frac{\pi i\hat\alpha^2}{\lambda}}\delta_{\hat\alpha\hat\beta}$

λ: lcm(# flavors)

always even in consistent cases

representations of $\tilde\Gamma_{2\lambda}$ finite metaplectic group

Note that $Y_{\hat\alpha}$ have modular weight $1/2$

Finally, one can also write the transf. properties for 4D matter superfields

$$\overset{6D}{\Omega} = \overset{4D}{\Phi} \otimes \overset{2D}{\psi}$$

Demanding that Ω does not transform (except by a numerical phase), we can choose

$$\Phi \xrightarrow{\tilde{\gamma}} \pm (c\tau + d)^{-1/2} \rho(\tilde{\gamma})\, \Phi$$

mod. weight $= -1/2$

with
$$[\rho(\tilde{S})]^{-1}_{j\ell} = -\frac{e^{i\pi(3M+1)/4}}{\sqrt{M}}\, e^{\frac{2\pi i j \ell}{M}}$$

$$[\rho(\tilde{T})]^{-1}_{j\ell} = e^{\pi i j\left(\frac{j}{M}+1\right)}\, \delta_{j,\ell}$$

that also build representations of $\tilde{\Gamma}_{2M}$

Note:
Seems to work without invoking SUSY
→ non-SUSY picture?

Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla (2021)

Where are we and what is still missing?

* Toroidal string compactifications provide a rich scheme of traditional and modular symmetries
  - Common origin for traditional, modular and CP-like symmetries
  - Eclectic picture: nontrivial mixing of traditional and modular flavor symmetries
  - Restrictions on modular weights, representations, Kähler potential --> more predictability!
  - In general, large flavor symmetries
  - Simplectic (multiple moduli) and metaplectic (fractional mod weight) symmetries arise

* To come
  - Phenomenology
  - Complete non-SUSY picture of flavor with modular symmetries
  - Wilson lines,...