Three Positive Pions in a Finite Volume

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Bethe Forum
Multihadron dynamics in a box
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[Many slides from Maxim Mai]
Outline

- Three-body dynamics in infinite volume
- The Finite-volume problems (application: 2-body)
- Three-body dynamics in finite volume
  - The 3-pion system at maximal isospin:
    Interpretation of recent lattice QCD data
3-body dynamics for mesons and baryons

Light mesons

- Important channel in GlueX @ JLab
- Finite volume spectrum from lattice QCD:
  - Lang, Leskovec, Mohler, Prelovsek (2014)
  - Woss, Thomas et al. [HadronSpectrum] (2018)
  - Hörz, Hanlon (2019), ...

Light baryons

- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: Lang et al. (2017)
Three-body Interactions with Isobars

Mai, Hu, M. D., Pilloni, Szczepaniak
3-body Unitarity

\[ \langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle \]

\[ \times \prod_{\ell=1}^3 \left[ \frac{d^4 k_\ell}{(2\pi)^4} (2\pi) \delta^+(k^2 - m^2) \right] (2\pi)^4 \delta^4 \left( P - \sum_{\ell=1}^3 k_\ell \right) \]

delta function sets all intermediate particles on-shell
3-body Unitarity

\[ \langle q_1, q_2, q_3 \mid (\hat{T} - \hat{T}^\dagger) \mid p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 \mid \hat{T}^\dagger \mid k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 \mid \hat{T} \mid p_1, p_2, p_3 \rangle \]

General Ansatz for the isobar-spectator interaction
→ B & τ are new unknown functions
3-body Unitarity

\[
\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle
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3-body Unitarity

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\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle
\]
3 → 3 scattering amplitude is a 3-dimensional integral equation

\[ \hat{T} = \hat{T}_c + \hat{T}_d \]

- Imaginary parts of \( B, S \) are fixed by unitarity/matching
- For simplicity \( v=\lambda \) (full relations available)

\[
\text{Disc } B(u) = 2\pi i \lambda^2 \frac{\delta \left( E_Q - \sqrt{m^2 + Q^2} \right)}{2\sqrt{m^2 + Q^2}}
\]
- Un-subtracted dispersion relation

\[
\langle q|B(s)|p \rangle = - \frac{\lambda^2}{2\sqrt{m^2+Q^2} \left( E_Q - \sqrt{m^2+Q^2+i\epsilon} \right)} + C
\]
- One-\( \pi \) exchange in TOPT → RESULT, NOT INPUT!

- One can map to field theory, but does not have to. Result is a-priori dispersive.
Here: Version in which isobar rewritten in on-shell $2 \rightarrow 2$ scattering amplitude $T_{22}$

\[
\langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle = \frac{1}{3!} \sum_{n=1}^{3} \sum_{m=1}^{3} T_{22}(\sigma(q_n)) \langle q_n | T(s) | p_m \rangle T_{22}(\sigma(p_m))
\]

\[
\langle q | T(s) | p \rangle = \langle q | C(s) | p \rangle + \frac{1}{m^2 - (P - p - q)^2 - i\epsilon}
\]

\[
- \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{2E_\ell} T_{22}(\sigma(\ell)) \left( \langle p | C(s) | \ell \rangle + \frac{1}{m^2 - (P - p - \ell)^2 - i\epsilon} \right) \langle \ell | T(s) | p \rangle
\]

(s-wave)
Application I: The $a_1(1260)$ lineshape

Sadasivan, M.D., Mai, very preliminary

- Recent efforts to study 3-body production beyond the “isobar approximation” (*)
  P. Magalhães, A. C. dos Reis et al., PRD84 (2011); Khmechandani, Martinez, Oset, PRC77 (2008); JPAC: Mikhasenko, Wunderlich et. al., JHEP (2019); Mikhasenko, Pilloni et. al., PRD98 (2018); A. Jackura et al., EPJC79 (2019); Jülich: Janssen et al., PRL (1993)

- Here: Full solution of three-body equation with **exact** three-body unitarity

- S- and D-waves in $\pi\rho$ included

\[ (*) \]

\[ \rho \]
\[ \pi \]
\[ \pi^+ \]
\[ \pi^- \]
\[ + \]
\[ g_S, g_D \]

\[ + \]
\[ \text{symmetrization of final states} \]

\[ (*) \] here meant in the sense of “no rescattering”, “no three-body unitarity”

Data: ALEPH coll. [hep-ex/0506072]
Application II: Dalitz Plots

\[ I^G(J^{PC}) = 1^{-}(1^{++}) \rightarrow (\pi\rho)_S, (\pi\rho)_D \]

\[ \sqrt{s} = 0.64 \text{ GeV} \]

\[ \sqrt{s} = 1.21 \text{ GeV} \]
From two to three particles in finite volume
The cubic lattice

- Side length $L$, periodic boundary conditions
  \[ \Psi(x) \rightleftharpoons \Psi(x + \hat{e}_i L) \]
  \[ \rightarrow \text{finite volume effects} \]
  \[ \rightarrow \text{Infinite volume } L \rightarrow \infty \text{ mapping} \]

- Lattice spacing $a$
  \[ \rightarrow \text{finite size effects} \]
  Modern lattice calculations:
  \[ a \approx 0.07 \text{ fm} \rightarrow p \approx 2.8 \text{ GeV} \]
  \[ \rightarrow (much) \text{ larger than typical hadronic scales;} \]
  not considered here.

- Unphysically large quark/hadron masses
  \[ \rightarrow (chiral) \text{ extrapolation required}. \]
How to derive the 2-body quantization condition

\[
\int \frac{d^3 q}{(2\pi)^3} \rightarrow \frac{1}{L^3} \sum_s \sum_{i=1}^{\vartheta(s)}
\]

Two-body unitarity

On-shell condition

Imaginary parts

Power-law fin-vol. effects

Lüscher

\[
p \cot \delta(p) = -8\pi \sqrt{s} \left( \tilde{G}(E) - \text{Re } G(E) \right)
\]
GWU lattice group*: All Isospins

* A. Alexandru, F.X. Lee et al.

Simultaneous fit with Inverse Amplitude Method (more later)

- Including correlation between energy eigenvalues, pion masses and pion decay constants
- Including correlations across energy eigenvalues & isospins
- Also other parameterizations (conformal mapping with Adler zero & unitarity, [Guo et al. (2018)])

**$m_\pi = 224$ MeV**

$I = 0, l = 0$

**Guo et al. (2018)**

**$m_\pi = 315$ MeV**

$p \cot \delta^{(0)}$ vs $p^2$ (GeV$^2$)

- $\eta = 1.0$
- $\eta = 1.17$
- $\eta = 1.33$

---

**$I = 1, l = 1$**

**Guo et al. (2016)**

---

**$I = 2, l = 0$**

**Culver et al. (2019)**

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[Culver et al., PRD100 (2019); Mai et al., arXiv:1908.01847 [hep-lat]]
GWU lattice group: Chiral Extrapolation

(Optional & model dependent)

Mai et al., arXiv:1908.01847 [hep-lat]

Scattering lengths and resonance poles:

<table>
<thead>
<tr>
<th>$m_{\pi}$ [MeV]</th>
<th>$m_{\pi} a^{l=0}_0$</th>
<th>$m_{\pi} a^{l=2}_0$</th>
<th>$m_\sigma$ [MeV]</th>
<th>$g_{\sigma\pi\pi}$ [MeV]</th>
<th>$m_\rho$ [MeV]</th>
<th>$g_{\rho\pi\pi}$ [MeV]</th>
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<tr>
<td>$\sim 315$</td>
<td>$+1.9008^{+0.0521}_{-0.0593}$</td>
<td>$-0.1538^{+0.0021}_{-0.0018}$</td>
<td>$+591^{+6}<em>{-5} - i109^{+4}</em>{-4}$</td>
<td>$533^{+2}_{-2}$</td>
<td>$+789^{+1}<em>{-1} - i20^{+0}</em>{-0}$</td>
<td>$226^{+2}_{-2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+502^{+4}<em>{-4} - i175^{+6}</em>{-5}$</td>
<td>$426^{+2}_{-2}$</td>
<td>$+738^{+2}<em>{-1} - i43^{+1}</em>{-1}$</td>
<td>$282^{+3}_{-2}$</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td>$+443^{+3}<em>{-3} - i221^{+6}</em>{-6}$</td>
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<td></td>
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<td>$397.8^{+0.6}_{-0.6}$</td>
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<tr>
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<td></td>
<td></td>
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<td>$+724^{+2}<em>{-4} - i67^{+1}</em>{-1}$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$323^{+5}_{-3}$</td>
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</tbody>
</table>
THREE-BODY AMPLITUDE IN A BOX

M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]
Lüscher-like formalism in $3 \rightarrow 3$ case is under investigation

Polejaeva/Rusetsky (2012)  
Briceño/Hansen/Sharpe (2014-)

Non-relativistic approaches based on dimer picture & effective field theory

Kreuzer, Griesshammer(2012), Hammer et al. (2016)

F. Romero, Rusetsky, Urbach et. al. (2018)

Equivalence of various 3-body formalisms; three-body unitarity for Hansen/Sharpe

Requirements

- 3-body systems involve (resonant) two-body sub-amplitudes: Construct such that 2-body information can be included
- Need extrapolations between different energies (problem of underdetermination)
- Allow for systematic improvement by allowing more and more quantum numbers as lattice data improve (problem of underdetermination)
- At least, **all** possible intermediate on-shell configurations must be identified and included to ensure all power-law finite-volume effects are taken account of.

⇒ This work: **Quantization condition from 3-body unitarity in isobar formulation**
Bulk properties of multi-hadron Systems (no resolution of microscopic properties)


Literature FinVol 3-body

Bulk properties of multi-hadron Systems (no resolution of microscopic properties)


FV Analysis of Hörz/Hanlon data

arXiv:1905.04277 [hep-lat]

- Fits of 2&3 body sector
- Including D-wave isobar
- Extraction of 3-body force

Two-body unitarity

On-shell condition

Imaginary parts

Power-law fin-vol. effects

Lüscher

\[ p \cot \delta(p) = -8\pi \sqrt{s} \left( \tilde{G}(E) - \text{Re} \ G(E) \right) \]

How to derive the 2-body quantization condition

Three-body?

Analogously!
Two-body unitarity

On-shell condition

Imaginary parts

Power-law fin-vol. effects

Lüschner

\[ p \cot \delta(p) = -8\pi \sqrt{s} \left( \tilde{G}(E) - \text{Re } G(E) \right) \]

Three-body unitarity

On-shell condition

Imaginary parts

Power-law fin-vol. effects

Quantization Condition

\[ \text{Det} \left( B_{uu'}^{\Gamma ss'}(W^2) + \frac{2E_s L^3}{\vartheta(s)} \tau_s(W^2)^{-1}\delta_{ss'}\delta_{uu'} \right) = 0 \]
Only exact three-body unitarity guarantees the cancellation of unphysical 1\textsuperscript{st} and 2\textsuperscript{nd} order poles.
Cancellations

\[ \overline{T} = T + \]

→ fin. vol. normalization of \( \delta \)-distribution!

\[ \overline{T}^A_{11}^+(s) = \tau_n(s)T^A_{nm} T^A_{11}^+(s)\tau_m(s) - 2E_n\tau_n(s)\frac{L^3}{\vartheta(n)}\delta_{nm} \]

\[ T^A_{nm}^+(s) = B^A_{nm}^+(s) - \frac{1}{L^3} \sum_{x \in \text{set}_8} \vartheta(x)B^A_{nx}^+(s)\frac{\tau_x(s)}{2E_x} T^A_{xm}^+(s) \]

\( B^A_{11}^+ \) singular at \( W^+ = E_m + E_n + E(q_{nj} + p_{mi}) \)

\( \tau_m^{-1} \) singular at \( W^{\pm\pm} = E_m \pm E(\frac{2\pi}{L}y) \pm E(\frac{2\pi}{L}y + p_{mi}) \) for \( y \in \mathbb{Z}^3 \)

– when isobar-momenta are discretized in the 3-body cms momenta

\[ \tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\ell \frac{\chi^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)} \]

Also: all 2nd order singularities in determinant cancel → All consequence of Manifest three-body unitarity
Numerical demonstration

[M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]]

- First numerical demonstration of three-body finite volume formalism
- 3 particles in finite volume: \( m=138 \text{ MeV}, L=3 \text{ fm} \)
- one S-wave isobar → two unknowns:
  - vertex(Isobar→2 stable particles)
  - subtraction constant (~mass)
- Project to \( \Gamma = A^{1+} \)
  → prediction of 3body energy-eigenlevels (\( C=0 \))

\[ T_{22} = \nu \tau \]

unphysical levels cancel out (exact proof available)
A physical system:
\[ \pi^+ \pi^+ \pi^+ \]

Mai, M.D., PRL 122 (2019), 062503
Three positive pions

• Maximal isospin: $\pi^+\pi^+\pi^+$
  ➢ LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  ➢ Repulsive channel
  ➢ $L=2.5 \text{ fm}$, $m_\pi=291/352/491/591 \text{ MeV}$

I. 2-body subchannel:
  ➢ one-channel problem: $\pi\pi$-system in S-wave, $I=2$
  ➢ 2-body amplitude consistent with 3-body one

Inverse Amplitude method


discretize (Lüscher) $\rightarrow$ predicted fin-vol. spectrum
II. 3-body spectrum

Remaining unknown: $C$

- genuine (momenta-dependent) 3-body “force”
- simplest case: $C_{qp} = c \delta^{(3)}(p - q)$

LatticeQCD results for ground level available for $\pi^+\pi^+\pi^+$
- Repulsive channel
- $L = 2.5 \text{ fm, } m_\pi = 291/352/491/591 \text{ MeV}$

Quantization condition:

$$\det \left( B_{uu'}^{\Gamma_{ss'}^{s}}(W^2) + \frac{2E_s L^3}{\vartheta(s)} \tau_s (W^2)^{-1} \delta_{ss'} \delta_{uu'} \right) = 0$$

NPLQCD, Detmold et al. (2008)
II. 3-body spectrum

Remaining unknown: $C$

- genuine (momenta-dependent) 3-body “force”
- simplest case: $C_{q\bar{p}} = c \delta^{(3)}(p-q)$

$\Gamma = A^+_1$

Fit $C$ to NPLQCD ground state level

$\rightarrow C = (0.2 \pm 1.5) \cdot 10^{-10}$
II. 3-body spectrum

Remaining unknown: $C$

- genuine (momenta-dependent) 3-body “force”
- simplest case: $C_{qp} = c \delta^{(3)}(p-q)$

**Quantization condition**

$$\text{Det} \left( B_{uu'}^{\Gamma_{ss'}}(W^2) + \frac{2E_s L^3}{\vartheta(s)} \tau_s(W^2)^{-1} \delta_{ss'} \delta_{uu'} \right) = 0$$

**Predict excited spectrum:**

→ novel pattern

1/1 of interacting/non-interacting lvls

→ all QC-poles are simple

→ chiral extrapolation to phys point

(under assumptions)
The Moving System $\pi^+ \pi^+ \pi^+$

Mai, M.D., Culver, Alexandru (in preparation)
New Lattice Data

Hörz, Hanlon, arXiv:1905.04277 [hep-lat]

Two-body spectrum $\pi^+\pi^+$

Three-body spectrum $\pi^+\pi^+\pi^+$

- **First lattice data on excited energy eigenvalues** from multi-pion operators → More reliable extraction of scattering eigenvalues
- D200 CLS ensemble (2+1) with improved Wilson fermions and tree-level Lüscher–Weisz gauge action; stochastic LapH method; $m_{\pi}=200$ MeV; $L=4.1$ fm
- High number of Wick contraction (20,679,840 diagrams) managed with novel method from quantum chemistry

See also talk by F. Romero López/ arXiv:1909.02973
Two-body spectrum: D-wave I

Hörz, Hanlon, arXiv:1905.04277 [hep-lat]

\[ \frac{E_{cm}}{m_\pi} \]

→ I=2 D-wave vanishes within uncertainties – what does IAM predict?
→ Caveat: Correlated $\chi^2$ reveals more information.
D-wave II: prediction

\[ 2m_\pi < \sigma^{1/2} < 4m_\pi \text{ (Lattice)} \]

\[ 2m_\pi < \sigma^{1/2} < 4m_\pi \text{ (Phys.)} \]

\[ m_\pi = 200 \text{ MeV} \]
- GW
- GL
- DP

\[ m_\pi = 139 \text{ MeV} \]
- GW
- GL
- DP

\[ \delta_2^2 \text{ [deg]} \]
- \[ \sigma^{1/2} \text{ [MeV]} \]

**GW:** GW global (arXiv:1908.01847 [hep-lat])

**GL:** Gasser, Leutwyler (Annals Phys. 158, 1984)

**DP:** Dobado, Peláez (PRD 56 (1997))

See also [Nebreda, Peláez, Ríos, PRD83 (2011)]
We may consider this as any suitable 2-body Parametrization (like, e.g., K-matrix with conformal mapping [D. Guo et al., GW Lattice, PRD 98, (2018)])
IAM predictions: Different LECs

(D-wave set to zero)

- Robust predictions of the 2-body spectrum irrespective of used LECs
- No sign of D-wave up to very high energies in irreps with S&D-wave mixing
- Ignore the vanishing $\pi^+\pi^+$ – D-wave, but keep the important $\pi^+$ – isobar D-wave

\[ \chi^2_{GL} \left( \sqrt{\sigma} \lesssim 4m_\pi \right) = 21(16) \]

(2-body, correlated; 11(10) data)
3-body Spectrum: Predictions (I)


- S-wave prediction good at threshold (like for NPLQCD data)
- S-wave prediction good at high energies → Energy dependence matched
- No sign of 3-body force (like for NPLQCD data)
- D-wave prediction qualitatively good → Relative* strength between S- and D-wave matched → Consequence that 3-body interaction dominated by exchange → Consequence of 3-body Unitarity
- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD

Technical note: Projection technique for 3-body systems to irreps from M.D., Hammer, Mai, Pang, Rusetsky, Wu PRD97 (2018)
### 3-body spectrum: Moving frames


\[ \tilde{P} = \sum_{i=1}^{3} \tilde{q}_i = 0 \]

\[ \tilde{P} = \sum_{i=1}^{3} \tilde{q}_i \neq 0 \]

<table>
<thead>
<tr>
<th>( A_{-1}^- (0) )</th>
<th>( E_{-1}^- (0) )</th>
<th>( A_2^- (1) )</th>
<th>( B_2^- (1) )</th>
<th>( A_2^- (2) )</th>
<th>( B_2^- (2) )</th>
<th>( A_2^- (3) )</th>
<th>( E^- (3) )</th>
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<tbody>
<tr>
<td>S</td>
<td>D</td>
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<td>S&amp;D</td>
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<td>S&amp;D</td>
<td>S&amp;D</td>
<td>S&amp;D</td>
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Low participating isobar-\( \pi^+ \) waves

→ Need to develop a framework for moving 3-body systems!
Moving frames for 3-body systems

**Lattice rest frame**

**3-body rest frame**

**2-body rest frame**

**Usually:** Explicit S- and D-wave projected parameterizations in coupled channels

**Here:** Boost of unprojected 3-body amplitude. *A-posteriori* projections with suitable Clebsch-Gordan coefficients → Requires plane-wave solution of scattering

\[
\tilde{P} = \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3
\]

\[
\langle L/2\pi \rangle \tilde{P} \in \{ (0, 0, 1), (0, 1, 1), (1, 1, 1) \}
\]

\[
\left\{ q = \tilde{q} + \left[ \left( \frac{\tilde{P}^0}{\sqrt{s}} - 1 \right) \frac{\tilde{q} \tilde{P}}{|\tilde{P}|^2} - \frac{\tilde{q}^0}{\sqrt{s}} \right] \tilde{P} \right\}
\]

**3-body summation:**

\[
\int \frac{d^3l}{(2\pi)^3} g(l) \rightarrow \int \frac{d^3\tilde{l}}{(2\pi)^3} g(1(\tilde{l})) \tilde{J}(\tilde{l}) \rightarrow \frac{1}{L^3} \sum_n g(1(\tilde{l})) \tilde{J}(\tilde{l})
\]

\[
\hat{T}(q(\tilde{q}), p(\tilde{p})) \rightarrow \text{3 → 3 boosted plane-wave amplitude}
\]

Poles → Eigenvalues
3-body spectrum: Complete Predictions


Hörz, Hanlon
Prediction from
2-body input (GL) &
three-body unitarity
Non-interacting

Central Result

\( \chi^2_{\text{GL}}(\sqrt{s} \lesssim 5m_\pi) = 10 \) (3-body, correlated; 11 data)

\( \chi^2_{2+3, \text{GL}} = 39(28) \) (2&3-body, correlated; 22(21) data)
GW Lattice Results

- nHYP-smeared clover fermions with mass-degenerate quark flavors ($N_f = 2$)
- $M_\pi = 227$ MeV and 315 MeV
- 3 elongated boxes
- Variational basis including $\pi^+\pi^+\pi^+$ operators

[Culver, Alexandru, Mai, M.D., preliminary]
3-body Unitarity

- 3-body unitarity dictates on-shell condition (exchange term & isobar propagator)
- On-shell condition dictates leading, power-law, finite-volume effects
- “Bare-bone” infinite-volume extrapolation tool (in spirit of Lüscher equation)
- Optional: Pion-mass extrapolation

The $\pi^+\pi^+\pi^+$ System

- First application to physical 3-body system (above threshold) [PRL 2019]
- NPLQCD threshold data well predicted, excited levels predicted; fit of C.
- First explanation of excited 3-body levels (data from Hörz/Hanlon)
- Consequences of three-body unitarity directly visible in data (S vs. D waves)
- First development and application of moving frames for 3-body systems

Outlook

→ Correlated fit to Hörz/Hanlon data
→ Implementation of spin isobars & multiple isobars
→ unequal masses
→ practical studies: $a_1(1260)$, Roper, exotics...
SPARES
$l=2$ $D$-wave at HadSpec Pion Masses

Figure from [Nebreda, Peláez, Ríos, PRD83 (2011)]
Data: [Dudek, Edwards, Peardon, Richards, Thomas, PRD83 (2011)]

Perturbative $O(p^4)$, $O(p^6)$ calculation
The Power of Unitarity

<table>
<thead>
<tr>
<th>Question: Does provide full imaginary part of all possible 3 → 3 transitions?</th>
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The Power of Unitarity

Question: Does provide full imaginary part of all possible 3 → 3 transitions?

Riddle 1

Riddle 2

Riddle 3

Riddle 4

Riddle 5
The Power of Unitarity

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Riddle 1

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Riddle 5

Answer: Yes. and are the only on-shell configurations in physical region. Three-body unitarity avoids many artificial complications of diagrammatic expansions.
Projection to irreps

- Lüscher formalism relies on regular 2→2 potentials
  - Now: manifestly singular interactions
  - Find generalization that projects also the interactions to the irreps of cubic symmetry, not only propagation

- Separation of variables
  - shells = sets of points related by $O_h$
  - Analogous to radial coordinate in infinite volume
  - Find the orthonormal basis for arbitrary functions defined on each point of a given shell.

- $J$ (inf. volume) → irreps (finite volume): $\Gamma \in \{A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm\}$
- Partial wave projection (inf. Volume) $\rightarrow$ Irrep. projection (fin.)

\[
f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{\mathbf{p}}) f_{\ell m}(p)
\]
\[
f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{\mathbf{p}}) f(\mathbf{p})
\]
\[
f^s(\hat{p}_j) = \sqrt{4\pi} \sum_{\Gamma_s} \sum_a f^s_{\Gamma a} \chi^s_{\Gamma a}(\hat{p}_j)
\]
\[
f_a^{\Gamma a} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{p}_j) \chi^s_{\Gamma a}(\hat{p}_j)
\]

(a is index u in quantization condition; Quantization condition has projection in incoming AND outgoing basis states with indices u, u')
Quantization Condition

\[ \text{Det} \left( B_{uu'}^{\Gamma ss'}(W^2) + \frac{2E_s L^3}{\vartheta(s)} \tau_s(W^2)^{-1} \delta_{ss'} \delta_{uu'} \right) = 0 \]

- Fix to 3→3 data
  - \( W \)- total energy
  - \( \vartheta \)- multiplicity
  - \( s/s' \)- shell index
  - \( u/u' \)- basis index
  - \( \Gamma \)- lattice volume
  - \( E_s \)- spect. energy
  - Determinant of \((s,u) \times (s',u')\) matrix at fixed \( W, \Gamma, L \)

\( \ell = 0 \)

- Fix to 2→2 data:
  \( T_{22} = v \tau v \)

- Not a Lüscher-like equation ("left": infinite volume, "right": finite volume)
- Instead: Fix parameters to lattice eigenvalues
- With parameters fixed, evaluate infinite-volume amplitude
- Same workflow as in many 2-body coupled-channel fits (see, e.g.,
  M.D., Meißner, Oset, Rusetsky, EPJA (2012))
Finite-volume & chiral extrapolations

QCD calculations in finite volume

- unphysical pion mass
- (periodic) boundary conditions
  → discrete momenta & discrete spectrum

Recipe for $2 \rightarrow 2$ scattering (e.g. $I=J=0$ $\pi\pi$ scattering)

LÜSCHER(1986)

- 1 eigenenergy ↔ 1 phase-shift in infinite volume
- also with coupled channels
  He et al. (2005)
  Doring, Prelovsek, HSC

CHIRAL EXTRAPOLATIONS

- $M_\pi$ dependence from NLO ChPT (IAM)
  Gasser, Leutwyler(1981)
  Dobado, Pelaez (1997)
- Extrapolation in flavor
  B. Hu, MD, R. Molina M. Mai et al. (2016)
GWU lattice group: the isoscalar sector

- nHYP-smeared clover fermions with mass-degenerate quark flavors ($N_f = 2$)
- $M_\pi = 227$ MeV and 315 MeV
- 3 elongated boxes
- Large variational basis including several meson-meson operators
- Moving frames
- Conformal mapping for $\sigma$ pole extraction
- Unitarized Chiral Perturbation Theory fits for chiral extrapolation:
  - chm1: $I = L = 0$, $M_\pi = 227, 315$ MeV
  - chm2: $I = L = 0, 1$, $M_\pi = 227, 315$ MeV

Chiral extrapolation and exp. data

$M_\pi = M_{\text{phys}}$

\[ f_0(980) \rightarrow \]

$M_\pi = 227$ MeV

$M_\pi = 315$ MeV
# Chiral extrapolation of $\sigma$ pole

$M_\pi = 138$ MeV

| Parametrization | Fitted data                  | $\text{Re } z^*$ | $-\text{Im } z^*$ | $|g|$  |
|----------------|------------------------------|------------------|-------------------|-------|
| chm1           | $\sigma_{227,315}$          | $440^{+60}_{-90}$| $240^{+20}_{-50}$ | $3.0^{+0.2}_{-0.6}$ |
| chm2           | $\sigma_{227} \rho_{227}$   | $430^{+20}_{-30}$| $250^{+30}_{-30}$ | $3.0^{+0.1}_{-0.1}$ |
| chm2           | $\sigma_{315} \rho_{315}$   | $460^{+10}_{-15}$| $210^{+40}_{-30}$ | $3.0^{+0.1}_{-0.1}$ |
| chm2           | $\sigma_{227,315} \rho_{227,315}$ | $440^{+10}_{-16}$| $240^{+20}_{-20}$ | $3.0^{+0.0}_{-0.0}$ |
| Ref. [1]       | experimental                | $449^{+22}_{-16}$| $275^{+12}_{-12}$ | $3.5^{+0.3}_{-0.2}$ |


[Consistent with conformal-mapping amplitude parametrization (model-independent, not shown)]
Residues

HadSpec, PRL (2017)

Hanhart, Pelaez, Rios, PRL (2008)

Pelaez, Rios, PRD (2010)

$|g_{\pi\rho\rho}|$ [GeV]

$M_\pi$ [GeV]

345 (GWU)

415 (GWU)
→ $\sigma$ becomes a (virtual) bound state @ $M_\pi = (345)\ 415\ MeV$