Form Factors and Trace Anomaly of Energy Momentum Tensor

Bao-Dong Sun
(South China Normal U., Bonn U., RUB)

Outline：

1. Introduction：nucleon case
2. GFFs for different spins & its Interpretation
3. Calculation in Free theories/Models/ChPT
4. Densities in sharply local wave packet/2D
5. Summary

Gravitational form factors（GFFs）

- Energy Momentum Tensor（EMT）

$$\hat{T}_C^{\mu\nu} = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial^\nu\phi_n - g^{\mu\nu}\mathcal{L}$$

$$\hat{T}^{\mu\nu}_{\text{grav}} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}} \longrightarrow$$

- nucleon GFFs：（Kobzarev & Okun 1962; Pagels 1966; Polyakov & Schweitzer, 2018）

$$\langle p', s'|\hat{T}^a_{\mu\nu}(x)|p, s\rangle = \bar{u}'\Big[A^a(t)\,\frac{\gamma_{\{\mu}P_{\nu\}}}{2} + B^a(t)\,\frac{i\,P_{\{\mu}\sigma_{\nu\}\rho}\Delta^\rho}{4m} + D^a(t)\,\frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{4m} + m\,\bar{c}^a(t)\,g_{\mu\nu}\Big]u\,e^{i(p'-p)x}$$

$(a = q, g)$

$t \to 0$

$A^a(t)$ → mass; $A^a(t)$, $B^a(t)$ → spin: external properties

$D^a(t)$ → D-term: “internal” property

“Druck”（Polyakov, 1999）

- GPDs ↔ GFFs（polynomiality）（Ji, 1996）

$$\int \mathrm{d}x\; x\,H^a(x,\xi,t) = A^a(t) + \xi^2 D^a(t)$$

$$\int \mathrm{d}x\; x\,E^a(x,\xi,t) = B^a(t) - \xi^2 D^a(t)$$

Ji sum $A^q(t) + B^q(t) = 2J^q(t)$

- DVCS @JLab@US-EIC @EicC (lower Q^2 v.s.)
- $\gamma^*\gamma \to \pi^0\pi^0$ @KEKB
- … …

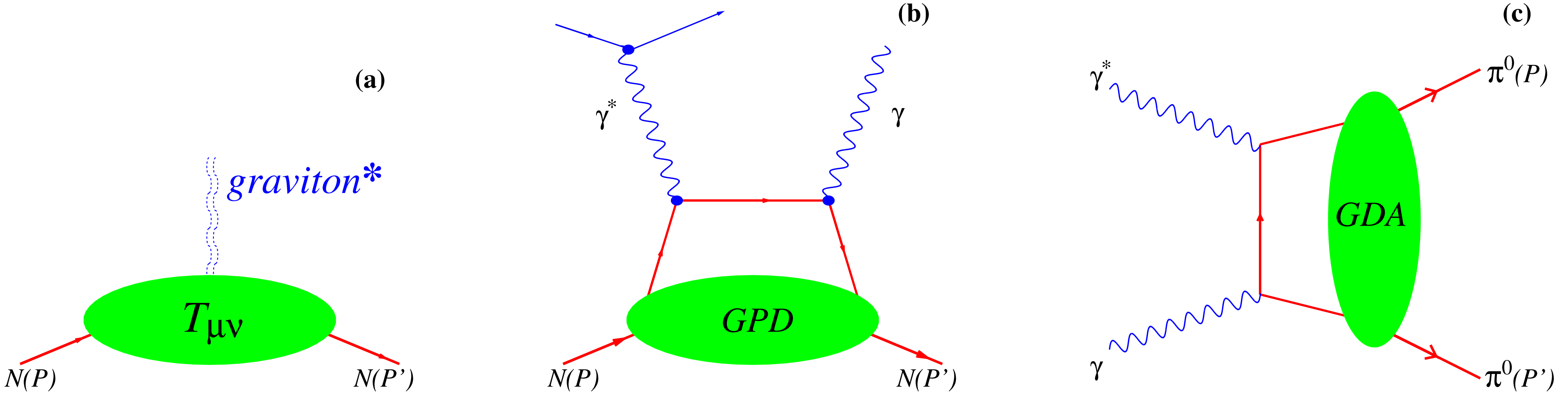
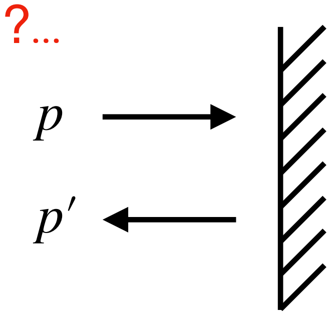


Figure 1. (a) A natural but impractical probe of EMT form factors is scattering off gravitons. (b) Hard-exclusive reactions like deeply virtual Compton scattering (DVCS) provide a realistic way to access EMT form factors through GPDs. Here one of the relevant tree-level diagrams is shown. (c) Information on the EMT structure of particles not available as targets, such as e.g. π^0, can also be accessed from studies of generalized distribution amplitudes (GDAs) which are an “analytic continuation” of GPDs to the crossed channel. The shown reaction $\gamma^*\gamma \to \pi^0\pi^0$ (and analog for other hadrons) can be studied in e^+e^- collisions.

Gravitational form factors（GFFs）

Problematic?...

Breit frame:



anti-collinear

$$2P = (p'+p) = (2E, \vec{0})$$
$$\Delta = (p'-p) = (0, \vec{\Delta})$$
$$t = \Delta^2$$

- Energy Momentum Tensor（EMT）

$$\hat{T}^{\mu\nu}_{C} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^\nu \phi_n - g^{\mu\nu}\mathcal{L}$$

$$\hat{T}^{\mu\nu}_{\rm grav} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm grav}}{\delta g^{\mu\nu}} \longrightarrow$$

- nucleon GFFs: （Kobzarev & Okun 1962; Pagels 1966; Polyakov & Schweitzer, 2018）

$$\langle p', s' | \hat{T}^a_{\mu\nu}(x) | p, s\rangle = \bar{u}' \Big[A^a(t) \frac{\gamma_{\{\mu}P_{\nu\}}}{2} + B^a(t) \frac{i\, P_{\{\mu}\sigma_{\nu\}\rho}\Delta^\rho}{4m} + D^a(t) \frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{4m} + m\, \bar{c}^a(t)\, g_{\mu\nu} \Big] u\, e^{i(p'-p)x}$$

$(a = q, g)$

$t \to 0$

→ mass } external properties

→ spin }

→ D-term "internal" property

"Druck"
（Polyakov, 1999）

- GPDs ↔ GFFs（polynomiality）（Ji, 1996）

$$\int \mathrm{d}x\; x\, H^a(x,\xi,t) = A^a(t) + \xi^2 D^a(t)$$
$$\int \mathrm{d}x\; x\, E^a(x,\xi,t) = B^a(t) - \xi^2 D^a(t)$$

Ji sum $A^q(t) + B^q(t) = 2J^q(t)$

- Mellin moments（Diehl, 2003; Belitsky, Radyushkin, 2005）

$$(P^+)^{n+1} \int \mathrm{d}x\, x^n \int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left[\bar{q}(-\tfrac{1}{2}z)\,\gamma^+ q(\tfrac{1}{2}z)\right]_{z^+=0,\, \boldsymbol{z}=0}$$
$$= \left(i\frac{\mathrm{d}}{\mathrm{d}z^-}\right)^n \left[\bar{q}(-\tfrac{1}{2}z)\,\gamma^+ q(\tfrac{1}{2}z)\right]\Big|_{z=0} = \bar{q}(0)\,\gamma^+ (i\overleftrightarrow{\partial}^+)^n\, q(0)$$

$\downarrow n \to 0$ $\quad\downarrow n \to 1$

probe $|N\rangle$ by $\hat{J}^\mu_{\rm em}$ $\quad \hat{T}^{\mu\nu}_{\rm grav}$

internal forces
（strong interaction etc.）

- {EM form factor, PDFs} ∈ GPDs

$$\int \mathrm{d}x\, H^q(x,\xi,t) = F_1^q(t)$$
$$\lim_{\Delta\to 0} H^q(x,\xi,t) = f_1^q(x)$$

GFFs of spin-0

Definition

$$\langle p'|\hat{T}^a_{\mu\nu}(x)|p\rangle = \left[2P_\mu P_\nu\, A^a(t) + \frac{1}{2}(\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2)\, D^a(t) + 2\; m^2\, \bar{c}^a(t)\, g_{\mu\nu}\right] e^{i(p'-p)x}\,. \qquad (7)$$

Polyakov & Schweitzer, 2018

free Klein-Gordon field $D_\pi = -1$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - V_0(\Phi), \qquad V_0(\Phi) = \frac{1}{2}m^2\Phi^2 \qquad (8)$$

$$\hat{T}^{\mu\nu}(x) = (\partial^\mu\Phi)(\partial^\nu\Phi) - g^{\mu\nu}\mathcal{L}, \qquad (10)$$

Hudson & Schweitzer, 2017
Collins, 1976

while...

$$T^{\mu\nu}_{\text{improve}} = T^{\mu\nu}_{\text{Eq.(10)}} + \theta^{\mu\nu}_{\text{improve}},$$

$$\theta^{\mu\nu}_{\text{improve}} = -h(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)\phi(x)^2, \qquad h = \frac{1}{4}\left(\frac{n-2}{n-1}\right), \qquad (16)$$

$$S_{\text{grav}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - V(\Phi) - \frac{1}{2}hR\Phi^2\right) \qquad (17)$$

$$D_{\text{interacting improved}} = -1 + 4h. \qquad (20)$$

$$D_\pi = -1 \;\rightarrow\; D_\pi^{\text{improve}} = -1 + 4h \;\rightarrow\; -\frac{1}{3}$$

1. cannot arbitrarily add "total derivatives" to EMT
2. h removes UV divergences up to three loops in dimensional regularization

Similar h terms as counterterms absorb power-counting violating in ChPT calc for spin 1/2, 1, 3/2 .

Alharazin, Djukanovic, Gegelia, Polyakov, 2020
Epelbaum, Gegelia, Meißner, Polyakov, 2021
Alharazin, Epelbaum, Gegelia, Meißner, BDS, 2023

GFFs of spin-1

- Definition: (Holstein, 2006; Cosyn, Cotogno, Freese, Lorcé, 2019; Cosyn, Freese, Pire, 2019; Polyakov, BDS, 2019)

$$
\begin{aligned}
\langle p', \sigma' | \hat{T}^a_{\mu\nu}(x) | p, \sigma \rangle = \Bigg[& 2P_\mu P_\nu \left(-\epsilon'^* \cdot \epsilon \, A^a_0(t) + \frac{\epsilon'^* \cdot P \, \epsilon \cdot P}{m^2} A^a_1(t) \right) \\
& + 2 \left[P_\mu (\epsilon'^*_\nu \, \epsilon \cdot P + \epsilon_\nu \, \epsilon'^* \cdot P) + P_\nu (\epsilon'^*_\mu \, \epsilon \cdot P + \epsilon_\mu \, \epsilon'^* \cdot P) \right] \; J^a(t) \\
& + \frac{1}{2} (\Delta_\mu \Delta_\nu - g_{\mu\nu} \Delta^2) \left(\epsilon'^* \cdot \epsilon \, D^a_0(t) + \frac{\epsilon'^* \cdot P \, \epsilon \cdot P}{m^2} D^a_1(t) \right) \\
& + \Big[\frac{1}{2} (\epsilon_\mu \epsilon'^*_\nu + \epsilon'^*_\mu \epsilon_\nu) \Delta^2 - (\epsilon'^*_\mu \Delta_\nu + \epsilon'^*_\nu \Delta_\mu) \, \epsilon \cdot P \\
& + (\epsilon_\mu \Delta_\nu + \epsilon_\nu \Delta_\mu) \, \epsilon'^* \cdot P - 4 g_{\mu\nu} \, \epsilon'^* \cdot P \, \epsilon \cdot P \Big] \; E^a(t) \\
& + \left(\epsilon_\mu \epsilon'^*_\nu + \epsilon'^*_\mu \epsilon_\nu - \frac{\epsilon'^* \cdot \epsilon}{2} g_{\mu\nu} \right) m^2 \, \bar{f}^a(t) \\
& + g_{\mu\nu} \left(\epsilon'^* \cdot \epsilon \, m^2 \, \bar{c}^a_0(t) \; + \epsilon'^* \cdot P \, \epsilon \cdot P \, \bar{c}^a_1(t) \right) \Bigg] \; e^{i(p'-p)x}
\end{aligned}
$$

6 conserving

3 non-conserving

- multipole expansion: (Polyakov, BDS, 2019)

$$
\langle \hat{T}^{00}_a(0) \rangle = 2m^2 \mathcal{E}^a_0(t) \, \delta_{\sigma'\sigma} + \hat{Q}^{kl} \, \Delta^k \Delta^l \; \mathcal{E}^a_2(t) \; ,
$$

$$
\langle \hat{T}^{0j}_a(0) \rangle = i \epsilon^{jkl} \hat{S}^k_{\sigma'\sigma} \Delta^l \, m \, \mathcal{J}^a(t) \; ,
$$

$$
\begin{aligned}
\langle \hat{T}^{ij}_a(0) \rangle = & \frac{1}{2} (\Delta^i \Delta^j - \delta^{ij} \vec{\Delta}^2) \mathcal{D}^a_0(t) \, \delta_{\sigma'\sigma} \\
& + \left(\Delta^j \Delta^k \hat{Q}^{ik} + \Delta^i \Delta^k \hat{Q}^{jk} - \vec{\Delta}^2 \hat{Q}^{ij} - \delta^{ij} \Delta^k \Delta^l \hat{Q}^{kl} \right) \; \mathcal{D}^a_2(t) \\
& + \frac{1}{2m^2} (\Delta^i \Delta^j - \delta^{ij} \vec{\Delta}^2) \Delta^k \Delta^l \hat{Q}^{kl} \; \mathcal{D}^a_3(t) \\
& + \text{non-conserving terms}
\end{aligned}
$$

✤ gravitational multipole form factors

$$
\mathcal{E}^a_0(t) = A^a_0(t) - \frac{t}{m^2} \frac{5}{12} A^a_0(t) + \cdots
$$

$$
\mathcal{E}^a_2(t) = -A^a_0(t) + 2J^a(t) - E^a(t) + \cdots
$$

$$
\mathcal{J}^a(t) = J^a(t) - \frac{t}{4m^2} \Big[J^a(t) - E^a(t) \Big] + \cdots
$$

→

$$
\mathcal{D}^a_0(t) = -D^a_0(t) + \frac{4}{3} E^a(t) + \cdots
$$

$$
\mathcal{D}^a_2(t) = -E^a(t)
$$

$$
\mathcal{D}^a_3(t) = \frac{1}{4} \Big[2D^a_0(t) - 2E^a(t) + D^a_1(t) \Big] + \cdots
$$

- spin operators, etc. ...

$$
\hat{S}_{\sigma'\sigma}^{\;\lambda} = \sqrt{S(S+1)} \; C^{S\sigma'}_{S\sigma 1\lambda}
$$

$$
\hat{Q}^{ij} = \frac{1}{2} \left[\hat{S}^i \hat{S}^j + \hat{S}^j \hat{S}^i - \frac{2}{3} S(S+1) \delta^{ij} \right]
$$

$$
\epsilon^\mu(p, \sigma) = \left(\frac{\vec{p} \cdot \hat{\epsilon}_\sigma}{m}, \hat{\epsilon}_\sigma + \frac{\vec{p} \cdot \hat{\epsilon}_\sigma}{m(m+E)} \vec{p} \right)
$$

(for $S = 1$)

GFFs of spin-3/2

Rarita-Schwinger spinor: $u^\mu = \sum C^{\frac{3}{2}\sigma}_{1\lambda\frac{1}{2}s} u_s(p)\epsilon^\mu_\lambda$

- Definition: (Cotogno, Lorcé, Lowdon, Morales, 2020; Kim, BDS, 2020)

$$\begin{aligned}\langle \hat{T}^{\mu\nu}_a(0)\rangle = -\overline{u}^{\alpha'}(p')\Bigg[&\frac{P^\mu P^\nu}{m}\left(g_{\alpha'\alpha}F^a_{1,0}(t) - \frac{\Delta_{\alpha'}\Delta_\alpha}{2m^2}F^a_{1,1}(t)\right)\\ &+ \frac{(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)}{4m}\left(g_{\alpha'\alpha}F^a_{2,0}(t) - \frac{\Delta_{\alpha'}\Delta_\alpha}{2m^2}F^a_{2,1}(t)\right)\\ &+ mg^{\mu\nu}\left(g_{\alpha'\alpha}F^a_{3,0}(t) - \frac{\Delta_{\alpha'}\Delta_\alpha}{2m^2}F^a_{3,1}(t)\right)\\ &+ \frac{i}{2}\frac{(P^\mu\sigma^{\nu\rho} + P^\nu\sigma^{\mu\rho})\Delta_\rho}{m}\left(g_{\alpha'\alpha}\underset{\text{(spin)}}{F^a_{4,0}(t)} - \frac{\Delta_{\alpha'}\Delta_\alpha}{2m^2}F^a_{4,1}(t)\right)\\ &- \frac{1}{m}(\Delta^\mu g^\nu_{\alpha'}\Delta_\alpha + \Delta^\nu g^\mu_{\alpha'}\Delta_\alpha + \Delta^\mu g^\nu_\alpha\Delta_{\alpha'} + \Delta^\nu g^\mu_\alpha\Delta_{\alpha'}\\ &- 2g^{\mu\nu}\Delta_{\alpha'}\Delta_\alpha - g^\mu_{\alpha'}g^\nu_\alpha\Delta^2 - g^\nu_{\alpha'}g^\mu_\alpha\Delta^2)F^a_{5,0}(t)\\ &+ m(g^\mu_{\alpha'}g^\nu_\alpha + g^\nu_{\alpha'}g^\mu_\alpha)F^a_{6,0}(t)\Bigg]u^\alpha(p,\sigma)\end{aligned}$$

7 conserving

3 non-conserving

- multipole expansion: (Cotogno, Lorcé, Lowdon, Morales, 2020) (Kim, BDS, 2020)

$$\langle \hat{T}^{00}_a(0)\rangle = 2mE\left[\mathcal{E}^a_0(t)\delta_{\sigma'\sigma} + \left(\frac{\sqrt{-t}}{m}\right)^2 \hat{Q}^{kl}_{\sigma'\sigma}Y^{kl}_2\mathcal{E}^a_2(t)\right]$$

$$\langle \hat{T}^{0i}_a(0)\rangle = 2mE\left[\frac{\sqrt{-t}}{m}i\epsilon^{ikl}Y^l_1\hat{S}^k_{\sigma'\sigma}\mathcal{J}^a_1(t) + \left(\frac{\sqrt{-t}}{m}\right)^3 i\epsilon^{ikl}Y^{lmn}_3\hat{O}^{kmn}_{\sigma'\sigma}\mathcal{J}^a_3(t)\right] \quad \longleftrightarrow$$

$$\begin{aligned}\langle \hat{T}^{ij}_a(0)\rangle = 2mE\Bigg[&\frac{1}{4m^2}(\Delta^i\Delta^j + \delta^{ij}\Delta^2)D^a_0(t)\delta_{\sigma'\sigma}\\ &+ \frac{1}{4m^4}\hat{Q}^{kl}_{\sigma'\sigma}(\Delta^i\Delta^j + \delta^{ij}\Delta^2)\Delta^k\Delta^l D^a_3(t)\\ &+ \frac{1}{2m^2}\left(\hat{Q}^{ik}_{\sigma'\sigma}\Delta^j\Delta^k + \hat{Q}^{jk}_{\sigma'\sigma}\Delta^i\Delta^k + \hat{Q}^{ij}_{\sigma'\sigma}\Delta^2 - \delta^{ij}\hat{Q}^{kl}_{\sigma'\sigma}\Delta^k\Delta^l\right)D^a_2(t)\\ &+ \text{non-conserving terms}\Bigg]\end{aligned}$$

Gluonic GFFs by MIT lattice group

Pefkou, Hackett, Shanahan, 2022

✤ gravitational multipole form factors

$$\mathcal{E}^a_0(t) = F^a_{1,0}(t) + F^a_{3,0}(t) - \frac{t}{m^2}\frac{5}{12}F^a_{1,0}(t) + \cdots$$

$$\mathcal{E}^a_2(t) = -\frac{1}{6}F^a_{1,0}(t) - \frac{1}{6}F^a_{1,1}(t) + \cdots$$

$$\mathcal{J}^a_1(t) = \frac{1}{3}F^a_{4,0}(t) - \frac{1}{3}F^a_{6,0}(t) + \cdots$$

$$\mathcal{J}^a_3(t) = -\frac{1}{6}\left[F^a_{4,0}(t) + F^a_{4,1}(t)\right] + \frac{t}{24m^2}F^a_{4,1}(t)$$

$$D^a_0(t) = F^a_{2,0}(t) - \frac{16}{3}F^a_{5,0}(t) + \cdots$$

$$D^a_2(t) = \frac{4}{3}F^a_{5,0}(t)$$

$$D^a_3(t) = -\frac{1}{6}F^a_{2,0}(t) - \frac{1}{6}F^a_{2,1}(t) + \cdots$$

- octupole operator:

$$\begin{aligned}\hat{O}^{ijk} =\frac{1}{6}\Bigg[&\hat{S}^i\hat{S}^j\hat{S}^k + \hat{S}^j\hat{S}^i\hat{S}^k + \hat{S}^k\hat{S}^j\hat{S}^i\\ &+ \hat{S}^j\hat{S}^k\hat{S}^i + \hat{S}^i\hat{S}^k\hat{S}^j + \hat{S}^k\hat{S}^i\hat{S}^j\\ &- \frac{6S(S+1)-2}{5}(\delta^{ij}\hat{S}^k + \delta^{ik}\hat{S}^j + \delta^{kj}\hat{S}^i)\Bigg]\end{aligned}$$

- n-rank irreducible tensors:

$$Y^{i_1 i_2 \ldots i_n}_n(\Omega_p) = \frac{(-1)^n}{(2n-1)!!}p^{n+1}\partial^{i_1}\partial^{i_2}\ldots\partial^{i_n}\frac{1}{p}$$

Interpretation：Static EMT

- Definition （Polyakov, 2003）

$$T^{\mu\nu}(\boldsymbol{r},\sigma',\sigma)=\sum_a T^{\mu\nu}_a(\boldsymbol{r},\sigma',\sigma)$$
$$=\sum_a\int\frac{d^3\Delta}{2E(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}\langle p',\sigma'|\hat{T}^{\mu\nu}_a(0)|p,\sigma\rangle$$

- energy（mass）densities

$$T^{00}(\boldsymbol{r},\sigma',\sigma)=\varepsilon_0(r)\delta_{\sigma'\sigma}+\varepsilon_2(r)\hat{Q}^{ij}_{\sigma'\sigma}Y^{ij}_2(\Omega_r)$$

- spin density

$$J^i(\boldsymbol{r},\sigma',\sigma)=\sum_a J^i_a(\boldsymbol{r},\sigma',\sigma)=\epsilon^{ijk}r^j\sum_a T^{0k}_a(\boldsymbol{r},\sigma',\sigma)$$

$$\rho_J(r)=-r\frac{d}{dr}\int\frac{d^3\Delta}{(2\pi)^3}e^{-\boldsymbol{\Delta}\cdot\boldsymbol{r}}\mathcal{J}_1(t)\quad\text{(averaged)}$$

（Kim, BDS, 2020）

✤ radii：（energy, spin, mechanical）

$$\langle r_E^2\rangle=\frac{1}{m}\int d^3r\ r^2\varepsilon_0(r)\qquad \left.\frac{dF_r}{dS_r}\right|_{\text{unp}}>0$$

$$\langle r_J^2\rangle=\frac{\int d^3r\,r^2\rho_J(r)}{\int d^3r\,\rho_J(r)}\qquad \nearrow\ (n=0)$$

$$\langle r_n^2\rangle_{\text{mech}}=\frac{\int d^3r\,r^2\left[p_n(r)+\frac{2}{3}s_n(r)\right]}{\int d^3r\ \left[p_n(r)+\frac{2}{3}s_n(r)\right]}$$

✤ energy deform by spin： （Kim, BDS, 2020）

$$\mathcal{Q}^{ij}_{\sigma'\sigma}=\frac{2}{15}\hat{Q}^{ij}_{\sigma'\sigma}\int d^3r\ r^2\varepsilon_2(r)$$

✤ generalized D-terms：（Panteleeva, Polyakov, 2020）

$$\mathcal{D}_n=m\int d^3r\,r^2p_n(r)=-\frac{4}{15}m\int d^3r\,r^2s_n(r)$$

（Kim, BDS, 2020）：

$$\mathcal{D}_0=D_0(0)\ \ (<0\text{ for stability!})$$

$$\mathcal{D}_2=D_2(0)+\frac{2}{m^2}\int_{-\infty}^{0}dt\,D_3(t)$$

$$\mathcal{D}_3=-\frac{5}{m^2}\int_{-\infty}^{0}dt\,D_3(t)$$

- pressure and shear forces：（“mechanical properties”）

$$T^{ij}(\boldsymbol{r},\sigma',\sigma)=p_0(r)\delta^{ij}\delta_{\sigma'\sigma}+s_0(r)Y^{ij}_2\delta_{\sigma'\sigma}+\left(p_2(r)+\frac{1}{3}p_3(r)-\frac{1}{9}s_3(r)\right)\hat{Q}^{ij}_{\sigma'\sigma}$$
$$+\left(s_2(r)-\frac{1}{2}p_3(r)+\frac{1}{6}s_3(r)\right)2\left[\hat{Q}^{ip}_{\sigma'\sigma}Y^{pj}_2+\hat{Q}^{jp}_{\sigma'\sigma}Y^{pi}_2-\delta^{ij}\hat{Q}^{pq}_{\sigma'\sigma}Y^{pq}_2\right]$$
$$+\hat{Q}^{pq}_{\sigma'\sigma}Y^{pq}_2\left[\left(\frac{2}{3}p_3(r)+\frac{1}{9}s_3(r)\right)\delta^{ij}+\left(\frac{1}{2}p_3(r)+\frac{5}{6}s_3(r)\right)Y^{ij}_2\right]$$

（Polyakov, BDS, 2019,
Panteleeva, Polyakov, 2020）

$p(r)$ and $s(r)$, normal/tangential force, stability conditions

- force acting on the area element $dS = \mathbf{dS_r \hat{e}_r} + \mathbf{dS_\theta \hat{e}_\theta} + \mathbf{dS_\phi \hat{e}_\phi}$

$$\frac{dF_r}{dS_r} = \delta_{\sigma'\sigma}\left(p_0(r) + \frac{2}{3}s_0(r)\right) + \hat{Q}^{rr}_{\sigma'\sigma}\left(p_2(r) + \frac{2}{3}s_2(r) + p_3(r) + \frac{2}{3}s_3(r)\right), \longrightarrow$$

normal force:

$$\frac{dF_\theta}{dS_r} = \hat{Q}^{\theta r}_{\sigma'\sigma}\left(p_2(r) + \frac{2}{3}s_2(r)\right), \qquad \frac{dF_\phi}{dS_r} = \hat{Q}^{\phi r}_{\sigma'\sigma}\left(p_2(r) + \frac{2}{3}s_2(r)\right), \longrightarrow$$

tangential force:

- stability condition (von Laue 1911): $\int d^3r\, p_n(r) = 0$

equilibrium relation ($\partial_\mu \hat{T}^{\mu\nu} = 0$):

$$\begin{cases} \dfrac{2}{3}\dfrac{ds_n(r)}{dr} + 2\dfrac{s_n(r)}{r} + \dfrac{dp_n(r)}{dr} = 0 \\ \downarrow \\ \int dr\, r^N s_n(r) = -\dfrac{3(N+1)}{2(N-2)} \int dr\, r^N p_n(r) \\ (\text{for } N > -1) \end{cases}$$

(Goeke, et al, 2007)

- local stability condition : $\left.\dfrac{dF_r}{dS_r}\right|_{\text{unp}} = p_0(r) + \dfrac{2}{3}s_0(r) \geq 0$

(unpolarized / spherically symmetric hadron)

(Polyakov & Schweitzer, 2018)

$$\downarrow$$

- D-term(unp): $\mathcal{D}_0 = m \int d^3r\, r^2 p_0(r) = -\dfrac{4}{15} m \int d^3r\, r^2 s_0(r) \leq 0$

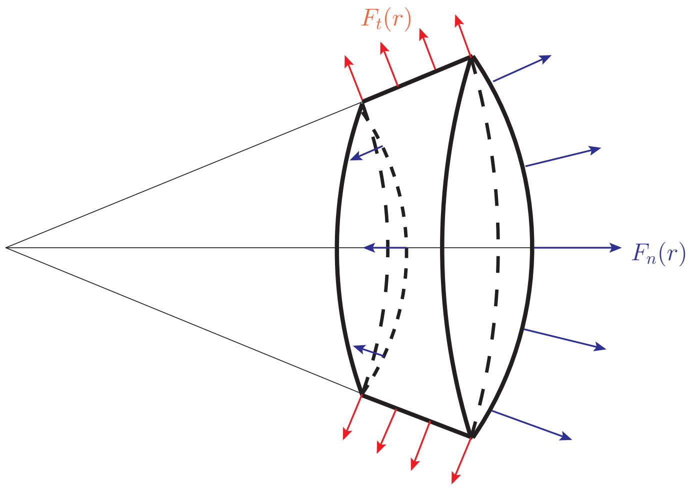
- dispersion relations (Polyakov 2003; Teryaev, 2005; Anikin, Teryaev , 2007; Diehl, Ivanov, 2007)

$$\mathcal{H}(\xi,t) = \int_{-1}^{1} dx \left(\frac{1}{\xi - x - i0} - \frac{1}{\xi + x - i0}\right) H(x,\xi,t)$$

$$\mathrm{Re}\mathcal{H}(\xi,t) = \Delta(t) + \frac{1}{\pi}\text{p.v.}\int_0^1 d\xi'\ \mathrm{Im}\mathcal{H}(\xi',t)\left(\frac{1}{\xi - \xi'} - \frac{1}{\xi + \xi'}\right)$$

$$\Delta(t) = \frac{4}{5}\sum_q e_q^2 D^q(t) + \sum_q e_q^2 d_3^q(t) + \ldots$$

(Gegenbauer polynomials)



$p(r)$ and $s(r)$: spin 1, 3/2

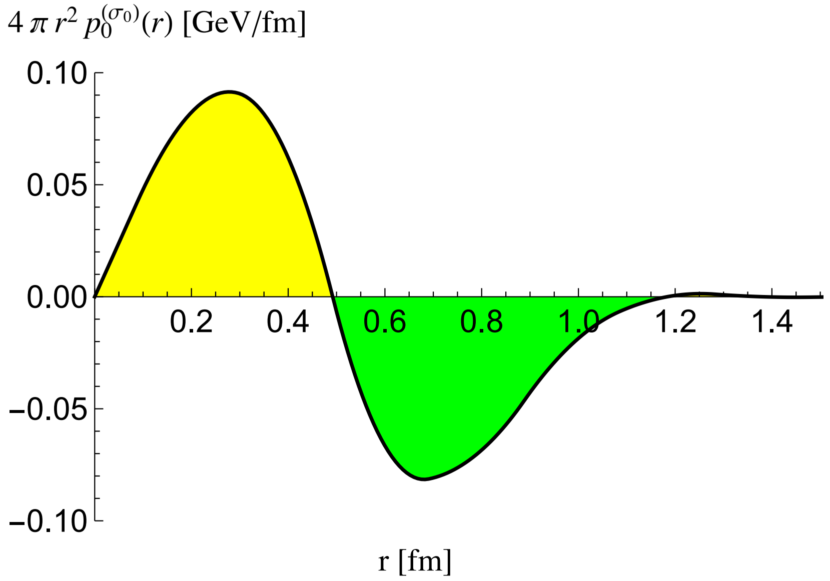
- equilibrium relation: $\frac{2}{3}\frac{ds_n(r)}{dr} + 2\frac{s_n(r)}{r} + \frac{dp_n(r)}{dr} = 0$
- solution in general (Polyakov & Schweitzer, 2018):

$$p_n(r) = \frac{1}{6m}\partial^2\tilde{\mathcal{D}}_n(r) = \frac{1}{6m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\tilde{\mathcal{D}}_n(r),$$
$$s_n(r) = -\frac{1}{4m}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\tilde{\mathcal{D}}_n(r),$$

- solution for spin 1, 3/2, (BDS, Dong, 2020; Kim, BDS, 2020)

$$\tilde{\mathcal{D}}_0(r) = \int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}D_0(t),$$
$$\tilde{\mathcal{D}}_2(r) = \int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}D_2(t) + \frac{1}{m^2}\left(\frac{d}{dr}\frac{d}{dr} - \frac{2}{r}\frac{d}{dr}\right)\int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}D_3(t),$$
$$\tilde{\mathcal{D}}_3(r) = -\frac{2}{m^2}\left(\frac{d}{dr}\frac{d}{dr} - \frac{3}{r}\frac{d}{dr}\right)\int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}D_3(t)$$

$\longrightarrow$ (valid for $J \geq 2$?)



for ρ meson in a quark model

(BDS, Dong, 2020)

- inverse to get $D_n(t)$

$$D_0(t) = 6m\int d^3r\,\frac{j_0(r\sqrt{-t})}{t}p_0(r),$$
$$D_2(t) = 2m\int d^3r\,\frac{j_2(r\sqrt{-t})}{t}\left(2s_2(r) - \frac{1}{2}p_3(r) + \frac{2}{3}s_3(r)\right),$$
$$D_3(t) = 4m^3\int d^3r\,\frac{j_4(r\sqrt{-t})}{t^2}\left(\frac{1}{2}p_3(r) + \frac{5}{6}s_3(r)\right)$$

$\longrightarrow$

✤ generalized D-terms (Kim, BDS, 2020)

$$\left\{\begin{aligned} \mathcal{D}_0 &= D_0(0) \ (< 0 \text{ for stability of unp hadron!}) \\ \mathcal{D}_2 &= D_2(0) + \frac{2}{m^2}\int_{-\infty}^{0} dt\, D_3(t) \\ \mathcal{D}_3 &= -\frac{5}{m^2}\int_{-\infty}^{0} dt\, D_3(t) \end{aligned}\right.$$

GFFs/densities in Free theory / Quark model / ChPT

FREE massive vector particle

(Holstein, 2006; Polyakov, BDS, 2019)

Table II: The free theory values of the total EMT FFs.

EMT FFs	$\mathcal{E}_0(t)$	$\mathcal{E}_2(t)$	$\mathcal{J}(t)$	$\mathcal{D}_0(t)$	$\mathcal{D}_2(t)$	$\mathcal{D}_3(t)$
free theory	1	0	1	$\frac{1}{3}-4h$	-1	0

- Proca Lagrangian + a non-minimal term (?):

$$S_{\rm grav}=\int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu+\frac{1}{2}hRA_\mu A^\mu\right) \longrightarrow$$

- conformal transformation: (Carroll, 2004; Dabrowski, 2009)

$$\tilde{g}_{\mu\nu}(x)=\Omega^2(x)g_{\mu\nu}(x),\quad \widetilde{m}=\Omega^{-1}m,$$
$$\widetilde{A}_\mu=A_\mu,\quad \widetilde{A}^\mu=\tilde{g}^{\mu\nu}\widetilde{A}_\nu=\Omega^{-2}A^\mu,$$
$$\widetilde{U}_{\mu\nu}=U_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$$

✤ all GFFs are t-independent: free of interaction

✤ $D_\pi=-1\rightarrow-\dfrac{1}{3}$: week interaction matters

✤ $D_{\rm fermion}=0\rightarrow\neq 0$: interaction!

✤ $D_\rho\leq 0\overset{?}{\leftrightarrow}h\geq\dfrac{1}{12}$: seems NOT allowed ...

Pagels, 1966; Novikov, Shifman, 1980; Hudson,Schweitzer, 2017; Polyakov & Schweitzer, 2018, etc.

- choices of S: conformal invariance (CI) (or not)

$$S^0_{\rm grav}=\int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu\right),\quad ({\rm CI})$$

$$S_{\rm grav}=\int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu+\frac{1}{2}hRA^2_\mu\right),\quad (\text{not CI for } h\neq 0) \longrightarrow$$

Ricci scalar term breaks CI !

$$S^2_{\rm grav}=\int d^4x\sqrt{-g}\left(\frac{1}{2}A_\mu\Box A_\mu-\frac{1}{2}A_\mu\nabla^\mu\nabla^\nu A_\nu+\frac{1}{2}m^2A^2_\mu\right),\quad (\text{not CI})$$

$(\text{with }\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu)$

$$S^3_{\rm grav}=\int d^4x\sqrt{-g}\left(\frac{1}{2}A_\mu\Box A_\mu-\frac{1}{2}A_\mu\nabla^\mu\nabla^\nu A_\nu+\frac{1}{2}m^2A^2_\mu-\frac{1}{2}R_{\mu\nu}A^\mu A^\nu\right),\quad (\text{CI and give same } D_0 \text{ as } S^0_{\rm grav}!)$$

- Riemann tensor $R_{\mu\nu\rho\sigma}$, Weyl tensor $C_{\mu\nu\rho\sigma}$, etc., but NO suitable mass-dim-4 terms!

ρ meson GFFs by a LC quark model

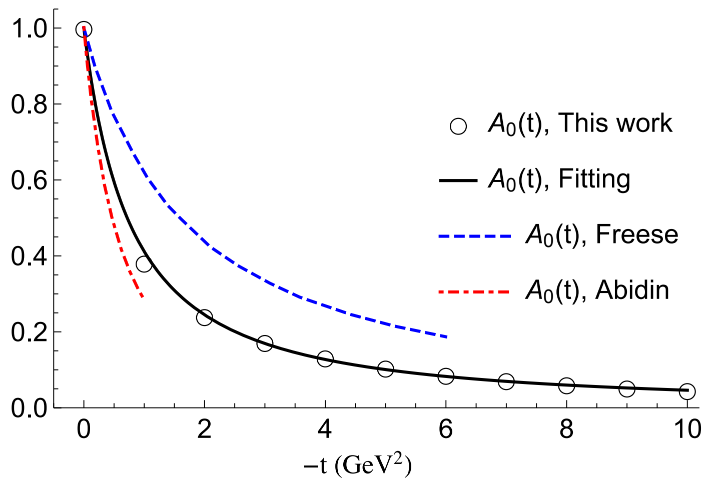
deuteron GPDs:（Berger, Cano, Diehl, Pire, 2001）

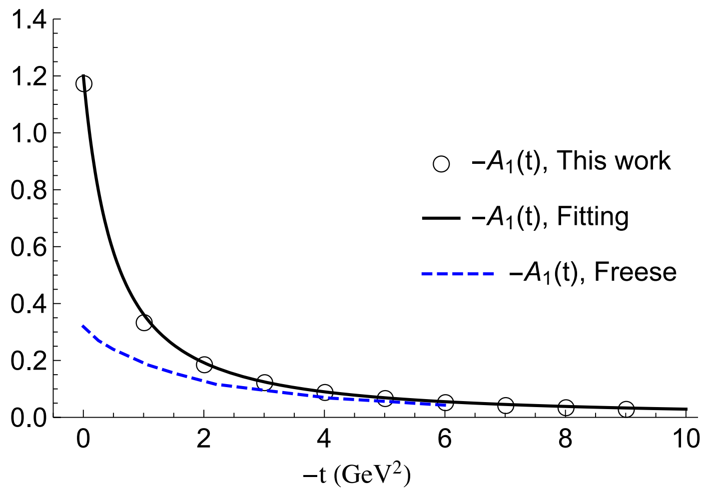
sum rules:（Cosyn, Freese, Pire, 2019. etc. ）

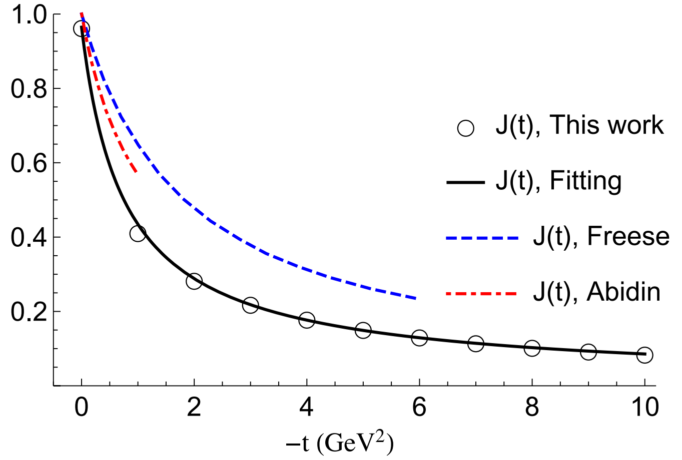
$D_\rho = -0.21 < 0$?

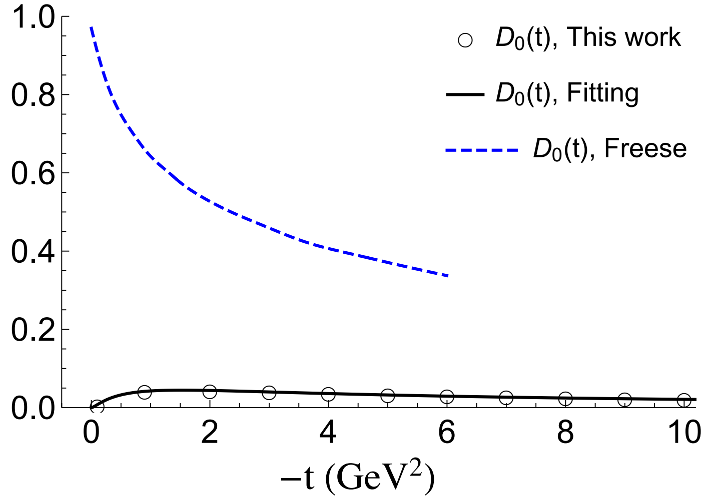
	$\sqrt{\langle r^2\rangle_{\rm mass}}$	$\sqrt{\langle r^2\rangle_{\rm elec.}}$	$\mathcal{Q}_{\rm mass}$	$\mathcal{Q}_{\rm elec.}$
AdS/QCD [30]	0.46	0.73		
NJL [31], Briet frame	0.45	0.67	-0.0224	-0.0200
NJL [31], Light Cone	0.32	0.45		
this work, Briet frame	0.53	0.72	-0.0322	-0.0212
this work, Light Cone	0.41			

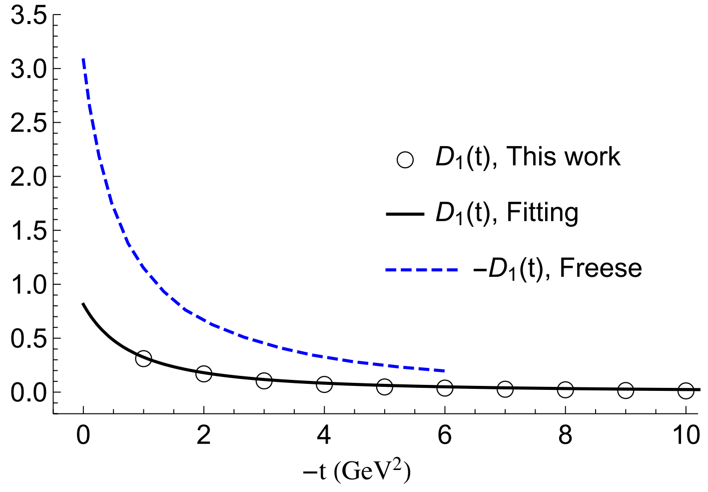
（Abidin et al, 2008; Freese et al, 2019; BDS, Dong, 2019）

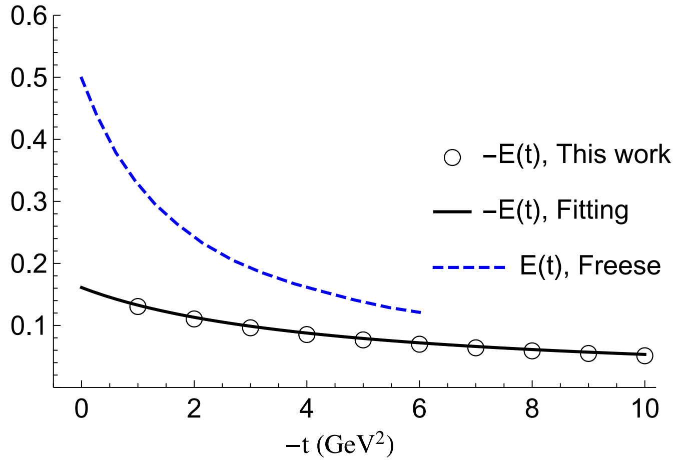






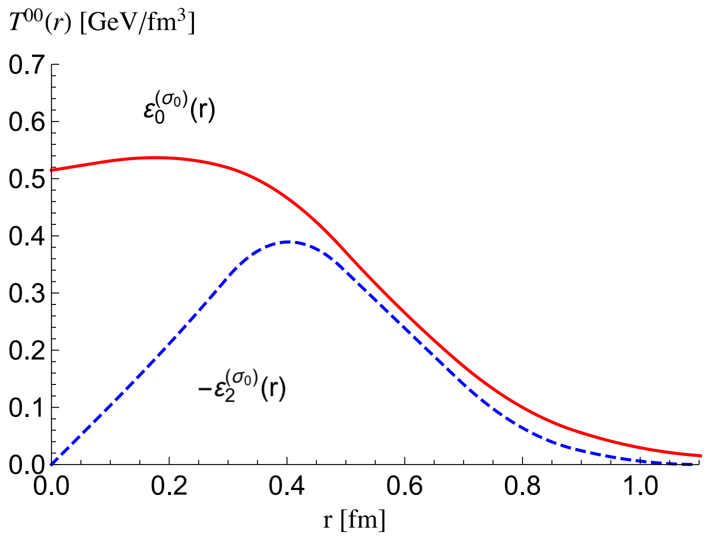




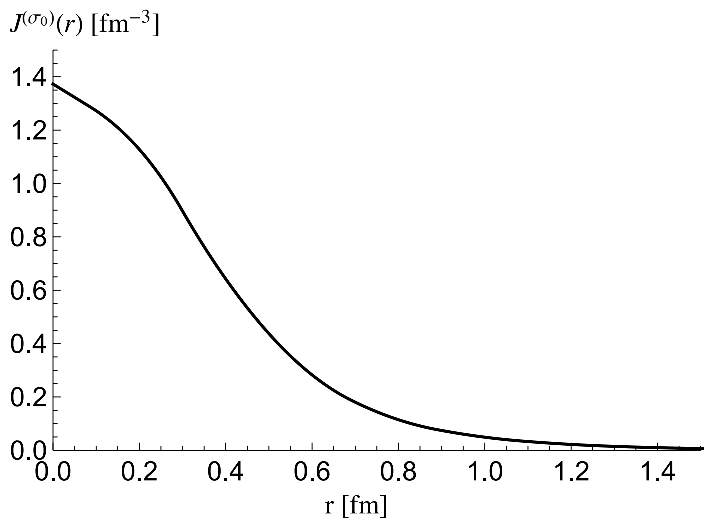


ρ meson densities by a quark model (BDS, Dong, 2017, 2020)

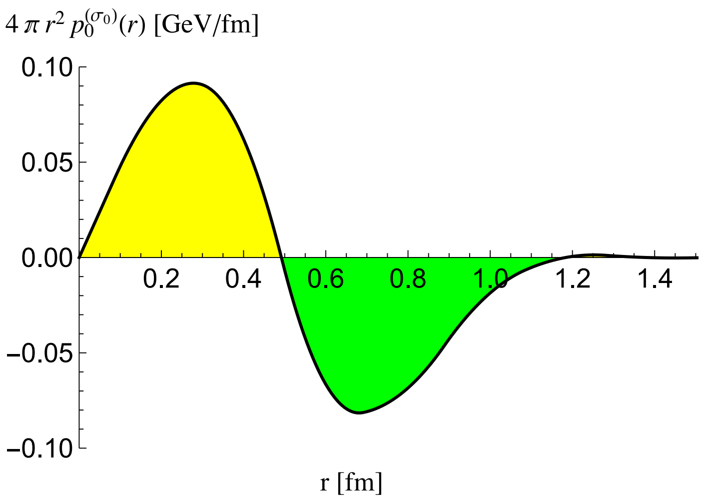
(energy/mass)

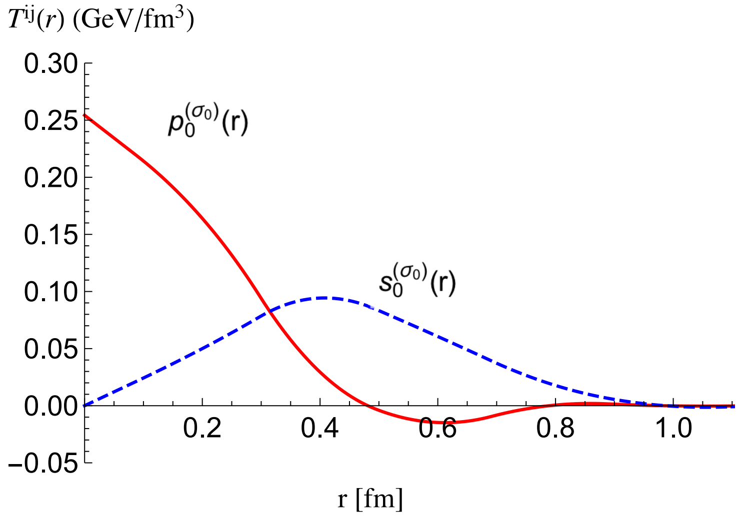


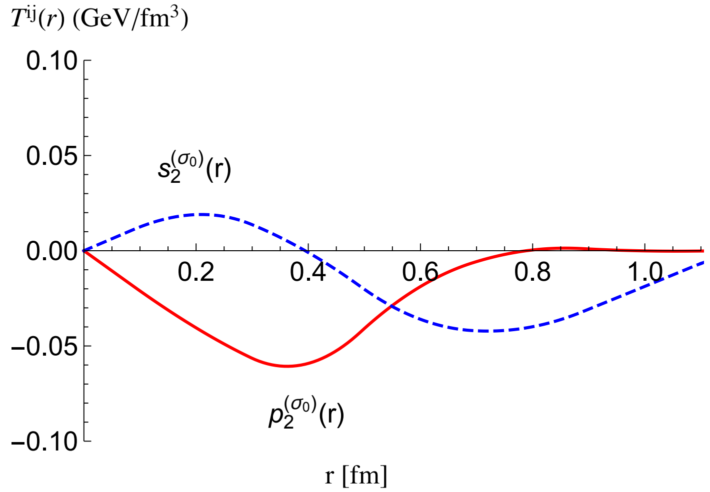
(spin)

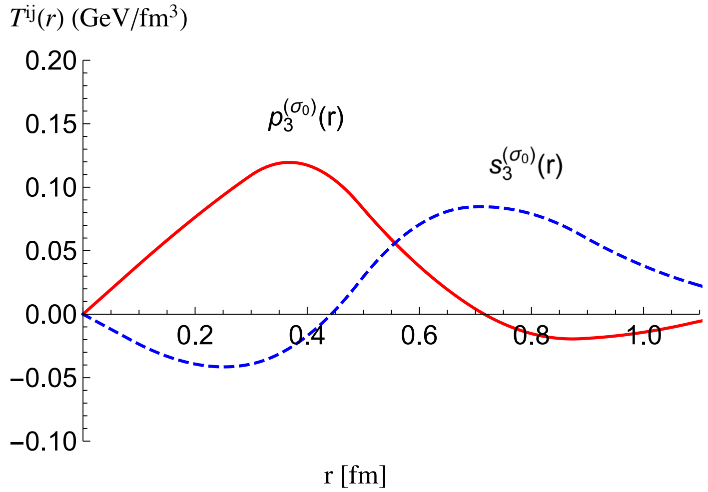


(pressure $p_0(r)$)









$$\frac{2}{3}\frac{ds_n(r)}{dr}+2\frac{s_n(r)}{r}+\frac{dp_n(r)}{dr}=0$$

✓

$$\int d^3r\,p_n(r)=0$$

✓

$$\left.\frac{dF_r}{dS_r}\right|_{\text{unp}}=p_0(r)+\frac{2}{3}s_0(r)\geq 0$$

?

Δ in quark-diquark model

Fu, BDS, Dong, 2201.08059

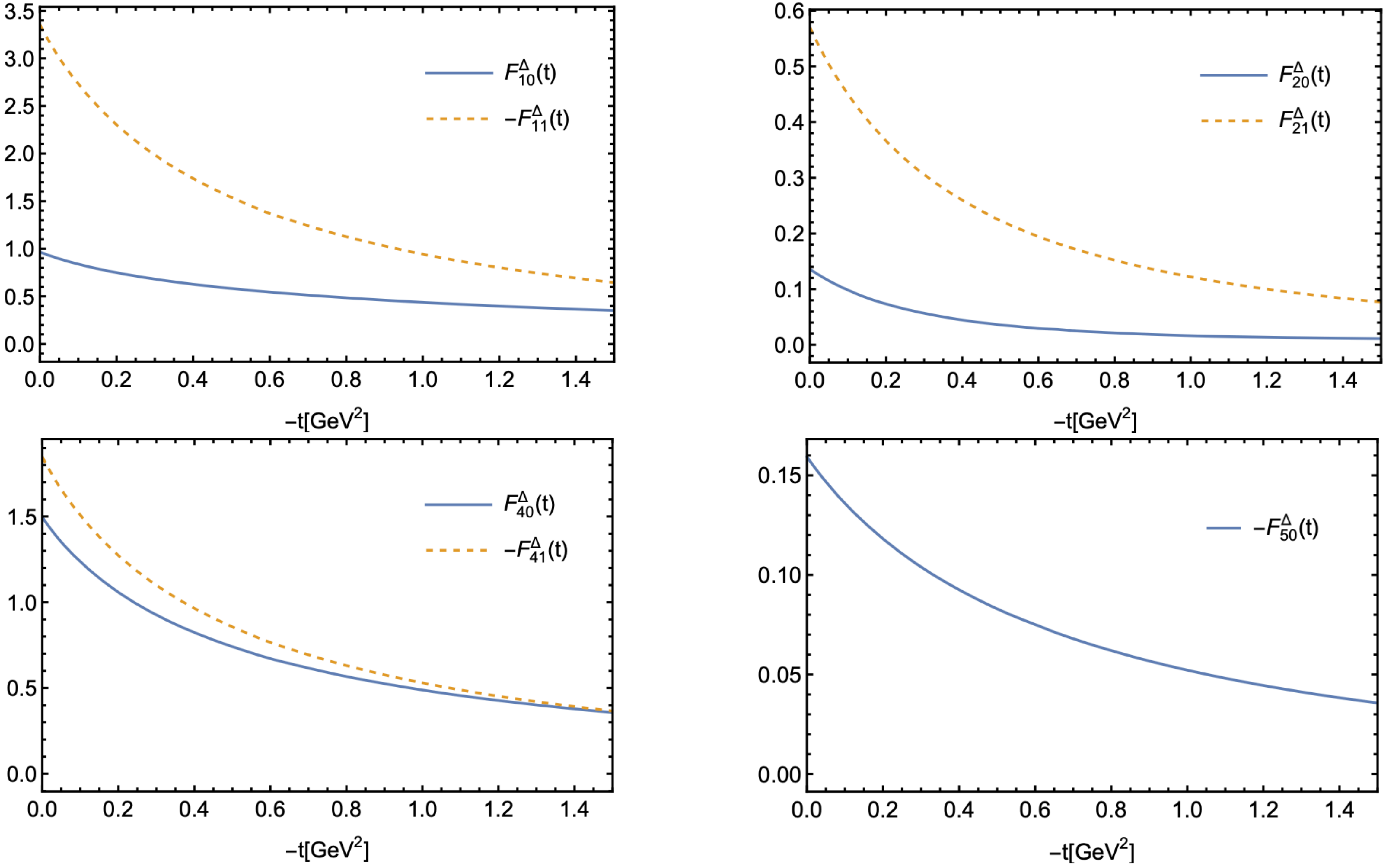


Figure 6: Calculated GFFs of $F^{T}_{10,11,20,21,40,41,50}$ as functions of $-t$ for Δ.

Δ in **quark-diquark model**

Fu, BDS, Dong, 2201.08059

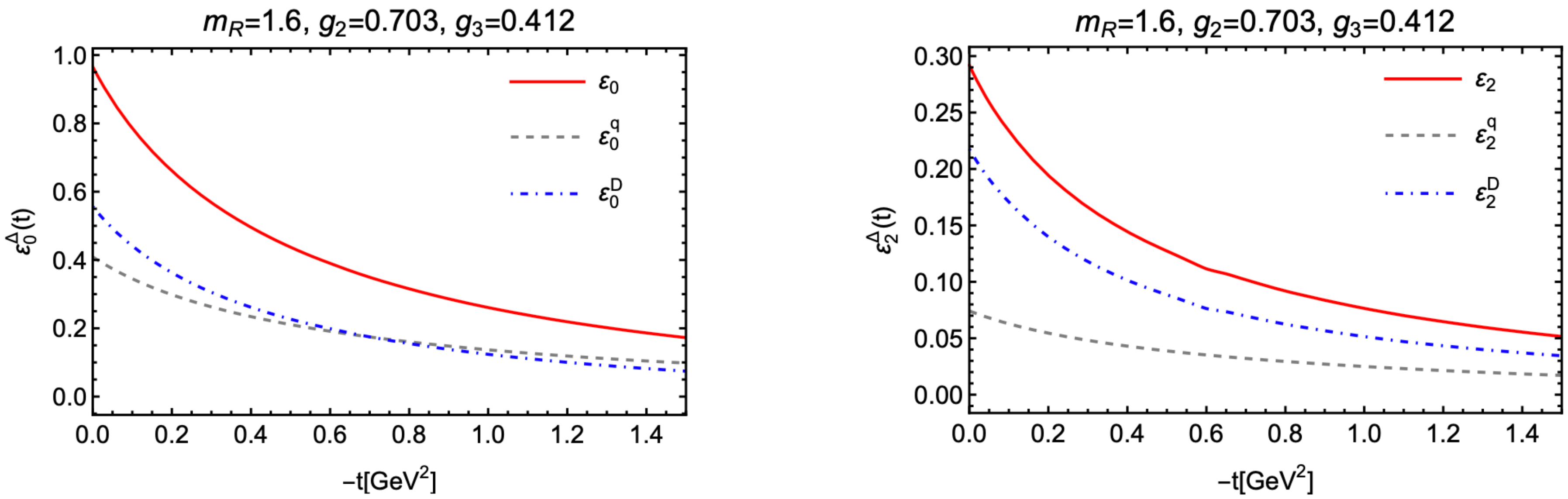


Figure 7: The calculated energy monopole form factor of the Δ as a function of $-t$ (left panel) and the energy quadrupole (right panel). The dashed, dashed-dotted and solid curves stand for the contributions from quark, diquark and their sum.

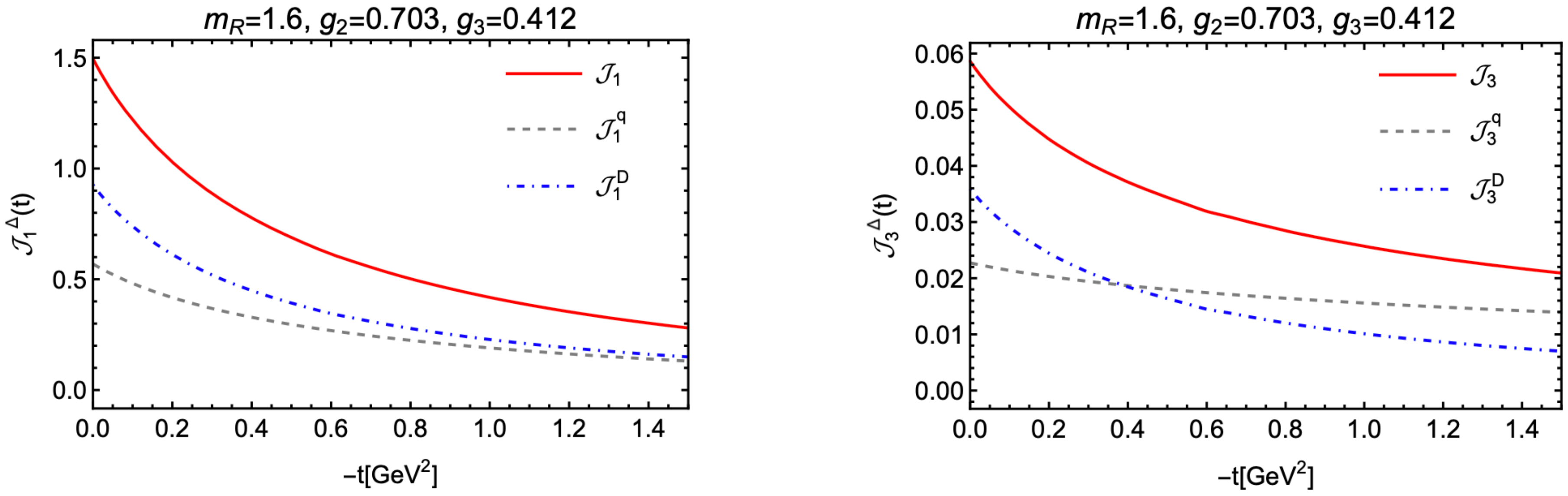
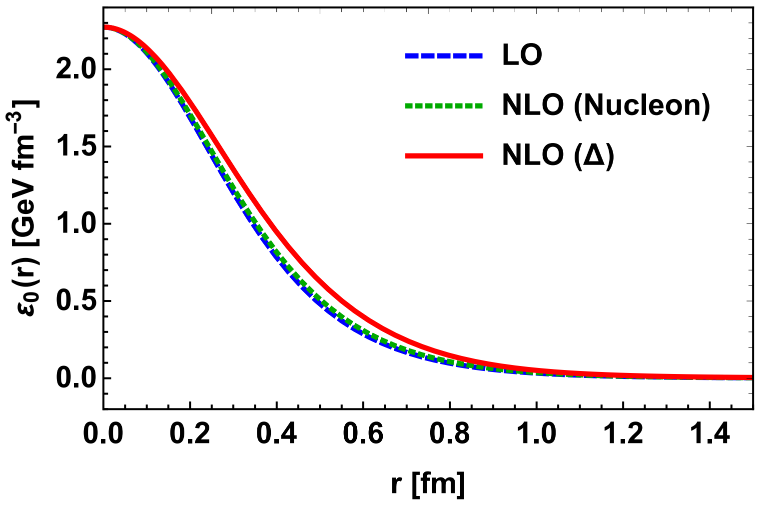


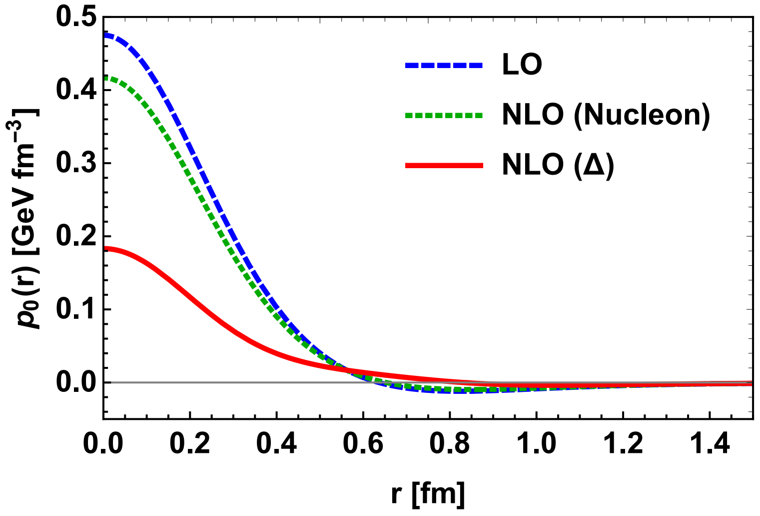
Figure 8: The angular momentum form factor of the Δ as a function of $-t$ (left panel), and the octupole angular momentum form factor (the right panel). The solid, dashed and dashed-dotted curves represent the total result, and the contributions of quark and diquark,respectively.

Δ densities by SU(2) Skyrme model (Kim, BDS, 2020)

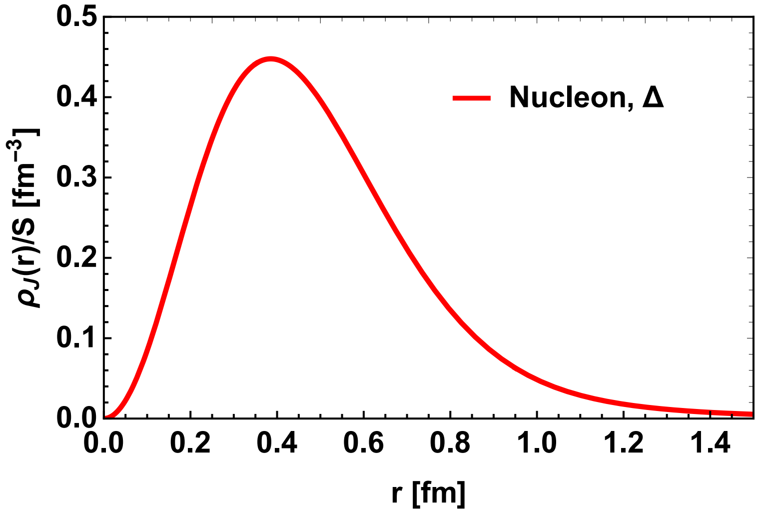
(energy/mass)

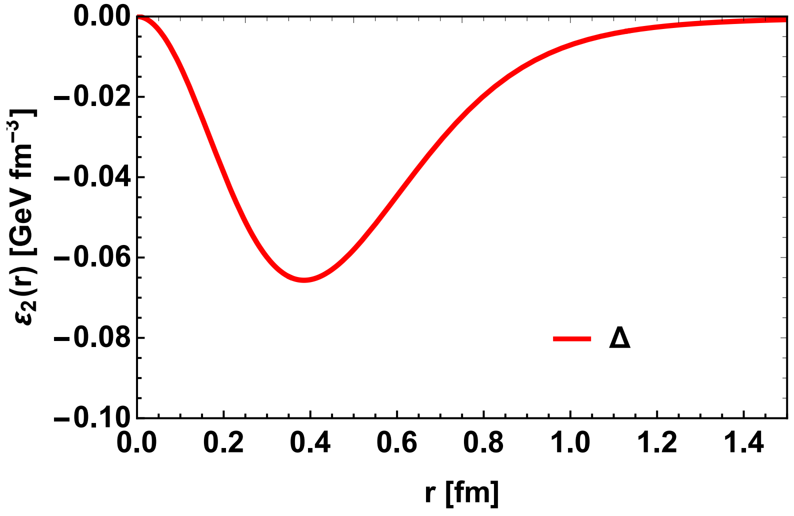


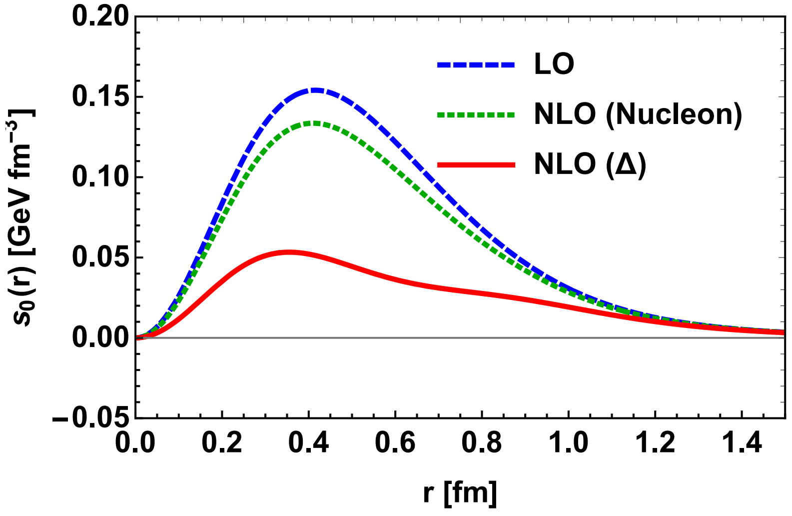
(pressure & shear forces: "mechanical")



(spin)







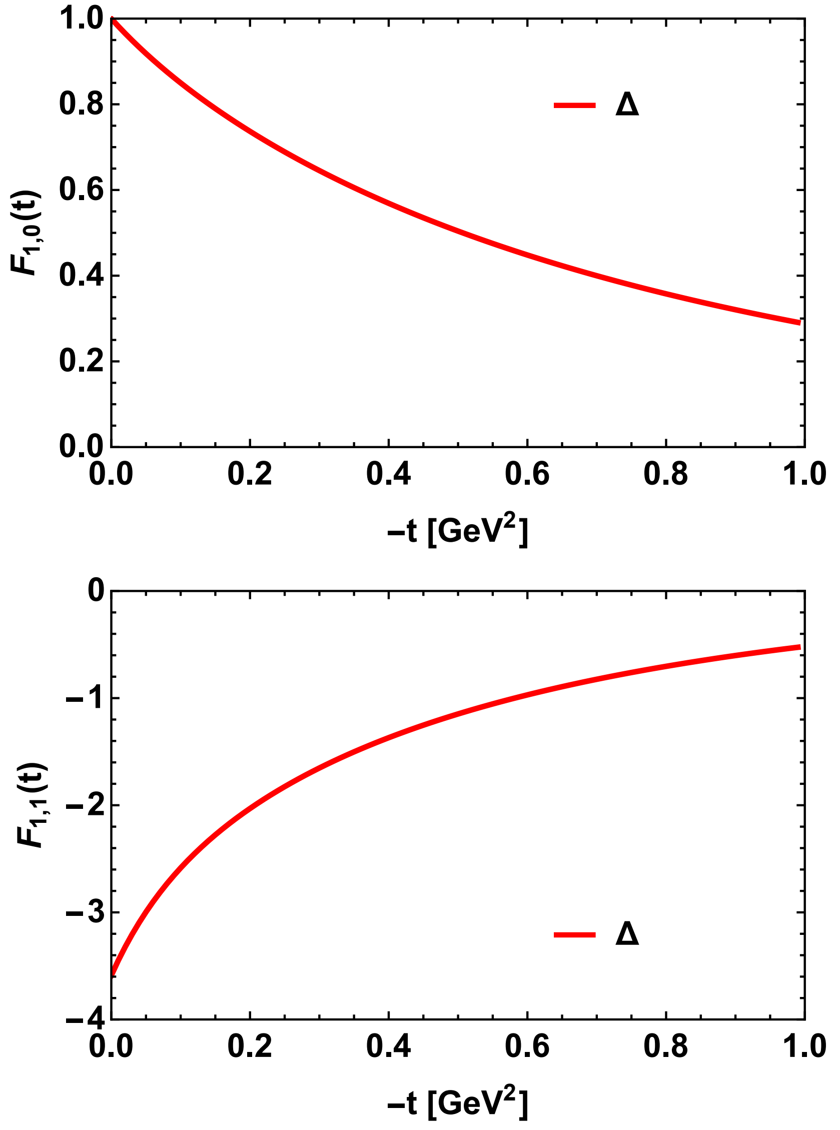
$$\langle r_J^2\rangle_{N,\Delta} = 0.92\,\mathrm{fm}^2$$

$$\langle r_E^2\rangle = 0.54\,\mathrm{fm}^2\;(\mathrm{LO})$$
$$\langle r_E^2\rangle = 0.57\,\mathrm{fm}^2\;(\mathrm{NLO,\ Nucleon})$$
$$\langle r_E^2\rangle = 0.64\,\mathrm{fm}^2\;(\mathrm{NLO,\ }\Delta)$$
$$\mathcal{Q}^{ij}_{\sigma'\sigma} = -0.0181\;Q^{ij}_{\sigma'\sigma}\mathrm{GeV}\cdot\mathrm{fm}^2$$

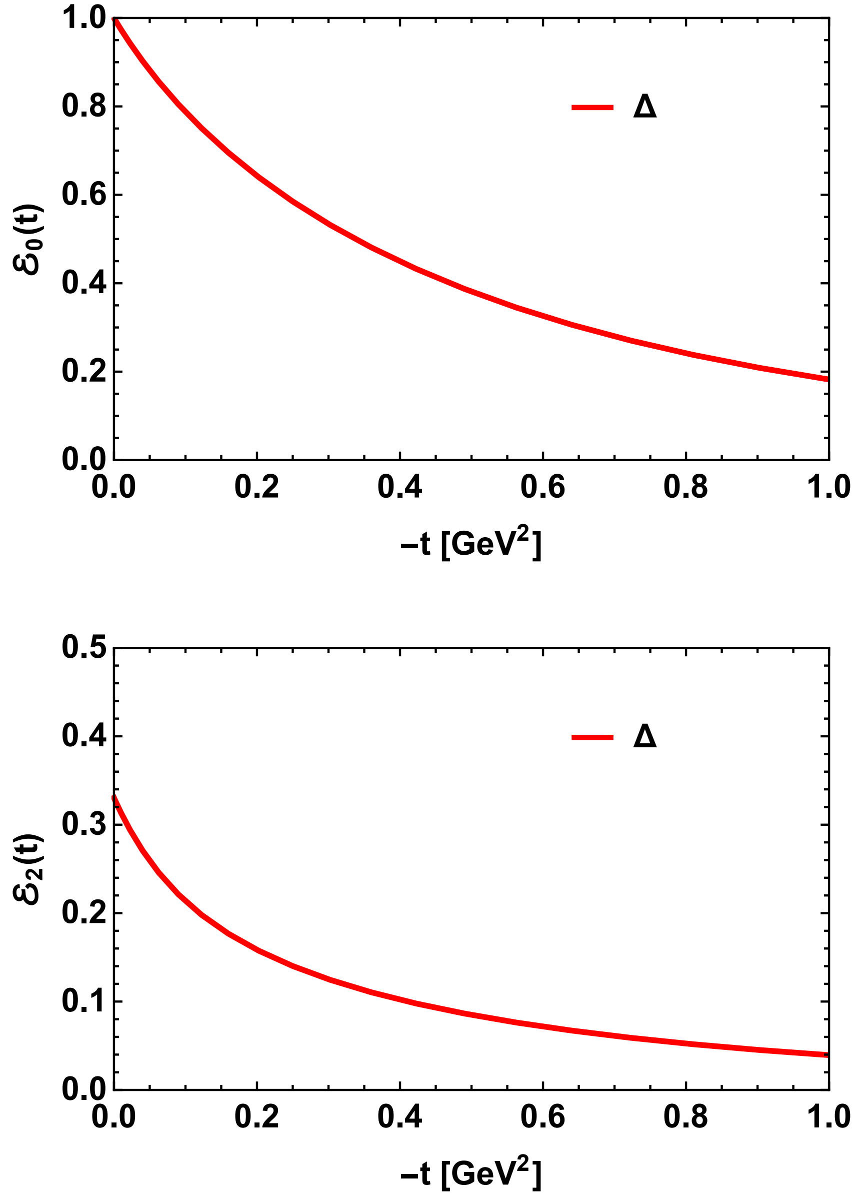
$$\langle r_0^2\rangle_{\mathrm{mech}} = 0.61\,\mathrm{fm}^2\;(\mathrm{LO})$$
$$\langle r_0^2\rangle_{\mathrm{mech}} = 0.63\,\mathrm{fm}^2\;(\mathrm{NLO,\ Nucleon})$$
$$\langle r_0^2\rangle_{\mathrm{mech}} = 0.85\,\mathrm{fm}^2\;(\mathrm{NLO,\ }\Delta)$$
$$\langle r_3^2\rangle_{\mathrm{mech}} = 0.33\,\mathrm{fm}^2$$

$$\mathcal{D}_0^{\Delta} = -3.53 < 0\;(\mathrm{stable!})$$ ✓
$$\mathcal{D}_0^{N} = -3.63$$
$$\mathcal{D}_2 = 0$$
$$\mathcal{D}_3 = -0.50$$

Δ GFFs/GMFFs by SU(2) Skyrme model (Kim, BDS, 2020)



$+ \dots$



$+ \dots$

larg-N_c behaviors:
(GMFFs)

$\mathcal{E}_0(t) \sim \mathcal{O}(N_c^0)$, $\mathcal{E}_2(t) \sim \mathcal{O}(N_c^0)$, $\mathcal{J}_0(t) \sim \mathcal{O}(N_c^0)$, $\mathcal{J}_3(t) \sim \mathcal{O}(N_c^0)$,

$D_0(t) \sim \mathcal{O}(N_c^2)$, $D_2(t) \sim \mathcal{O}(N_c^0)$, $D_3(t) \sim \mathcal{O}(N_c^2)$

✓

Δ GFFs in chiral perturbation theory（ChPT）

Alharazin, Epelbaum, Gegelia, Meissner, Sun, EPJC.82.907

Δ GFFs in chiral perturbation theory（ChPT）

Δ fields（Rarita-Schwinger）

One way::

$$u^{\alpha}(p,\lambda)=\sum_{\rho,\sigma}C^{\frac{3}{2}\lambda}_{1\rho,\frac{1}{2}\sigma}\epsilon^{\alpha}(p,\rho)u(p,\sigma)$$

The other：

$$\Psi_{\mu}(x)=\sum_{s_\Delta}\int\frac{d^3p}{(2\pi)^3}\frac{M_\Delta}{E}\left[b\left(\vec{p},s_\Delta\right)u_\mu\left(\vec{p},s_\Delta\right)e^{-ip\cdot x}+d^{\dagger}\left(\vec{p},s_\Delta\right)v_\mu\left(\vec{p},s_\Delta\right)e^{ip\cdot x}\right]$$

$$\Psi^1_\mu=\frac{1}{\sqrt{2}}\left[\Delta^{++}-\frac{1}{\sqrt{3}}\Delta^0,\frac{1}{\sqrt{3}}\Delta^+-\Delta^-\right]^T_\mu,$$

$$\Psi^2_\mu=-\frac{i}{\sqrt{2}}\left[\Delta^{++}+\frac{1}{\sqrt{3}}\Delta^0,\frac{1}{\sqrt{3}}\Delta^++\Delta^-\right]^T_\mu,$$

$$\Psi^3_\mu=\sqrt{\frac{2}{3}}\left[\Delta^+,\Delta^0\right]^T_\mu.$$

Δ in chiral perturbation theory（ChPT）

Building blocks：

$$
\begin{aligned}
D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \,, \\
\nabla_\mu \Psi^i_\nu &= \nabla^{ij}_\mu \Psi^j_\nu = \left[\delta^{ij}\partial_\mu + \delta^{ij}\Gamma_\mu - i\delta^{ij} v^{(s)}_\mu - i\epsilon^{ijk}\mathrm{Tr}\left(\tau^k \Gamma_\mu\right) + \frac{i}{2}\delta^{ij}\omega^{ab}_\mu \sigma_{ab}\right]\Psi^j_\nu - \Gamma^\alpha_{\mu\nu}\Psi^i_\alpha, \\
\nabla_\mu \bar{\Psi}^i_\nu &= \nabla^{ij}_\mu \Psi^j_\nu = \bar{\Psi}^j_\nu \left[\delta^{ij}\partial_\mu - \delta^{ij}\Gamma_\mu + i\delta^{ij} v^{(s)}_\mu + i\epsilon^{ijk}\mathrm{Tr}\left(\tau^k \Gamma_\mu\right) - \frac{i}{2}\delta^{ij}\omega^{ab}_\mu \sigma_{ab}\right] + \bar{\Psi}^i_\alpha \Gamma^\alpha_{\mu\nu}, \\
\nabla_\mu \Psi &= \partial_\mu \Psi + \frac{i}{2}\,\omega^{ab}_\mu \sigma_{ab}\Psi + \left(\Gamma_\mu - i v^{(s)}_\mu\right)\Psi \,, \\
\nabla_\mu \bar{\Psi} &= \partial_\mu \bar{\Psi} - \frac{i}{2}\,\bar{\Psi}\,\sigma_{ab}\,\omega^{ab}_\mu - \bar{\Psi}\left(\Gamma_\mu - i v^{(s)}_\mu\right) \,, \\
u_\mu &= i\left[u^\dagger \partial_\mu u - u \partial_\mu u^\dagger - i(u^\dagger v_\mu u - u v_\mu u^\dagger)\right] \,, \\
\chi &= 2B_0(s + ip) \,, \\
\Gamma_\mu &= \frac{1}{2}\left[u^\dagger \partial_\mu u + u \partial_\mu u^\dagger - i(u^\dagger v_\mu u + u v_\mu u^\dagger)\right] \,, \\
\omega^{ab}_\mu &= -g^{\nu\lambda} e^a_\lambda \left(\partial_\mu e^b_\nu - e^b_\sigma \Gamma^\sigma_{\mu\nu}\right) , \\
\Gamma^\lambda_{\alpha\beta} &= \frac{1}{2}\,g^{\lambda\sigma}\left(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right) \,, \\
R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \,, \\
R &= g^{\mu\nu} R^\lambda{}_{\mu\lambda\nu} \,.
\end{aligned}
$$

e^a_μ is vielbein gravitational fields：

$$
\begin{aligned}
e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}, &\quad e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu}, \\
e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}, &\quad e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}.
\end{aligned}
$$

Actions

$$
\begin{aligned}
S_{\pi}^{(2)} &= \int d^4x\sqrt{-g}\,\left\{\frac{F^2}{4}\,g^{\mu\nu}\,\mathrm{Tr}(D_\mu U(D_\nu U)^\dagger) + \frac{F^2}{4}\,\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right\}\,, \\
S_{\pi\mathrm{N}}^{(1)} &= \int d^4x\sqrt{-g}\left\{\bar\Psi\, i\gamma^\mu \overset{\leftrightarrow}{\nabla}_\mu \Psi - m\bar\Psi\Psi + \frac{g_A}{2}\,\bar\Psi\gamma^\mu\gamma_5 u_\mu \Psi\right\}, \\
S_{\pi\Delta}^{(1)} &= -\int d^4x\sqrt{-g}\Big[g^{\mu\nu}\,\bar\Psi^i_\mu\, i\gamma^\alpha\overset{\leftrightarrow}{\nabla}_\alpha\Psi^i_\nu - m_\Delta\, g^{\mu\nu}\bar\Psi^i_\mu\Psi^i_\nu - g^{\lambda\sigma}\left(\bar\Psi^i_\mu i\gamma^\mu\overset{\leftrightarrow}{\nabla}_\lambda\Psi^i_\sigma + \bar\Psi^i_\lambda i\gamma^\mu\overset{\leftrightarrow}{\nabla}_\sigma\Psi^i_\mu\right) \\
&+\; i\bar\Psi^i_\mu\gamma^\mu\gamma^\alpha\gamma^\nu\overset{\leftrightarrow}{\nabla}_\alpha\Psi^i_\nu + m_\Delta\bar\Psi^i_\mu\gamma^\mu\gamma^\nu\Psi^i_\nu + \frac{g_1}{2}\,g^{\mu\nu}\bar\Psi^i_\mu u_\alpha\gamma^\alpha\gamma_5\Psi^i_\nu + \frac{g_2}{2}\bar\Psi^i_\mu\,(u^\mu\gamma^\nu + u^\nu\gamma^\mu)\,\gamma_5\Psi^i_\nu \\
&+\; \frac{g_3}{2}\bar\Psi^i_\mu u_\alpha\gamma^\mu\gamma^\alpha\gamma_5\gamma^\nu\Psi^i_\nu\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
S_{\pi N\Delta}^{(1)} &= -\int d^4x\sqrt{-g}\; g_{\pi N\Delta}\;\bar\Psi_{\mu,i}\,(g^{\mu\nu} - \gamma^\mu\gamma^\nu)\,u_{\nu,i}\Psi + \mathrm{H.c.} \\
S_{\pi\Delta,a}^{(2)} &= \int d^4x\sqrt{-g}\,a_1\,\bar\Psi^i_\mu\Theta^{\mu\alpha}(z)\,\langle\chi_+\rangle\, g_{\alpha\beta}\Theta^{\beta\nu}(z')\Psi^i_\nu
\end{aligned}
$$

where： $\gamma_\mu \equiv e^a_\mu\gamma_a$

Actions

$$\begin{aligned} S^{(2)}_{\pi\Delta,b} \;=\; & \int d^4x\sqrt{-g}\Bigg[h_1 R\; g^{\alpha\beta}\bar{\Psi}^i_\alpha\Psi^i_\beta + h_2 R\;\bar{\Psi}^i_\alpha\gamma^\alpha\gamma^\beta\Psi^i_\beta + ih_3 R\left(g^{\alpha\lambda}\bar{\Psi}^i_\alpha\gamma^\beta\overrightarrow{\nabla}_\lambda\Psi^i_\beta - g^{\beta\lambda}\bar{\Psi}^i_\alpha\gamma^\alpha\overleftarrow{\nabla}_\lambda\Psi^i_\beta\right) \\ + \; & h_4 R^{\mu\nu}\;\bar{\Psi}^i_\mu\Psi^i_\nu + 2ih_5 R^{\mu\nu}\; g^{\alpha\beta}\bar{\Psi}^i_\alpha\gamma_\mu\overleftrightarrow{\nabla}_\nu\Psi^i_\beta + ih_6 R^{\mu\nu}g^{\alpha\beta}\left(\bar{\Psi}^i_\alpha\gamma_\mu\overrightarrow{\nabla}_\beta\Psi^i_\nu - \bar{\Psi}^i_\nu\gamma_\mu\overleftarrow{\nabla}_\beta\Psi^i_\alpha\right) \\ + \; & ih_7 R^{\mu\nu}\left(\bar{\Psi}^i_\alpha\gamma^\alpha\overrightarrow{\nabla}_\mu\Psi^i_\nu - \bar{\Psi}^i_\nu\gamma^\alpha\overleftarrow{\nabla}_\mu\Psi^i_\alpha\right) + h_8 R^{\mu\nu}\left(\bar{\Psi}^i_\alpha\gamma^\alpha\gamma_\mu\Psi^i_\nu + \bar{\Psi}^i_\nu\gamma_\mu\gamma^\alpha\Psi^i_\alpha\right) \\ + \; & ih_9 R^{\mu\nu}\left(\bar{\Psi}^i_\kappa\gamma^\kappa\gamma^\alpha\gamma_\mu\overrightarrow{\nabla}_\nu\Psi^i_\alpha - \bar{\Psi}^i_\alpha\gamma_\mu\gamma^\alpha\gamma^\kappa\overleftarrow{\nabla}_\nu\Psi^i_\kappa\right) + ih_{10}R^{\mu\nu\alpha\beta}\bar{\Psi}^i_\alpha\sigma_{\mu\nu}\Psi^i_\beta \\ + \; & i\left[h_{11}\;R^{\mu\nu\alpha\beta} + h_{12}\;R^{\mu\alpha\nu\beta}\right]\left(\bar{\Psi}^i_\alpha\gamma_\mu\overrightarrow{\nabla}_\nu\Psi^i_\beta - \bar{\Psi}^i_\beta\gamma_\mu\overleftarrow{\nabla}_\nu\Psi^i_\alpha\right) + h_{13}R^{\mu\alpha\nu\beta}\bar{\Psi}^i_\alpha\gamma_\mu\gamma_\nu\Psi^i_\beta \\ + \; & i\left[h_{14}\;R^{\mu\nu\alpha\beta} + h_{15}\;R^{\mu\alpha\nu\beta}\right]\left(\bar{\Psi}^i_\kappa\gamma^\kappa\gamma_\mu\gamma_\nu\overrightarrow{\nabla}_\alpha\Psi^i_\beta - \bar{\Psi}^i_\beta\gamma_\nu\gamma_\mu\gamma^\kappa\overleftarrow{\nabla}_\alpha\Psi^i_\kappa\right)\Bigg], \end{aligned}$$

$$T_{\mu\nu}(g,\psi) \;=\; \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}$$

$$T_{\mu\nu}(g,\psi) \;=\; \frac{1}{2e}\left[\frac{\delta S}{\delta e^{a\mu}}\,e^a_\nu + \frac{\delta S}{\delta e^{a\nu}}\,e^a_\mu\right]$$

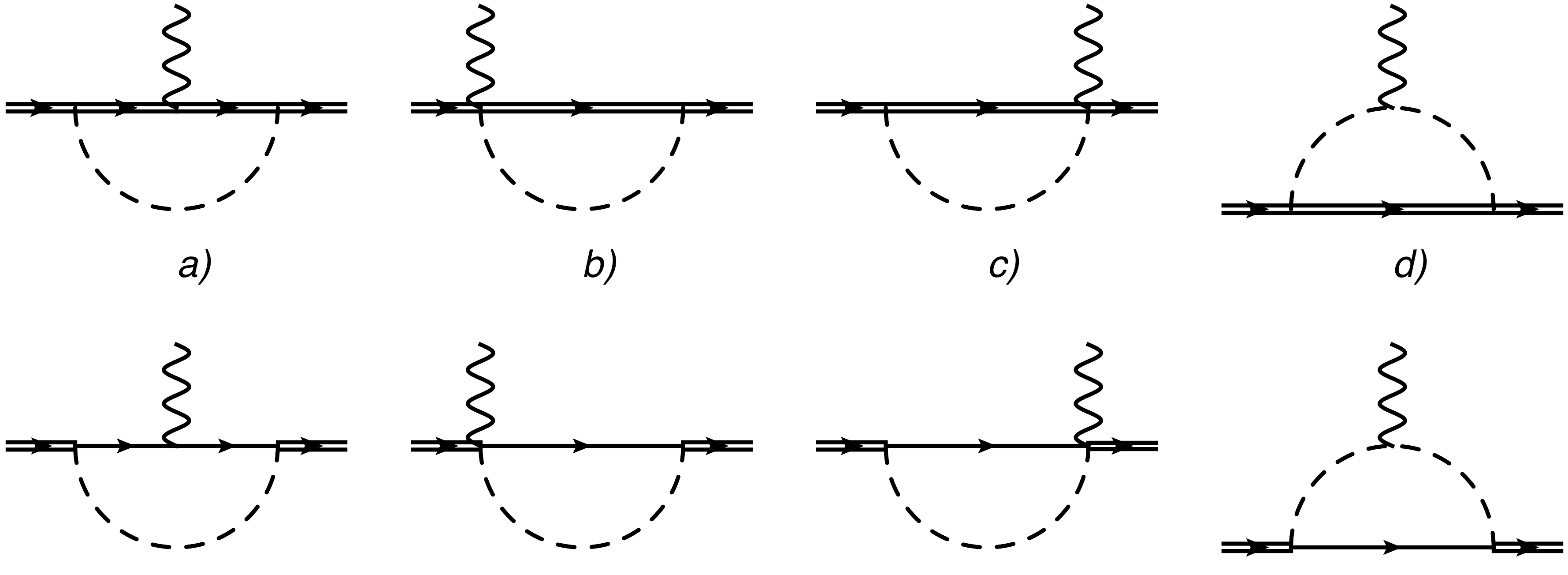
EMTs in ChPT

$$
\begin{aligned}
T^{(2)}_{\pi,\mu\nu} &= \frac{F^2}{4}\,\mathrm{Tr}(D_\mu U (D_\nu U)^\dagger) - \frac{\eta_{\mu\nu}}{2}\left\{\frac{F^2}{4}\,\mathrm{Tr}(D^\alpha U (D_\alpha U)^\dagger) + \frac{F^2}{4}\,\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right\} + (\mu\leftrightarrow\nu)
\end{aligned}
$$

$$
\begin{aligned}
T^{(1)}_{\pi N,\mu\nu} &= \frac{i}{2}\bar\Psi\,\gamma_\mu \overleftrightarrow{D}_\nu \Psi + \frac{g_A}{4}\bar\Psi\,\gamma_\mu\gamma_5 u_\nu \Psi - \frac{\eta_{\mu\nu}}{2}\left(\bar\Psi\, i\gamma^\alpha \overleftrightarrow{D}_\alpha \Psi - m\bar\Psi\Psi + \frac{g_A}{2}\,\bar\Psi\gamma^\alpha\gamma_5 u_\alpha \Psi\right) + (\mu\leftrightarrow\nu)
\end{aligned}
$$

$$
\begin{aligned}
T^{(1)}_{\pi\Delta,\mu\nu} &= -\bar\Psi^i_\mu\, i\gamma^\alpha \overleftrightarrow{D}_\alpha \Psi^i_\nu + \bar\Psi^i_\alpha\, i\gamma^\alpha \overleftrightarrow{D}_\mu \Psi^i_\nu + \bar\Psi^i_\mu\, i\gamma^\alpha \overleftrightarrow{D}_\nu \Psi^i_\alpha + m_\Delta \bar\Psi^i_\mu \Psi^i_\nu - \frac{i}{2}\,\bar\Psi^i_\alpha\,\gamma_\mu \overleftrightarrow{D}_\nu \Psi^{i\alpha} \\
&+ \frac{i}{2}\Big(\bar\Psi^i_\mu\,\gamma_\nu \overleftrightarrow{D}_\alpha \Psi^{i\alpha} + \bar\Psi^{i\alpha}\,\gamma_\nu \overleftrightarrow{D}_\alpha \Psi^i_\mu - \bar\Psi^i_\mu\,\gamma_\nu\gamma^\alpha\gamma_\beta \overleftrightarrow{D}_\alpha \Psi^{i,\beta} - \bar\Psi^i_\alpha\gamma^\alpha\gamma_\nu\gamma^\beta \overleftrightarrow{D}_\mu \Psi^i_\beta - \bar\Psi^i_\alpha\gamma^\alpha\gamma^\beta\gamma_\nu \overleftrightarrow{D}_\beta \Psi^i_\mu\Big) \\
&+ \frac{i}{4}\,\partial^\lambda\Big[\bar\Psi^{i,\alpha}\Big(\gamma_\mu \eta_{\lambda[\alpha}\eta_{\beta]\mu} + \eta_{\lambda\mu}\eta_{\nu[\alpha}\gamma_{\beta]} + \eta_{\mu\nu}\eta_{\lambda[\beta}\gamma_{\alpha]}\Big)\Psi^{i,\beta}\Big] - \frac{m_\Delta}{2}\,\left(\bar\Psi^i_\mu\,\gamma_\nu\gamma^\alpha\Psi^i_\alpha + \bar\Psi^i_\alpha\,\gamma^\alpha\gamma_\nu\Psi^i_\mu\right) \\
&- \frac{g_1}{4}\,[2\bar\Psi^i_\mu u_\alpha\gamma^\alpha\gamma_5\Psi^i_\nu + \bar\Psi^{i,\alpha}u_\mu\gamma_\nu\gamma_5\Psi^i_\alpha] - \frac{g_2}{4}\,[2\bar\Psi^i_\mu u_\nu\gamma^\alpha\gamma_5\Psi^i_\alpha + 2\bar\Psi^i_\alpha u_\nu\gamma^\alpha\gamma_5\Psi^i_\mu \\
&+ \bar\Psi^{i,\alpha}u_\alpha\gamma_\nu\gamma_5\Psi^i_\mu + \bar\Psi^i_\mu u_\alpha\gamma_\nu\gamma_5\Psi^{i\alpha}] - \frac{g_3}{4}\,[\bar\Psi^i_\mu u_\alpha\gamma_\nu\gamma^\alpha\gamma_5\gamma^\beta\Psi^i_\beta + \bar\Psi^i_\beta u_\alpha\gamma^\beta\gamma^\alpha\gamma_5\gamma_\nu\Psi^i_\mu \\
&+ \bar\Psi^i_\alpha u_\mu\gamma^\alpha\gamma_\nu\gamma_5\gamma^\beta\Psi^i_\beta] + \frac{\eta_{\mu\nu}}{2}\Big[\bar\Psi^i_\alpha\, i\gamma^\beta \overleftrightarrow{D}_\beta \Psi^{i\alpha} - m_\Delta\,\bar\Psi^i_\alpha\Psi^{i\alpha} - \bar\Psi^i_\alpha i\gamma^\alpha \overleftrightarrow{D}_\beta \Psi^{i\beta} - \bar\Psi^{i\alpha} i\gamma^\beta \overleftrightarrow{D}_\alpha \Psi^i_\beta \\
&+ i\bar\Psi^i_\rho\gamma^\rho\gamma^\alpha\gamma^\lambda \overleftrightarrow{D}_\alpha \Psi^i_\lambda + m_\Delta\bar\Psi^i_\alpha\gamma^\alpha\gamma^\beta\Psi^i_\beta + \frac{g_1}{2}\,\bar\Psi^i_\beta u_\alpha\gamma^\alpha\gamma_5\Psi^{i\beta} + \frac{g_2}{2}\bar\Psi^{i\alpha}\,(u_\alpha\gamma_\beta + u_\beta\gamma_\alpha)\,\gamma_5\Psi^{i\beta} \\
&+ \frac{g_3}{2}\bar\Psi^i_\alpha u_\beta\gamma^\alpha\gamma^\beta\gamma_5\gamma^\lambda\Psi^i_\lambda\Big] + (\mu\leftrightarrow\nu)\ , \\
T^{(1)}_{\pi N\Delta,\mu\nu} &= \frac{1}{2}g_{\pi N\Delta}\ \eta_{\mu\nu}\left[\bar\Psi^i_\alpha u^\alpha_i\Psi + \bar\Psi u^\alpha_i\Psi^i_\alpha - \bar\Psi^i_\alpha\gamma^\alpha\gamma^\beta u^i_\beta\Psi - \bar\Psi\gamma^\beta\gamma^\alpha u^i_\beta\Psi^i_\alpha\right] - g_{\pi N\Delta}\ \left(\bar\Psi^i_\mu u^i_\nu\Psi + \bar\Psi u^i_\nu\Psi^i_\mu\right) \\
&+ \frac{1}{2}g_{\pi N\Delta}\ \left[\bar\Psi^i_\mu\gamma_\nu\gamma^\alpha u^i_\alpha\Psi + \bar\Psi^i_\alpha\gamma^\alpha\gamma_\mu u^i_\nu\Psi + \bar\Psi\gamma^\alpha\gamma_\nu u^i_\alpha\Psi^i_\mu + \bar\Psi\gamma_\mu\gamma^\alpha u^i_\nu\Psi^i_\alpha\right] + (\mu\leftrightarrow\nu)\,, \\
T^{(2)}_{\pi\Delta,a,\mu\nu} &= a_1\ \bar\Psi^i_\mu\,\langle\chi_+\rangle\,\Psi^i_\nu + \frac{\tilde z}{2}a_1\ \left(\bar\Psi^i_\mu\gamma_\nu\gamma^\alpha\,\langle\chi_+\rangle\,\Psi^i_\alpha + \bar\Psi^i_\alpha\gamma^\alpha\gamma_\mu\,\langle\chi_+\rangle\,\Psi^i_\nu\right) - \frac{a_1}{2}\eta_{\mu\nu}\,[\bar\Psi^i_\alpha\,\langle\chi_+\rangle\,\Psi^{i\alpha} \\
&+ \tilde z\ \bar\Psi^i_\alpha\gamma^\alpha\gamma^\beta\,\langle\chi_+\rangle\,\Psi^i_\beta] + (\mu\leftrightarrow\nu)\,,
\end{aligned}
$$

$$\begin{aligned}
T^{(2)}_{\pi\Delta,b,\mu\nu} \;=\;& h_1\left(\eta_{\mu\nu}\partial_\lambda\partial^\lambda-\partial_\mu\partial_\nu\right)\bar{\Psi}^i_\alpha\Psi^{i\alpha}+\frac{h_4}{2}\left[\partial^\lambda\partial_\lambda\left(\bar{\Psi}^i_\nu\Psi^i_\mu\right)+\eta_{\mu\nu}\partial^\alpha\partial^\beta\left(\bar{\Psi}^i_\beta\Psi^i_\alpha\right)\right.\\
-\;& \left.\partial^\lambda\partial_\mu\left(\bar{\Psi}^i_{(\lambda}\Psi^i_{\nu)}\right)\right]+ih_5\left[\partial^\lambda\partial_\lambda\left(\bar{\Psi}^i_\alpha\gamma_\mu\overset{\leftrightarrow}{D}_\nu\Psi^{i\alpha}\right)+\eta_{\mu\nu}\partial^\kappa\partial^\beta\left(\bar{\Psi}^i_\alpha\gamma_\beta\overset{\leftrightarrow}{D}_\kappa\Psi^{i\alpha}\right)-\partial^\lambda\partial_\mu\left(\bar{\Psi}^i_\alpha\gamma_{(\lambda}\overset{\leftrightarrow}{D}_{\nu)}\Psi^{i\alpha}\right)\right]\\
+\;& \frac{ih_6}{2}\left[\partial^\lambda\partial_\lambda\left(\bar{\Psi}^{i\alpha}\gamma_\mu\vec{D}_\alpha\Psi^i_\nu-\bar{\Psi}_\nu\gamma_\mu\overset{\leftarrow}{D}_\alpha\Psi^{i\alpha}\right)+\eta_{\mu\nu}\partial^\kappa\partial^\beta\left(\bar{\Psi}^{i\alpha}\gamma_\beta\vec{D}_\alpha\Psi^i_\kappa-\bar{\Psi}^i_\kappa\gamma_\beta\overset{\leftarrow}{D}_\alpha\Psi^{i\alpha}\right)\right.\\
-\;& \left.\partial^\lambda\partial_\mu\left(\bar{\Psi}^{i\alpha}\gamma_{(\lambda}\vec{D}_\alpha\Psi^i_{\nu)}-\bar{\Psi}^i_{(\nu}\gamma_{\lambda)}\overset{\leftarrow}{D}_\alpha\Psi^{i\alpha}\right)\right]+ih_{10}\;\partial^\kappa\partial^\beta\left(\bar{\Psi}^i_\kappa\sigma_{\beta\nu}\Psi^i_\mu-\bar{\Psi}^i_\mu\sigma_{\beta\nu}\Psi^i_\kappa\right)\\
+\;& \frac{ih_{11}}{2}\partial^\kappa\partial^\beta\left[\bar{\Psi}^i_\kappa\gamma_\beta\vec{D}_\mu\Psi^i_\nu-\bar{\Psi}^i_\kappa\gamma_\nu\vec{D}_\beta\Psi^i_\mu+\bar{\Psi}^i_\mu\gamma_\nu\vec{D}_\beta\Psi^i_\kappa-\bar{\Psi}^i_\nu\gamma_\beta\vec{D}_\mu\Psi^i_\kappa-\bar{\Psi}^i_\nu\gamma_\beta\overset{\leftarrow}{D}_\mu\Psi^i_\kappa+\bar{\Psi}^i_\mu\gamma_\nu\overset{\leftarrow}{D}_\beta\Psi^i_\kappa\right.\\
-\;& \left.\bar{\Psi}^i_\kappa\gamma_\nu\overset{\leftarrow}{D}_\beta\Psi^i_\mu+\bar{\Psi}^i_\kappa\gamma_\beta\overset{\leftarrow}{D}_\mu\Psi^i_\nu\right]+\frac{ih_{12}}{2}\partial^\kappa\partial^\beta\left[\bar{\Psi}^i_\mu\gamma_\beta\vec{D}_\kappa\Psi^i_\nu-\bar{\Psi}^i_\beta\gamma_\nu\vec{D}_\kappa\Psi^i_\mu+\bar{\Psi}^i_\beta\gamma_\nu\vec{D}_\mu\Psi^i_\kappa-\bar{\Psi}^i_\mu\gamma_\beta\vec{D}_\nu\Psi^i_\kappa\right.\\
-\;& \left.\bar{\Psi}^i_\nu\gamma_\beta\overset{\leftarrow}{D}_\kappa\Psi^i_\mu+\bar{\Psi}^i_\mu\gamma_\nu\overset{\leftarrow}{D}_\kappa\Psi^i_\beta-\bar{\Psi}^i_\kappa\gamma_\nu\overset{\leftarrow}{D}_\mu\Psi^i_\beta+\bar{\Psi}^i_\kappa\gamma_\beta\overset{\leftarrow}{D}_\nu\Psi^i_\mu\right]+\frac{h_{13}}{2}\partial^\kappa\partial^\beta\left[\eta_{\mu\nu}\bar{\Psi}^i_\beta\Psi^i_\kappa-\bar{\Psi}^i_\beta\gamma_\nu\gamma_\kappa\Psi^i_\mu\right.\\
-\;& \left.\bar{\Psi}^i_\mu\gamma_\beta\gamma_\nu\Psi^i_\kappa+\bar{\Psi}^i_\mu\gamma_\beta\gamma_\kappa\Psi^i_\nu\right]+(\mu\leftrightarrow\nu)\;.
\end{aligned}$$



a) b) c) d)

e) f) g) h)

GFFs at Tree order

$$
\begin{aligned}
F_{1,0,\mathrm{tree}}(t) &= 1-\frac{t}{m_\Delta^2}+\frac{t\,(2h_5 m_\Delta+2h_{10}-h_{13})}{m_\Delta}-\frac{(-2h_6+2h_{11}+h_{12})\,t^2}{2m_\Delta^2}\,,\\
F_{1,1,\mathrm{tree}}(t) &= -4-4m_\Delta\,(h_{12}m_\Delta-2h_{10}+h_{13})+(4h_6-2\,(2h_{11}+h_{12}))\,t\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{2,0,\mathrm{tree}}(t) &= -2-4\,(2h_1-2h_{10}+h_{13})\,m_\Delta+(2h_6-2h_{11}-h_{12})\,t\,,\\
F_{2,1,\mathrm{tree}}(t) &= 0\,,\\
F_{4,0,\mathrm{tree}}(t) &= \frac{3}{2}-\frac{t}{2m_\Delta^2}+t\left(\frac{h_{10}}{m_\Delta}-\frac{h_{13}}{2m_\Delta}+h_5-h_6+h_{11}+\frac{h_{12}}{2}\right)-\frac{(-2h_6+2h_{11}+h_{12})\,t^2}{4m_\Delta^2}\,,\\
F_{4,1,\mathrm{tree}}(t) &= -2-2m_\Delta\,(h_{12}m_\Delta-2h_{10}+h_{13})+(2h_6-2h_{11}-h_{12})\,t\,,\\
F_{5,0,\mathrm{tree}}(t) &= -\frac{1}{2}+\frac{1}{2}\,(h_4+4h_{10}-h_{13})\,m_\Delta+\frac{1}{4}\,(2h_6-2h_{11}-h_{12})\,t\,,
\end{aligned}
$$

h_i's also provide/contains counter terms（δh_i's）under EOMS scheme：

$$
\begin{aligned}
\delta h_1 &= \frac{\delta h_{12} m_N}{2}-\frac{\left(1575\,g_{\pi N\Delta}^2+172\,g_1^2\right)m_N}{207360\pi^2F^2}\,,\\
\delta h_4 &= -2\,\delta h_{10}-\delta h_{12}m_N-\frac{m_N(45\,g_{\pi N\Delta}^2+2336\,g_1^2)}{51840\pi^2F^2}\,,\\
\delta h_5 &= -\frac{\delta h_{12}}{2}-\frac{11(135\,g_{\pi N\Delta}^2+124\,g_1^2)}{207360\pi^2F^2}\,,\\
\delta h_{13} &= 2\,\delta h_{10}-\delta h_{12}m_N+\frac{\left(9\,g_{\pi N\Delta}^2+490\,g_1^2\right)m_N}{10368\pi^2F^2}\,.
\end{aligned}
$$

GFFs at One-Loop order（$t = 0$）

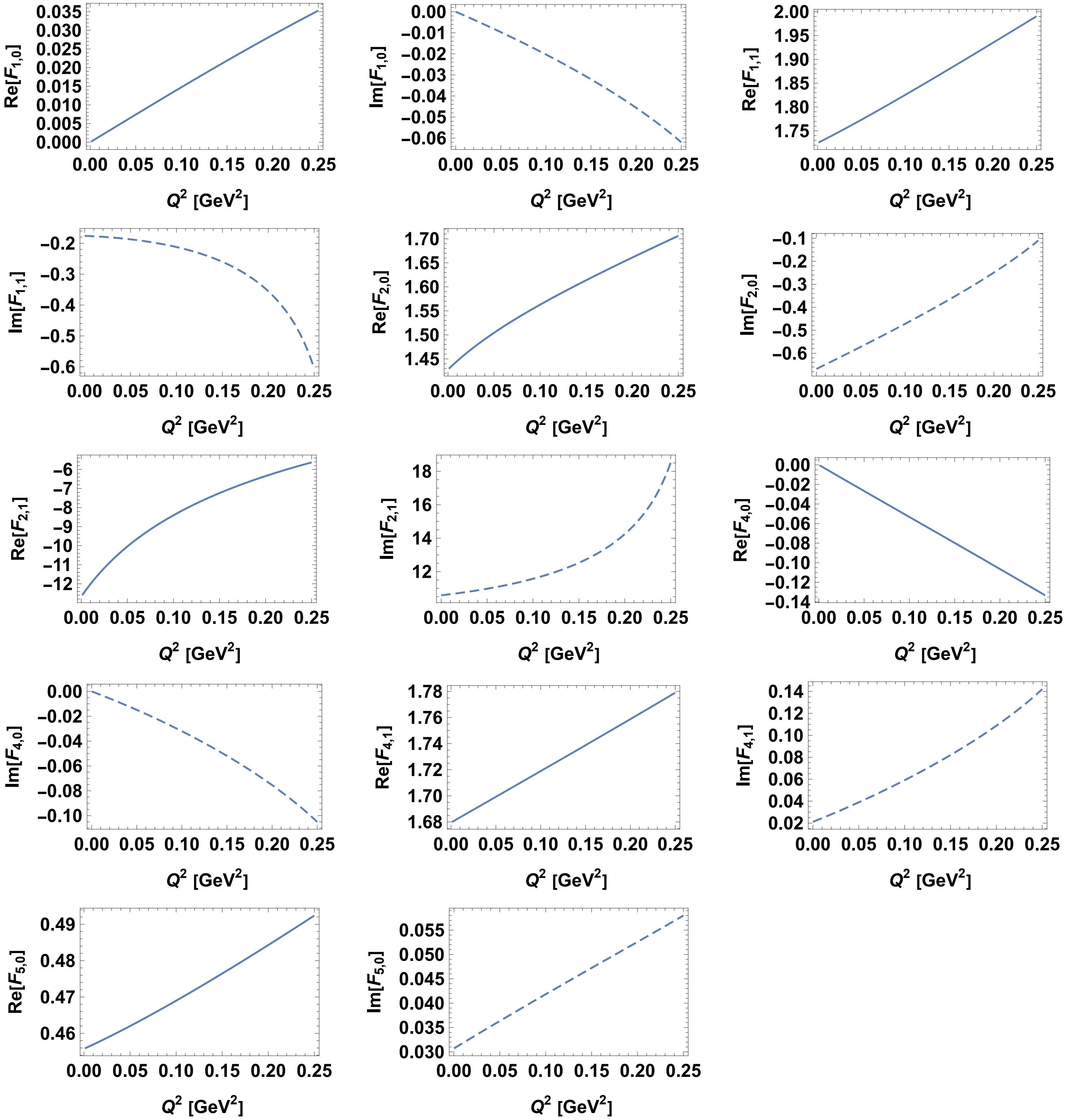
$$
\begin{aligned}
F_{1,0,\mathrm{loop}}(0) &= 0\,, \\
F_{1,1,\mathrm{loop}}(0) &= -\frac{5g_1^2 m_N(3\pi M - 49\delta)}{648\pi^2 F^2} \\
&+ \frac{g_{\pi N\Delta}^2 m_N}{144\pi^2 F^2\left(M^2-\delta^2\right)}\Bigg(-53\delta^3 + 24\delta\left(M^2-\delta^2\right)\ln\frac{M}{m_N} + 24i\pi\delta^2\sqrt{\delta^2-M^2} - 12i\pi M^2\sqrt{\delta^2-M^2} \\
&+ 12\left(M^2-2\delta^2\right)\sqrt{\delta^2-M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M} + 53\delta M^2\Bigg) + \mathcal{O}(\epsilon^2)\,, \\
F_{2,0,\mathrm{loop}}(0) &= -\frac{g_1^2 m_N(25\pi M - 1068\delta)}{2160\pi^2 F^2} \\
&+ \frac{g_{\pi N\Delta}^2 m_N\left(29\delta + 48\delta\ln\frac{M}{m_N} - 48i\pi\sqrt{\delta^2-M^2} + 48\sqrt{\delta^2-M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M}\right)}{288\pi^2 F^2} + \mathcal{O}(\epsilon^2)\,, \\
F_{2,1,\mathrm{loop}}(0) &= -\frac{g_1^2 m_N^3}{54\pi F^2 M} + \frac{g_{\pi N\Delta}^2 M m_N^3\sqrt{\frac{\delta^2}{M^2}-1}\left(\ln\left(\sqrt{\frac{\delta^2}{M^2}-1}+\frac{\delta}{M}\right)-i\pi\right)}{15\pi^2 F^2\left(M^2-\delta^2\right)} + \mathcal{O}(\epsilon^0)\,, \\
F_{4,0,\mathrm{loop}}(0) &= 0\,, \\
F_{4,1,\mathrm{loop}}(0) &= \frac{5g_{\pi N\Delta}^2 m_N^2}{576\pi^2 F^2} + \frac{235 g_1^2 m_N^2}{2592\pi^2 F^2} + \mathcal{O}(\epsilon)\,, \\
F_{5,0,\mathrm{loop}}(0) &= -\frac{g_1^2 m_N(150\pi M - 3323\delta)}{25920\pi^2 F^2} \\
&+ \frac{g_{\pi N\Delta}^2 m_N\left(5\delta + 2\delta\ln\frac{M}{m_N} - 2i\pi\sqrt{\delta^2-M^2} + 2\sqrt{\delta^2-M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M}\right)}{96\pi^2 F^2} + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
$$

Slopes of the GFFs

$$F_{i,j}(t) \;=\; F_{i,j}(0) + s_{F_{i,j}} t + \mathcal{O}(t^2)$$

$$
\begin{aligned}
s_{F_{1,0}} &= \frac{g_1^2(8\delta - 255\pi M)}{10368\pi^2 F^2 m_N} \\
&+ \frac{g_{\pi N\Delta}^2}{576\pi^2 F^2 m_N\left(M^2-\delta^2\right)}\Bigg(25\delta(\delta^2 - M^2) + 24\delta\left(\delta^2 - M^2\right)\ln\frac{M}{m_N} - 12i\pi(2\delta^2 - M^2)\sqrt{\delta^2 - M^2} \\
&- 12\left(M^2 - 2\delta^2\right)\sqrt{\delta^2 - M^2}\ln\frac{\delta + \sqrt{\delta^2 - M^2}}{M}\Bigg) + \mathcal{O}(\epsilon^2)\,, \\
s_{F_{1,1}} &= \frac{g_1^2 m_N}{432\pi F^2 M} + \frac{g_{\pi N\Delta}^2 m_N\left(\delta^3 + M^2\left(-\delta + i\pi\sqrt{\delta^2 - M^2}\right) - M^2\sqrt{\delta^2 - M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M}\right)}{120\pi^2 F^2\left(M^2 - \delta^2\right)^2} + \mathcal{O}(\epsilon^0)\,, \\
s_{F_{2,0}} &= -\frac{g_1^2 m_N}{108\pi F^2 M} + \frac{g_{\pi N\Delta}^2 m_N\left(\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M} - i\pi\right)}{60\pi^2 F^2\sqrt{\delta^2 - M^2}} + \mathcal{O}(\epsilon^0)\,, \\
s_{F_{2,1}} &= \frac{g_{\pi N\Delta}^2 m_N^3\left(-\delta^3 + M^2\left(\delta - i\pi\sqrt{\delta^2 - M^2}\right) + M^2\sqrt{\delta^2 - M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M}\right)}{140\pi^2 F^2 M^2\left(M^2 - \delta^2\right)^2} - \frac{g_1^2 m_N^3}{504\pi F^2 M^3} + \mathcal{O}(\epsilon^{-2})\,, \\
s_{F_{4,0}} &= \frac{g_{\pi N\Delta}^2\left(163\delta^2 - 96\left(M^2 - \delta^2\right)\ln\frac{M}{m_N} - 96i\pi\delta\sqrt{\delta^2 - M^2} + 96\delta\sqrt{\delta^2 - M^2}\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M} - 163M^2\right)}{4608\pi^2 F^2\left(M^2 - \delta^2\right)} \\
&+ \frac{g_1^2\left(877 - 150\ln\frac{M}{m_N}\right)}{25920\pi^2 F^2} + \mathcal{O}(\epsilon)\,, \\
s_{F_{4,1}} &= 0 + \mathcal{O}(\epsilon^{-1})\,, \\
s_{F_{5,0}} &= \frac{g_1^2 m_N}{3456\pi F^2 M} + \frac{g_{\pi N\Delta}^2 m_N\left(\ln\frac{\delta+\sqrt{\delta^2-M^2}}{M} - i\pi\right)}{960\pi^2 F^2\sqrt{\delta^2 - M^2}} + \mathcal{O}(\epsilon^0)\,.
\end{aligned}
$$

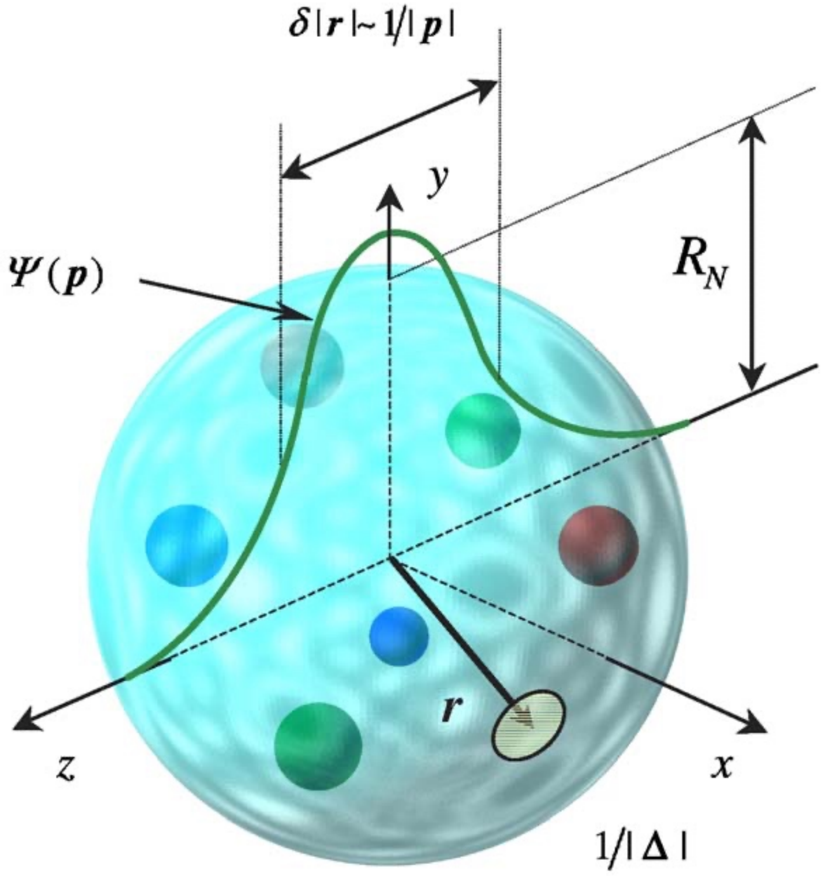
One-Loop contributions to GFFs



EM & EMT densities

in local wave packet / in 2D

for charge, magnetic, mass, spin, internal forces



Belitsky, Radyushkin, 2005.

$$1, \delta|r| \ll R_N$$

$$2, \delta|r| \ll 1/|\mathbf{\Delta}| \qquad \rightarrow \qquad 1/R_N \ll |\mathbf{\Delta}| \ll |\mathbf{p}| \ll \mathbf{M_N}$$

$$3, \delta|r| \gg 1/M_N \qquad\qquad (\delta|r| \sim 1/|\mathbf{p}|)$$

In limit: $|\Psi(\boldsymbol{p})|^2 = \frac{(2\pi)^3}{2M_N}\delta^{(3)}(\boldsymbol{p})$

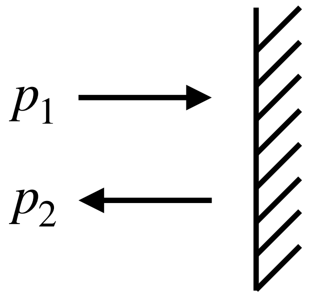
$$F(\mathbf{\Delta}) \equiv \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\mathbf{\Delta}\cdot\boldsymbol{r}}\rho(\boldsymbol{r})$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\Psi^*(\boldsymbol{p}-\tfrac{1}{2}\mathbf{\Delta})\Psi(\boldsymbol{p}+\tfrac{1}{2}\mathbf{\Delta})\langle\boldsymbol{p}-\tfrac{1}{2}\mathbf{\Delta}|j^0(0)|\boldsymbol{p}+\tfrac{1}{2}\mathbf{\Delta}\rangle \longrightarrow \quad 2M_N F(\mathbf{\Delta}) = \langle -\tfrac{1}{2}\mathbf{\Delta}|j_0(0)|\tfrac{1}{2}\mathbf{\Delta}\rangle$$

$$\longrightarrow \quad R_N^2 = \langle\boldsymbol{r}^2\rangle_E \equiv \int \mathrm{d}^3\boldsymbol{r}\boldsymbol{r}^2\rho(\boldsymbol{r}) = -6\frac{\partial G_E(-\mathbf{\Delta}^2)}{\partial\mathbf{\Delta}^2}$$

Breit frame:

$$p_1 = (E, \tfrac{1}{2}\mathbf{\Delta}), \qquad p_2 = (E, -\tfrac{1}{2}\mathbf{\Delta})$$



$$2P = (p_1 + p_2) = (2E, \mathbf{0})$$
$$\Delta = (p_2 - p_1) = (0, \mathbf{\Delta})$$
$$t = \Delta^2$$

hydrogen atom: $R_{\text{atom}}M_{\text{atom}} \sim M_{\text{atom}}/(m_e\alpha_{\text{em}}) \sim 10^5$

But: nucleon: $M_N R_N \sim 4$

Hudson, Schweitzer, 2017

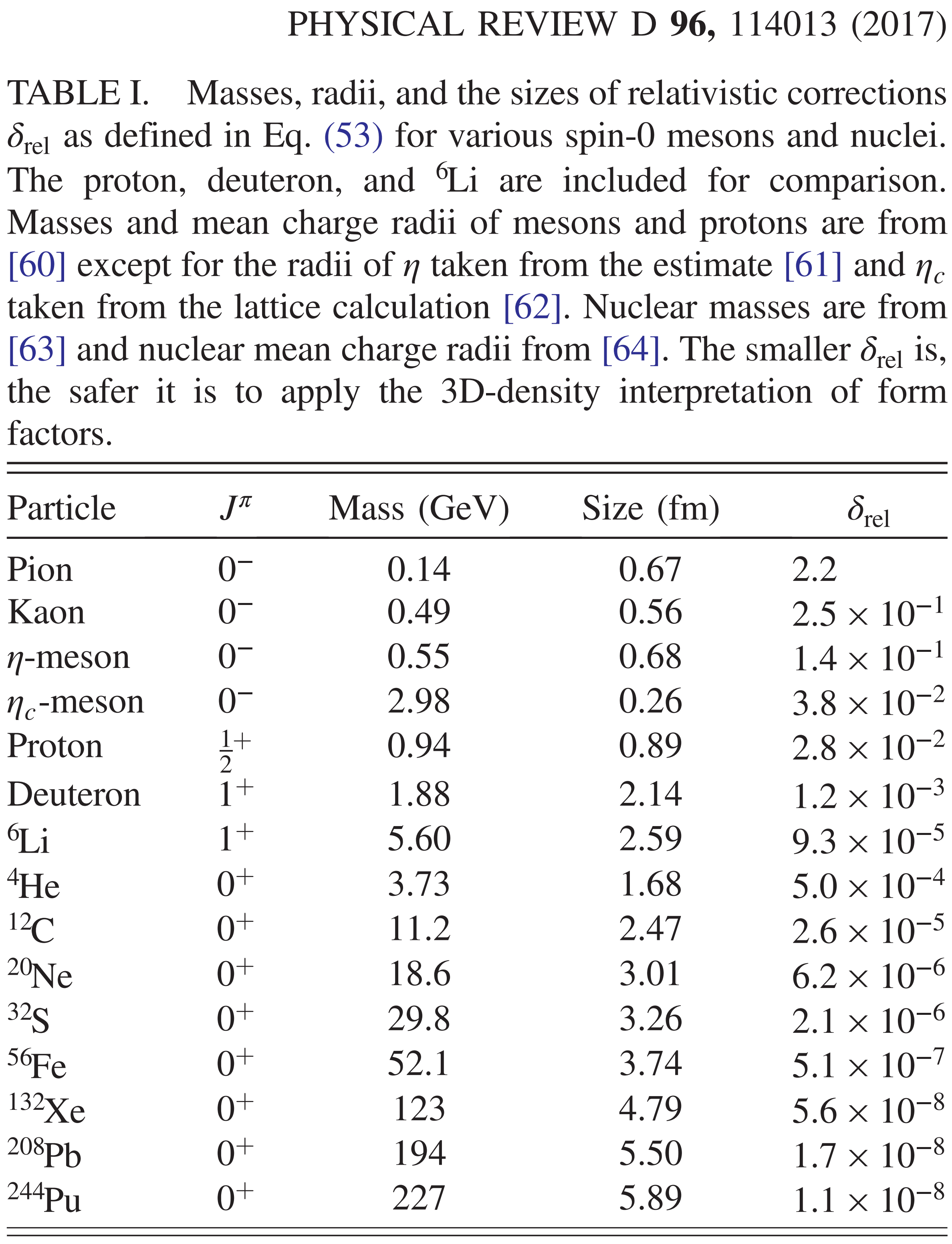


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TABLE I. Masses, radii, and the sizes of relativistic corrections $\delta_{\rm rel}$ as defined in Eq. (53) for various spin-0 mesons and nuclei. The proton, deuteron, and ^{6}Li are included for comparison. Masses and mean charge radii of mesons and protons are from [60] except for the radii of η taken from the estimate [61] and η_c taken from the lattice calculation [62]. Nuclear masses are from [63] and nuclear mean charge radii from [64]. The smaller $\delta_{\rm rel}$ is, the safer it is to apply the 3D-density interpretation of form factors.

Particle	J^π	Mass (GeV)	Size (fm)	$\delta_{\rm rel}$
Pion	0^-	0.14	0.67	2.2
Kaon	0^-	0.49	0.56	2.5×10^{-1}
η-meson	0^-	0.55	0.68	1.4×10^{-1}
η_c-meson	0^-	2.98	0.26	3.8×10^{-2}
Proton	$\frac{1}{2}^+$	0.94	0.89	2.8×10^{-2}
Deuteron	1^+	1.88	2.14	1.2×10^{-3}
^{6}Li	1^+	5.60	2.59	9.3×10^{-5}
^{4}He	0^+	3.73	1.68	5.0×10^{-4}
^{12}C	0^+	11.2	2.47	2.6×10^{-5}
^{20}Ne	0^+	18.6	3.01	6.2×10^{-6}
^{32}S	0^+	29.8	3.26	2.1×10^{-6}
^{56}Fe	0^+	52.1	3.74	5.1×10^{-7}
^{132}Xe	0^+	123	4.79	5.6×10^{-8}
^{208}Pb	0^+	194	5.50	1.7×10^{-8}
^{244}Pu	0^+	227	5.89	1.1×10^{-8}

Criticisms on Breit frame: Burkardt, 2000.Miller, 2007. Jaffe 2021. etc..

Localized Wave Packet

Epelbaum, Gegelia, Lange, Meißner, Polyakov, PRL 2022
Panteleeva, Epelbaum, Gegelia, Meißner, 2022
Alharazin, BDS, Epelbaum, Gegelia, Meißner, 2022

Heisenberg-picture:

$$|\Phi, \mathbf{X}, s\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\, \phi(s, \mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{X}} |p, s\rangle, \tag{2}$$

Normalization in ZAMF:
zero average momentum frame

$$\int d^3p\, |\phi(s, \mathbf{p})|^2 = 1\,. \tag{3}$$

Spherically Sym & Dimensionless:

$$\phi(\mathbf{p}) = R^{3/2}\, \tilde{\phi}(R\mathbf{p})\,, \tag{4}$$

EM parameterization:
(spin-3/2)

$$\begin{aligned}\langle p_f, s'|J_\mu|p_i, s\rangle \;&=\; -\bar{u}^\beta(p_f, s')\Bigg[\frac{P_\mu}{m}\left(g_{\alpha\beta}F^V_{1,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F^V_{1,1}(q^2)\right) \\ &\quad + \frac{i}{2m}\sigma_{\mu\rho}q^\rho \left(g_{\alpha\beta}F^V_{2,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F^V_{2,1}(q^2)\right)\Bigg] u^\alpha(p_i, s)\,,\end{aligned} \tag{5}$$

localize:

$$\begin{aligned} j^\mu_\phi(s', s, \mathbf{r}) \;&\equiv\; \langle\Phi, \mathbf{X}, s'|\hat{J}^\mu(\mathbf{x}, 0)|\Phi, \mathbf{X}, s\rangle \\ &= -\int \frac{d^3P\, d^3q}{(2\pi)^3\sqrt{4EE'}}\, \bar{u}^\beta\left(P + \frac{q}{2}, \sigma'\right)\Bigg[\frac{P_\mu}{m}\left(g_{\alpha\beta}F^V_{1,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F^V_{1,1}(q^2)\right) \\ &+ \frac{i}{2m}\sigma_{\mu\rho}q^\rho\left(g_{\alpha\beta}F^V_{2,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F^V_{2,1}(q^2)\right)\Bigg] u^\alpha\left(P - \frac{q}{2}, \sigma\right)\phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right)\phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right)e^{-i\mathbf{q}\cdot\mathbf{r}}, \end{aligned} \tag{8}$$

$\mathbf{P} \equiv \mathbf{Q}/R, \quad R \to 0$ Only large $\mathbf{P}$ region contributes!

“Naive” Breit Frame is problematic:
first $m \to \infty$ then $R \to 0$

Charge and Magnetic densities

Alharazin, BDS, Epelbaum, Gegelia, Meißner, 2022

Using multipole expansion:

$$\begin{aligned} j^0_\phi(s',s,\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{1}{4\pi} \int d^2\hat{n} \left\{ \mathcal{Z}_0(-q_\perp^2)\, \delta_{s's} + \left[\mathcal{Z}_1(-q_\perp^2)\, \hat{n}^k \hat{n}^l + \mathcal{Z}_2(-q_\perp^2)\, \frac{q_\perp^k q_\perp^l}{m^2} \right] \hat{Q}^{kl}_{s's} \right\} && (11) \\ &= \rho_0^C(r)\, \delta_{s's} + \rho_2^C(r)\, Y_2^{kl}(\Omega_r)\, \hat{Q}^{kl}_{s's}, && (12) \end{aligned}$$

Monopole Quadrupole

$$\begin{aligned} j^i_\phi(s',s,\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \frac{i}{4\pi} \int d^2\hat{n} \Big\{ \left[\mathcal{A}_0(-q_\perp^2)\, \hat{n}^i \hat{n}^l \epsilon^{kln} + \mathcal{A}_1(-q_\perp^2)\, (\, \delta^{kl} - \hat{n}^k \hat{n}^l)\, \epsilon^{iln} \right] \frac{q_\perp^n}{m} \hat{S}^k_{s's} \\ &+ \left[\left(\mathcal{A}_2(-q_\perp^2)\, \hat{n}^t \hat{n}^z + \mathcal{A}_3(-q_\perp^2)\, \frac{q_\perp^t q_\perp^z}{m^2} \right) \hat{n}^i \hat{n}^l \epsilon^{kln} \right. \\ &+ \left. \left(\mathcal{A}_4(-q_\perp^2) \hat{n}^t \hat{n}^z + \mathcal{A}_5(-q_\perp^2)\, \frac{q_\perp^t q_\perp^z}{m^2} \right) \epsilon^{iln} \left(\delta^{kl} - \hat{n}^k \hat{n}^l \right) \right] \frac{q_\perp^n}{m} \hat{O}^{ktz}_{s's} \Big\} && (13) \\ &= i\epsilon^{ikn} \hat{S}^k_{s's} Y_1^n \frac{1}{m} \frac{d}{dr} \rho_1^M(r) \;\; + i\epsilon^{ikn} \hat{O}^{ktz}_{s's} Y_3^{ntz} \frac{r^3}{m^3} \left(\frac{1}{r} \frac{d}{dr} \right)^3 \rho_3^M(r)\,, && (14) \end{aligned}$$

Dipole Octupole

"Naive" Breit Frame:

$$\begin{aligned} j^0_{\rm naive}(s',s,\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \left\{ \left[F^V_{1,0}(-\mathbf{q}^2) + \frac{\mathbf{q}^2}{6m^2} F^V_{1,1}(-\mathbf{q}^2) \right] \delta_{s's} - F^V_{1,1}(-\mathbf{q}^2)\, \frac{q^k q^l}{6m^2} \hat{Q}^{kl}_{s's} \right\}, \\ j^i_{\rm naive}(s',s,\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} i\epsilon^{ikn} \frac{q^n}{3m} \left\{ \left[F^V_{2,0}(-\mathbf{q}^2) + \frac{\mathbf{q}^2}{5m^2} F^V_{2,1}(-\mathbf{q}^2) \right] \hat{S}^k_{s's} - F^V_{2,1}(-\mathbf{q}^2)\, \frac{q^l q^z}{2m^2} \hat{O}^{klz}_{s's} \right\}. && (19) \end{aligned}$$

EMT densities

Alharazin, BDS, Epelbaum, Gegelia, Meißner, 2022
Freese, Miller, 2022
Panteleeva, Epelbaum, Gegelia, Meißner, 2022

$$\begin{aligned} t_\phi^{\mu\nu}(\mathbf{r}) &\equiv \langle \Phi, \mathbf{X}, s' | \hat{T}^{\mu\nu}(\mathbf{x}, 0) | \Phi, \mathbf{X}, s \rangle \\ &= -\int \frac{d^3P\, d^3q}{(2\pi)^3\sqrt{4EE'}}\, \bar{u}^\beta\left(P + \frac{q}{2}, \sigma'\right) \Bigg[\frac{P_\mu P_\nu}{m} \left(g_{\alpha\beta} F_{1,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F_{1,1}(q^2) \right) \\ &+ \frac{q_\mu q_\nu - \eta_{\mu\nu} q^2}{4m} \left(g_{\alpha\beta} F_{2,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F_{2,1}(q^2) \right) + \frac{i}{2} \frac{(P_\mu \sigma_{\nu\rho} + P_\nu \sigma_{\mu\rho})\, q^\rho}{m} \left(g_{\alpha\beta} F_{4,0}(q^2) - \frac{q_\alpha q_\beta}{2m^2}\, F_{4,1}(q^2) \right) \\ &- \frac{1}{m}\, (g_{\nu\beta} q_\mu q_\alpha + g_{\mu\beta} q_\nu q_\alpha + g_{\nu\alpha} q_\mu q_\beta + g_{\mu\alpha} q_\nu q_\beta - 2 g_{\mu\nu} q_\alpha q_\beta \\ &- g_{\mu\beta} g_{\nu\alpha} q^2 - g_{\nu\beta} g_{\mu\alpha} q^2)\, F_{5,0}(q^2) \Bigg] u^\alpha\left(P - \frac{q}{2}, \sigma\right) \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}}. \end{aligned} \tag{21}$$

Using multipole expansion：

$$t_\phi^{00}(s', s, \mathbf{r}) = N_{\phi,R} \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int d^2\hat{n} \left\{ \mathcal{E}_0(q_\perp^2)\, \delta_{s's} + \left[\mathcal{E}_1(q_\perp^2)\, \hat{n}^k \hat{n}^l + \mathcal{E}_2(q_\perp^2)\, \frac{q_\perp^k q_\perp^l}{m^2} \right] \hat{Q}^{kl}_{s's} \right\}, \tag{22a}$$

$$\begin{aligned} t_\phi^{0i}(s', s, \mathbf{r}) &= i\, N_{\phi,R} \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int d^2\hat{n}\, \Bigg\{ \left[\mathcal{C}_0(q_\perp^2)\, \epsilon^{kln} \hat{n}^l \hat{n}^i + \mathcal{C}_1(q_\perp^2)\, \epsilon^{iln} \left(\delta^{kl} - \hat{n}^k \hat{n}^l \right) \right] \frac{q_\perp^n}{m}\, \hat{S}^k_{s's} \\ &+ \Bigg[\left(\mathcal{C}_2(q_\perp^2) \hat{n}^t \hat{n}^z + \mathcal{C}_3(q_\perp^2) \frac{q_\perp^t q_\perp^z}{m^2} \right) \epsilon^{kln} \hat{n}^l \hat{n}^i \\ &+ \left(\mathcal{C}_4(q_\perp^2) \hat{n}^t \hat{n}^z + \mathcal{C}_5(q_\perp^2) s \frac{q_\perp^t q_\perp^z}{m^2} \right) \epsilon^{iln} \left(\delta^{kl} - \hat{n}^k \hat{n}^l \right) \Bigg] \frac{q_\perp^n}{m}\, \hat{O}^{ktz}_{s's} \Bigg\}, \end{aligned} \tag{22b}$$

$$t_\phi^{ij}(s', s, \mathbf{r}) = t^{ij}_{\phi,0}(s', s, \mathbf{r}) + t^{ij}_{\phi,2}(s', s, \mathbf{r}), \tag{22c}$$

$\sim 1/R$
motion of system

$\sim R$
internal pressure ＆ shear forces
（needs higher order contributions）

$$N_{\phi,R} = \frac{1}{R} \int_0^\infty dQ\, Q^3 |\tilde{\phi}(|\mathbf{Q}|)|^2\,,$$

$$N_{\phi,R,2} = \frac{m^2 R}{2} \int_0^\infty dQ\, Q |\tilde{\phi}(|\mathbf{Q}|)|^2\,.$$

Pressure and Shear Forces

Alharazin, BDS, Epelbaum, Gegelia, Meißner, 2022

$$\tilde{t}^{ij}_{\phi,2}(s',s,\mathbf{r}) \longrightarrow \begin{aligned} p_0(r) &= \tilde{v}_0\,(r) - \frac{1}{6m^2}\partial^2 w_0\,(r)\,, & s_0(r) &= -\frac{1}{2m^2} r \frac{d}{dr}\frac{1}{r}\frac{d}{dr} w_0\,(r)\,, \\ p_2(r) &= 0, & s_2(r) &= 0, \\ p_3(r) &= m^2 \tilde{v}_1\,(r) - \frac{1}{6}\partial^2 w_1\,(r)\,, & s_3(r) &= -\frac{1}{2} r \frac{d}{dr}\frac{1}{r}\frac{d}{dr} w_1\,(r)\,, \end{aligned} \qquad (37)$$

Conservation of EMT:

$$\partial_\mu t^{\mu\nu}_\phi(s',s,\mathbf{r},t)|_{t=0} = \partial_0 t^{0\nu}_\phi(s',s,\mathbf{r},t)|_{t=0} + \partial_i t^{i\nu}_\phi(s',s,\mathbf{r},t)|_{t=0} = 0. \qquad (39)$$

Breit Frame only has 2nd term

differential eqs:

$$p_n'(r) + \frac{2}{3}s_n'(r) + \frac{2}{r}s_n(r) = h_n'\,(r)\,, \quad \text{with } n = 0,2,3, \qquad (40)$$

von Laue stability condition:

$$\int d^3r\, p_n(r) \;=\; 0, \quad \text{with} \;\; n = 0,2,3, \qquad (42)$$

as long as $\lim_{q_\perp^2\to 0}\left(q_\perp^2\right)^\delta F_{2,0}\left(-q_\perp^2\right) = 0$ and $\lim_{q_\perp^2\to 0}\left(q_\perp^2\right)^\delta F_{2,1}\left(-q_\perp^2\right) = 0$, for $\delta > 0$.

generalized D-terms:

$$\mathcal{D}_n \;=\; -\frac{4}{15}m^2\int \mathrm{d}^3r\, r^2 s_n(r) = m^2 \int \mathrm{d}^3r\, r^2\left[p_n\,(r) - h_n\,(r)\right], \quad \text{with} \;\; n = 0,2,3. \qquad (43)$$

internal forces:

$$\frac{dF_r}{dS_r} \;=\; N_{\phi,R,2}\Bigg[\left(p_0(r) + \frac{2}{3}s_0(r)\right)\delta_{s's} + \left(p_2(r) + \frac{2}{3}s_2(r)\right)\hat{Q}^{rr}_{s's} + \ldots\;, \qquad \frac{dF_\theta}{dS_r} = \ldots\;, \qquad \frac{dF_\varphi}{dS_r} = \ldots$$

long-range behavior under ChPT

Alharazin, BDS, Epelbaum, Gegelia, Meißner, 2022

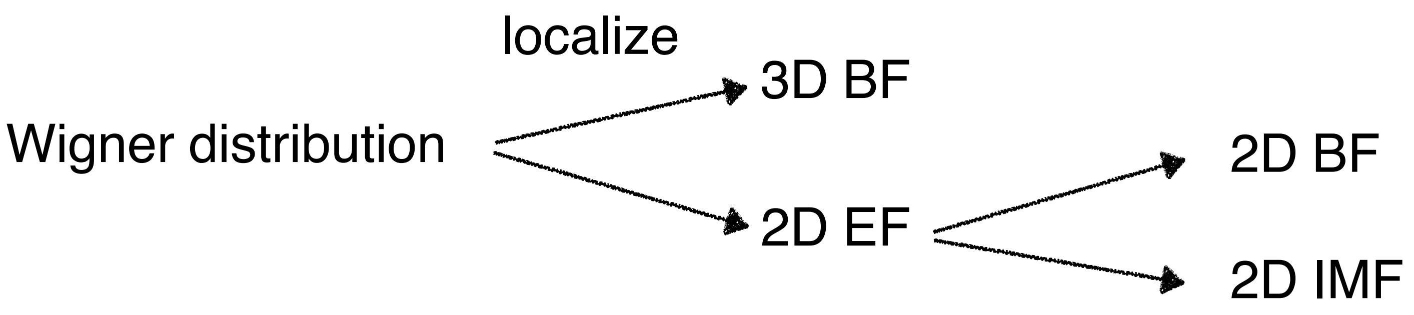
$$\begin{aligned}
\rho_0^E(r) &= \frac{25g_1^2}{1536F^2m_\Delta}\frac{1}{r^6} - \frac{10g_1^2}{81\pi^2F^2m_\Delta^2}\frac{1}{r^7} + \mathcal{O}\left(\frac{1}{r^8}\right), && (51)\\
\rho_2^E(r) &= \frac{35g_1^2}{6144F^2m_\Delta}\frac{1}{r^6} + \frac{35g_1^2}{162\pi^2F^2m_\Delta^2}\frac{1}{r^7} + \mathcal{O}\left(\frac{1}{r^8}\right), && (52)\\
\rho_1^J(r) &= \frac{5g_1^2}{162\pi^2F^2m_\Delta}\frac{1}{r^5} - \frac{125g_1^2}{3072F^2m_\Delta^2}\frac{1}{r^6} + \mathcal{O}\left(\frac{1}{r^7}\right), && (53)\\
\rho_3^J(r) &= -\frac{625g_1^2}{24576F^2m_\Delta^2}\frac{1}{r^6} + \frac{5g_1^2}{54\pi^2F^2m_\Delta^3}\frac{1}{r^7} + \mathcal{O}\left(\frac{1}{r^8}\right), && (54)\\
p_0(r) &= -\frac{25g_1^2}{2304F^2m_\Delta}\frac{1}{r^6} - \frac{75g_1^2}{1024F^2m_\Delta^3}\frac{1}{r^8} + \mathcal{O}\left(\frac{1}{r^9}\right), && (55)\\
s_0(r) &= \frac{5g_1^2}{96F^2m_\Delta}\frac{1}{r^6} + \frac{15g_1^2}{64F^2m_\Delta^3}\frac{1}{r^8} + \mathcal{O}\left(\frac{1}{r^8}\right), && (56)\\
p_3(r) &= \frac{85g_1^2m_\Delta}{221184F^2}\frac{1}{r^4} - \frac{155g_1^2}{196608F^2m_\Delta}\frac{1}{r^6} + \mathcal{O}\left(\frac{1}{r^8}\right), && (57)\\
s_3(r) &= -\frac{25g_1^2m_\Delta}{9216F^2}\frac{1}{r^4} + \frac{15g_1^2}{4096F^2m_\Delta}\frac{1}{r^6} + \mathcal{O}\left(\frac{1}{r^8}\right). && (58)
\end{aligned}$$

general stability conditions: $\rho_0^E(r) > 0$ and $\frac{2}{3}s_0(r) + p_0(r) > 0$

Note: necessary but not sufficient for a system to be stable

Panteleeva, Polyakov, 2021. Freese, Miller, 2022.
Kim, BDS, Fu, Kim, 2022.

Another way: 3D → 2D



Elastic Frame (EF)

Lorcé, PRL 2020

3D BF: $2P = (p_1 + p_2) = (2E, \mathbf{0})$
$\Delta = (p_2 - p_1) = (0, \mathbf{\Delta})$

2D EF: $2P = (2E, \mathbf{0}_\perp, P_z)$
$\Delta = (0, \mathbf{\Delta}_\perp, 0)$ ⟶ 2D BF: $P = (E, \mathbf{0})$, $\Delta = (0, \mathbf{\Delta}_\perp, 0)$

↓

2D IMF: $2P = (2E, \mathbf{0}_\perp, P_z \to \infty)$
$\Delta = (0, \mathbf{\Delta}_\perp, 0)$

$$\langle \hat{T}^{\mu\nu}(\boldsymbol{r})\rangle = \int \frac{d^3\boldsymbol{P}}{(2\pi)^3} \int d^3\boldsymbol{R}\, W(\boldsymbol{R}, \boldsymbol{P}) \langle \hat{T}^{\mu\nu}(\boldsymbol{r})\rangle_{\boldsymbol{R},\boldsymbol{P}}, \tag{20}$$

Wigner distribution:

$$\begin{aligned} W(\boldsymbol{R}, \boldsymbol{P}) &= \int \frac{d^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{R}} \tilde{\psi}^* \left(\boldsymbol{P} + \frac{\boldsymbol{\Delta}}{2}\right) \tilde{\psi}\left(\boldsymbol{P} - \frac{\boldsymbol{\Delta}}{2}\right) \\ &= \int d^3\boldsymbol{z}\, e^{-i\boldsymbol{z}\cdot\boldsymbol{P}} \psi^* \left(\boldsymbol{R} - \frac{\boldsymbol{z}}{2}\right) \psi\left(\boldsymbol{R} + \frac{\boldsymbol{z}}{2}\right). \end{aligned} \tag{21}$$

Abel transformation:

$$A[g](x_\perp) = \mathcal{G}(x_\perp) = \int_{x_\perp}^{\infty} \frac{dr}{r} \frac{g(r)}{\sqrt{r^2 - x_\perp^2}}, \quad g(r) = -\frac{2}{\pi} r^2 \int_r^{\infty} dx_\perp \frac{d\mathcal{G}(x_\perp)}{dx_\perp} \frac{g(r)}{\sqrt{x_\perp^2 - r^2}}. \tag{68}$$

e.g.:

$$\int dx_z \langle \hat{T}^{00}(0)\rangle_{-\boldsymbol{r},\boldsymbol{0}} = \int dx_z\, \varepsilon_0(r)\delta_{\lambda'\lambda} = \varepsilon_0^{(2\mathrm{D})}(x_\perp)\delta_{\lambda'\lambda},$$

$$\varepsilon_0^{(2\mathrm{D})}(x_\perp) = \int_{x_\perp}^{\infty} dr \frac{2r\varepsilon_0(r)}{\sqrt{r^2 - x_\perp^2}}.$$

1. Fully relativistic description in 2D IMF (avoid relativistic corrections)
2. 2D BF vs 2D IMF shows relativistic effects
3. Operation is not invertible

2D BF: $\Delta_z \to 0$

Kim, Sun, Fu, Kim, 2022

$$\langle p',\lambda'|\hat{T}^{00}(0)|p,\lambda\rangle = 2m^2\mathcal{E}_{(0,1)}(t)\delta_{\sigma'\sigma} + 2m^2\mathcal{E}_{(0,0)}(t)\delta_{\lambda'3}\delta_{\lambda3} + 4m^2\tau\mathcal{E}_2(t)\hat{Q}^{kl}X_2^{kl}(\theta_{\Delta_\perp}).$$

$$\langle p',\lambda'|\hat{T}^{0i}(0)|p,\lambda\rangle = 2m^2\sqrt{\tau}i\epsilon^{3li}\hat{S}^3_{\lambda'\lambda}X_1^l(\theta_{\Delta_\perp})\mathcal{J}_1(t),$$

$$\begin{aligned}\langle p',\lambda'|\hat{T}^{ij}(0)|p,\lambda\rangle &= 2m^2\tau\left[\left(\frac{1}{3}\mathcal{D}_2(t)-\frac{1}{2}\mathcal{D}_0(t)\right)\delta_{\sigma'\sigma}+\left(-\frac{2}{3}\mathcal{D}_2(t)-\frac{1}{2}\mathcal{D}_0(t)\right)\delta_{\lambda'3}\delta_{\lambda3}\right]\delta^{ij}\\ &+2m^2\tau X_2^{ij}(\theta_{\Delta_\perp})\delta_{\lambda'\lambda}\mathcal{D}_0(t)+4m^2\tau\left[\hat{Q}^{ik}X_2^{jk}(\theta_{\Delta_\perp})+\hat{Q}^{jk}X^{ik}(\theta_{\Delta_\perp})-\hat{Q}^{lm}X_2^{lm}(\theta_{\Delta_\perp})\delta^{ij}\right]\mathcal{D}_2(t)\\ &+8m^2\tau^2\hat{Q}^{lm}\left(X_2^{lm}(\theta_{\Delta_\perp})+\frac{1}{2}\delta^{lm}\right)\left(X_2^{ij}(\theta_{\Delta_\perp})-\frac{1}{2}\delta^{ij}\right)\mathcal{D}_3(t)\end{aligned}$$

2D n-rank irreducible tensors:

$$X_0(\theta_{x_\perp}) := 1,\quad X_1^i(\theta_{x_\perp}) = \frac{x_\perp^i}{x_\perp},\quad X_2^{ij}(\theta_{x_\perp}) = \frac{x_\perp^i x_\perp^j}{x_\perp^2} - \frac{1}{2}\delta^{ij}.$$

$$X_n^{i_1\ldots i_n}(\theta_{x_\perp}) = \frac{(-1)^{n+1}}{(2n-2)!!}x_\perp^n\partial^{i_1}\ldots\partial^{i_n}\ln x_\perp$$

3D n-rank irreducible tensors:

$$Y_0(\Omega_r) = 1,\quad Y_1^i(\Omega_r) = \frac{r^i}{r},\quad Y_2^{ij}(\Omega_r) = \frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}$$

$$Y_n^{i_1 i_2\ldots i_n}(\Omega_r) = \frac{(-1)^n}{(2n-1)!!}r^{n+1}\partial^{i_1}\partial^{i_2}\cdots\partial^{i_n}\frac{1}{r}$$

2D IMF: $P_z \to \infty$

3D BF: $2P = (p_1 + p_2) = (2E, \mathbf{0})$
$\Delta = (p_2 - p_1) = (0, \mathbf{\Delta})$

2D BF: $\Delta = (0, \mathbf{\Delta}_\perp, 0)$

2D IMF: $2P = (2E, \mathbf{0}_\perp, P_z \to \infty)$

$$
\begin{aligned}
\langle p', \lambda' | \hat{T}^{00}(0) | p, \lambda \rangle &= 2P_z^2 \mathcal{E}^{\mathrm{IMF}}_{(0,0)}(t) \delta_{\lambda' 3} \delta_{3\lambda} + 2P_z^2 \mathcal{E}^{\mathrm{IMF}}_{(0,1)}(t) \delta_{\sigma' \sigma} \\
&+ 2P_z^2 \sqrt{\tau} \mathcal{E}^{\mathrm{IMF}}_1(t) i\epsilon^{3jk} \hat{S}^j_{\lambda' \lambda} X_1^k(\theta_{\Delta_\perp}) + 4P_z^2 \tau \mathcal{E}^{\mathrm{IMF}}_2(t) \hat{Q}^{kl} X_2^{kl}(\theta_{\Delta_\perp}), \\
\langle p', \lambda' | \hat{T}^{0i}_a(0) | p, \lambda \rangle &= 2mP_z \sqrt{\tau} i\epsilon^{3li} \hat{S}^3_{\lambda' \lambda} X_1^l(\theta_{\Delta_\perp}) \mathcal{J}^{\mathrm{IMF}}_1(t) + 4mP_z \tau \left(X_2^{ik}(\theta_{\Delta_\perp}) - \frac{1}{2} \delta^{ik} \right) \mathcal{J}^{\mathrm{IMF}}_2(t) \hat{Q}^{3k}, \\
\langle p', \lambda' | \hat{T}^{ij}(0) | p, \lambda \rangle &= 2m^2 \tau \left[\left(\frac{1}{3} \mathcal{D}^{\mathrm{IMF}}_2 - \frac{1}{2} \mathcal{D}^{\mathrm{IMF}}_{(0,1)} \right) \delta_{\sigma' \sigma} + \left(-\frac{2}{3} \mathcal{D}^{\mathrm{IMF}}_2 - \frac{1}{2} \mathcal{D}^{\mathrm{IMF}}_{(0,0)} \right) \delta_{\lambda' 3} \delta_{\lambda 3} \right] \delta^{ij} \\
&+ 2m^2 \tau X_2^{ij}(\theta_{\Delta_\perp}) \left[\delta_{\sigma' \sigma} \mathcal{D}^{\mathrm{IMF}}_{(0,1)}(t) + \delta_{\lambda' 3} \delta_{\lambda 3} \mathcal{D}^{\mathrm{IMF}}_{(0,0)}(t) \right] \\
&+ 4m^2 \tau \left[\hat{Q}^{ik} X_2^{jk}(\theta_{\Delta_\perp}) + \hat{Q}^{jk} X_2^{ik}(\theta_{\Delta_\perp}) - \hat{Q}^{lm} X_2^{lm}(\theta_{\Delta_\perp}) \delta^{ij} \right] \mathcal{D}^{\mathrm{IMF}}_2(t) \\
&+ 8m^2 \tau^{3/2} i\epsilon^{lm3} \hat{S}^l X_1^m(\theta_{\Delta_\perp}) \left(X_2^{ij}(\theta_{\Delta_\perp}) - \frac{1}{2} \delta^{ij} \right) \mathcal{D}^{\mathrm{IMF}}_1(t) \\
&+ 8m^2 \tau^2 \hat{Q}^{lm} \left(X_2^{lm}(\theta_{\Delta_\perp}) + \frac{1}{2} \delta^{lm} \right) \left(X_2^{ij}(\theta_{\Delta_\perp}) - \frac{1}{2} \delta^{ij} \right) \mathcal{D}^{\mathrm{IMF}}_3(t),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}^{\mathrm{IMF}}_{(0,0)}(t) &= \frac{1}{3(1+\tau)^2} \left[12\tau \mathcal{J}_1 - 3(\tau - 1)\mathcal{E}_0 + \tau(2 + 4\tau)\mathcal{E}_2 + \tau(\tau - 1)(3\mathcal{D}_0 - 2\mathcal{D}_2) - 4\tau^2(1 + 2\tau)\mathcal{D}_3 \right], \\
\mathcal{E}^{\mathrm{IMF}}_{(0,1)}(t) &= \frac{1}{3(1+\tau)^2} \left[6\tau \mathcal{J}_1 + 3\mathcal{E}_0 - \tau \mathcal{E}_2 - 3\tau \mathcal{D}_0 - \tau \mathcal{D}_2 - 3\tau^2 \mathcal{D}_2 + 2\tau^2 \mathcal{D}_3 \right],
\end{aligned}
$$

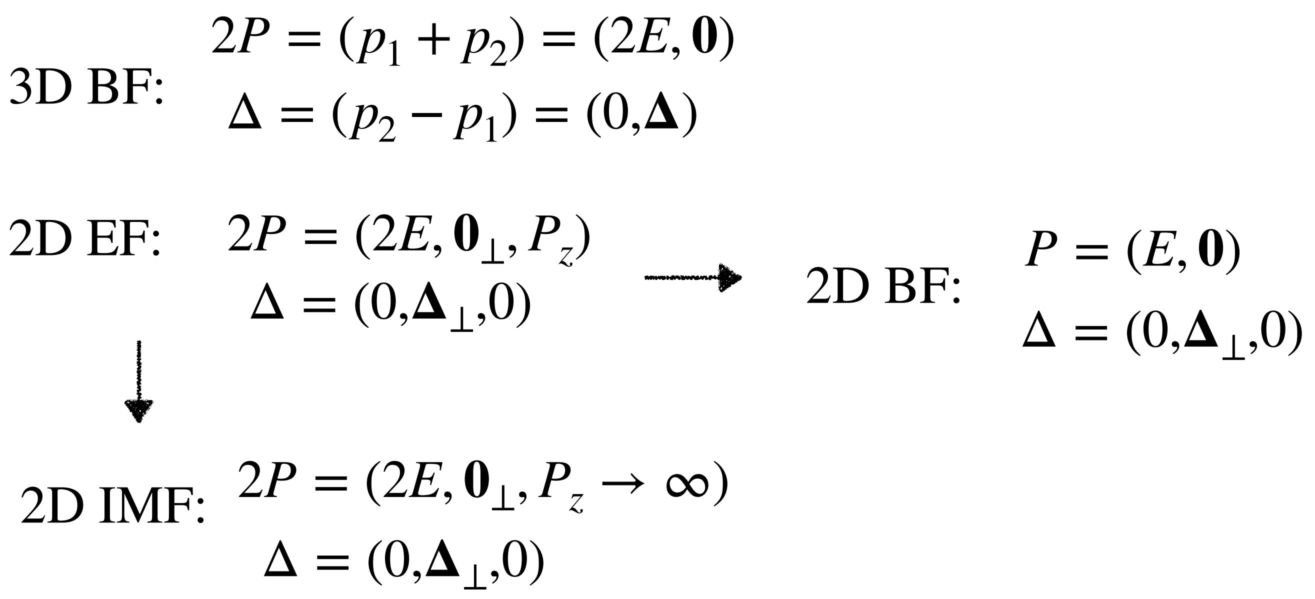
$$
\mathcal{D}^{\mathrm{IMF}}_{(0,1)}(t) = \mathcal{D}_0 + \frac{\tau}{3} G_W, \ \mathcal{D}^{\mathrm{IMF}}_{(0,0)}(t) = \mathcal{D}_0 + \frac{4\tau}{3} G_W,
$$

$$
\mathcal{D}^{\mathrm{IMF}}_1(t) = \frac{1}{4} G_W, \ \mathcal{D}^{\mathrm{IMF}}_2(t) = \mathcal{D}_2, \ \mathcal{D}^{\mathrm{IMF}}_3(t) = \mathcal{D}_3 - \frac{1}{4} G_W,
$$

1, monopole mix with quadrupole

2, D-term FFs do Wigner spin rotation

EMT distributions



3D BF: $2P = (p_1 + p_2) = (2E, \mathbf{0})$

$\Delta = (p_2 - p_1) = (0, \boldsymbol{\Delta})$

2D EF: $2P = (2E, \mathbf{0}_\perp, P_z)$

$\Delta = (0, \boldsymbol{\Delta}_\perp, 0)$

2D BF: $P = (E, \mathbf{0})$

$\Delta = (0, \boldsymbol{\Delta}_\perp, 0)$

2D IMF: $2P = (2E, \mathbf{0}_\perp, P_z \to \infty)$

$\Delta = (0, \boldsymbol{\Delta}_\perp, 0)$

3D BF:

$$T^{\mu\nu}_{\rm BF}(\boldsymbol{x}, \lambda', \lambda) = \langle \hat{T}^{\mu\nu}(0) \rangle_{-\boldsymbol{x},\boldsymbol{0}} = \int \frac{d^3\boldsymbol{\Delta}}{2P_0(2\pi)^3} e^{-i\boldsymbol{x}\cdot\boldsymbol{\Delta}} \langle p', \lambda' | \hat{T}^{\mu\nu}(0) | p, \lambda \rangle$$

2D EF:

$$T^{\mu\nu}_{\rm EF}(\boldsymbol{x}_\perp, P_z, \lambda', \lambda) := \int dx_z \langle \hat{T}^{\mu\nu}(0) \rangle_{-\boldsymbol{r},\boldsymbol{0}} = \int \frac{d^2\boldsymbol{\Delta}_\perp}{2P_0(2\pi)^2} e^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{\Delta}_\perp} \langle p', \lambda' | \hat{T}^{\mu\nu}(0) | p, \lambda \rangle \Big|_{\Delta_z = 0}.$$

2D IMF:

$$T^{00}_{\rm IMF}(\boldsymbol{x}_\perp, \lambda', \lambda) := T^{00}_{\rm EF}(\boldsymbol{x}_\perp, P_z, \lambda', \lambda) \frac{m}{P_0} \Big|_{P_z \to \infty},$$

$$T^{ij}_{\rm IMF}(\boldsymbol{x}_\perp, \lambda', \lambda) := T^{ij}_{\rm EF}(\boldsymbol{x}_\perp, P_z, \lambda', \lambda) \frac{P_0}{m} \Big|_{P_z \to \infty}.$$

Note: Need to divide Lorentz factors in order to be normalized to be its mass m, instead of its momentum P_z

$$G(t) = \frac{G(0)}{(1-t/\Lambda^2)^4}.$$

$$T^{00}_{\mathrm{EF}}(\boldsymbol{x}_\perp, 0, \lambda', \lambda) = \delta_{3\lambda}\delta_{\lambda' 3}\varepsilon^{(2\mathrm{D})}_{(0,0)}(x_\perp) + \delta_{\sigma'\sigma}\varepsilon^{(2\mathrm{D})}_{(0,1)}(x_\perp) + \hat{Q}^{ij}X^{ij}_2(\theta_{x_\perp})\varepsilon^{(2\mathrm{D})}_2(x_\perp)$$

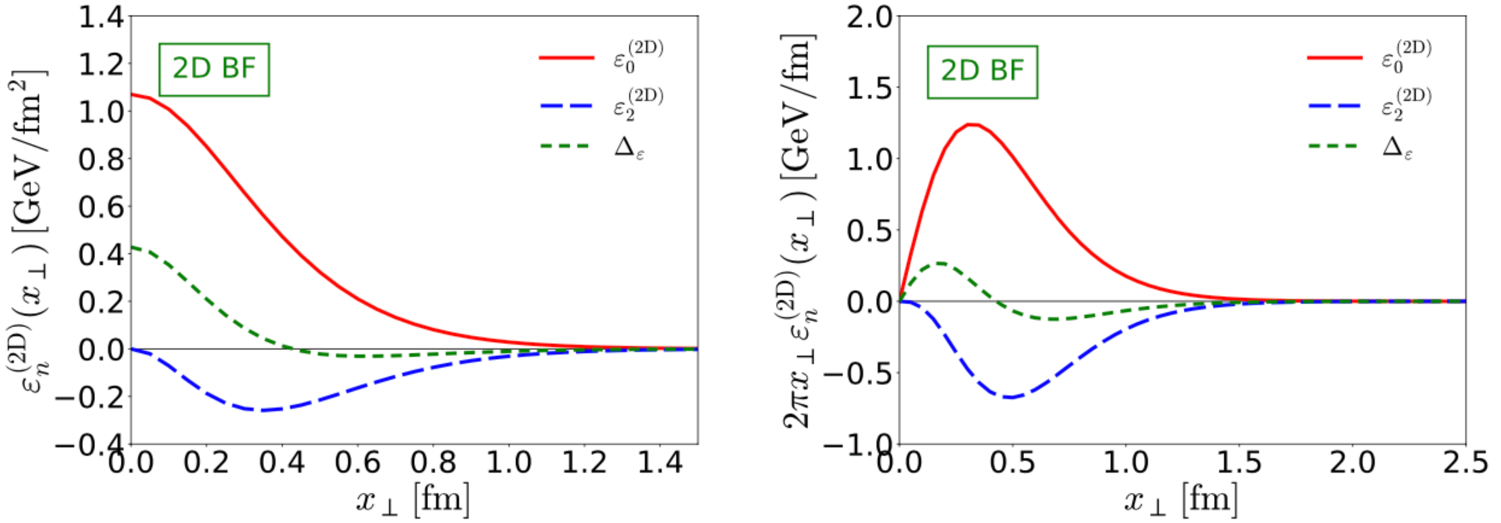


FIG. 1. Mass distributions of a spin-1 particle in the 2D Breit frame. In the left panel, the solid, dashed, and short-dashed curves draw the numerical results for $\varepsilon_0^{(2\mathrm{D})}$, $\varepsilon_2^{(2\mathrm{D})}$, and Δ_ε defined in Eq. (70) and Eq. (73), respectively. In the right panel, those weighted by $2\pi x_\perp$ are exhibited.

$$T^{00}_{\mathrm{IMF}}(\boldsymbol{x}_\perp, \lambda', \lambda) = \delta_{3\lambda}\delta_{\lambda' 3}\varepsilon^{\mathrm{IMF}}_{(0,0)}(x_\perp) + \delta_{\sigma'\sigma}\varepsilon^{\mathrm{IMF}}_{(0,1)}(x_\perp) + \epsilon^{3jk}\hat{S}^j X^k_1(\theta_{x_\perp})\varepsilon^{\mathrm{IMF}}_1(x_\perp) + \hat{Q}^{ij}X^{ij}_2(\theta_{x_\perp})\varepsilon^{\mathrm{IMF}}_2(x_\perp),$$

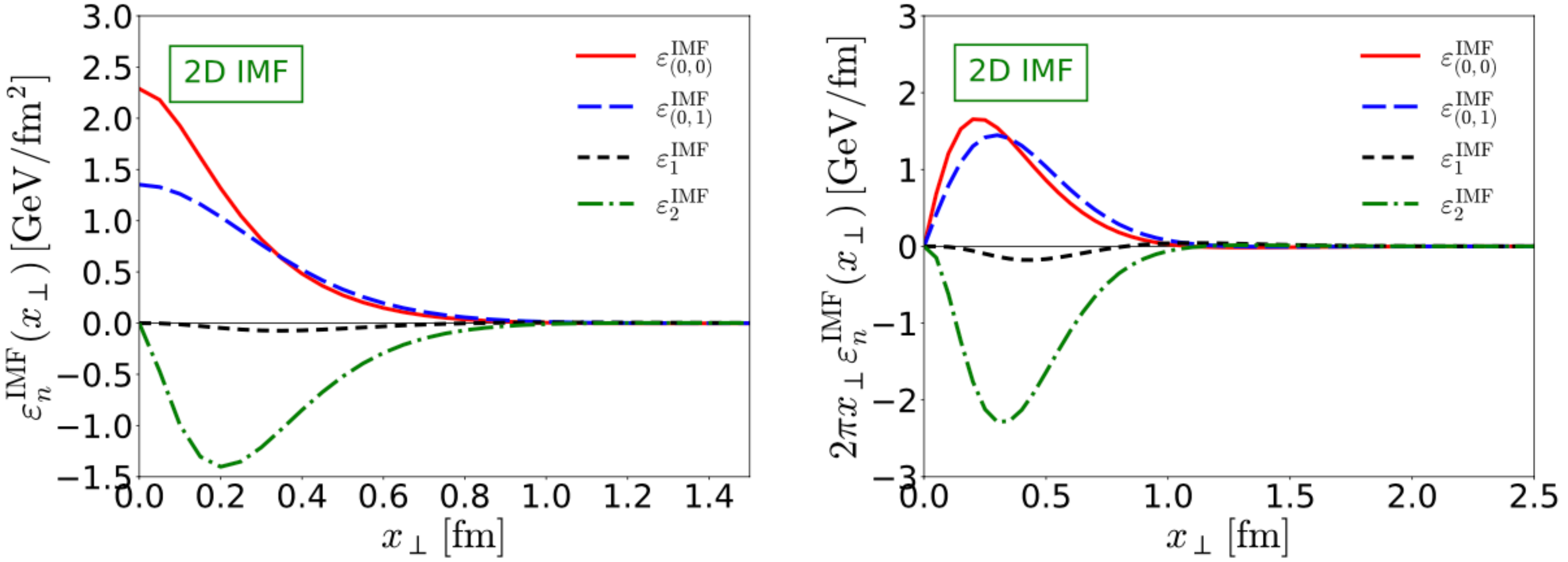


FIG. 3. Transverse mass densities of a spin-1 particle in the 2D infinite-momentum frame. The solid and dashed curves draw the 2D mass densities $\varepsilon^{\mathrm{IMF}}_{(0,0)}$ and $\varepsilon^{\mathrm{IMF}}_{(0,1)}$, whereas the short-dashed and dot-dashed ones depict $\varepsilon^{\mathrm{IMF}}_1$ and $\varepsilon^{\mathrm{IMF}}_2$. The expressions for these mass densities are given in Eq. (55). In the right panel, we draw those weighted by $2\pi x_\perp$.

Spin density:

$$J^3_{\text{IMF}}(\boldsymbol{x}_\perp,\lambda',\lambda) = \epsilon^{3jk}x^j_\perp T^{0k}_{\text{EF}}(\boldsymbol{x}_\perp,P_z,\lambda',\lambda)\Big|_{P_z\to\infty}$$

$$= \hat{S}^3_{\lambda'\lambda}\int\frac{d^2\boldsymbol{\Delta}}{(2\pi)^2}e^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{\Delta}_\perp}\left[\mathcal{J}_1(t)+t\frac{d\mathcal{J}_1(t)}{dt}\right]$$

$$+ i\epsilon^{3jl}\hat{Q}^{3l}_{\lambda'\lambda}\frac{1}{2m}\int\frac{d^2\boldsymbol{\Delta}}{(2\pi)^2}e^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{\Delta}_\perp}\Delta^j\left[3\mathcal{J}_2(t)+2t\frac{d\mathcal{J}_2(t)}{dt}\right]$$

Pressure and shear force:

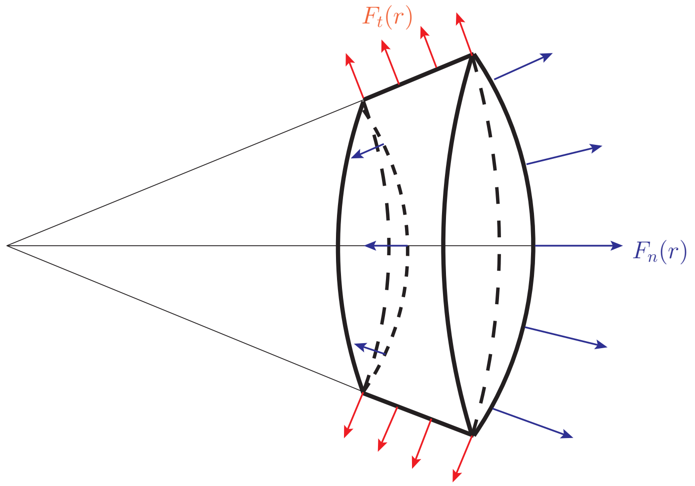
$$T^{ij}_{\text{IMF}}(\boldsymbol{x}_\perp,\lambda',\lambda) = \left(p^{\text{IMF}}_{(0,1)}(x_\perp)-\frac{2}{3}p^{\text{IMF}}_2(x_\perp)\right)\delta_{\sigma'\sigma}\delta^{ij}+s^{\text{IMF}}_{(0,1)}(x_\perp)\delta_{\sigma'\sigma}X^{ij}_2(\theta_{x_\perp})$$

$$+\left(p^{\text{IMF}}_{(0,0)}(x_\perp)+\frac{4}{3}p^{\text{IMF}}_2(x_\perp)\right)\delta_{\lambda'3}\delta_{\lambda3}\delta^{ij}+s^{\text{IMF}}_{(0,0)}(x_\perp)\delta_{\lambda'3}\delta_{\lambda3}X^{ij}_2(\theta_{x_\perp})$$

$$+2s^{\text{IMF}}_2(x_\perp)\left[\hat{Q}^{ip}X^{pj}_2(\theta_{x_\perp})+\hat{Q}^{jp}X^{pi}_2(\theta_{x_\perp})-\delta^{ij}\hat{Q}^{pq}X^{pq}_2(\theta_{x_\perp})\right]$$

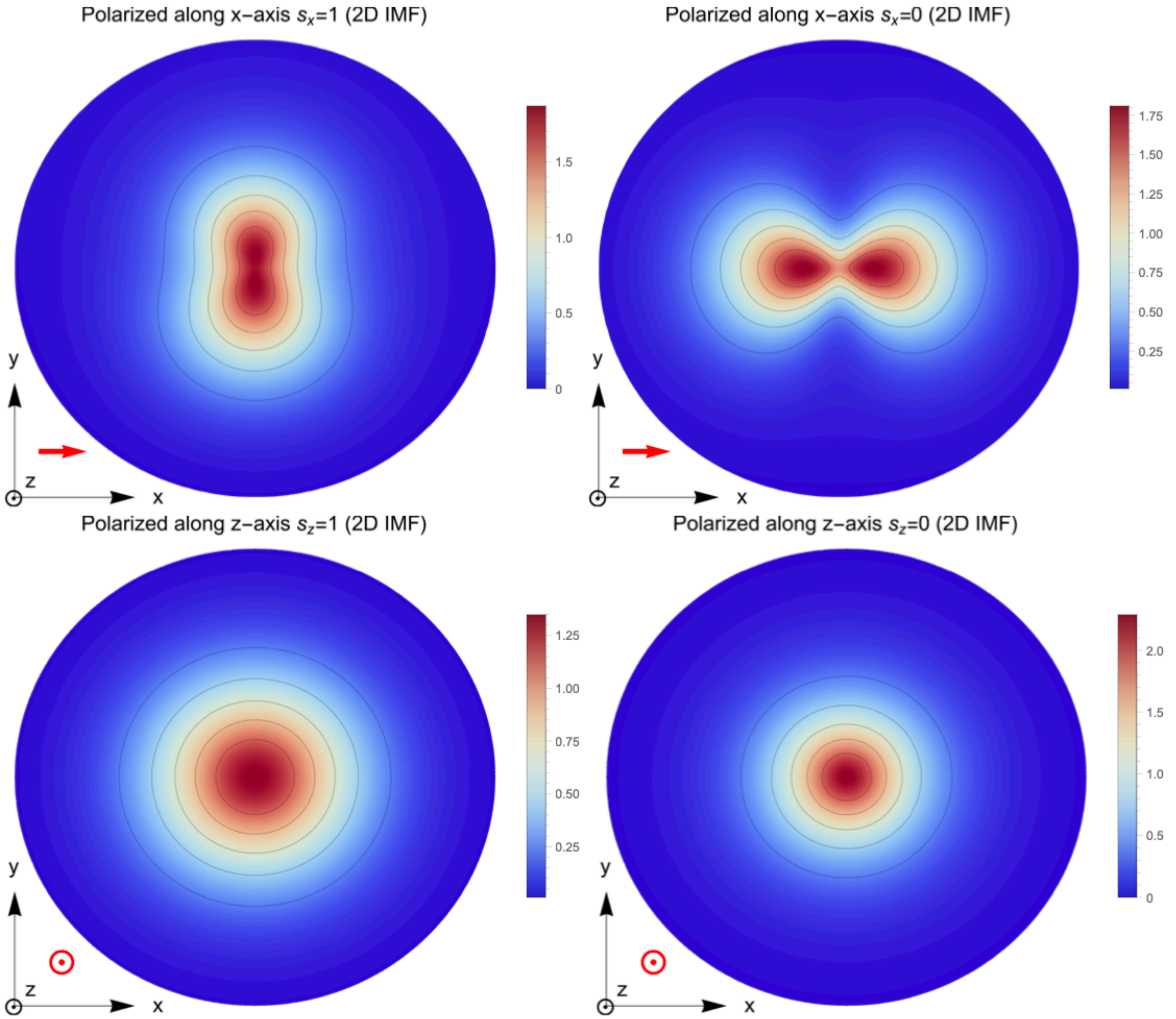
$$-\frac{1}{m^2}\hat{Q}^{pq}\partial^p\partial^q\left(s^{\text{IMF}}_3(x_\perp)X^{ij}_2(\theta_{x_\perp})+p^{\text{IMF}}_3(x_\perp)\delta^{ij}\right)$$

$$-\frac{2}{m}\epsilon^{lm3}\hat{S}^l\partial^m\left(s^{\text{IMF}}_1(x_\perp)X^{ij}_2(\theta_{x_\perp})+p^{\text{IMF}}_1(x_\perp)\delta^{ij}\right).$$



Internal local force fields:

$$\hat{\boldsymbol{x}}^i_\perp T^{ij}_{\text{IMF}}(\boldsymbol{x}_\perp,\lambda',\lambda) = \frac{dF_r}{dS_r}\hat{\boldsymbol{x}}^j_\perp+\frac{dF_\theta}{dS_r}\hat{\boldsymbol{\theta}}^j_\perp,\quad \hat{\boldsymbol{\theta}}^i_\perp T^{ij}_{\text{IMF}}(\boldsymbol{x}_\perp,\lambda',\lambda) = \frac{dF_r}{dS_\theta}\hat{\boldsymbol{x}}^j_\perp+\frac{dF_\theta}{dS_\theta}\hat{\boldsymbol{\theta}}^j_\perp$$

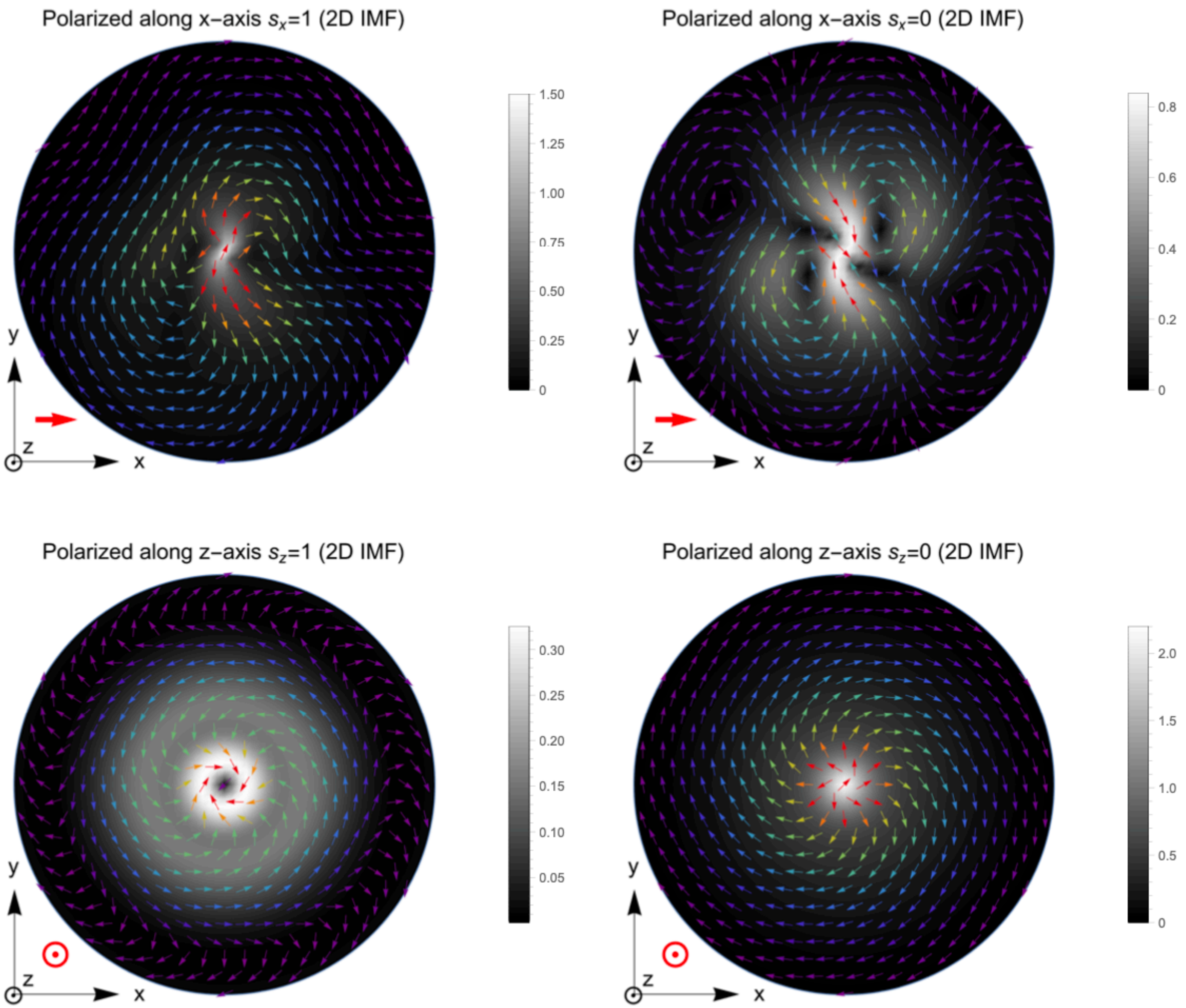

Polarized along x–axis s_x=1 (2D IMF)

Polarized along x–axis s_x=0 (2D IMF)

Polarized along z–axis s_z=1 (2D IMF)

Polarized along z–axis s_z=0 (2D IMF)

FIG. 5. $T^{00}(x_\perp)$ visualized in the 2D IMF by choosing a specific polarization. In the upper-left (upper-right) panel, we draw the mass distribution when the spin-1 particle is polarized with $s_x = 1$ ($s_x = 0$). In the lower-left (lower-right) panel, we illustrate that with $s_z = 1$ ($s_z = 0$).



Polarized along x–axis s_x=1 (2D IMF)

Polarized along x–axis s_x=0 (2D IMF)

Polarized along z–axis s_z=1 (2D IMF)

Polarized along z–axis s_z=0 (2D IMF)

FIG. 9. Strong force fields inside a fastly moving spi-1 particle ($P_z = \infty$) are visualized in the 2D plane when the target is polarized along x- and z-axis.

Summary

- Parameterization for matrix elements of EMT, defining GFFs, which are related to the fundamental properties, mass, spin, D-term. It's interpretation are given in terms of static densities.

- Estimation for GFFs are done by using Skyrme model, quark model and ChPT. Therefore the corresponding densities can be obtained for mass, spin, internal forces (pressure and shear forces.)

- Breit frame has problem when defining densities for light hadrons. By using localized wave packet in the ZAMF, one can bypass the problem. Another way is going to two-dimension (2D). Mechanical structure of a spin-1 particle are investigated in 3D BF, 2D BF and 2D IMF. They related by the Abel transformations.

Outlook

- ChPT calculation for $N - \Delta$ transition FFs
- Density interpretation for transition processes?

Many Thanks for your attention!