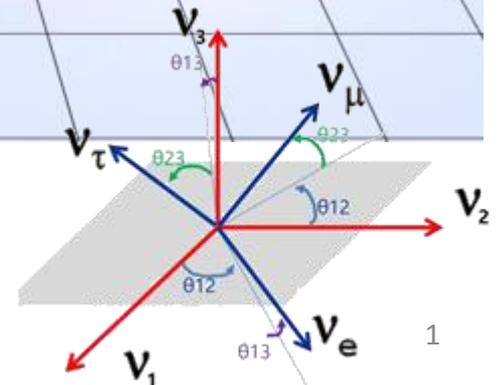# Recent development in modular flavor symmetry
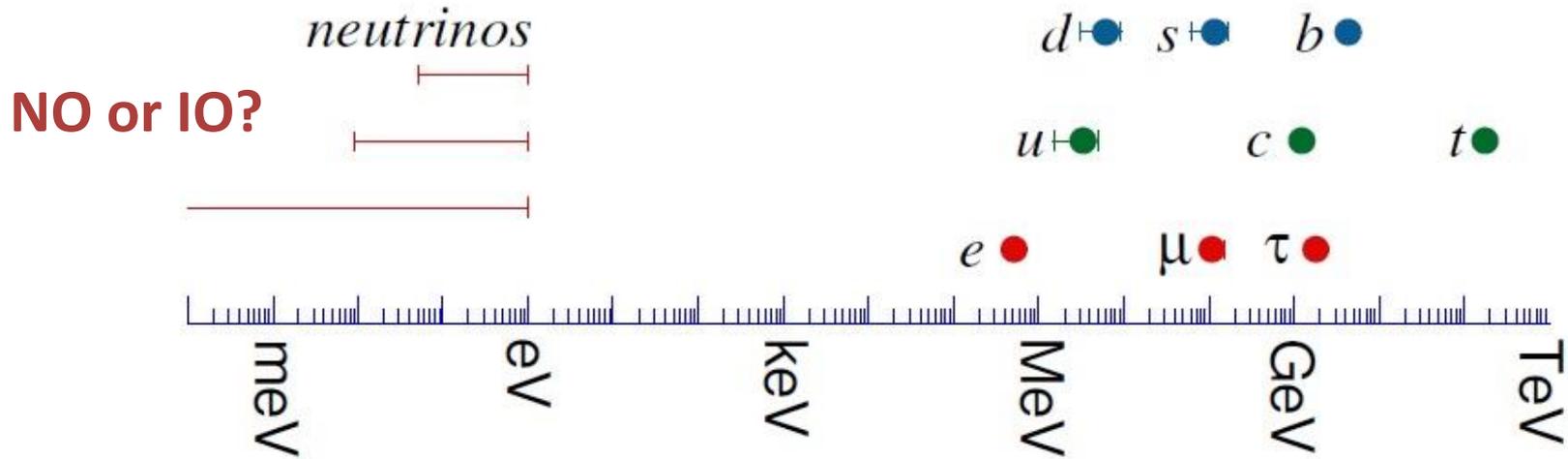
Gui-Jun Ding

University of Science and Technology of China

Virtual mini-workshop on modular flavor symmetries, May 3rd, 2021, Bethe Center for Theoretical Physics

Collaboration with Peng Chen, Ferruccio Feruglio, Stephen F. King , Cai-Chang Li, Xiang-Gan Liu, Jun-Nan Lu, Chang-Yuan Yao, Ye-Ling Zhou

Mysteries of masses and mixings in SM

neutrinos

d s b

NO or IO?

u c t

e μ τ

meV   eV   keV   MeV   GeV   TeV

Quark mixings are small

CKM

$$|V| = \begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{ccc} d & s & b \\ \end{array}$$

Lepton mixings are large

MNS

$$|U| = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$

In SM, the fermion masses and flavor mixing are determined by Yukawa coupling constants which are unconstrained by gauge symmetry.

"Who orderd that ?"

**What is the principle to control flavors of quarks/leptons ?**

Isidor Issac Rabi

Symmetry as a guiding principle to flavor puzzle

The fundamental principle underlying the fermion masses and flavor mixing structure is unkonwn so far. Symmetry can help to reduce the number of free parameters in the Yukawa coupling.

➢ **GUTs:** connecting quarks and leptons,but no explanation of the flavor structure.

➢ **Flavor symmetry**

Flavor symmetry(horizontal)
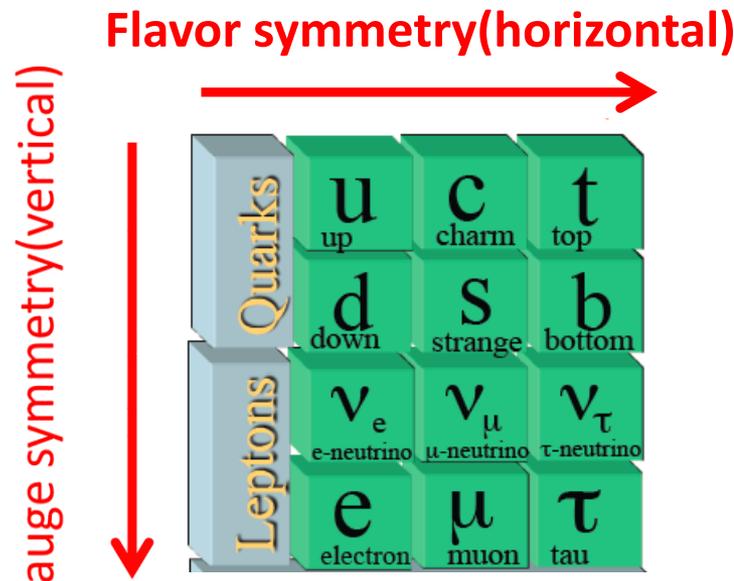
$$\mathcal{L}_m = -Y_{ij}^e(\langle\Phi_e\rangle)\overline{L_i}He_{Rj} - \frac{1}{2}Y_{ij}^\nu(\langle\Phi_\nu\rangle)\overline{L_i^c}HH^T L_j$$

Modular flavor symmetry: flavor symmetry arises from modular group，the modulus τ is the unique flavon $<\Phi_e>=<\Phi_\nu>=\tau$ and Yukawa couplings are modular forms. **[Feruglio, 1706.08749]**

Modular invariant theory

For *N*=1 global SUSY, the modular invariant action [Ferrara et al, 1989; Feruglio, 1706.08749]

$$S = \int d^4x d^2\theta d^2\bar{\theta}\, K(\Phi_I, \bar{\Phi}_I, \tau, \bar{\tau}) + \int d^4x d^2\theta\, W(\Phi_I, \tau) + \text{h.c.}$$

➤ Kahler potential (**not fixed by symmetry**) [Chen,Sanchez,Ratz,1909.06910]

Minimal: $K = -h\Lambda^2 \ln(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\Phi_I|^2$ ⟶ **kinetic terms**

➤ Modular invariant superpotential

$$W = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau) \Phi_{I_1} \Phi_{I_2} \ldots \Phi_{I_n}$$

$Y_{I_1 I_2 \ldots I_n}(\tau)$ are modular forms

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d},$$

$$\Phi_I \to (c\tau + d)^{-k_I} \rho_I(\gamma)\Phi_I$$

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

Modular invariance requires

$$\begin{cases} k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n} \\ \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1 \end{cases}$$

Modular forms

Modular forms are **holomorphic** functions transforming under

$$Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \quad \forall\, \gamma \in \bar{\Gamma}(N)$$

N: level, positive integer

k: modular weight, **even integer**

$$\bar{\Gamma}(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\, \gamma \in \bar{\Gamma},\ \gamma = I(\mathrm{mod}\ N) \right\}$$

Modular forms of weight k and level N form a linear space, they can be decomposed into irreducible representations of finite modular group,

$$Y_i(\gamma\tau) = (c\tau + d)^k \rho_{ij}(\gamma) Y_j(\tau), \quad \gamma \in \bar{\Gamma}$$

[Feruglio, 1706.08749]

ρ is unitary representation of $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N) = \left\{ S, T \mid S^2 = (ST)^3 = T^N = 1 \right\}$

➢ **Inhomogeneous** finite modular group **Γ$_N$**

N	$d_{2k}(\Gamma(N))$	$|\Gamma_N|$	Γ_N
2	$k+1$	6	S_3
3	$2k+1$	12	A_4
4	$4k+1$	24	S_4
5	$10k+1$	60	A_5
6	$12k$	72	$S_3 \times A_4$
7	$28k-2$	168	$\Sigma(168)$

[Kobayashi et al, arXiv:1803.10391]

[Feruglio, 1706.08749]

[Penedo, Petcov, arXiv:1806.11040]

[Novichkov,Penedo,Petcov,Titov,arXiv:1812.02158; Ding,King,Liu,arXiv:1903.12588]

[Ding,King, Li,Zhou,arXiv:2004.12662]

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Extending modular forms: even weight →integral weight

➢ The weights of SL(2,Z) modular forms are **even**

$$Y\left(S^2\tau\right) = (-1)^k Y(\tau) = Y(\tau)$$

$$(-1)^k = 1$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$R = S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

k should be **even** otherwise $Y(\tau)=0$

➢ Integral weight modular forms of level N

$$Y\left(\gamma\tau\right) = (c\tau + d)^k Y(\tau), \quad \forall\, \gamma \in \Gamma(N)$$

$$Y\left(\gamma\tau\right) = (c\tau + d)^k \rho_r(\gamma) Y(\tau), \quad \gamma \in SL(2,Z) \equiv \Gamma$$

[Liu,Ding,1907.01488]

ρ is unitary representation of the **homogeneous** finite modular group:

$$\Gamma'_N \equiv \Gamma / \Gamma(N)$$

$$\Gamma'_N \cong \left\{ S, T \mid S^2 = R, \ (ST)^3 = R^2 = T^N = 1, \ TR = RT \right\}$$

Γ'_N is the double cover of $\Gamma_N \cong \Gamma'_N / \{1, R\}$

● Constraint on modular form

Choosing $\gamma = S^2 = -1_2$

$$Y(S^2\tau) = Y(\tau) = (-1)^k \rho_r(S^2) Y(\tau)$$

$$(-1)^k \rho_r(S^2) = 1$$

- The integral weight modular forms are homogeneous polynomials of the lowest weight one modular forms which can be constructed from the Dedekind eta function and Klein forms.

N=3:
$$\mathcal{M}_k(\Gamma(3)) = \bigoplus_{a+b=k,\,a,b\geq 0} \mathbb{C}\frac{\eta^{3a}(3\tau)\eta^{3b}(\tau/3)}{\eta^k(\tau)}$$

N=4:
$$\mathcal{M}_k(\Gamma(4)) = \bigoplus_{a+b=2k,\,a,b\geq 0} \mathbb{C}\frac{\eta^{2b-2a}(4\tau)\eta^{5a-b}(2\tau)}{\eta^{2a}(\tau)}$$

N=5:
$$\mathcal{M}_k(\Gamma(5)) = \bigoplus_{a+b=5k,\,a,b\geq 0} \mathbb{C}\frac{\eta^{15k}(5\tau)}{\eta^{3k}(\tau)}\mathfrak{k}^a_{\frac{1}{5},\frac{0}{5}}(5\tau)\mathfrak{k}^b_{\frac{2}{5},\frac{0}{5}}(5\tau)$$

N	$\dim\mathcal{M}_k(\Gamma(N))$	$\Gamma'_N \equiv \Gamma/\Gamma(N)$	$\lvert\Gamma'_N\rvert$	Irreps of lowest weight
2	$k/2+1\,(k\ \text{even})$	S_3	6	—
3	$k+1$	T'	24	**2**
4	$2k+1$	S'_4	48	$\hat{\mathbf{3}}'$
5	$5k+1$	A'_5	120	**6**

[Liu,Ding,1907.01488]

[Novichkov, Penedo, Petcov, 2006.03058; Liu,Yao, Ding,2006.10722]

[Novichkov, Penedo, Petcov,2006.03058; Yao, Liu, Ding, 2011.03501]

modular forms: integral weight →half integral (fractional) weight

Defining half integral weight modular forms:

$$Y(\gamma\tau) = (c\tau + d)^k Y(\tau) \implies Y(\gamma\tau) = (c\tau + d)^{k/2} Y(\tau)$$

➢ $J_{k/2}(\gamma,\tau) = (c\tau+d)^{k/2}$ is **not** the automorphy factor, the cocycle condition is not fulfilled

$$J_{k/2}(\gamma_1\gamma_2, \tau) = \zeta_{k/2}^{-1}(\gamma_1, \gamma_2) J_{k/2}(\gamma_1, \gamma_2\tau) J_{k/2}(\gamma_2, \tau)$$

two-cocycle: $\quad \zeta_{k/2}(\gamma_1, \gamma_2) \in \{1, e^{\pi i k}\}$

It is necessary to extend the SL(2,Z) to the metaplectic cover group Mp(2,Z) by including two branches of the complex square root. [G. Shimura,Annals of Mathematics 1973]

$$\widetilde{\Gamma} = \left\{ \widetilde{\gamma} = (\gamma, \phi(\gamma,\tau)) \,\Big|\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ \phi(\gamma,\tau)^2 = (c\tau+d) \right\}, \quad \phi(\gamma,\tau) = \pm(c\tau + d)^{1/2}$$

✓ multiplication law

$$(\gamma_1, \phi(\gamma_1,\tau))(\gamma_2, \phi(\gamma_2,\tau)) = (\gamma_1\gamma_2, \phi(\gamma_1, \gamma_2\tau)\phi(\gamma_2,\tau))$$

✓ generators

$$\widetilde{S} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, -\sqrt{-\tau} \right), \quad \widetilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \implies \boxed{\widetilde{S}^8 = (\widetilde{S}\widetilde{T})^3 = 1}$$

✓ The principal congruence subgroup Γ(4N)

$$\Gamma(4N) \cong \widetilde{\Gamma}(4N) = \left\{ \widetilde{h} = (h, v(h)J_{1/2}(h,\tau)) \mid h \in \Gamma(4N) \right\},$$

$$v(h) = \left(\frac{c}{d}\right) = \pm 1 \mapsto \text{Kronecker symbol}.$$

- ➢ finite metaplectic group $\tilde{\Gamma}_{4N} \equiv \tilde{\Gamma}/\tilde{\Gamma}(4N)$

$$\tilde{\Gamma}_{4N} \cong \left\{ \tilde{S}, \tilde{T} \mid \tilde{S}^8 = (\tilde{S}\tilde{T})^3 = 1, \quad \tilde{T}^{4N} = 1 \right\}$$

$$\tilde{T}^{4N} = \left(\begin{pmatrix} 1 & 4N \\ 0 & 1 \end{pmatrix}, 1 \right) \in \tilde{\Gamma}(4N)$$

Additional conditions are required for N≥2

- ➢ Modular form of half-integral weight **k/2** : **a multiplier $v(h)$ is needed!!!**

$$Y(h\tau) = v^k(h)(c\tau + d)^{k/2} Y(\tau), \quad \forall\, \tilde{h} \in \tilde{\Gamma}(4N)$$

$$Y(\gamma\tau) = \phi^k(\gamma,\tau)\rho_r(\tilde{\gamma})Y(\tau), \quad \forall\, \tilde{\gamma} \in \tilde{\Gamma}$$

[Liu, Yao, Qu, Ding, 2007.13706]

Half integral weight modular forms can be arranged into irreps ρ_r of $\tilde{\Gamma}_{4N}$

Matter fields: $\psi \rightarrow \phi^{-k_\psi}(\gamma,\tau)\rho_r(\tilde{\gamma})\psi$ The modular weight k_ψ can not be arbitrary.

- ➢ **fractional** weight modular forms

N	weight r	$\dim \mathcal{M}_r(\Gamma(N))$	$\mathcal{M}_r(\Gamma(N))\|_{k=1}$	$\tilde{\Gamma}_N$	$\|\tilde{\Gamma}_N\|$	GAP ID
4	$k/2$	$k+1$	$\left\{ \theta_3(0\|2\tau),\ \theta_2(0\|2\tau) \right\}$	$\tilde{S}_4$	96	[96,67]
5	$k/5$	$k+1$	$\left\{ f_1^{(5)}(\tau),\ f_3^{(5)}(\tau) \right\}$	$Z_5 \times \Gamma_5'$	600	[600,54]
7	$2k/7$	$\begin{cases} 4k-2 \ (\text{for } k \geq 2) \\ 3 \ (\text{for } k=1) \end{cases}$	$\left\{ f_1^{(7)}(\tau),\ f_3^{(7)}(\tau),\ f_5^{(7)}(\tau) \right\}$	$Z_7 \times \Gamma_7$	1176	[1176,212]
9	$k/3$	$\begin{cases} 9k-9 \ (\text{for } k \geq 3) \\ 10 \ (\text{for } k=2) \\ 4 \ (\text{for } k=1) \end{cases}$	$\left\{ f_1^{(9)}(\tau),\ f_3^{(9)}(\tau),\ f_5^{(9)}(\tau),\ f_7^{(9)}(\tau) \right\}$	$\tilde{\Gamma}_9$	1944	[1944,2976]

complex moduli: single->multiple

➢ Superstring theory requires 10D spacetime, our universe is 4D and thus the extra **6D** sapce is compact.

Compactification

10D Minkowski ➜ **4D Minkowski** **X** **6d Calabi-Yau**
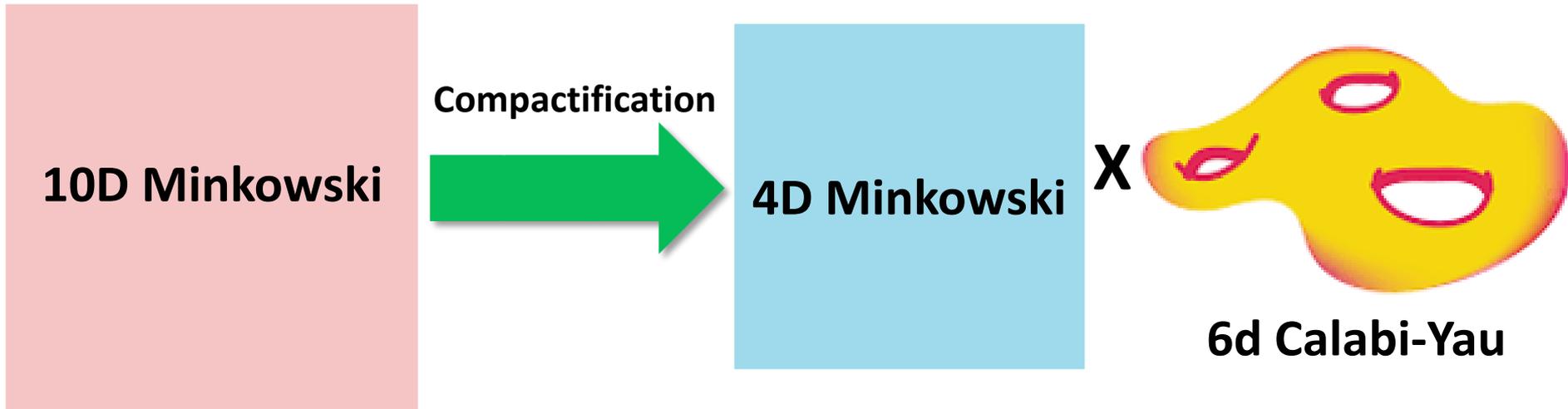
➢ The 4D effective Lagrangian by integrating over 6D depends on the structure of compact space which is parametrized by several moduli

$$S = \int d^4x d^6y \; \mathcal{L}_{10D} \Rightarrow \int d^4x \; \mathcal{L}_{\text{eff}}(\varphi, \tau_i)$$

[factorizable moduli: Varzielas, King and Zhou,1906.02208]

τ

τ_1 τ_2 τ_3
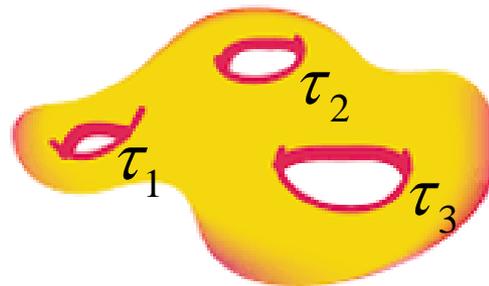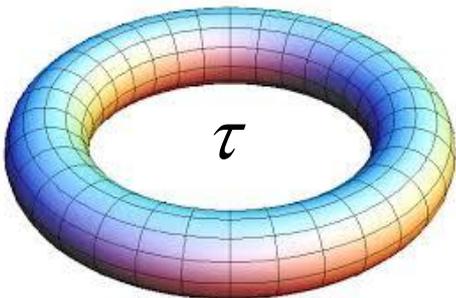
Symplectic (Siegel) modular group and modular form

Modular group $SL(2,\mathbb{Z}) \equiv \Gamma$

Siegel modular group $Sp(2g,\mathbb{Z}) \equiv \Gamma_g$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z},$$

$$ad - bc = 1$$

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A,B,C,D \in GL(g,\mathbb{Z})$$

$$\gamma^t J \gamma = J, \quad J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

$$H_1 = \left\{ \tau \in \mathbb{C} \mid \mathbf{Im}\,\tau > \mathbf{0} \right\}$$

$$H_g = \left\{ \tau \in GL(g,\mathbb{C}) \mid \tau^t = \tau, \mathbf{Im}\,\tau > \mathbf{0} \right\}$$

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

$$\tau \to \gamma\tau = (A\tau + B)(C\tau + D)^{-1}$$

$$\Gamma(N) = \left\{ \gamma \in SL(2,\mathbb{Z}) \mid \gamma = 1_2 \ (\mathrm{mod}\ N) \right\}$$

$$\Gamma_g(N) = \left\{ \gamma \in Sp(2g,\mathbb{Z}) \mid \gamma = 1_{2g} \ (\mathrm{mod}\ N) \right\}$$

$$\Gamma'_N = SL(2,\mathbb{Z})/\Gamma(N)$$

$$\Gamma_{g,N} = Sp(2g,\mathbb{Z})/\Gamma_g(N)$$

$$Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \quad \gamma \in \Gamma(N)$$

$$Y(\gamma\tau) = [\det(C\tau + D)]^k Y(\tau), \quad \gamma \in \Gamma_g(N)$$

More general: modular form→automorphic forms

[Ding, Feruglio, Liu, 2010.07952]

Invariant loci in moduli space

For the lowest nontrivial $g=2$ and level $n=2$, the finite Siegel modular group $\Gamma_{2,2}$ is isomorphic to S_6: two 1-dim, four 5-dim, two 9-dim, two 10-dim and one 16-dim irreducible representations. **No 3-dim representation to accommodate the 3 generation of fermions.**

➢ We go to subspace of Siegel upper half plane to obtain smaller finite modular group with 3-dim rep. All inequivalent modular spaces were calssified by **Gottschling,1961**

✓ **two-dimensional**

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad N_2(H) \cong (S_3 \times S_3) \rtimes Z_2$$

$$\begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix}, \qquad N_2(H) = S_4 \times Z_2$$

[This 2-dim subspace can be obtained in string theory: **Baur, Kade, Nilles, Sanchez, Vaudrevange, 2012.09586**]

[For model building, see: **Ding, Feruglio, Liu, 2010.07952**]

✓ **one-dimensional**

$$\begin{pmatrix} i & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix},$$

$$\begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_1 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}.$$

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Extending modular symmetry with gCP

➤ A unique CP transformation consistent with modular symmetry

[Novichkov, Petcov et al, 1905.11970; Baur, Nilles et al,1901.03251]

$$\tau \rightarrow -\tau^*$$

➤ CP transformations correspond to the automorphism $u(\gamma)$ of modular group. The outer automorphism of Γ can be generated by

$$u(\gamma) = \chi(\gamma)U\gamma U^{-1} = \chi(\gamma)\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

[I. Reiner,1955]

$\chi(\gamma)$ is called character and it a homomorphism of Γ into $\{\pm 1\}$

$$\text{1st:} \quad \chi_1(S) = \chi_1(T) = 1$$

$$\text{2nd:} \quad \chi_2(S) = \chi_2(T) = -1$$

➤ CP action on matter superfield

$$\varphi(x) \xrightarrow{\mathcal{CP}} X_{\mathbf{r}}\overline{\varphi}(x_{\mathcal{P}})$$

X_r is a unitary matrix in flavor space

Applying the consistency condition chain $\mathcal{CP} \rightarrow \gamma \rightarrow \mathcal{CP}^{-1}$

$$X_r \rho_r^*(\gamma) X_r^{-1} = \chi(\gamma)^{-k_\varphi} \rho_r(u(\gamma))$$

✓ The 1st gCP for the trivial character $\chi_1(\gamma)=1$

$$X_r \rho_r^*(S) X_r^{-1} = \rho_r(S^{-1}), \quad X_r \rho_r^*(T) X_r^{-1} = \rho_r(T^{-1})$$

In **symmetric** basis: the representation matrices of S and T are symmetric

$$X_\mathbf{r} = 1_\mathbf{r} \quad \Longrightarrow \quad g_i^* = g_i \quad \textbf{real couplings}$$

✓ The 2nd gCP for the character $\chi_2(S)=\chi_2(T)=-1$

$$X_r \rho_r^*(S) X_r^{-1} = \sigma \, \rho_r(S^{-1}), \quad X_r \rho_r^*(T) X_r^{-1} = \sigma \, \rho_r(T^{-1}), \quad \sigma \equiv (-1)^{-k_\varphi} \rho_r(S^2)$$

The **2nd** gCP depending on the weight k_φ and representation ρ_r assignment. It reduces to 1st gCP for σ=1, and the 2nd CP transformation can possibly be defined for **σ=-1** requiring

[Novichkov,Penedo,Petcov, 2006.03058]

① the level N is even
② the dimension of representation ρ_r is even
③ tr($\rho_r(T)$)=0 and additionally tr($\rho_r(S)$)=0 for g=1

Phenomenology: to implement the 2nd gCP, both left and right handed matter fields should be assigned to 1⊕2 under finite modular group, the resulting fermion mass matrices to be block-diagonal and some mixing angles are vanishing.

➢ CP conserved values of modulus: $-\tau^*=\gamma\tau$
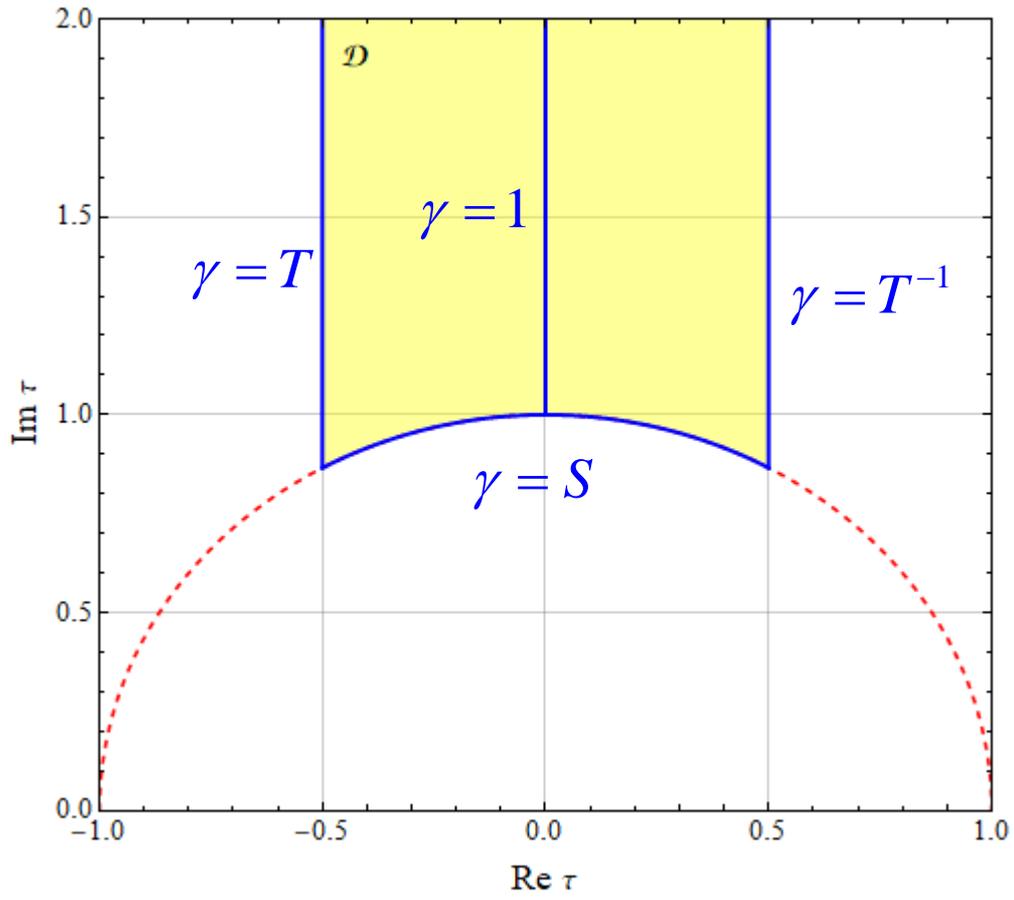
If τ is imaginary or at the boundary of the fundamental region, certain residual CP symmetry would be preserved. After including gCP, τ is the only source of CP violation.

Implications of residual CP symmetry

$$\mathcal{W} = -E_i^c \mathcal{Y}_{ij}^e(\tau) L_j H_d - \frac{1}{2\Lambda} L_i \mathcal{Y}_{ij}^\nu(\tau) L_j H_u H_u$$

Modular invariance requires the Yukawa couplings $\mathcal{Y}^e(\tau)$ and $\mathcal{Y}^\nu(\tau)$ transform as

$$\mathcal{Y}^e(\gamma\tau) = \mathcal{D}^{k_{H_d}} \rho_{H_d}^\dagger(\gamma) \, \mathcal{D}^{k_{E^c}} \rho_{E^c}^*(\gamma) \, \mathcal{Y}^e(\tau) \, \rho_L^\dagger(\gamma) \mathcal{D}^{k_L}$$

$$\mathcal{Y}^\nu(\gamma\tau) = \mathcal{D}^{2k_{H_u}} \rho_{H_u}^{2\dagger}(\gamma) \, \mathcal{D}^{k_L} \rho_L^*(\gamma) \, \mathcal{Y}^\nu(\tau) \, \rho_L^\dagger(\gamma) \mathcal{D}^{k_L}$$

$$\boxed{\mathcal{D} = c\tau + d}$$

CP invariance implies

$$\mathcal{Y}^e(-\tau^*) = X_{H_d}^\dagger \, X_{E^c}^* \, \mathcal{Y}^{e*}(\tau) \, X_L^\dagger, \quad \mathcal{Y}^\nu(-\tau^*) = X_{H_u}^{2\dagger} \, X_L^* \, \mathcal{Y}^{\nu*}(\tau) \, X_L^\dagger.$$

At the CP conserving point $-\tau^* = \gamma\tau$, the charged lepton and neutrino mass matrices are invariant under a **common** CP transfromation

[Novichkov, Petcov et al, 1905.11970; Ding, Feruglio, Liu, 2102.06716]

$$\Omega_L^\dagger M_e^\dagger(\tau) M_e(\tau) \Omega_L = \left[M_e^\dagger(\tau) M_e(\tau) \right]^*,$$

$$\Omega_L^\dagger M_\nu^\dagger(\tau) M_\nu(\tau) \Omega_L = \left[M_\nu^\dagger(\tau) M_\nu(\tau) \right]^*.$$

$$\Omega_L = \rho_L^\dagger(\gamma) \mathcal{D}^{k_L} X_L$$

The values of moduli τ should departure from the CP conserving point to generate non-trivial CP phases as well as non-vanishing CP violation in leptogenesis.

Modular symmetry merges abelian and non-abelian flavor symmetry

➤ Modular symmetry origin of texture zero

Odd weight modular forms are in the ``**spinor" irreps** $\rho_r(S^2)=-1$, even weight modular forms are in the ``**vector" irreps** $\rho_r(S^2)=+1$

For example: level N=3

Assignment: $\quad Q_D \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \sim 2 \text{ (or } 2', 2''), \quad Q_3 \sim 1 \text{ (or } 1', 1'')$ **Left-handed quarks**

Mass matrix: $\quad q_D^c \equiv \begin{pmatrix} q_1^c \\ q_2^c \end{pmatrix} \sim 2 \text{ (or } 2', 2''), \quad q_3^c \sim 1 \text{ (or } 1', 1'')$ **Right-handed quarks**

vanishing for $k_{Q_D} + k_{q_D^c}$ odd

vanishing for $k_{Q_D} + k_{q_3^c}$ even

$$M_q = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

[Lu, Liu, Ding, 1912.07573]

vanishing for $k_{Q_3} + k_{q_D^c}$ even

vanishing for $k_{Q_3} + k_{q_3^c}$ odd

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Five texture zeros of quark mass matrices can be achieved from the Γ′$_3$=T′ modular symmetry

$$\text{Case } \mathcal{A}: \quad M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \text{Case } \mathcal{B}: \quad M_q = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}$$

$$\text{Case } \mathcal{C}: \quad M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix}, \quad \text{Case } \mathcal{D}: \quad M_q = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix}$$

$$\text{Case } \mathcal{E}: \quad M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}.$$

[Lu, Liu, Ding, 1912.07573]

New features:

✓ texture zeros are exact and they are enforced by the structure of modular forms

✓ Non-vanishing entries are correlated with each other

➤ weighton mechanism--- modular weights play the role the FN charge **[King, King, 2002.00969]**

✓ The benchmark A_4 modular model:

$L \sim (3, 1), e^c \sim (1, 1), \mu^c \sim (1'', 1), \tau^c \sim (1', 1), N^c \sim (3, 1), H_{u,d} \sim (1,0)$ **[Feruglio, 1706.08749]**

$$\mathcal{W}_e = \alpha e^c (LY_3^{(2)})_\mathbf{1} H_d + \beta \mu^c (LY_3^{(2)})_{\mathbf{1}'} H_d + \gamma \tau^c (LY_3^{(2)})_{\mathbf{1}''} H_d$$

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix} \overset{\tau \to i\infty}{\longmapsto} \mathcal{Y}_e \propto \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \Rightarrow \alpha \ll \beta \ll \gamma$$

✓ Improved model including weighton which is SM and A4 singlet with non-zero modular weight

$L \sim (3, 1), e^c \sim (1, -3), \mu^c \sim (1'', -1), \tau^c \sim (1', 0), N^c \sim (3, 1), H_{u,d} \sim (1,0), \phi \sim (1,1)$

$$\mathcal{W}_e = \alpha e^c \tilde{\phi}^4 (LY_3^{(2)})_\mathbf{1} H_d + \beta \mu^c \tilde{\phi}^2 (LY_3^{(2)})_{\mathbf{1}'} H_d + \gamma \tau^c \tilde{\phi} (LY_3^{(2)})_{\mathbf{1}''} H_d, \quad \tilde{\phi} \equiv \frac{\phi}{\Lambda}$$

$$\mathcal{Y}_e = \begin{pmatrix} \alpha \tilde{\phi}^4 Y_1 & \alpha \tilde{\phi}^4 Y_3 & \alpha \tilde{\phi}^4 Y_2 \\ \beta \tilde{\phi}^2 Y_2 & \beta \tilde{\phi}^2 Y_1 & \beta \tilde{\phi}^2 Y_3 \\ \gamma \tilde{\phi} Y_3 & \gamma \tilde{\phi} Y_2 & \gamma \tilde{\phi} Y_1 \end{pmatrix} \overset{\tau \to i\infty}{\longmapsto} \mathcal{Y}_e \propto \begin{pmatrix} \alpha \tilde{\phi}^4 & 0 & 0 \\ 0 & \beta \tilde{\phi}^2 & 0 \\ 0 & 0 & \gamma \tilde{\phi} \end{pmatrix} \Rightarrow \alpha \sim \beta \sim \gamma$$

Another possibility: hierarchical fermion mass matrices could arise from the proximity of the modulus to certain fixed point.

[Okada,Tanimoto, 2009.14242; Feruglio, Gherardi, Romanino,Titov, 2101.08718 ; Novichkov,Penedo,Petcov,2102.07488]

Quark-lepton unification based on double cover of S_4

	L	(e^c, μ^c, τ^c)	N^c	Q	(u^c, c^c, t^c)	(d^c, s^c, b^c)	$H_{u,d}$
$\Gamma_4' \equiv S_4'$	3	$(1, 1, \hat{1}')$	3	3	$(1, 1, \hat{1}')$	$(1', \hat{1}, \hat{1}')$	1
k_I	2	$(2, 0, 1)$	0	k_Q	$(4 - k_Q, 6 - k_Q, 3 - k_Q)$	$(4 - k_Q, 5 - k_Q, 5 - k_Q)$	0

We include the gCP symmetry such that all coupling constants are real

Lepton sector: **[Liu, Yao, Ding, 2006.10722]**

$$\mathcal{W}_e = \alpha_e (E_1^c L Y_3^{(4)})_1 H_d + \beta_e (E_2^c L Y_3^{(2)})_1 H_d + \gamma_e (E_3^c L Y_{\hat{3}}^{(3)})_1 H_d \,,$$

$$\mathcal{W}_\nu = g_1 (N^c L Y_2^{(2)})_1 H_u + g_2 (N^c L Y_3^{(2)})_1 H_u + \Lambda (N^c N^c)_1 \,,$$

$$M_e = \begin{pmatrix} \alpha_e Y_4^{(4)} & \alpha_e Y_6^{(4)} & \alpha_e Y_5^{(4)} \\ \beta_e Y_3^{(2)} & \beta_e Y_5^{(2)} & \beta_e Y_4^{(2)} \\ \gamma_e Y_2^{(3)} & \gamma_e Y_4^{(3)} & \gamma_e Y_3^{(3)} \end{pmatrix} v_d \,, \qquad M_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda \,,$$

$$M_D = \begin{pmatrix} 0 & g_1 Y_1^{(2)} - g_2 Y_5^{(2)} & g_1 Y_2^{(2)} + g_2 Y_4^{(2)} \\ g_1 Y_1^{(2)} + g_2 Y_5^{(2)} & g_1 Y_2^{(2)} & - g_2 Y_3^{(2)} \\ g_1 Y_2^{(2)} - g_2 Y_4^{(2)} & g_2 Y_3^{(2)} & g_1 Y_1^{(2)} \end{pmatrix} v_u \,.$$

Charged lepton masses: $\alpha, \ \beta, \ \gamma$

Light neutrino mass matrix : $\dfrac{g_1^2 v_u^2}{\Lambda}, \ \dfrac{g_2}{g_1}, \ \tau$

Quark sector:

$$W_u = \alpha_u(u^c Q Y_{\mathbf{3}}^{(4)})_{\mathbf{1}} H_u + \beta_u(c^c Q Y_{\mathbf{3},I}^{(6)})_{\mathbf{1}} H_u + \gamma_u(c^c Q Y_{\mathbf{3},II}^{(6)})_{\mathbf{1}} H_u + \delta_u(t^c Q Y_{\hat{\mathbf{3}}}^{(3)})_{\mathbf{1}} H_u$$

$$W_d = \alpha_d(d^c Q Y_{\mathbf{3}'}^{(4)})_{\mathbf{1}} H_d + \beta_d(s^c Q Y_{\hat{\mathbf{3}}',I}^{(5)})_{\mathbf{1}} H_d + \gamma_d(s^c Q Y_{\hat{\mathbf{3}}',II}^{(5)})_{\mathbf{1}} H_d + \delta_d(b^c Q Y_{\hat{\mathbf{3}}}^{(5)})_{\mathbf{1}} H_d$$

$$M_u = \begin{pmatrix} \alpha_u Y_4^{(4)} & \alpha_u Y_6^{(4)} & \alpha_u Y_5^{(4)} \\ \beta_u Y_5^{(6)} + \gamma_u Y_8^{(6)} & \beta_u Y_7^{(6)} + \gamma_u Y_{10}^{(6)} & \beta_u Y_6^{(6)} + \gamma_u Y_9^{(6)} \\ \delta_u Y_2^{(3)} & \delta_u Y_4^{(3)} & \delta_u Y_3^{(3)} \end{pmatrix} v_u$$

➡️

$$M_d = \begin{pmatrix} \alpha_d Y_7^{(4)} & \alpha_d Y_9^{(4)} & \alpha_d Y_8^{(4)} \\ \beta_d Y_6^{(5)} + \gamma_d Y_9^{(5)} & \beta_d Y_8^{(5)} + \gamma_d Y_{11}^{(5)} & \beta_d Y_7^{(5)} + \gamma_d Y_{10}^{(5)} \\ \delta_d Y_3^{(5)} & \delta_d Y_5^{(5)} & \delta_d Y_4^{(5)} \end{pmatrix} v_d$$

8 real coupling constants: $\alpha_{u,d}, \ \beta_{u,d}, \ \gamma_{u,d}, \ \delta_{u,d}$

The complex modulus τ is common in both quark and lepton sectors, and it is the unique source breaking modular symmetry and CP

$$\langle \tau \rangle = -0.2123 + 1.5201i$$

Best fit values of input parameters:

$$\beta_u/\alpha_u = 325.6502, \quad \gamma_u/\alpha_u = 2427.3101, \quad \delta_u/\alpha_u = 219.3019,$$

$$\alpha_u v_u = 2.7758 \times 10^{-5} \text{ GeV}, \quad \beta_d/\alpha_d = 466.6990, \quad \gamma_d/\alpha_d = -234.0473,$$

$$\delta_d/\alpha_d = 2.3388, \quad \alpha_d v_d = 1.72111 \times 10^{-5} \text{ GeV}, \quad \beta_e/\alpha_e = 0.0187,$$

$$\gamma_e/\alpha_e = 0.1466, \quad g_2/g_1 = 0.6834, \quad \alpha_e v_d = 16.8880 \text{MeV}, \quad g_1^2 v_u^2/\Lambda = 0.3043 \text{ meV}.$$

Predictions: almost all observables are within the 1σ regions

$$\theta^q_{12} = 0.22752 \,, \quad \theta^q_{13} = 0.003379 \,, \quad \theta^q_{23} = 0.038886 \,, \quad \delta^q_{CP} = 75.9958° \,,$$

$$m_u/m_c = 0.001929 \,, \quad m_c/m_t = 0.002725 \,, \quad m_d/m_s = 0.050345 \,, \quad m_s/m_b = 0.017726 \,,$$

$$\sin^2 \theta^l_{12} = 0.34981 \,, \quad \sin^2 \theta^l_{13} = 0.02193 \,, \quad \sin^2 \theta^l_{23} = 0.56393 \,,$$

$$\delta^l_{CP} = 266.1824° \,, \quad \alpha_{21} = 1.1482\pi \,, \quad \alpha_{31} = 0.1522\pi \,,$$

$$m_1 = 3.5269 \text{ meV} \,, \quad m_2 = 9.2919 \text{ meV} \,, \quad m_3 = 50.2404 \text{ meV} \,,$$

$$\sum_i m_i = 63.0592 \text{ meV} \,, \quad m_{\beta\beta} = 2.5480 \text{ meV} \,.$$

✓ The model uses **15** parameters including to describe the masses and mixing of both quark and lepton sectors: **12** masses+**6** mixing angles+**3** CP phases.

✓ The predictions for neutrino masses, mixing angles and CP violation phases are compatible with the experimental data of neutrino oscillation and cosmology. Precise measurements of θ_{23} , δ_{CP} and the effective mass $m_{\beta\beta}$ in 0ν2β decay can exclude this model.

✓ More free parameters are involved in quark-lepton unification models with A_4. **[Okada, Tanimoto, 1905.13421,2005.00775, 2012.01688; Yao,Lu,Ding,2012.13390]**.

Modular GUT
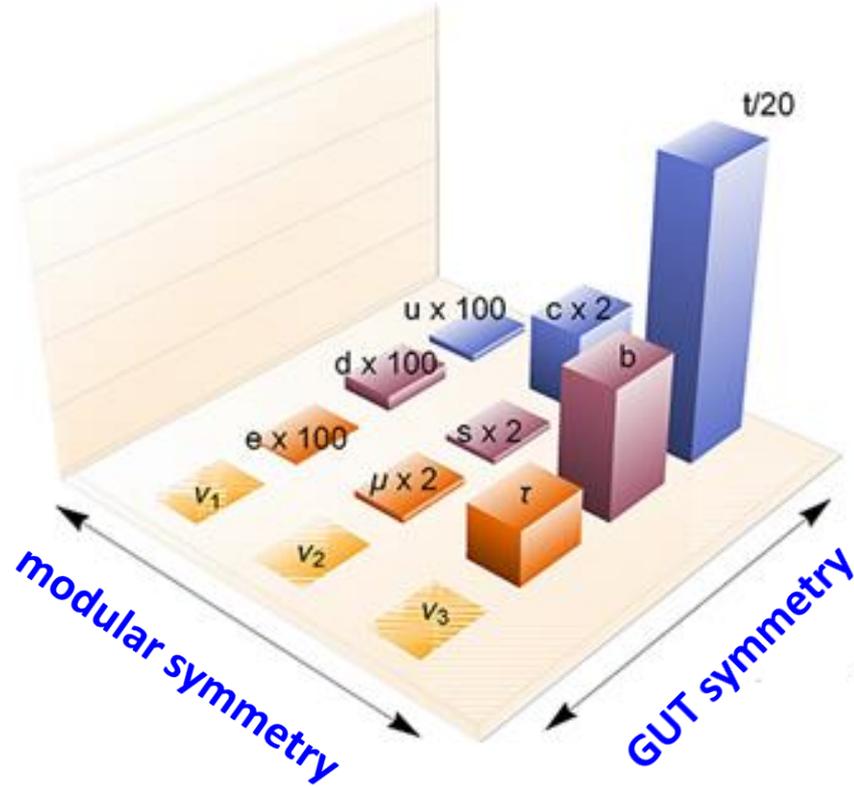
Each family fits into the SU(5) multiplets

$$N = \nu^c \sim \mathbf{1}, \quad \overline{F} = \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ -\nu \end{pmatrix} \sim \overline{\mathbf{5}}, \quad T = \begin{pmatrix} 0 & u_b^c & -u_g^c & -u_r & -d_r \\ -u_b^c & 0 & u_r^c & -u_g & -d_g \\ u_g^c & -u_r^c & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & e^c \\ d_r & d_g & d_b & -e^c & 0 \end{pmatrix} \sim \mathbf{10}$$

The most general superpotential for quark and lepton masses

$$\mathcal{W} = NNf_M(Y) + N\overline{F}H_5 f_N(Y) + \overline{F}TH_{\overline{5}}f_D(Y) + \overline{F}TH_{\overline{45}}f'_D(Y) + TTH_5 f_U(Y)$$

Here $f_M(Y)$, $f_N(Y)$, $f_D(Y)$, $f'_D(Y)$ and $f_U(Y)$, are modular forms determined by the representation and weight assignments of the matter fields. There are **7** types of A4x SU(5) modular models classified according to the A4 reps of matter fields:

	N	$\overline{F}$	T
Type-I	**3**	**3**	$\mathbf{1(1',1'')}$
Type-II	**3**	$\mathbf{1(1',1'')}$	**3**
Type-III	$\mathbf{1(1',1'')}$	**3**	**3**
Type-IV	**3**	**3**	**3**
Type-V	$\mathbf{1(1',1'')}$	**3**	$\mathbf{1(1',1'')}$
Type-VI	$N_{1,2} \sim \mathbf{1(1',1'')}$	**3**	**3**
Type-VII	$N_{1,2} \sim \mathbf{1(1',1'')}$	**3**	$\mathbf{1(1',1'')}$

[Chen,Ding,King,2101.12724]

Classification of modular A_4 x SU(5) GUT

The most general superpotential for quark and lepton masses

$$\mathcal{W} = NNf_M(Y) + N\overline{F}H_5 f_N(Y) + \overline{F}TH_{\overline{5}}f_D(Y) + \overline{F}TH_{\overline{45}}f_D'(Y) + TTH_5 f_U(Y)$$

Here $f_M(Y)$, $f_N(Y)$, $f_D(Y)$, $f_D'(Y)$ and $f_U(Y)$, are modular forms determined by the representation and weight assignments of the matter fields. There are **7** types of A4x SU(5) modular models classified according to the A_4 reps of matter fields:

Type-I	#P	$(\rho_N, \rho_F, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	k_N	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
$\mathcal{I}_1$	18	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	1	1	$(1,1,3)$
$\mathcal{I}_2$	22	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	0	2	$(2,0,4)$
$\mathcal{I}_3$	22	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	0	2	$(0,2,4)$
$\mathcal{I}_4$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	0	2	$(0,0,4)$
$\mathcal{I}_5$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	0	2	$(0,2,4)$
$\mathcal{I}_6$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	0	2	$(2,2,4)$
$\mathcal{I}_7$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1'}, \mathbf{1}, \mathbf{1''})$	0	2	$(0,0,4)$
$\mathcal{I}_8$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	2	0	$(2,2,2)$
$\mathcal{I}_9$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	0	2	$(2,2,4)$
$\mathcal{I}_{10}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''})$	0	2	$(0,2,4)$
$\mathcal{I}_{11}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	0	2	$(2,2,4)$
$\mathcal{I}_{12}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	0	2	$(2,4,2)$
$\mathcal{I}_{13}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	2	0	$(2,2,4)$
$\mathcal{I}_{14}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	2	0	$(2,4,4)$
$\mathcal{I}_{15}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	2	0	$(2,2,4)$
$\mathcal{I}_{16}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	2	0	$(2,4,2)$
$\mathcal{I}_{17}$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	2	0	$(2,4,4)$

Type-II	#P	$(\rho_N, \rho_T, \rho_{F_1}, \rho_{F_2}, \rho_{F_3})$	k_N	$(k_{F_1}, k_{F_2}, k_{F_3})$	k_T
$\mathcal{II}_1$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'})$	1	$(1,3,1)$	3
$\mathcal{II}_2$	24	$(\mathbf{3}, \mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1})$	1	$(1,3,1)$	3

Type-VII	#P	$(\rho_F, \rho_{N_1}, \rho_{N_2}, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	(k_{N_1}, k_{N_2})	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
$\mathcal{VII}_1$	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(1,3)$	3	$(1,1,3)$
$\mathcal{VII}_2$	22	$(\mathbf{3}, \mathbf{1'}, \mathbf{1}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(1,3)$	3	$(1,1,3)$
$\mathcal{VII}_3$	22	$(\mathbf{3}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(1,3)$	3	$(1,1,3)$
$\mathcal{VII}_4$	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(0,2)$	2	$(0,2,4)$
$\mathcal{VII}_5$	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(0,2)$	2	$(0,2,4)$
$\mathcal{VII}_6$	23	$(\mathbf{3}, \mathbf{1}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(2,2)$	2	$(0,2,4)$

| Type-V	#P	$(\rho_F, \rho_{N_1}, \rho_{N_2}, \rho_{N_3}, \rho_{T_1}, \rho_{T_2}, \rho_{T_3})$	$(k_{N_1}, k_{N_2}, k_{N_3})$	k_F	$(k_{T_1}, k_{T_2}, k_{T_3})$
$\mathcal{V}_1$	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(1,3,3)$	1	$(1,1,3)$
$\mathcal{V}_2$	22	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(3,1,3)$	1	$(1,1,3)$
$\mathcal{V}_3$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	$(0,2,2)$	2	$(2,2,2)$
$\mathcal{V}_4$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	$(4,2,2)$	0	$(2,2,2)$
$\mathcal{V}_5$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	$(4,2,2)$	0	$(2,2,4)$
$\mathcal{V}_6$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(4,2,2)$	0	$(2,4,2)$
$\mathcal{V}_7$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(4,2,2)$	0	$(2,4,4)$
$\mathcal{V}_8$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	$(2,4,2)$	0	$(2,2,2)$
$\mathcal{V}_9$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	$(2,4,2)$	0	$(2,2,4)$
$\mathcal{V}_{10}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(2,4,2)$	0	$(2,4,2)$
$\mathcal{V}_{11}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(2,4,2)$	0	$(2,4,4)$
$\mathcal{V}_{12}$	24	$(\mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	$(2,4,2)$	0	$(2,2,2)$
$\mathcal{V}_{13}$	24	$(\mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1''}, \mathbf{1}, \mathbf{1'})$	$(2,4,2)$	0	$(2,4,4)$
$\mathcal{V}_{14}$	24	$(\mathbf{3}, \mathbf{1''}, \mathbf{1''}, \mathbf{1}, \mathbf{1''}, \mathbf{1}, \mathbf{1})$	$(2,4,2)$	0	$(2,2,4)$
$\mathcal{V}_{15}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''})$	$(0,0,0)$	2	$(0,0,4)$
$\mathcal{V}_{16}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1})$	$(0,0,0)$	2	$(0,0,4)$
$\mathcal{V}_{17}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1})$	$(0,0,0)$	4	$(0,0,4)$
$\mathcal{V}_{18}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1})$	$(2,0,0)$	2	$(0,0,4)$
$\mathcal{V}_{19}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''})$	$(0,0,0)$	4	$(0,0,4)$
$\mathcal{V}_{20}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''})$	$(2,0,0)$	2	$(0,0,4)$
$\mathcal{V}_{21}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	$(0,0,0)$	2	$(2,0,4)$
$\mathcal{V}_{22}$	24	$(\mathbf{3}, \mathbf{1}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	$(2,0,0)$	2	$(2,0,4)$
$\mathcal{V}_{23}$	24	$(\mathbf{3}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(0,0,0)$	2	$(0,2,4)$
$\mathcal{V}_{24}$	24	$(\mathbf{3}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(2,0,0)$	2	$(0,2,4)$
$\mathcal{V}_{25}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1})$	$(0,2,0)$	2	$(0,0,4)$
$\mathcal{V}_{26}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''})$	$(0,2,0)$	2	$(0,0,4)$
$\mathcal{V}_{27}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'}, \mathbf{1'})$	$(0,2,0)$	2	$(2,0,4)$
$\mathcal{V}_{28}$	24	$(\mathbf{3}, \mathbf{1'}, \mathbf{1'}, \mathbf{1''}, \mathbf{1''}, \mathbf{1''}, \mathbf{1'})$	$(0,2,0)$	2	$(0,2,4)$

53 models with small number of free parameters

✓ Predictions for the lepton Dirac CP phase
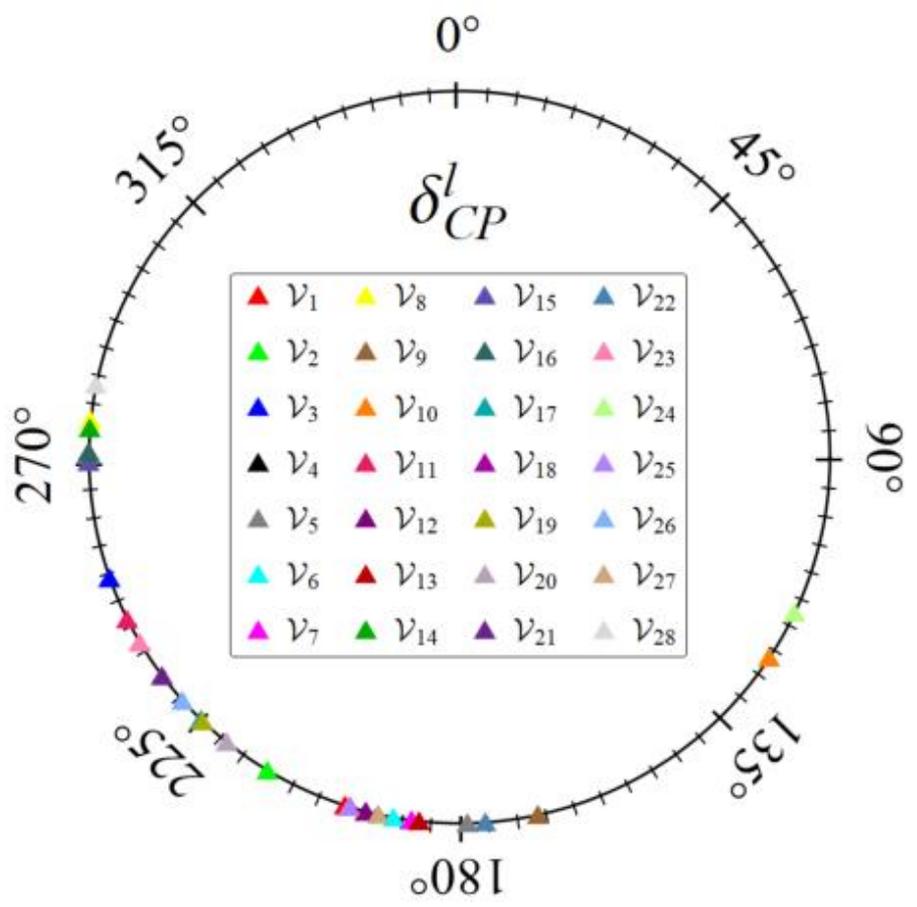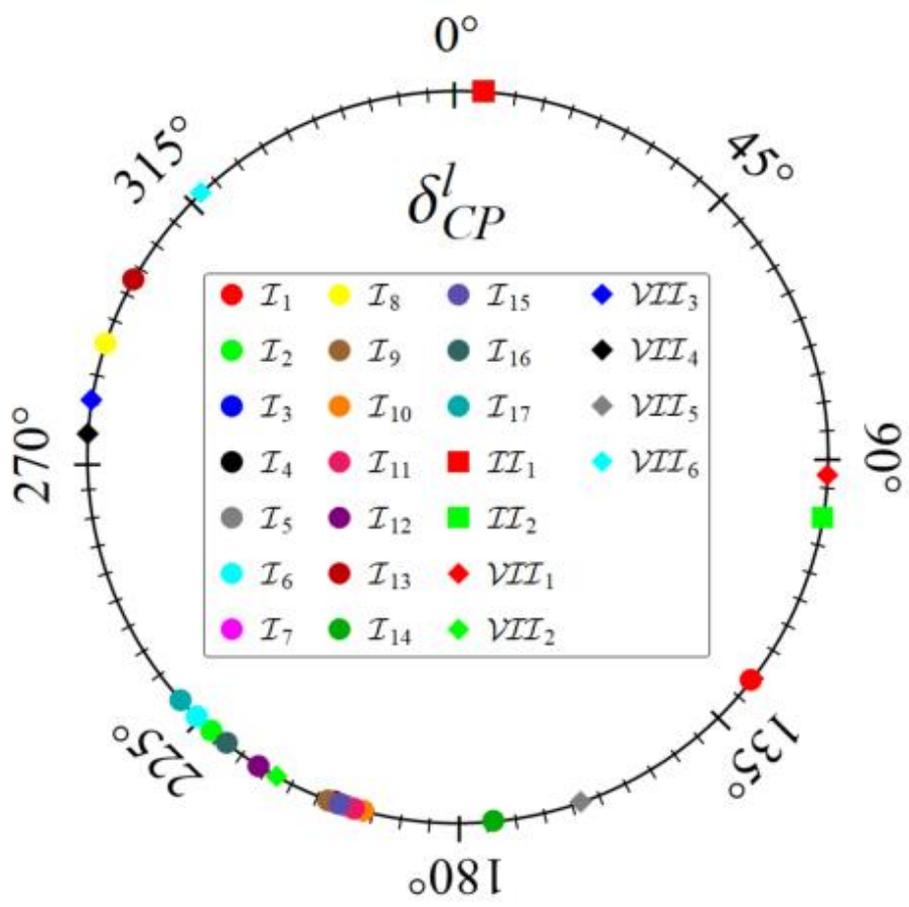
[Chen,Ding,King,2101.12724]

✓ Predictions for the neutrinoless double decay
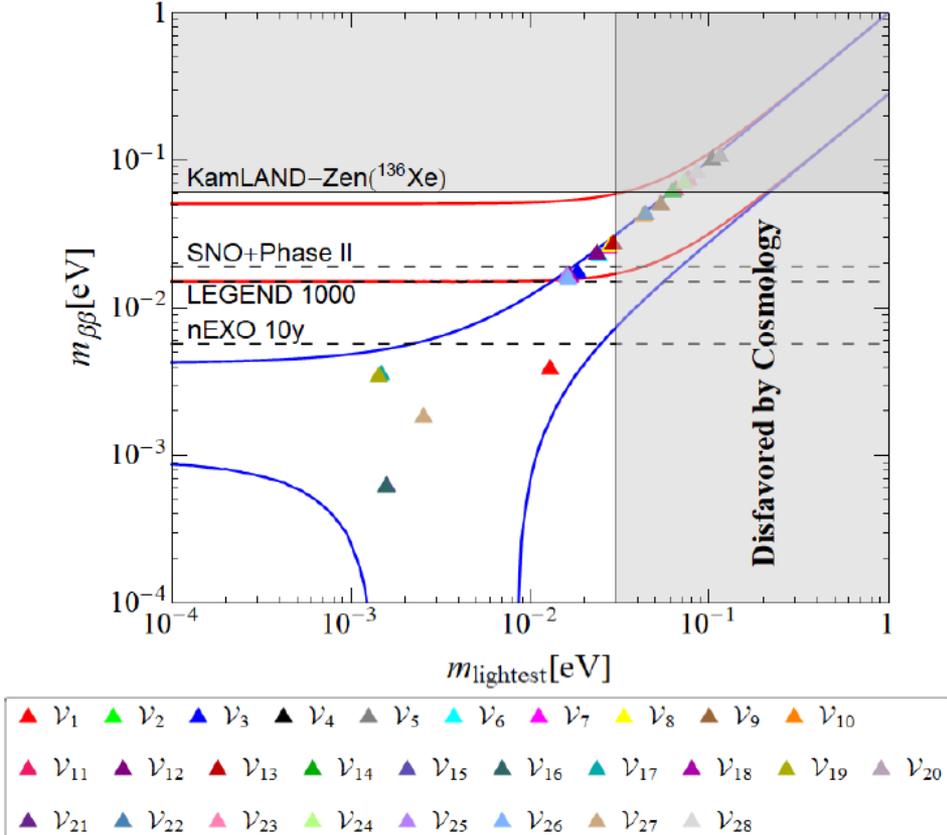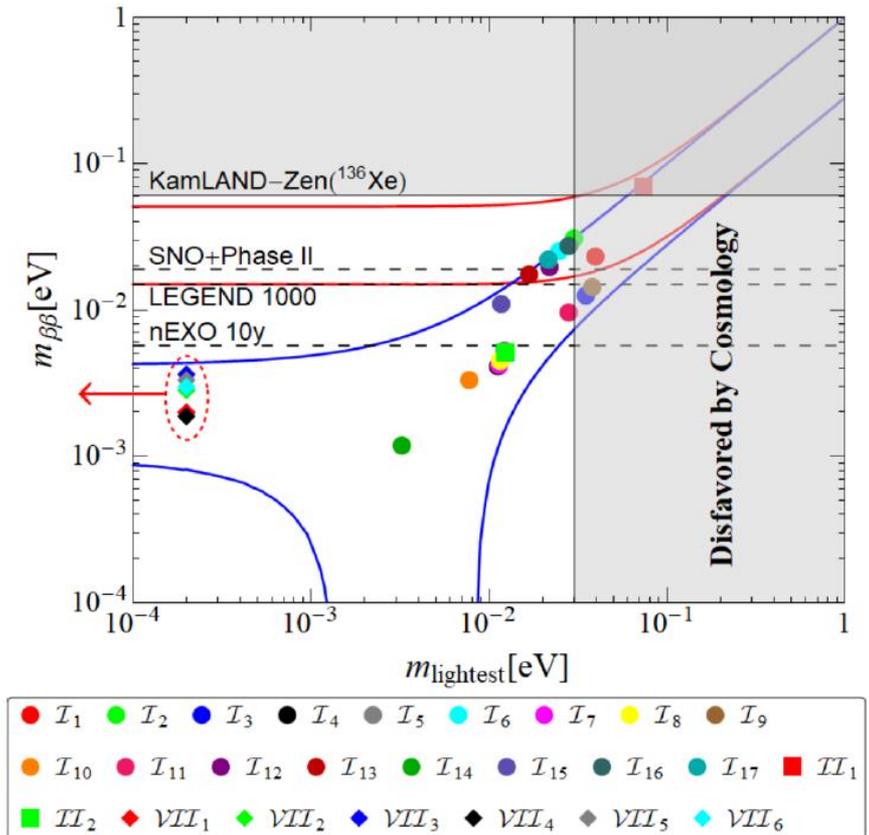
[Chen,Ding,King,2101.12724]

I apologize for missing your contributions.

Summary

- Neutrino oscillation calls for convincing model of neutrino masses and mixings, with testable and confirmed predictions

- Modular symmetry is a new promising approach to the fermion masses and flavor mixing puzzles. Modular symmetry can naturally give rise to texture zero, the masses and mixing patterns of quark and lepton can be explained simultaneously, and simple predictive modular models could be constructed.

- Modular symmetry is still at the early stage of its development, many aspects still need to be understood.

Thank you for your attention!