Three Positive Pions in a Finite Volume

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Jefferson Lab

Bethe Forum

Multihadron dynamics in a box

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MA 7156/1-1

[Many slides from **Maxim Mai**]

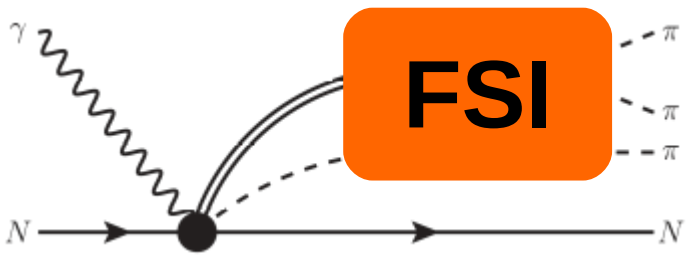
Outline

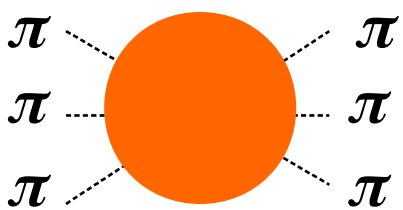
- Three-body dynamics in infinite volume
- The Finite-volume problems (application: 2-body)
- **Three-body dynamics in finite volume**
  - The 3-pion system at maximal isospin:

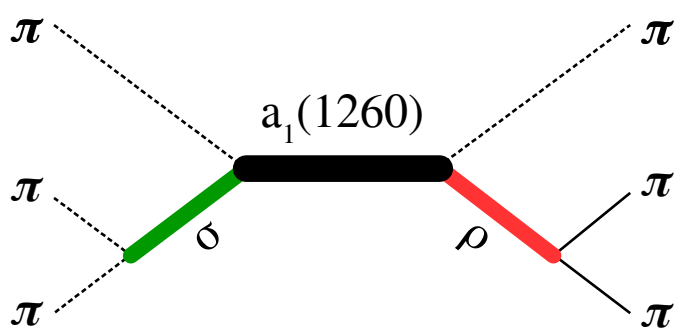
    Interpretation of recent lattice QCD data

3-body dynamics for mesons and baryons

Light mesons

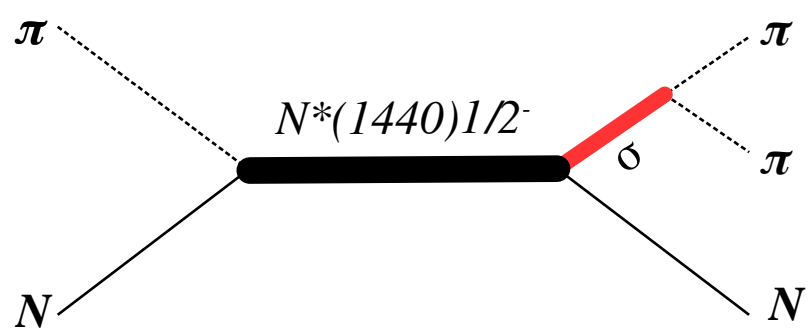






- Important channel in GlueX @ JLab
- Finite volume spectrum from lattice QCD:
  Lang, Leskovec, Mohler, Prelovsek (2014)
  Woss, Thomas et al. [HadronSpectrum] (2018)
  Hörz, Hanlon (2019), ...

Light baryons

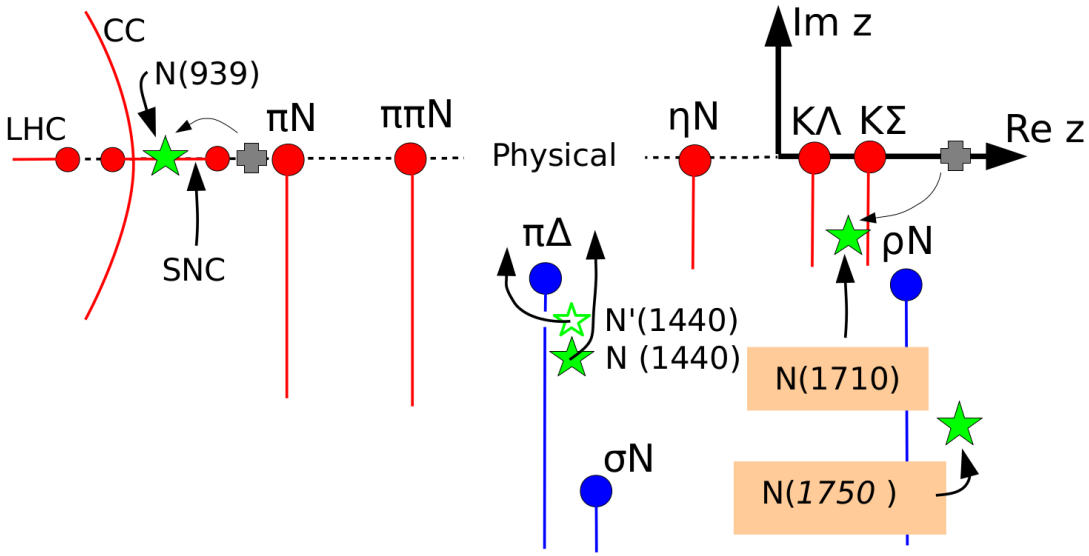


- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

Three-body Interactions with Isobars

Mai, Hu, M. D., Pilloni, Szczepaniak

Eur. Phys. J. A53 (2017) 177



3-body Unitarity

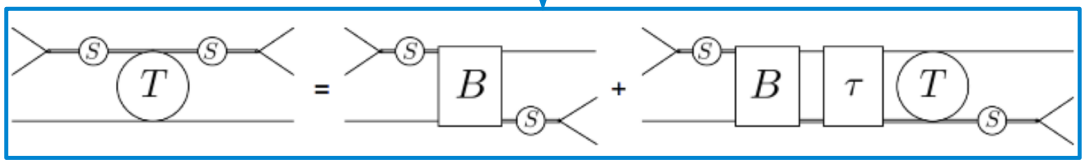
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4 k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2 - m^2)}\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)$$

delta function sets all intermediate particles on-shell

3-body Unitarity

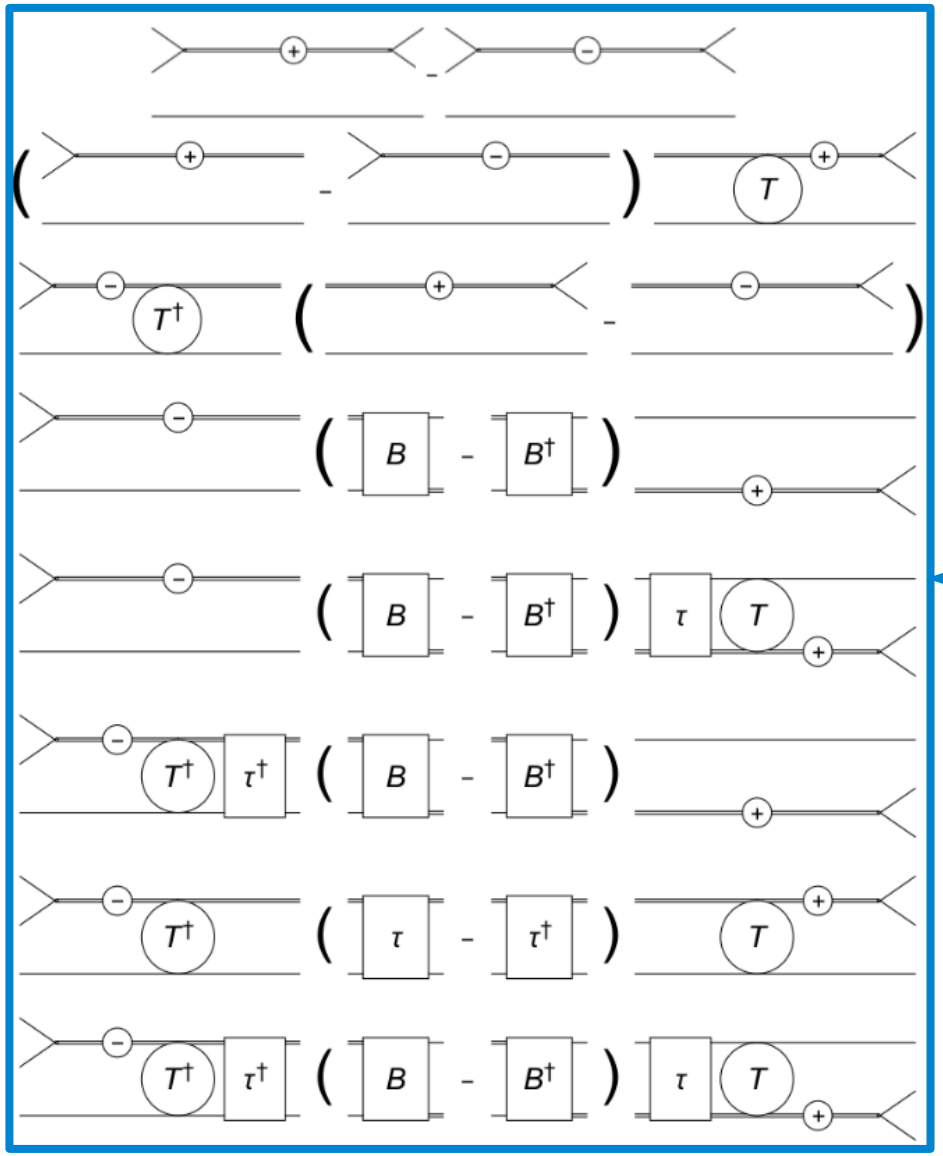
$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$

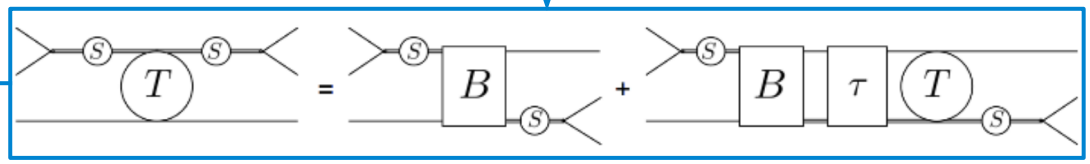


General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

3-body Unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$





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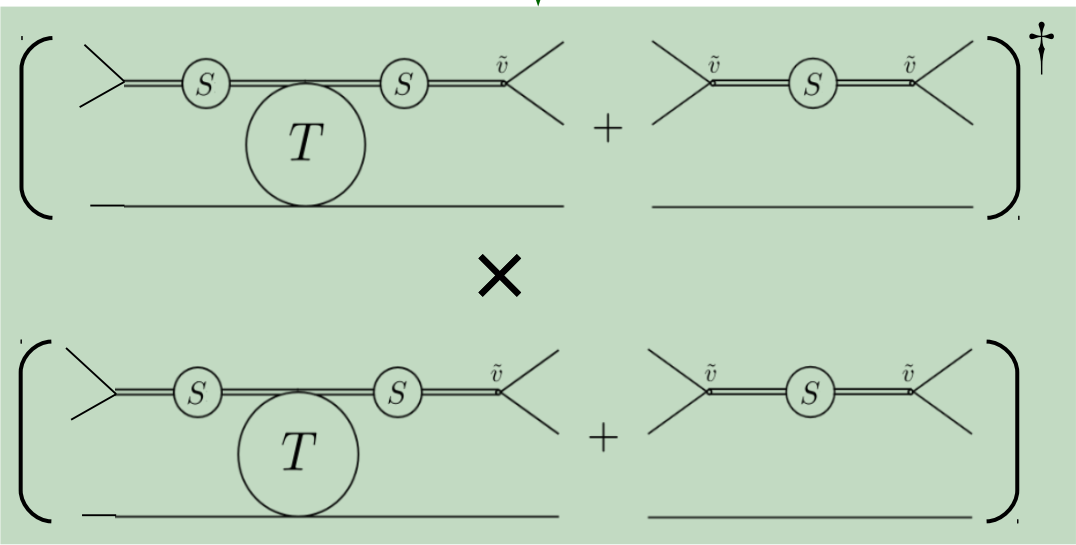

3-body Unitarity

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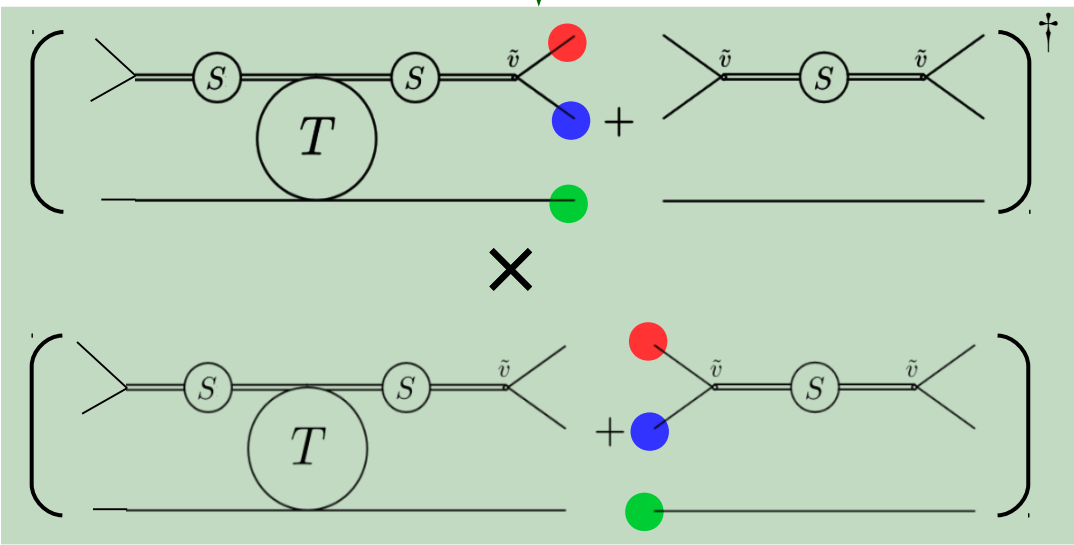

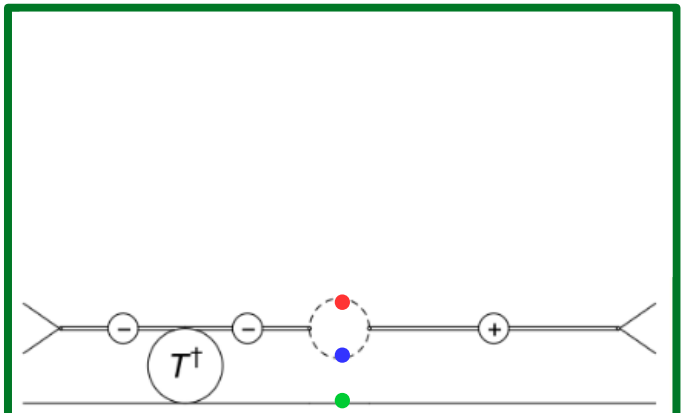

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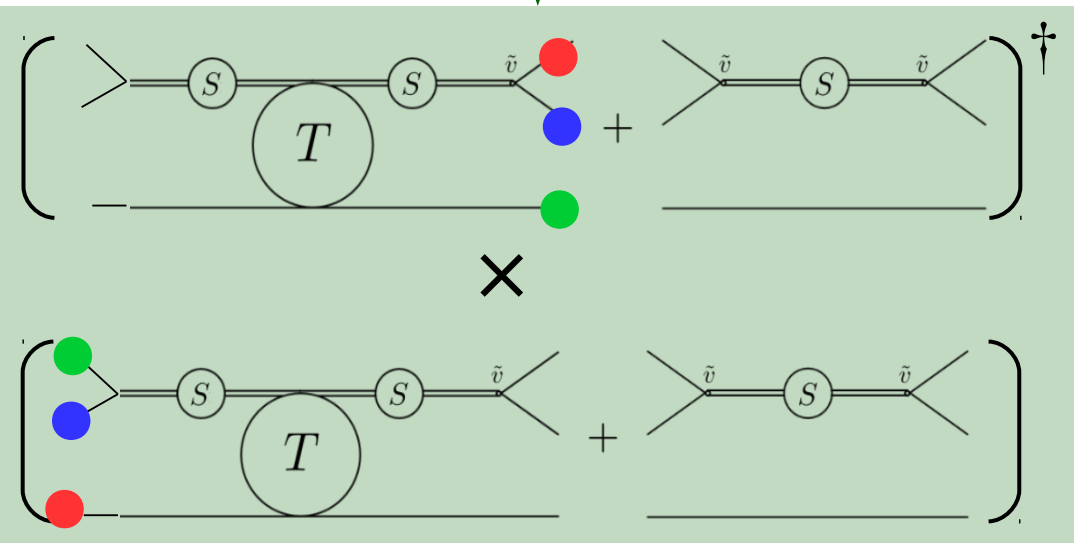

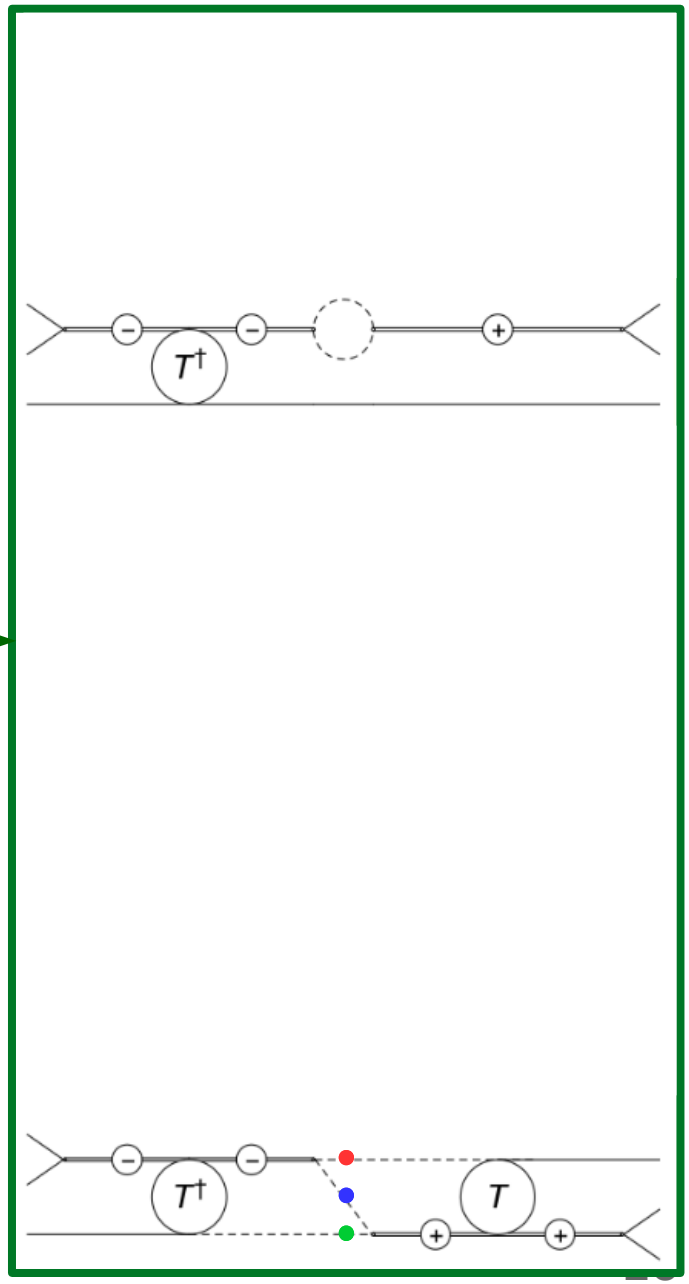

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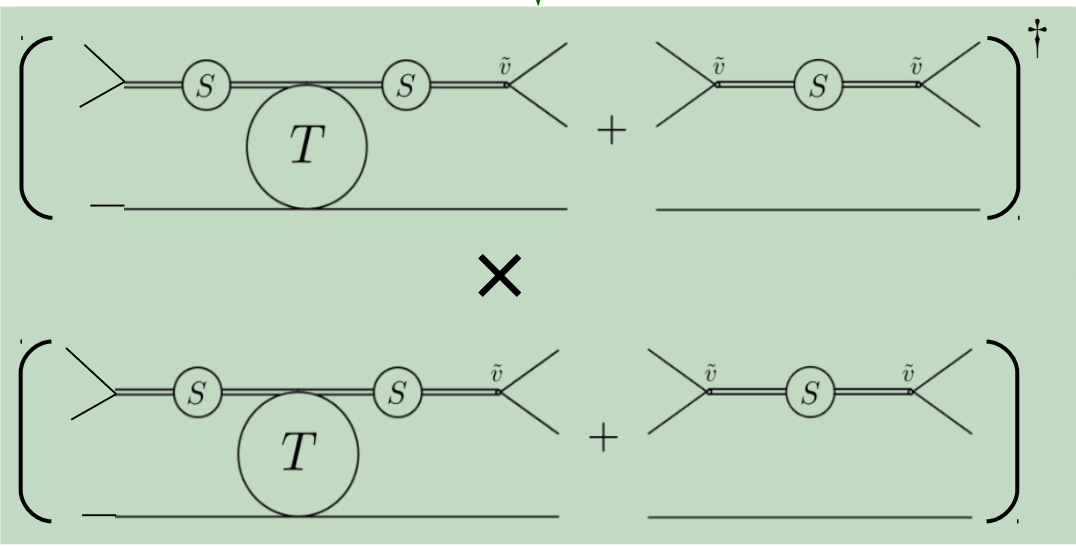
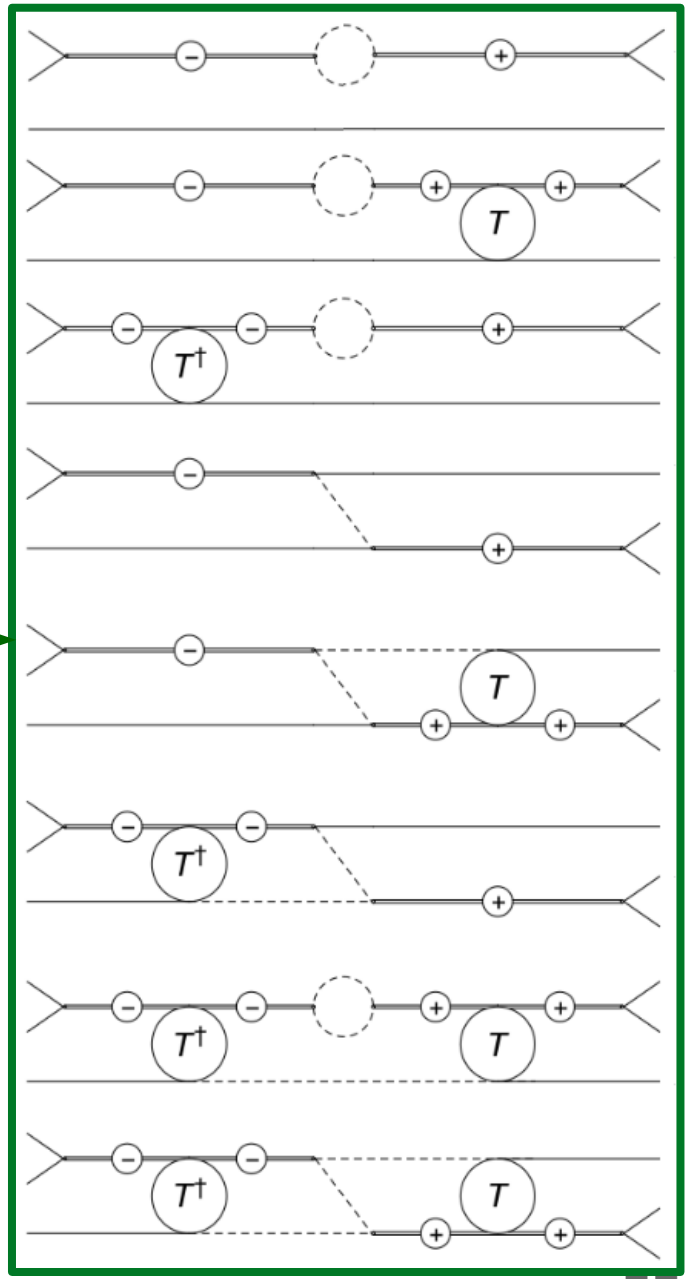


8 topologies



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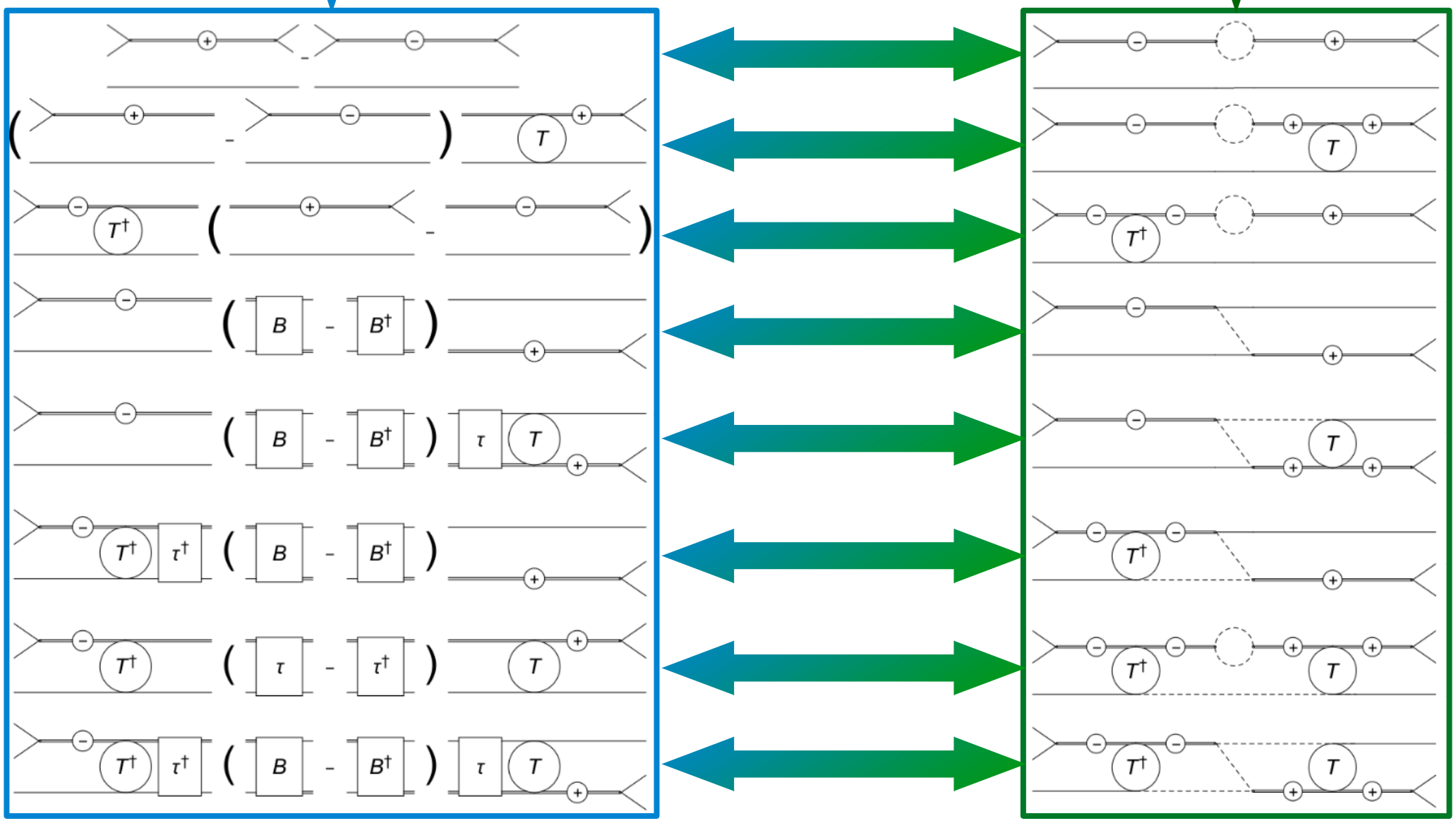
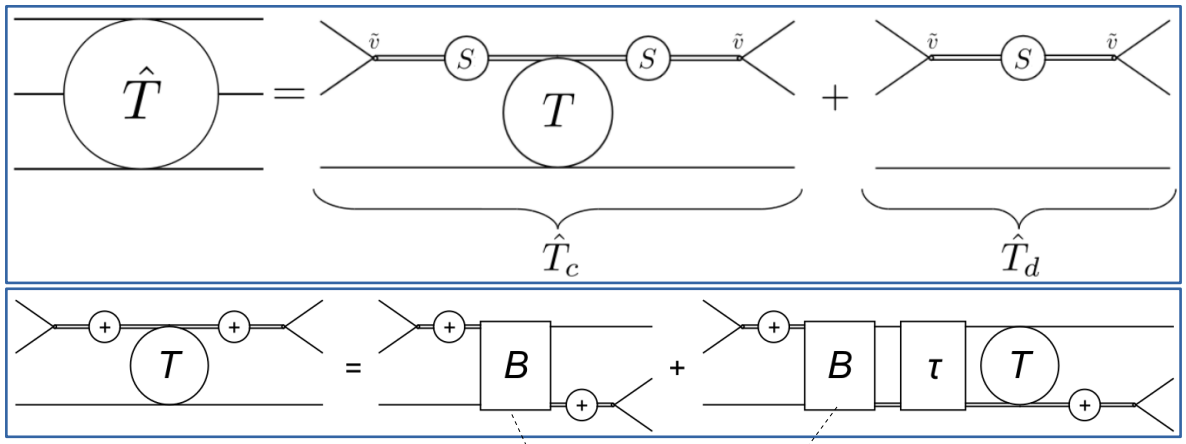


Scattering amplitude I

3 → 3 scattering amplitude is a 3-dimensional integral equation



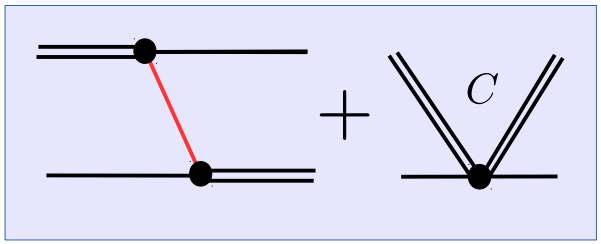
– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

– For simplicity ***v=λ*** (full relations available)

$$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$



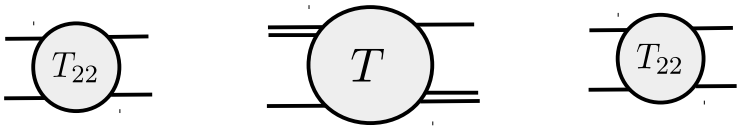
- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One can map to field theory, but does not have to. Result is a-priori dispersive.

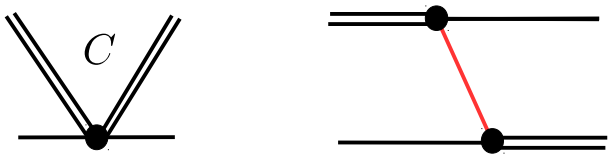
Scattering amplitude II

Here: Version in which isobar rewritten in on-shell 2→ 2 scattering amplitude T_{22}

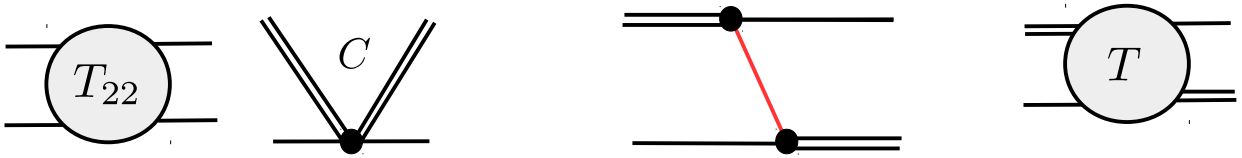
$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle = \frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$



$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2-(P-p-q)^2-i\epsilon}$$



$$-\int \frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2E_\ell}T_{22}(\sigma(\ell))\left(\langle p|C(s)|\ell\rangle + \frac{1}{m^2-(P-p-\ell)^2-i\epsilon}\right)\langle \ell|T(s)|p\rangle$$



(s-wave)

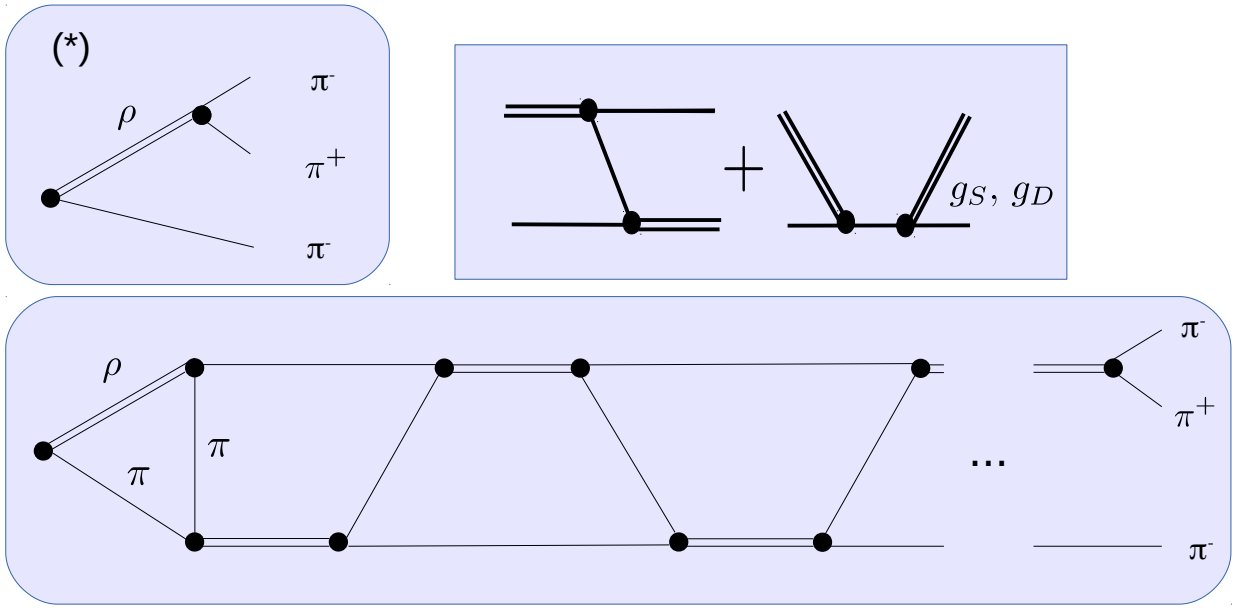
Application I: The $a_1(1260)$ lineshape

Sadasivan, M.D., Mai, very preliminary

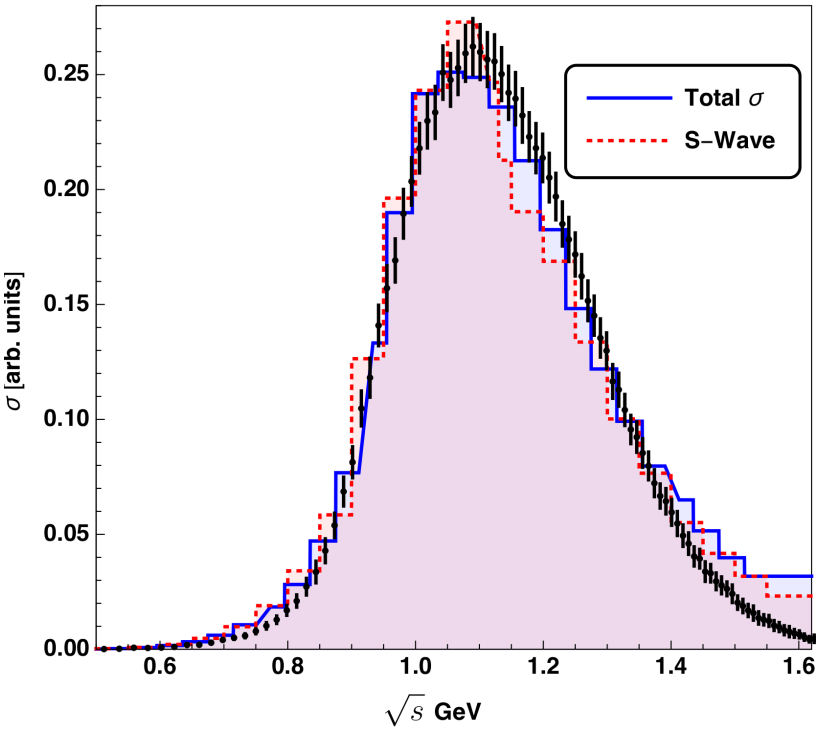
- Recent efforts to study 3-body production beyond the "isobar approximation" (*)

  P. Magalhães, A. C. dos Reis et al., PRD84 (2011); Khmechandani, Martinez, Oset, PRC77 (2008); JPAC: Mikhasenko, Wunderlich et. al., JHEP (2019); Mikhasenko, Pilloni et. al., PRD98 (2018); A. Jackura et al., EPJC79 (2019); Jülich: Janssen et al., PRL (1993)

- Here: Full solution of three-body equation with **exact** three-body unitarity
- S- and D-waves in $\pi\rho$ included



+ symmetrization of final states



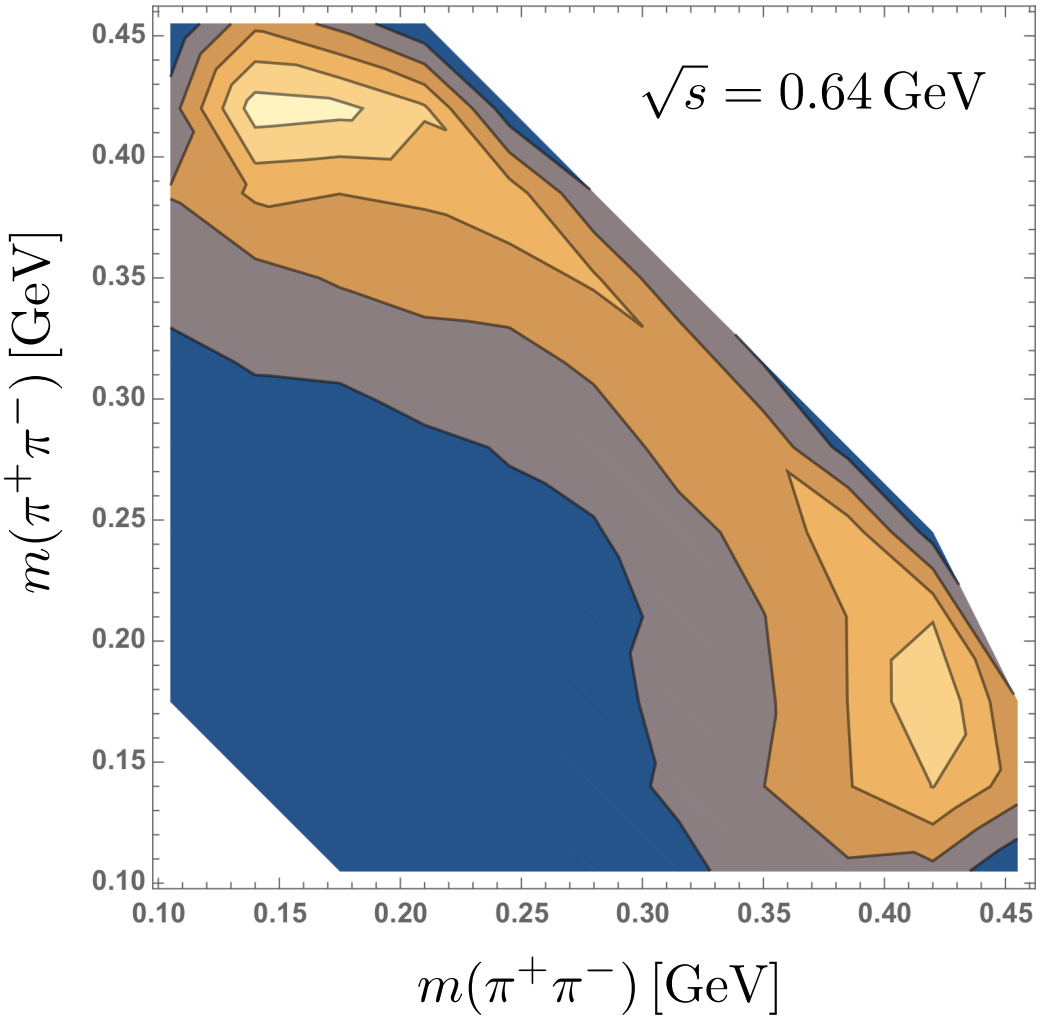
Data: ALEPH coll. [hep-ex/0506072]

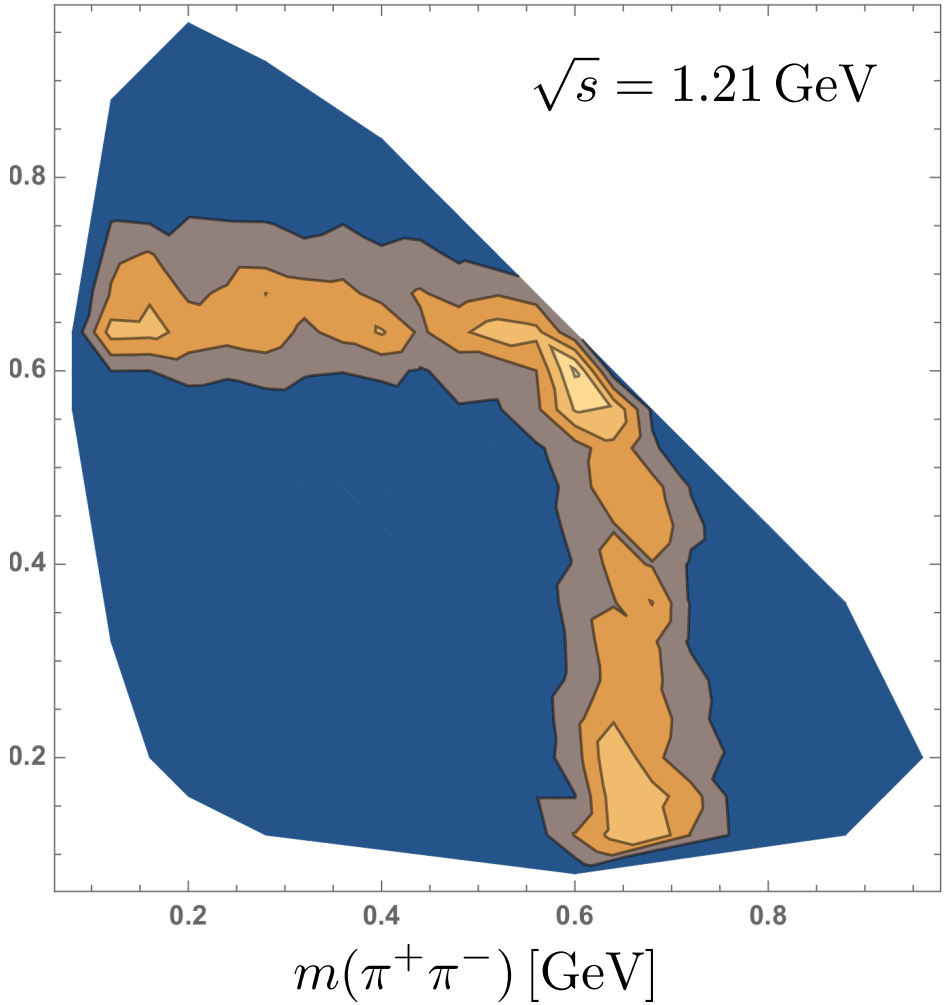
(*) here meant in the sense of "no rescattering", "no three-body unitarity"

Application II: Dalitz Plots

Sadasivan, M.D., Mai, enormously preliminary

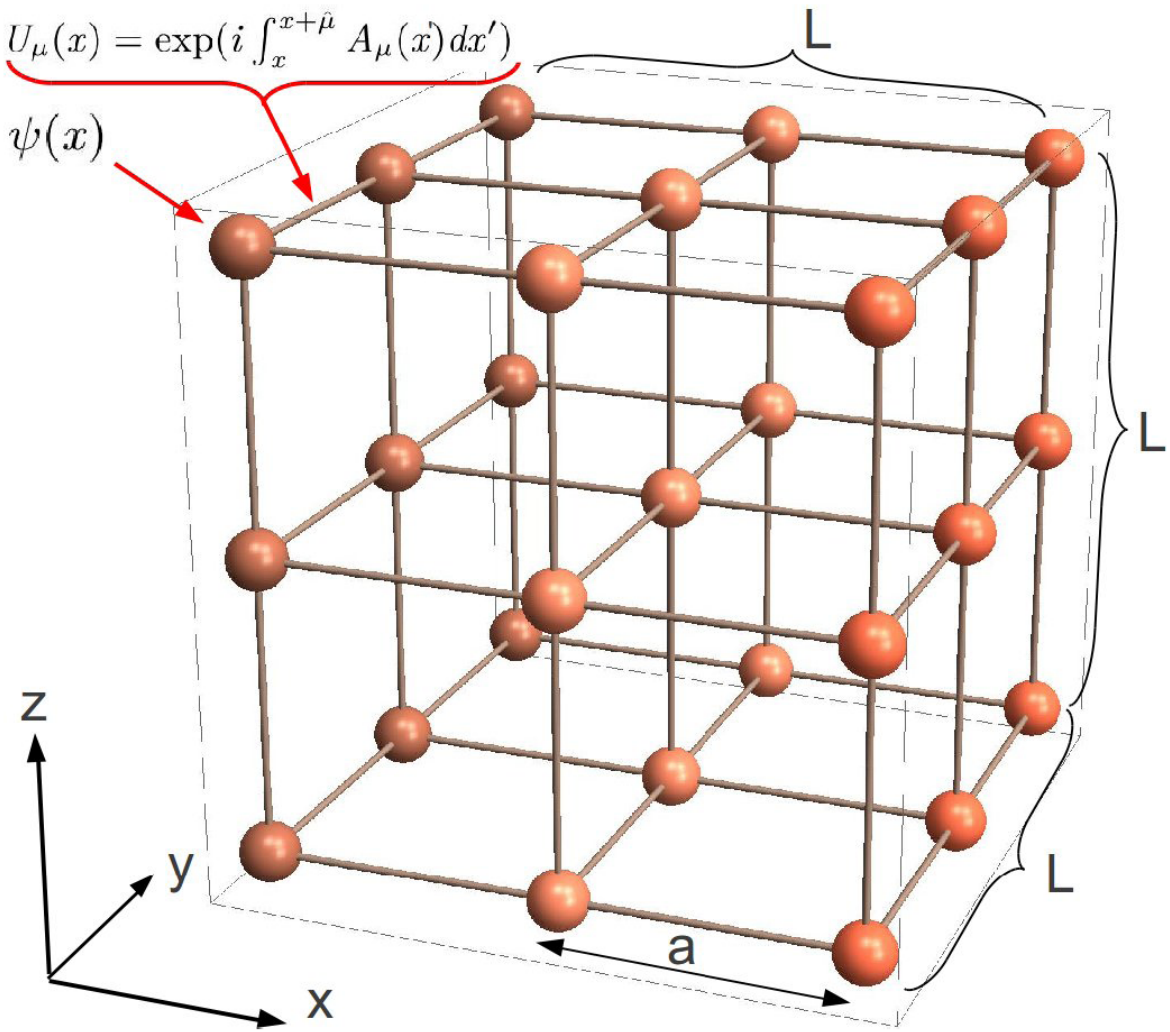
$$I^G(J^{PC}) = 1^-(1^{++}) \to (\pi\rho)_S,\ (\pi\rho)_D$$





From two to three particles in finite volume

The cubic lattice



- Side length L,
  periodic boundary conditions
  $\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L)$
  $\rightarrow$ finite volume effects
  $\rightarrow$ Infinite volume $L \rightarrow \infty$ mapping

- Lattice spacing a
  $\rightarrow$ finite size effects
  Modern lattice calculations:
  $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV
  $\rightarrow$ (much) larger than typical hadronic scales;

  not considered here.

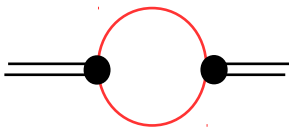
- Unphysically large quark/hadron masses
  $\rightarrow$ (chiral) extrapolation required.

How to derive the 2-body quanti-zation condition

Two-body unitarity

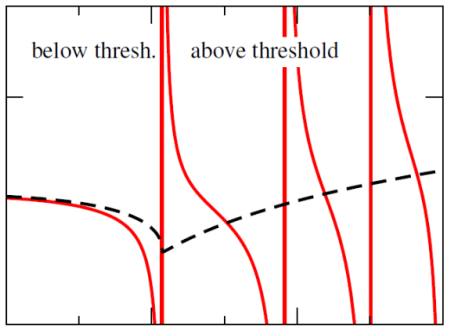
↓

On-shell condition



↓

Imaginary parts



Infinite → Fin. Vol

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_s \sum_{i=1}^{\vartheta(s)}$$

↓

Power-law fin-vol. effects

↓

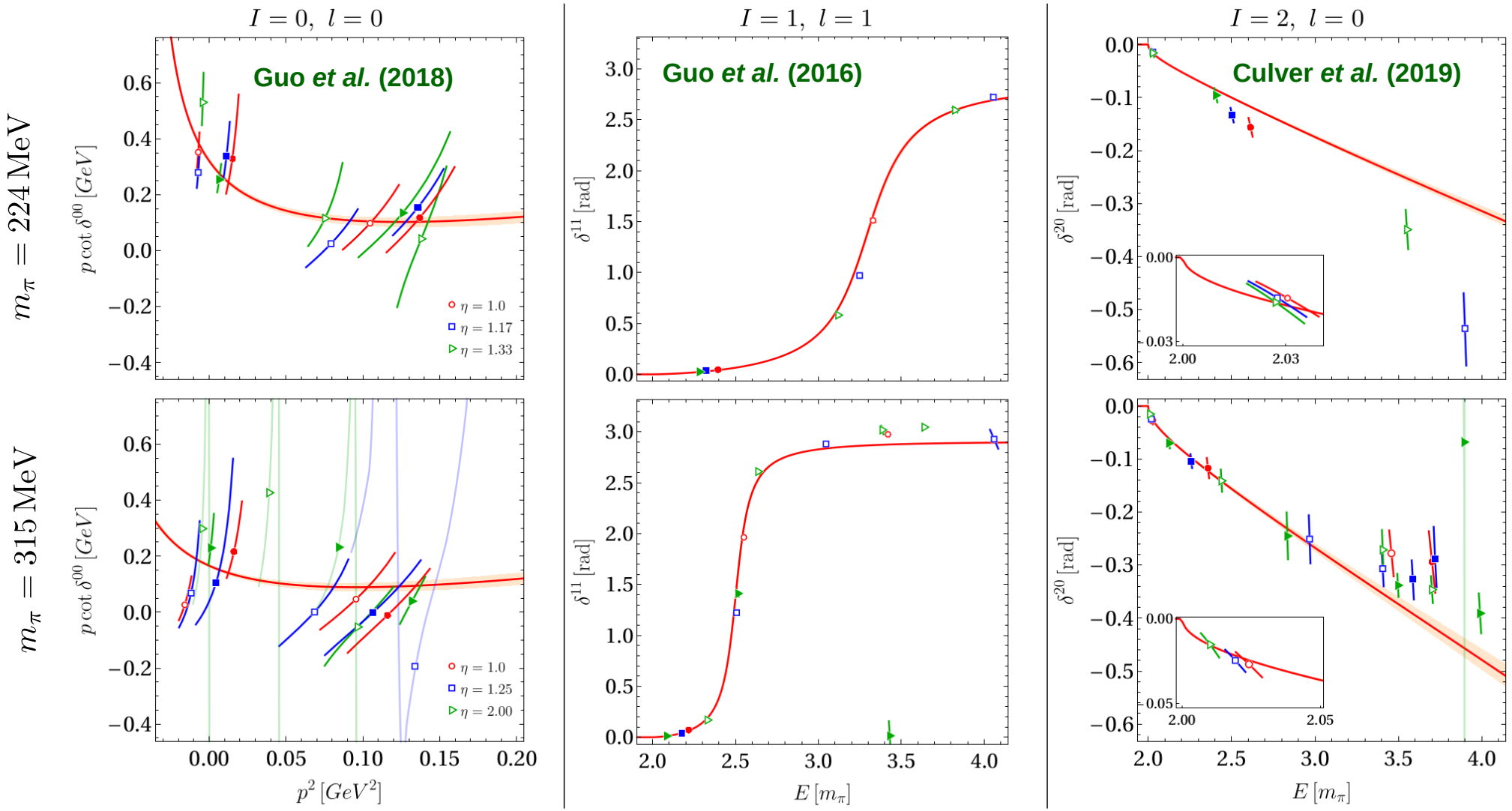
Lüscher

$$p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)$$

GWU lattice group*: All Isospins

* A. Alexandru, F.X. Lee *et al.*

[Culver et al., PRD100 (2019); Mai *et al.*, arXiv:1908.01847 [hep-lat]]

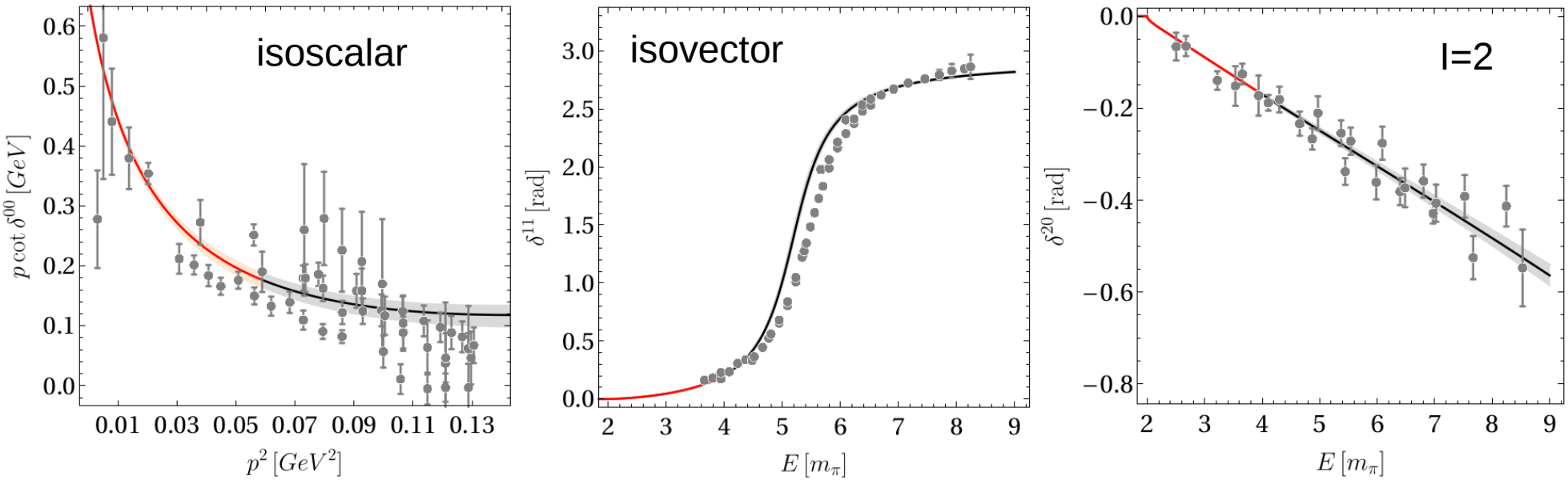


- Simultaneous fit with Inverse Amplitude Method (more later)
- Including correlation between energy eigenvalues, pion masses and pion decay constants
- Including correlations across energy eigenvalues & isospins
- Also other parameterizations (conformal mapping with Adler zero & unitarity, **[Guo *et al.* (2018)]**)

GWU lattice group: Chiral Extrapolation

(Optional & model dependent)

[Mai et al., arXiv:1908.01847 [hep-lat]]



Scattering lengths and resonance poles:

m_π [MeV]	~ 315	~ 224	139
$m_\pi\, a_0^{I=0}$	$+1.9008^{+0.0521}_{-0.0593}$	$+0.6985^{+0.0010}_{-0.0015}$	$+0.2132^{+0.0008}_{-0.0009}$
$m_\pi\, a_0^{I=2}$	$-0.1538^{+0.0021}_{-0.0018}$	$-0.0952^{+0.0010}_{-0.0009}$	$-0.0433^{+0.0002}_{-0.0002}$
m_σ [MeV]	$+591^{+6}_{-5} - i109^{+4}_{-4}$	$+502^{+4}_{-4} - i175^{+6}_{-5}$	$+443^{+3}_{-3} - i221^{+6}_{-6}$
$g_{\sigma\pi\pi}$ [MeV]	533^{+2}_{-2}	426^{+2}_{-2}	$397.8^{+0.6}_{-0.6}$
m_ρ [MeV]	$+789^{+1}_{-1} - i20^{+0}_{-0}$	$+738^{+2}_{-1} - i43^{+1}_{-1}$	$+724^{+2}_{-4} - i67^{+1}_{-1}$
$g_{\rho\pi\pi}$ [MeV]	226^{+2}_{-2}	282^{+3}_{-2}	323^{+5}_{-3}

THREE-BODY AMPLITUDE IN A BOX

M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]

Overview

Lüscher-like formalism in 3 → 3 case is under investigation

Polejaeva/Rusetsky (2012)

Briceño/Hansen/Sharpe (2014-)

Non-relativistic approaches based on dimer picture & effective field theory

Kreuzer, Griesshammer(2012), Hammer et al. (2016)

F. Romero, Rusetsky, Urbach et. al. (2018)

Equivalence of various 3-body formalisms; three-body unitarity for Hansen/Sharpe

Requirements **Jackura et al. (2019) [JPAC], Briceño et al. (2019)**

- 3-body systems involve (resonant) two-body sub-amplitudes: Construct such that 2-body information can be included
- Need extrapolations between different energies (problem of underdetermination)
- Allow for systematic improvement by allowing more and more quantum numbers as lattice data improve (problem of underdetermination)
- At least, **all** possible intermediate on-shell configurations must be identified and included to ensure all power-law finite-volume effects are taken account of.

$\Longrightarrow$ This work: **Quantization condition from 3-body unitarity in isobar formulation**

(incomplete) Literature FinVol 3-body

D. Agadjanov, M. Döring, M. Mai, U.-G. Meißner, and A. Rusetsky, JHEP **06**, 043 (2016), arXiv:1603.07205 [hep-lat].

J. Bulava and M. T. Hansen, (2019), arXiv:1903.11735 [hep-lat].

M. T. Hansen, H. B. Meyer, and D. Robaina, Phys. Rev. **D96**, 094513 (2017), arXiv:1704.08993 [hep-lat].

S. Hashimoto, PTEP **2017**, 053B03 (2017), arXiv:1703.01881 [hep-lat].

Bulk properties of multi-hadron Systems (no resolution of microscopic properties)

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P. Klos, S. König, H. W. Hammer, J. E. Lynn, and A. Schwenk, Phys. Rev. **C98**, 034004 (2018), arXiv:1805.02029 [nucl-th].

Y. Meng, C. Liu, U.-G. Meißner, and A. Rusetsky, Phys. Rev. **D98**, 014508 (2018), arXiv:1712.08464 [hep-lat].

H.-W. Hammer, J.-Y. Pang, and A. Rusetsky, JHEP **09**, 109 (2017), arXiv:1706.07700 [hep-lat].

U.-G. Meißner, G. Ríos, and A. Rusetsky, Phys. Rev. Lett. **114**, 091602 (2015), [Erratum: Phys. Rev. Lett.117,no.6,069902(2016)], arXiv:1412.4969 [hep-lat].

M. Mai and M. Döring, Phys. Rev. Lett. **122**, 062503 (2019), arXiv:1807.04746 [hep-lat].

P. Guo, M. Döring, and A. P. Szczepaniak, Phys. Rev. **D98**, 094502 (2018), arXiv:1810.01261 [hep-lat].

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F. Romero-López, S. R. Sharpe, T. D. Blanton, R. A. Briceño, and M. T. Hansen, (2019), arXiv:1908.02411 [hep-lat].

F. Romero-López, A. Rusetsky, and C. Urbach, Eur. Phys. J. **C78**, 846 (2018), arXiv:1806.02367 [hep-lat].

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FV Analysis of Hörz/Hanlon data
arXiv:1905.04277 [hep-lat]

- Fits of 2&3 body sector
- Including D-wave isobar
- Extraction of 3-body force

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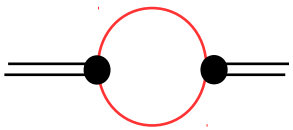
How to derive the 2-body quanti-zation condition

Two-body unitarity

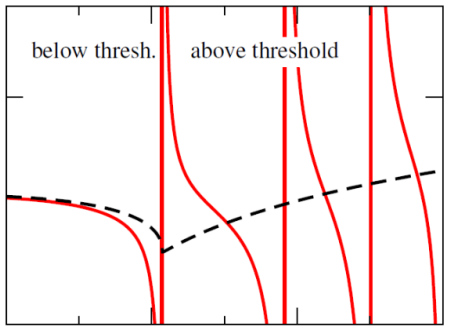
↓

On-shell condition



↓

Imaginary parts



Infinite

→ Fin. Vol

↓

Power-law fin-vol. effects

↓

Lüscher

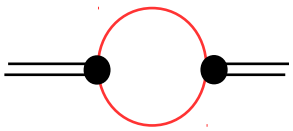
$$p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)$$

Three-body?

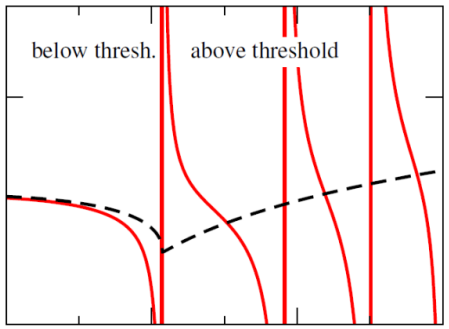
Analogously!

Two-body unitarity

On-shell condition



Imaginary parts



Infinite

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Power-law fin-vol. effects

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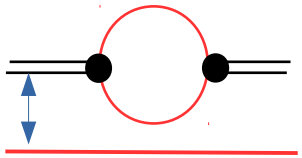
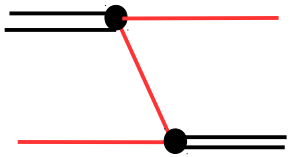
$$p \cot \delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)$$

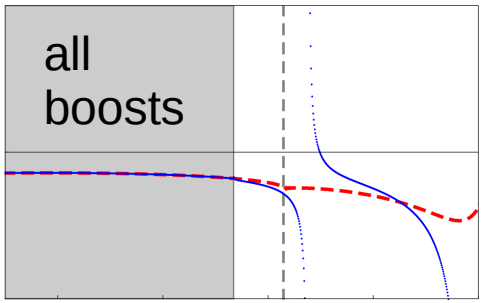
Three-body unitarity

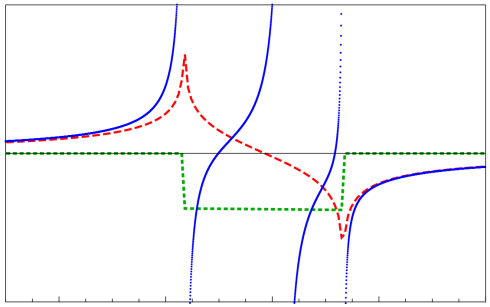
On-shell condition





Imaginary parts





Power-law fin-vol. effects

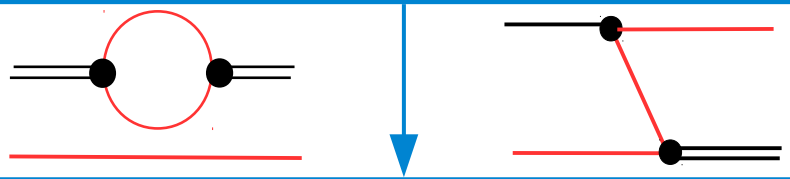
Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{ss'}\delta_{uu'}\right) = \mathbf{0}$$

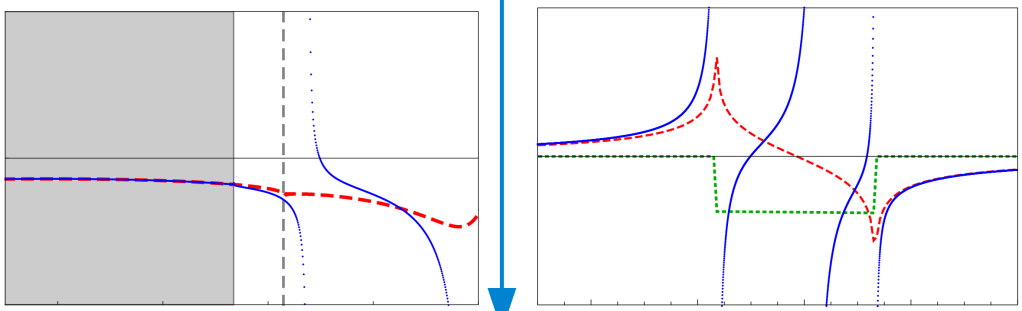
Only exact three-body unitarity guarantees the cancellation of unphysical 1st and 2nd order poles

Three-body unitarity

On-shell condition



Imaginary parts



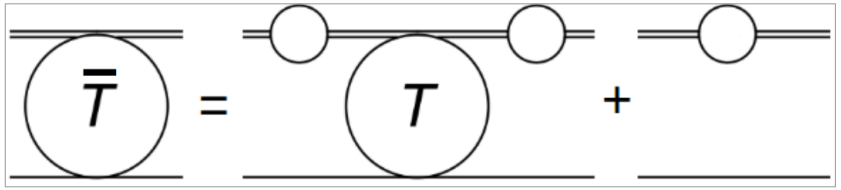
Power-law fin-vol. effects

Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{ss'}\delta_{uu'}\right) = \mathbf{0}$$

.

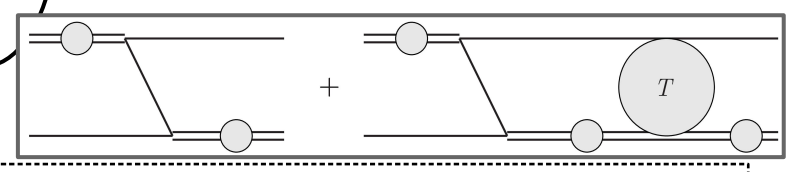
Cancellations



→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s) \tau_m(s) - 2E_n \tau_n(s) \frac{L^3}{\vartheta(n)} \delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3} \sum_{x \in set_8} \vartheta(x) B^{A_1^+}_{nx}(s) \frac{\tau_x(s)}{2E_x} T^{A_1^+}_{xm}(s)$$



$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

– when isobar-momenta are discretized in the 3-body cms momenta

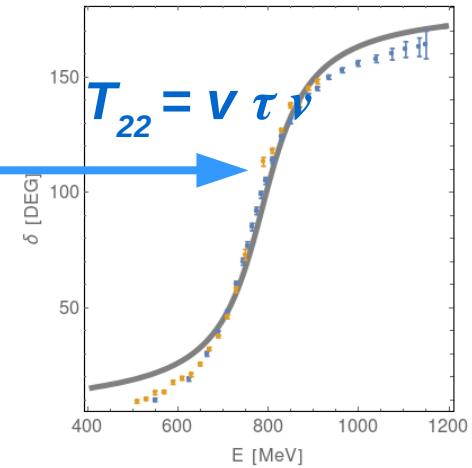
$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\boldsymbol{\ell} \frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

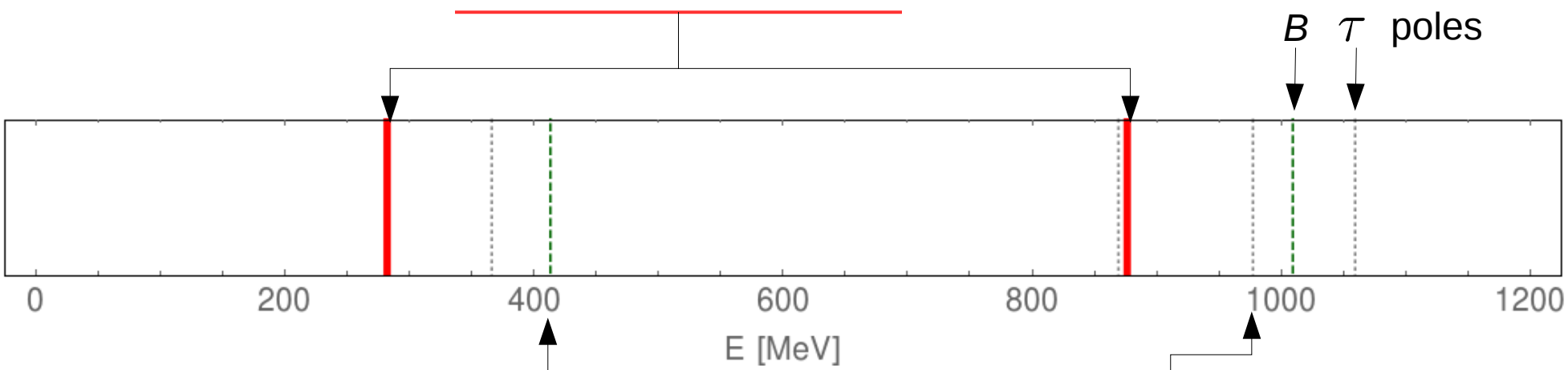
Also: all 2nd order singularities in determinant cancel → All consequence of Manifest three-body unitarity

Numerical demonstration

[M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]]

- First numerical demonstration of three-body finite volume formalism
- 3 particles in finite volume: ***m=138 MeV, L=3 fm***
- one S-wave isobar → two unknowns:
  - – vertex(Isobar→2 stable particles)
  - – subtraction constant (~mass)
- **Project to $\Gamma = A^{1+}$**
  - → **prediction of 3body energy-eigenlevels (C=0)**





unphysical levels cancel out (exact proof available)

A physical system:

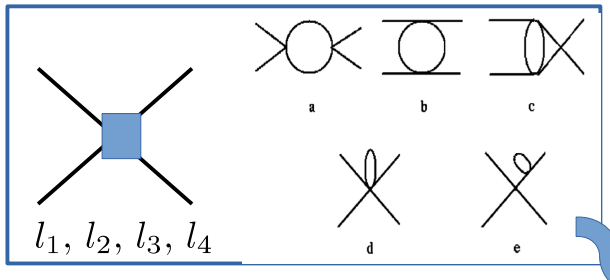
$\pi^+\pi^+\pi^+$

Mai, M.D., PRL 122 (2019), 062503

Three positive pions

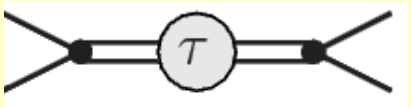
- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

NPLQCD, Detmold et al.(2008)



Inverse Amplitude method

Truong(1988), Peláez (1999), Gómez Nicola, Peláez (2002),...

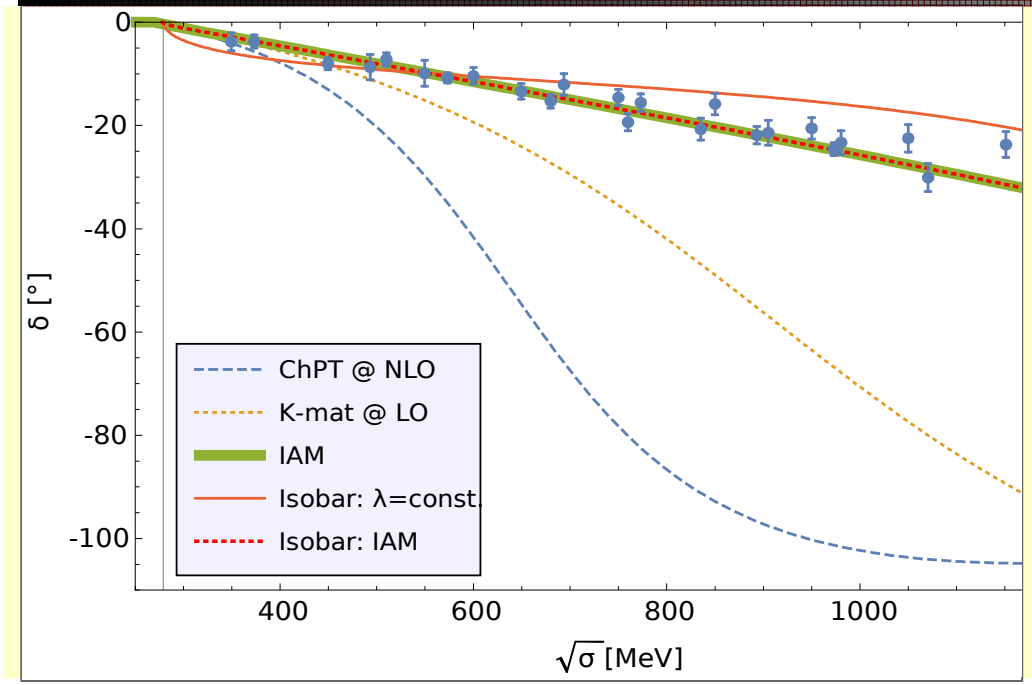


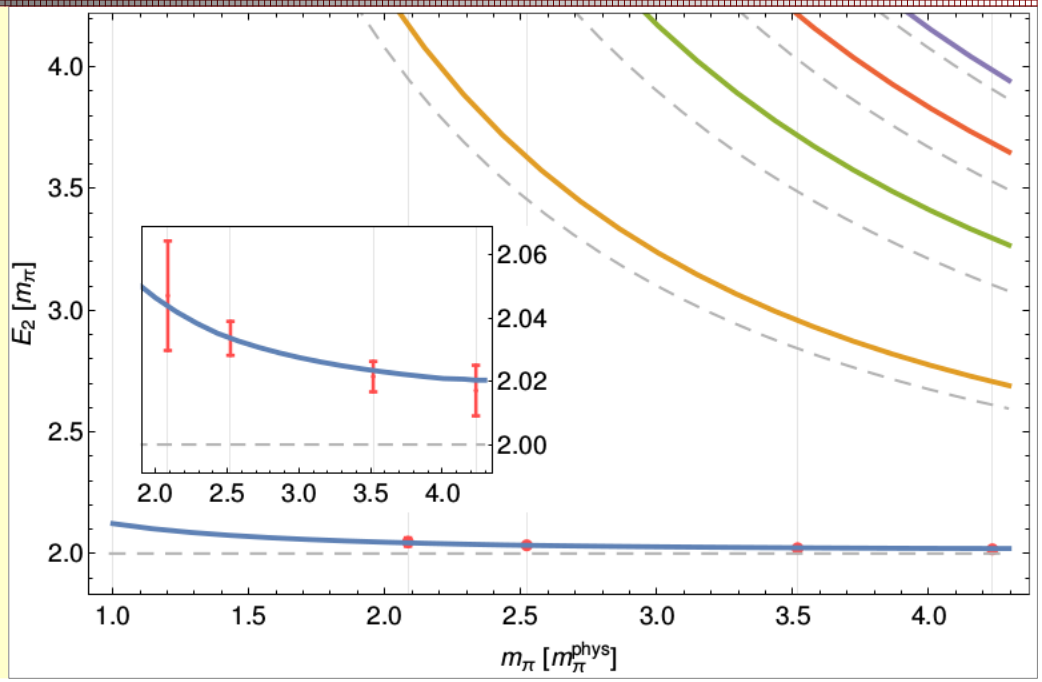
$$\frac{T_{\rm LO}^2}{T_{\rm LO} - T_{\rm NLO}}$$

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one

discretize (Lüscher) → predicted fin-vol. spectrum





- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

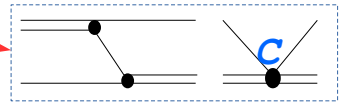
NPLQCD, Detmold et al.(2008)

II. 3-body spectrum

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{ss'}\delta_{uu'}\right) = \mathbf{0}$$

Remaining unknown: ***C***



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $C_{qp} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$

- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

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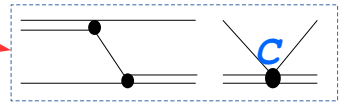

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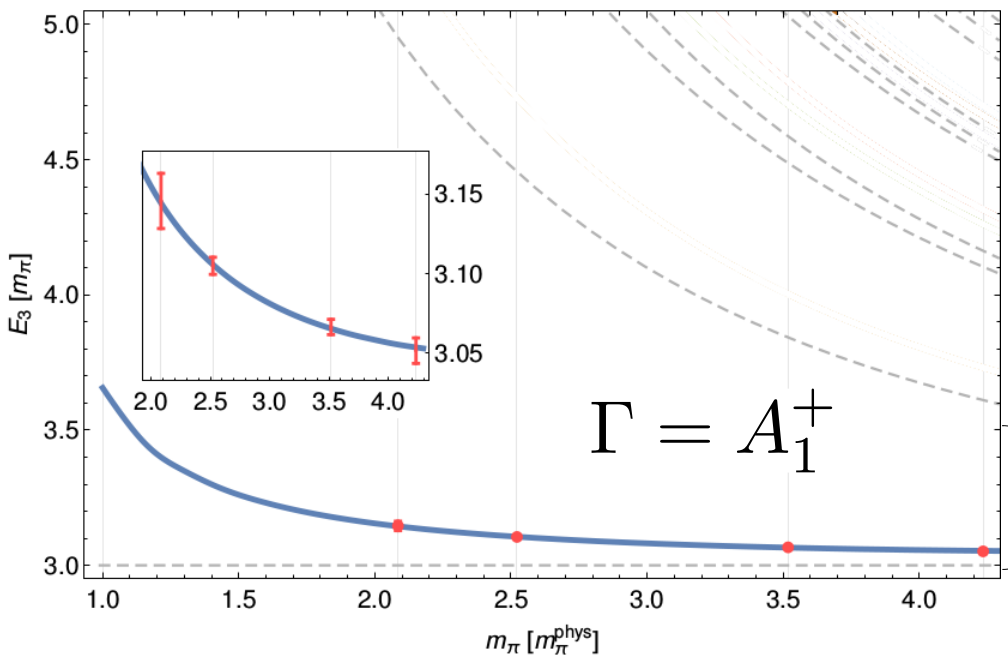


Fit ***C*** to NPLQCD ground state level

→ ***C=(0.2 ± 1.5)·10^{-10}***

First prediction of excited levels for physical system

- Maximal isospin: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$
  - Repulsive channel
  - ***L=2.5 fm, m_π=291/352/491/591 MeV***

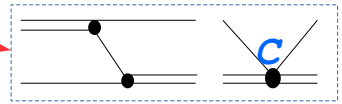
NPLQCD, Detmold et al.(2008)

II. 3-body spectrum

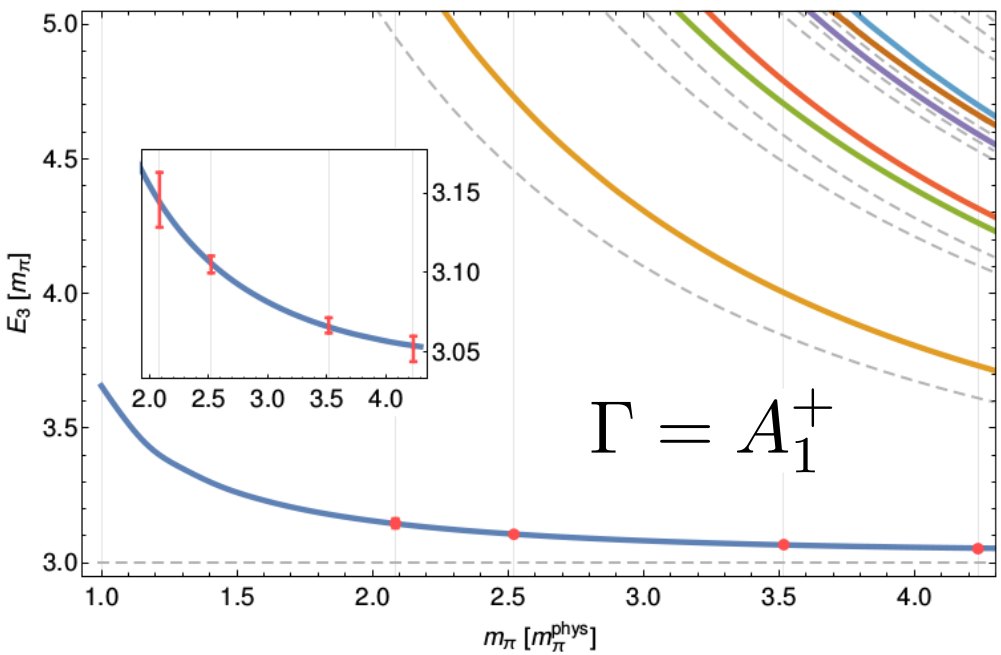
QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W}^2) + \frac{2E_s L^3}{\vartheta(s)}\tau_s(\mathbf{W}^2)^{-1}\delta_{ss'}\delta_{uu'}\right) = 0$$

Remaining unknown: c



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $C_{qp} = c\,\delta^{(3)}(p-q)$



Predict excited spectrum:

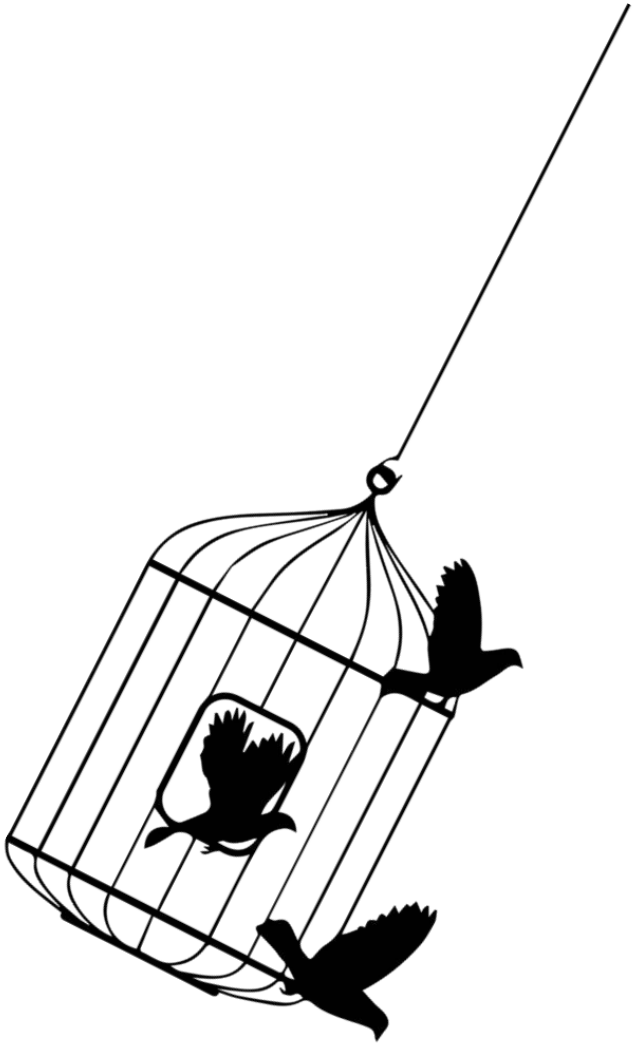
→ novel pattern

1/1 of interacting/non-interacting lvls

→ all QC-poles are simple

→ chiral extrapolation to phys point

(under assumptions)



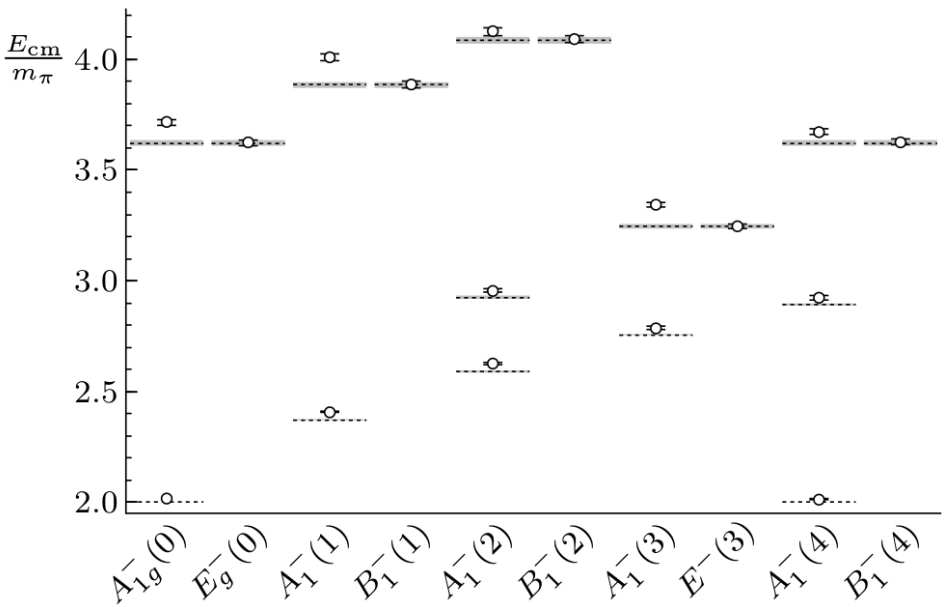
The Moving $\pi^+\pi^+\pi^+$ System

Mai, M.D., Culver, Alexandru
(in preparation)

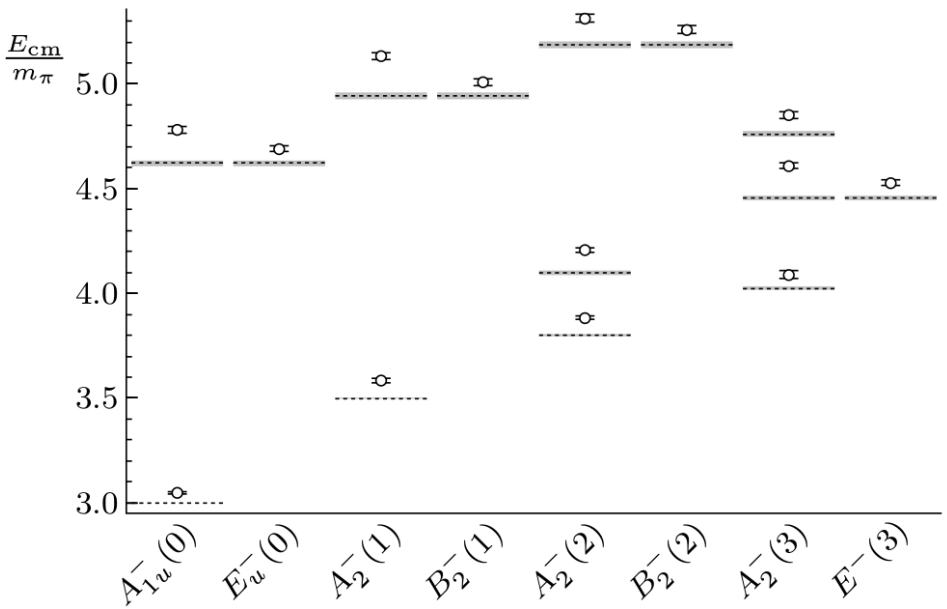
New Lattice Data

Hörz, Hanlon, arXiv:1905.04277 [hep-lat]

Two-body spectrum $\pi^+\pi^+$



Three-body spectrum $\pi^+\pi^+\pi^+$

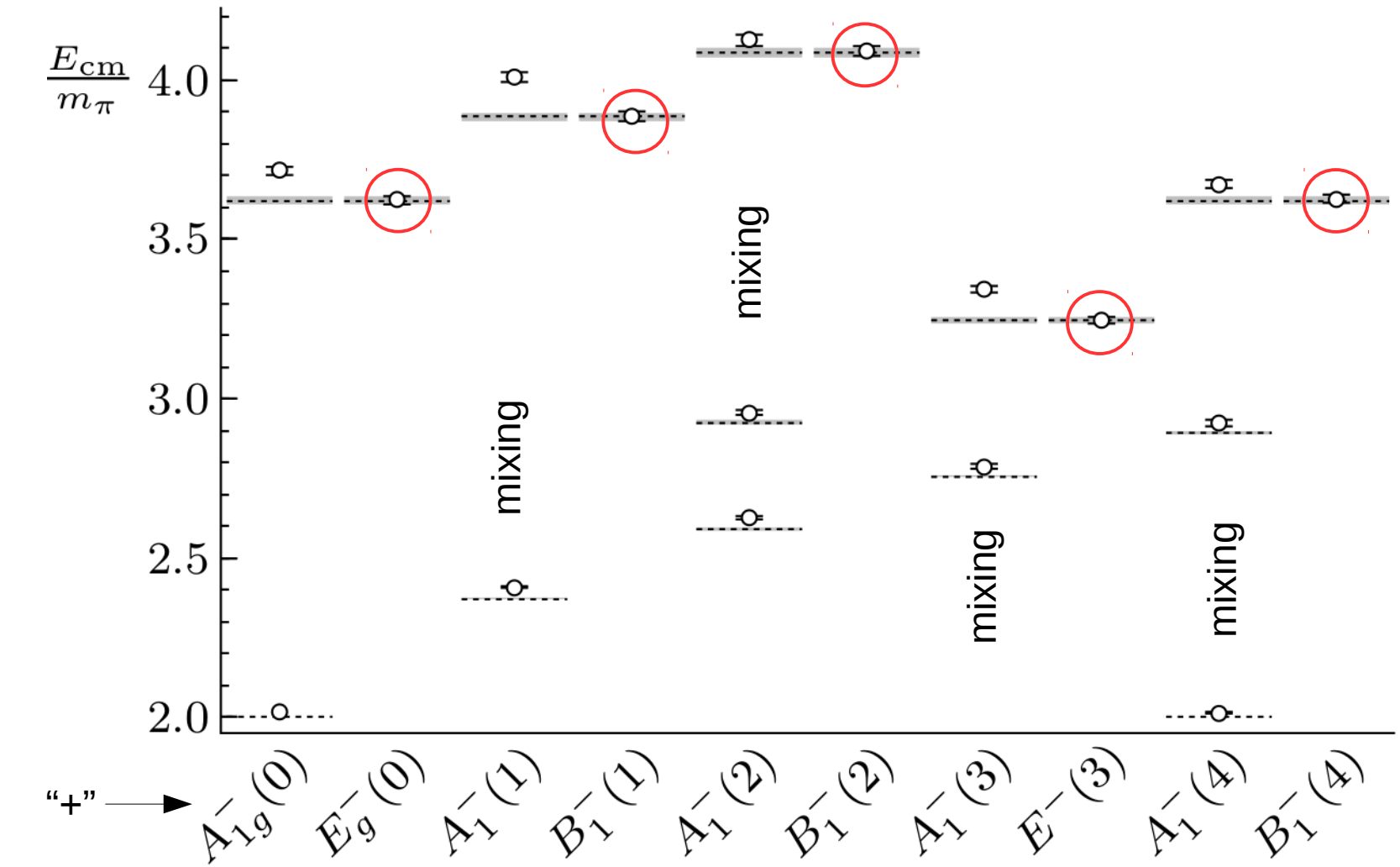


- **First lattice data on excited energy eigenvalues** from multi-pion operators
  → More reliable extraction of scattering eigenvalues
- D200 CLS ensemble (2+1) with improved Wilson fermions and tree-level Lüscher–Weisz gauge action; stochastic LapH method; m_π=200 MeV; **L=4.1 fm**
- High number of Wick contraction (20,679,840 diagrams) managed with novel method from quantum chemistry

See also talk by **F. Romero López/ arXiv:1909.02973**

Two-body spectrum: D-wave I

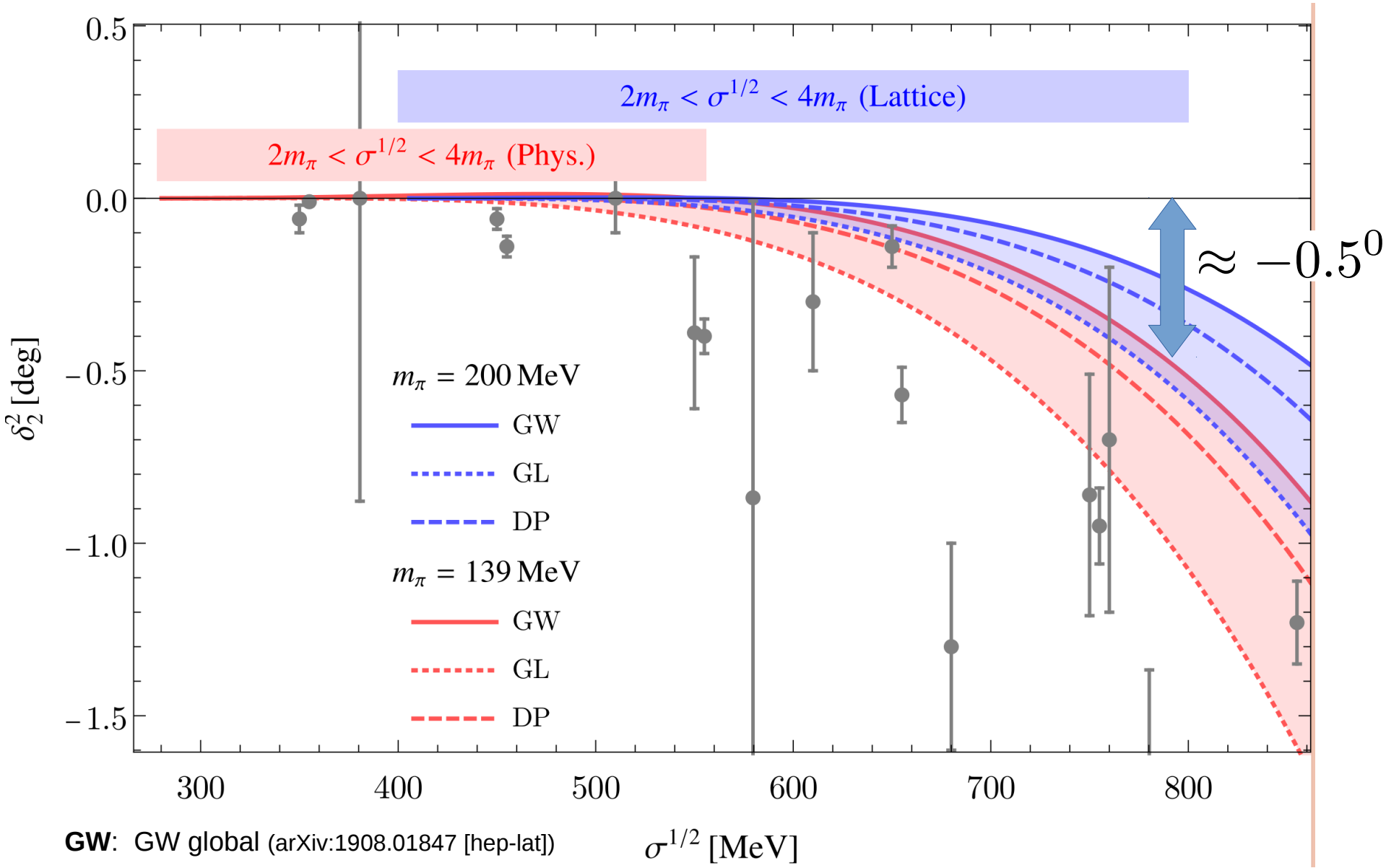
Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



PW $\ell < 3$: S D S&D D S&D D S&D D S&D D

→ I=2 D-wave vanishes within uncertainties – what does IAM predict?
→ Caveat: Correlated χ^2 reveals more information .

D-wave II: prediction



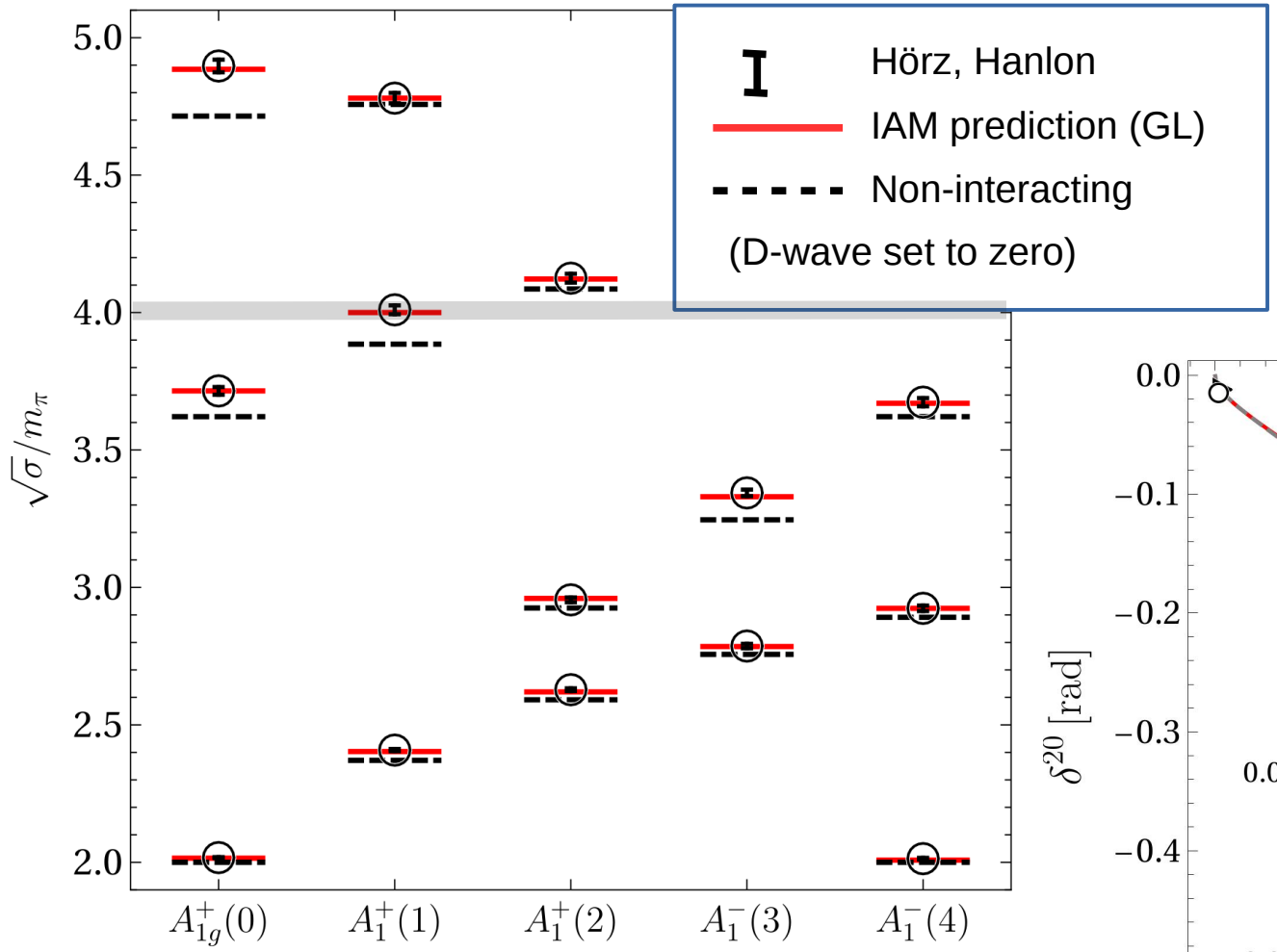
GW: GW global (arXiv:1908.01847 [hep-lat])

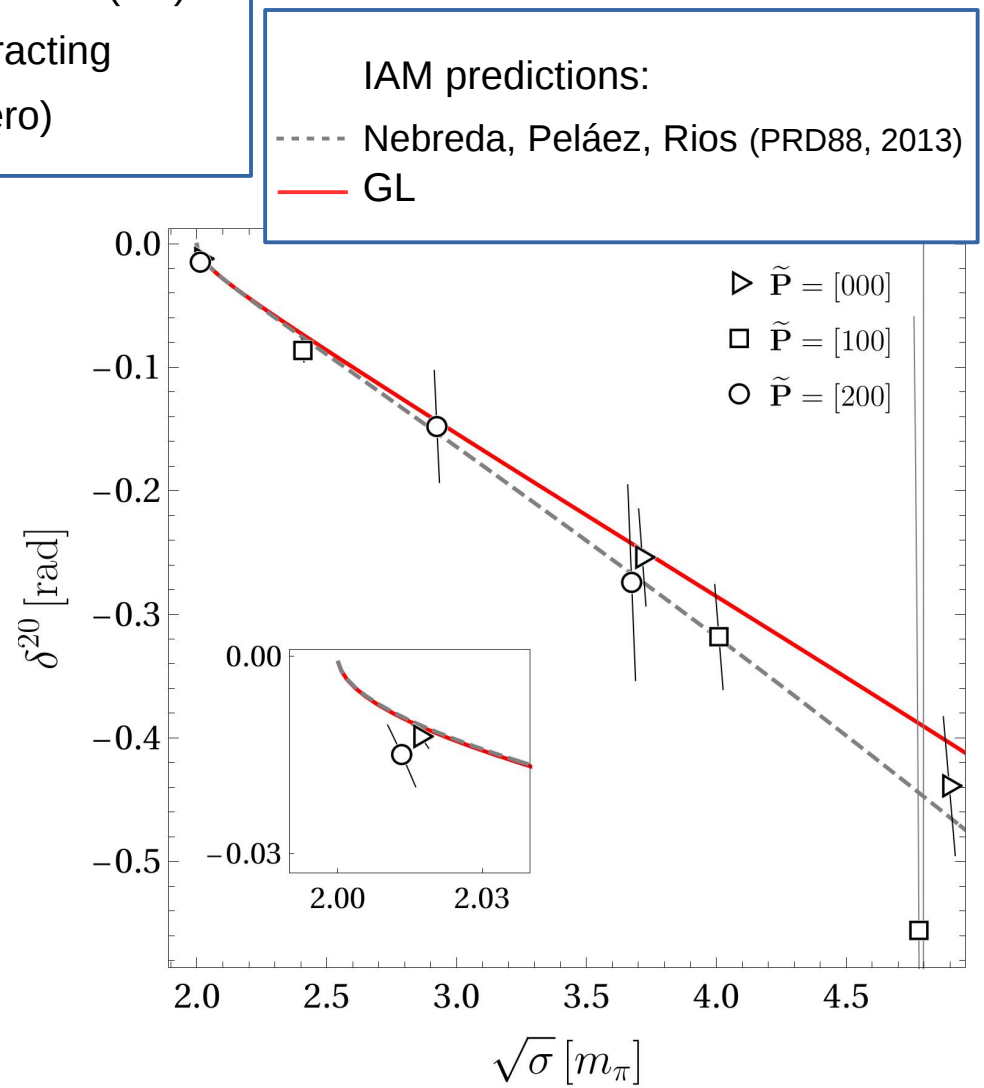
GL: Gasser, Leutwyler (Annals Phys. 158, 1984)

DP: Dobado, Peláez (PRD 56 (1997))

See also [Nebreda, Peláez, Ríos, PRD83 (2011)]

IAM predictions 2-body spectrum

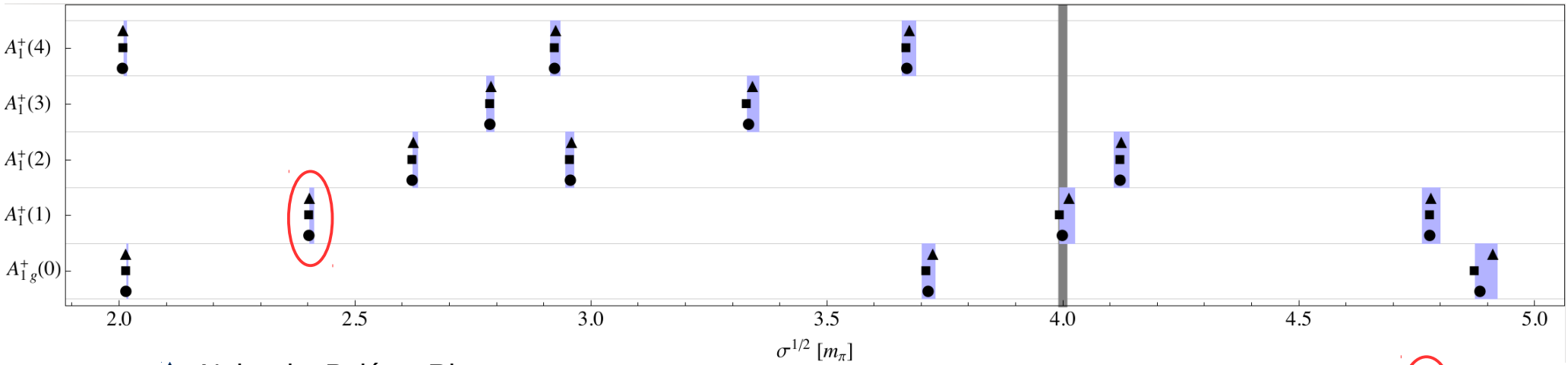




→ We may consider this as *any* suitable 2-body Parametrization (like, e.g., K-matrix with conformal mapping **[D. Guo et al., GW Lattice, PRD 98, (2018)]**)

IAM predictions: Different LECs

(D-wave set to zero)

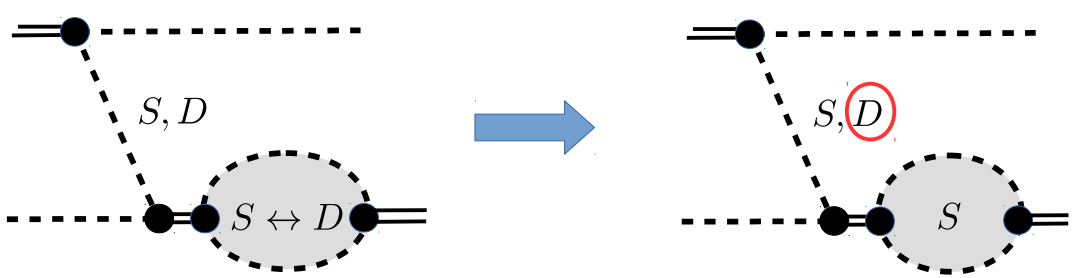


- ▲ Nebreda, Peláez, Rios (PRD88, 2013)
- ■ GW global (arXiv:1908.01847 [hep-lat])
- ● Gasser, Leutwyler (Annals Phys. 158, 1984)

$$\chi^2_{\rm GL}(\sqrt{\sigma} \lesssim 4m_\pi) = 21(16)$$

(2-body, correlated; 11(10) data)

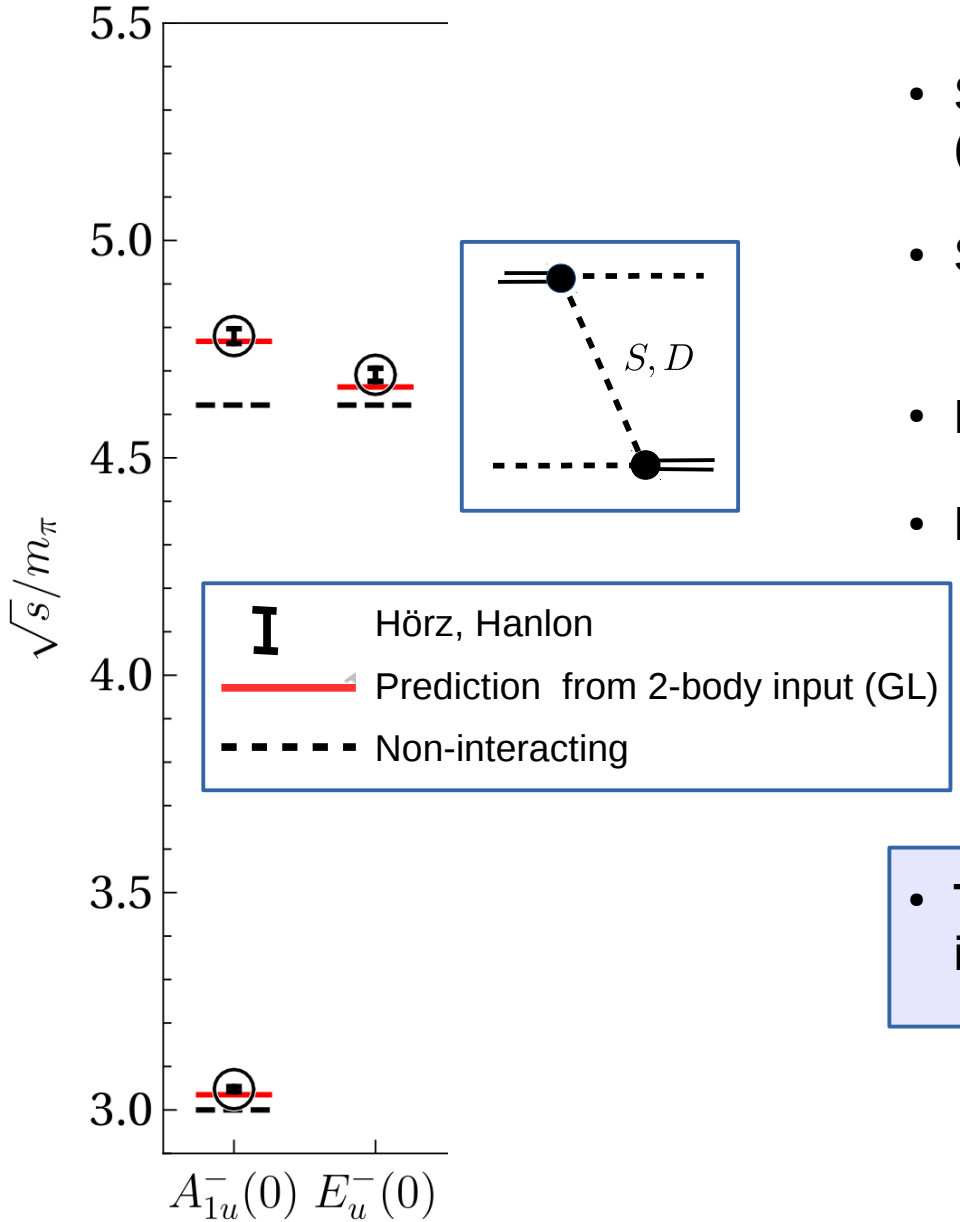
- Robust predictions of the 2-body spectrum irrespective of used LECs
- No sign of D-wave up to very high energies in irreps with S&D-wave mixing
- Ignore the vanishing $\pi^+\pi^+$ – D-wave, but keep the important π^+ – isobar D-wave



"in-flight transitions"

3-body Spectrum: Predictions (I)

Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



S **D** (lowest participating wave)

- S-wave prediction good at threshold (like for NPLQCD data)
- S-wave prediction good at high energies
  → **Energy dependence matched**
- No sign of 3-body force (like for NPLQCD data)
- D-wave prediction qualitatively good
  → **Relative* strength between S- and D-wave matched**
  → **Consequence that 3-body interaction dominated by exchange**
  → **Consequence of 3-body Unitarity**

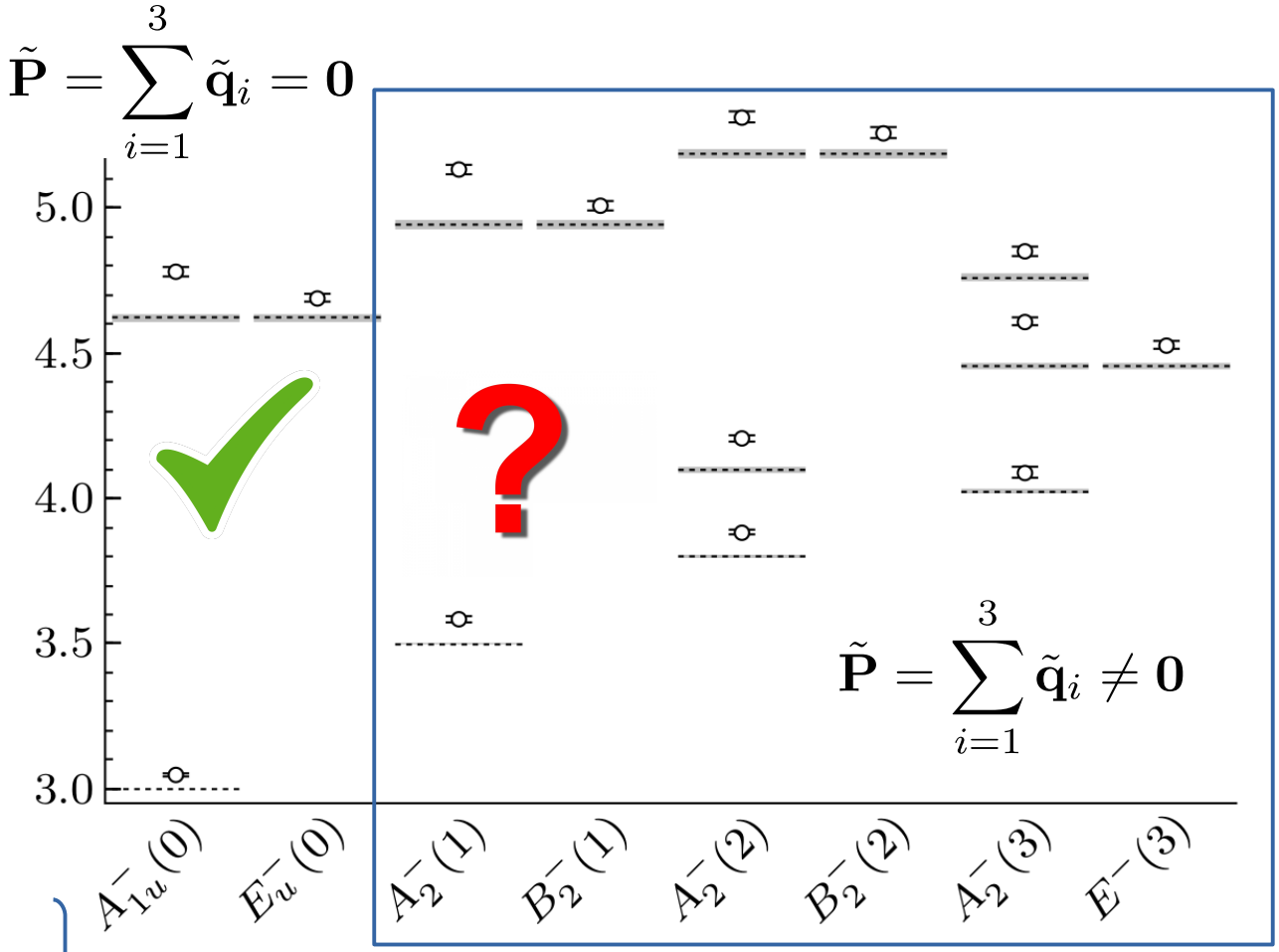
- **Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD**

*and absolute

Technical note: Projection technique for 3-body systems to irreps from M.D., Hammer, Mai, Pang, Rusetsky, Wu PRD97 (2018)

3-body spectrum: Moving frames

Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]

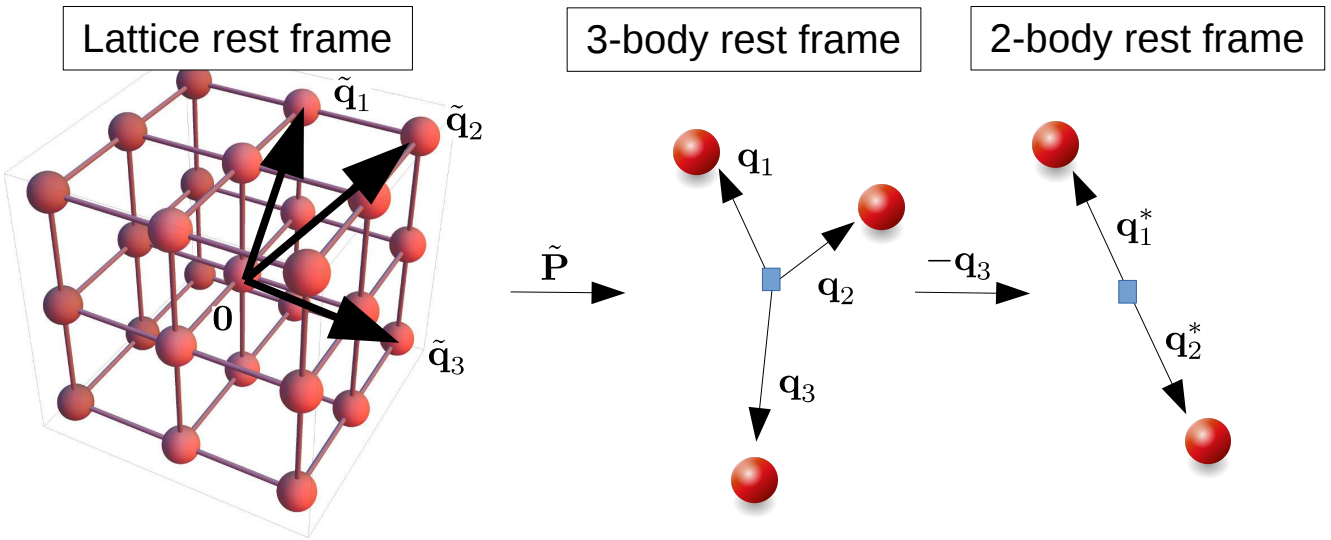


Low participating isobar-π^+ waves

S	D	S&D	S&D	S&D	S&D	S&D	S&D

→ Need to develop a framework for moving 3-body systems!

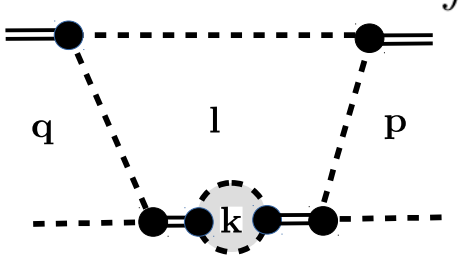
Moving frames for 3-body systems



Usually: Explicit S- and D-wave projected parameterizations in coupled channels
Here: Boost of unprojected 3-body amplitude. A-posteriori projections with suitable Clebsch-Gordan coefficients → Requires plane-wave solution of scattering

$$\left.\begin{array}{l}\tilde{\mathbf{P}} = \tilde{\mathbf{q}}_1 + \tilde{\mathbf{q}}_2 + \tilde{\mathbf{q}}_3 = \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3 \\ (L/2\pi)\tilde{\mathbf{P}} \in \{(0,0,1),\ (0,1,1),\ (1,1,1)\}\end{array}\right\} \mathbf{q} = \tilde{\mathbf{q}} + \left[\left(\frac{\tilde{P}^0}{\sqrt{s}} - 1\right)\frac{\tilde{\mathbf{q}}\tilde{\mathbf{P}}}{|\tilde{\mathbf{P}}^2|} - \frac{\tilde{q}^0}{\sqrt{s}}\right]\tilde{\mathbf{P}}$$

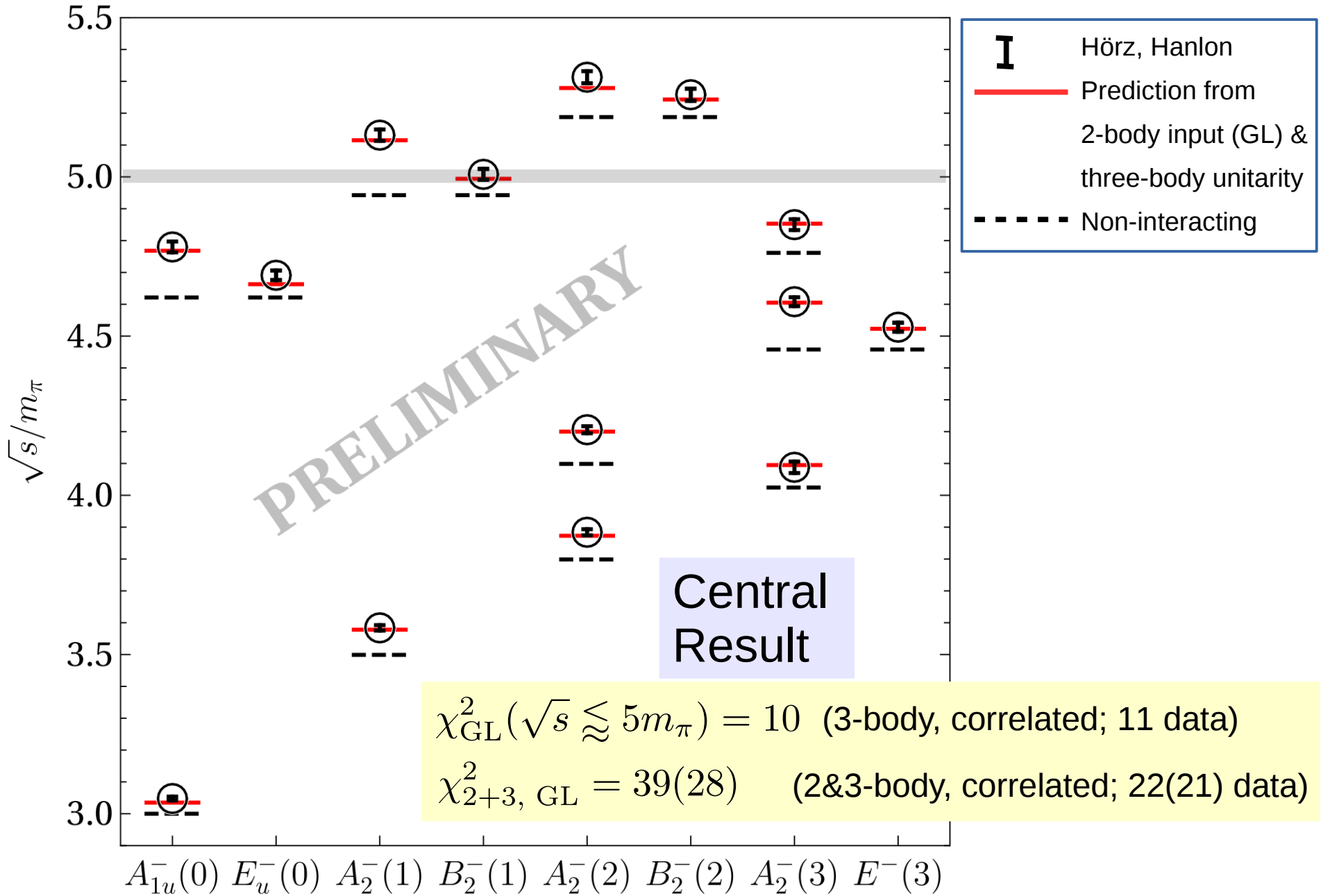
3-body summation: $$\int \frac{d^3\mathbf{l}}{(2\pi)^3} g(\mathbf{l}) \to \int \frac{d^3\tilde{\mathbf{l}}}{(2\pi)^3} g(\mathbf{l}(\tilde{\mathbf{l}}))\, \tilde{J}(\tilde{\mathbf{l}}) \to \frac{1}{L^3}\sum_{\mathbf{n}} g(\mathbf{l}(\tilde{\mathbf{l}}))\, \tilde{J}(\tilde{\mathbf{l}})$$



$\hat{T}(\mathbf{q}(\tilde{\mathbf{q}}), \mathbf{p}(\tilde{\mathbf{p}}))$ 3→3 boosted plane-wave amplitude
Poles → Eigenvalues

3-body spectrum: Complete Predictions

Data: Hörz, Hanlon, arXiv:1905.04277 [hep-lat]



Central Result

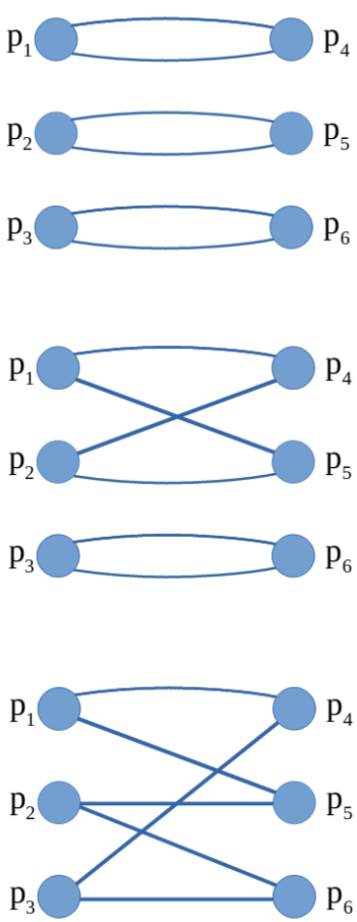
$\chi^2_{\rm GL}(\sqrt{s} \lesssim 5m_\pi) = 10$ (3-body, correlated; 11 data)

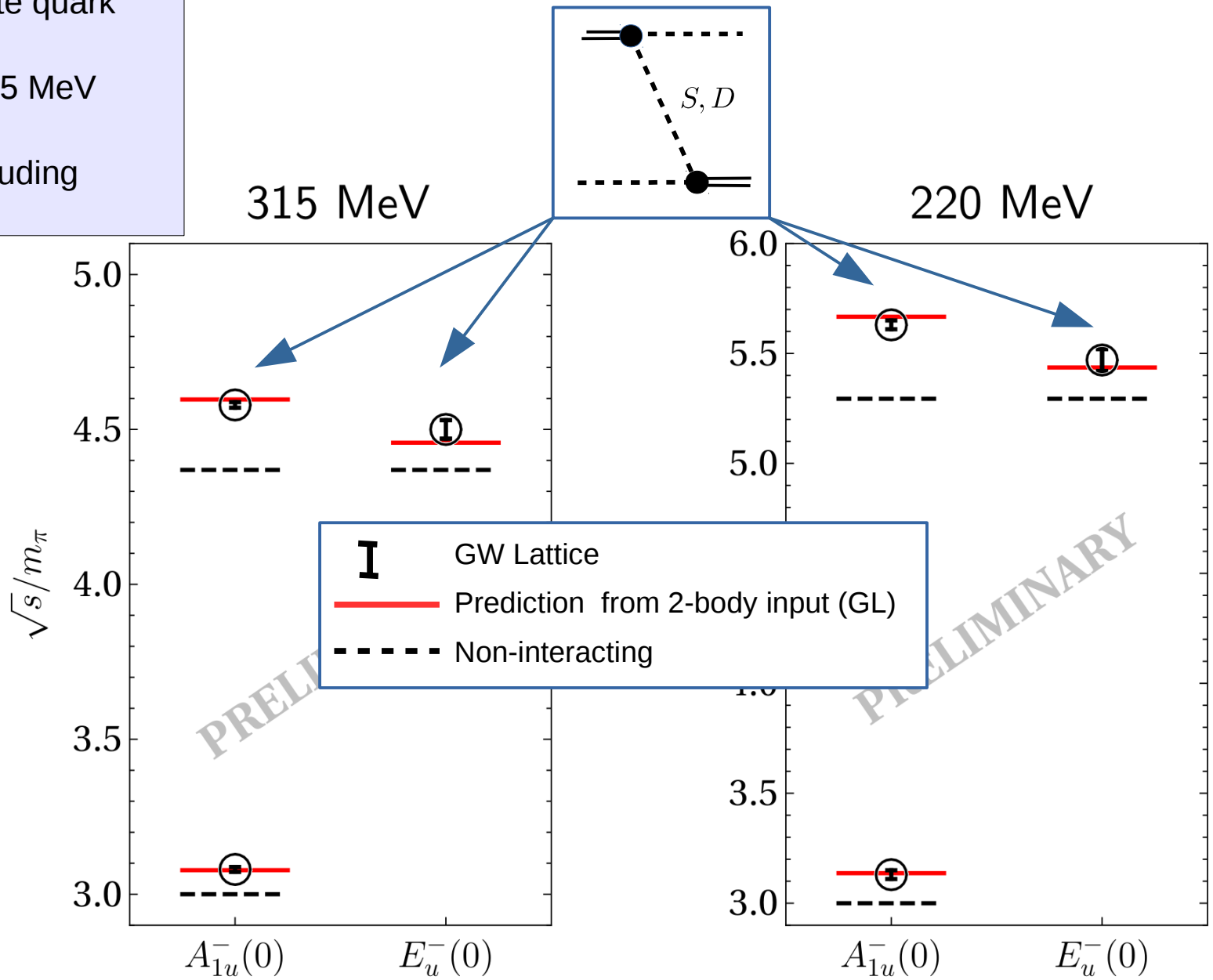
$\chi^2_{2+3,\ \rm GL} = 39(28)$ (2&3-body, correlated; 22(21) data)

GW Lattice Results

- nHYP-smeared clover fermions with mass-degenerate quark flavors (N_f = 2)
- M_π=227 MeV and 315 MeV
- 3 elongated boxes
- Variational basis including $\pi^+\pi^+\pi^+$ operators

[Culver, Alexandru, Mai, M.D., preliminary]

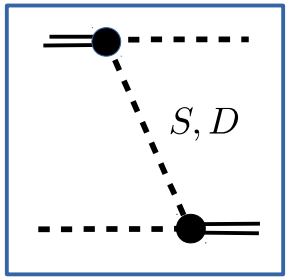




Summary

3-body Unitarity

- 3-body unitarity dictates on-shell condition (exchange term & isobar propagator)
- On-shell condition dictates leading, power-law, finite-volume effects
- “Bare-bone”infinite-volume extrapolation tool (in spirit of Lüscher equation)
- Optional: Pion-mass extrapolation



The $\pi^+\pi^+\pi^+$ System

- First application to physical 3-body system (above threshold) [PRL 2019]
- NPLQCD threshold data well predicted, excited levels predicted; fit of C.
- First explanation of excited 3-body levels (data from Hörz/Hanlon)
- Consequences of three-body unitarity directly visible in data (S vs. D waves)
- First development and application of moving frames for 3-body systems

OUTLOOK

→ Correlated fit to Hörz/Hanlon data

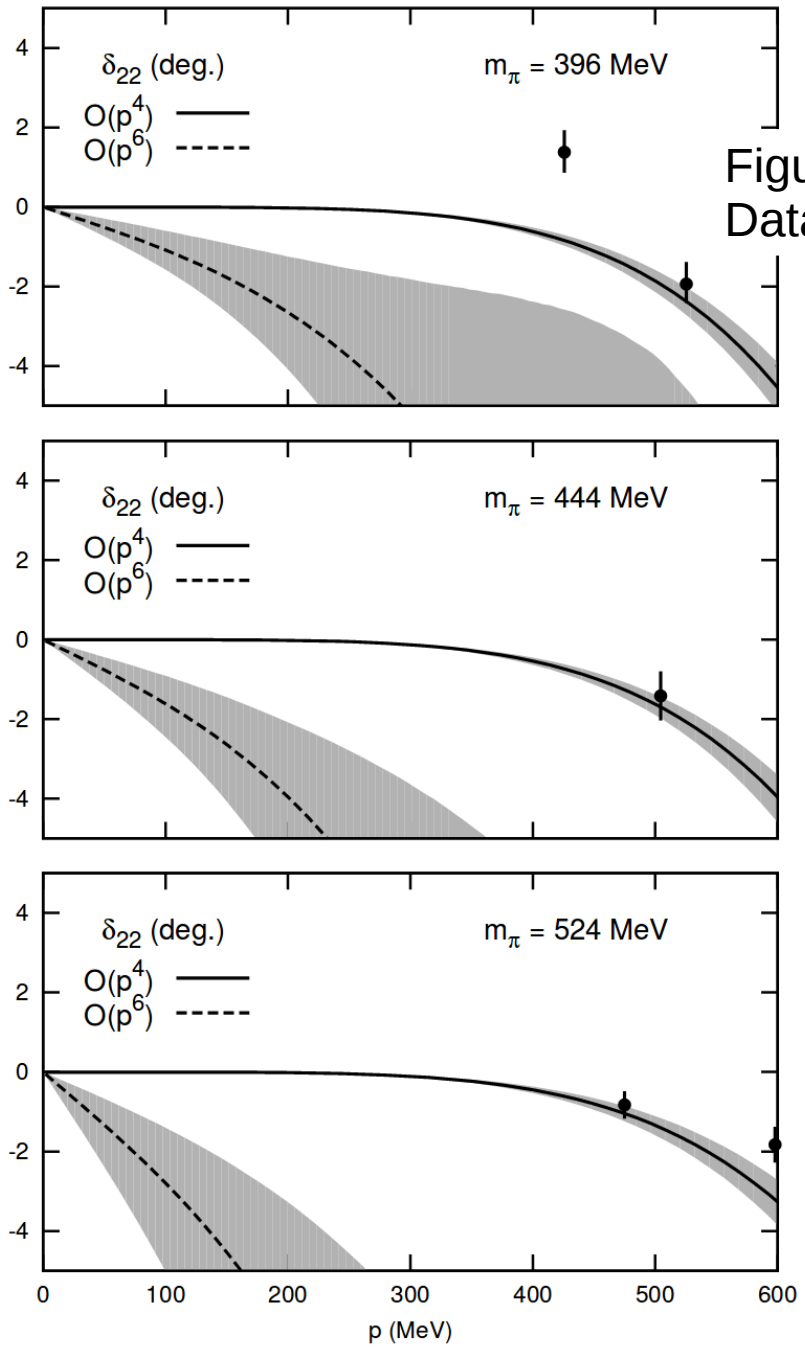
→ Implementation of spin isobars & multiple isobars

→ unequal masses

→ practical studies: a_1(1260), Roper, exotics...

SPARES

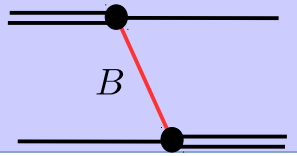
I=2 *D*-wave at HadSpec Pion Masses



Perturbative $O(p^4)$, $O(p^6)$ calculation

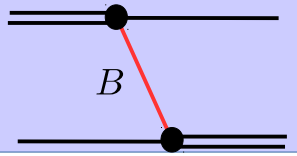
Figure from [Nebreda, Peláez, Ríos, PRD83 (2011)]
Data: [Dudek, Edwards, Peardon, Richards, Thomas, PRD83 (2011)]

The Power of Unitarity

Question: Does 

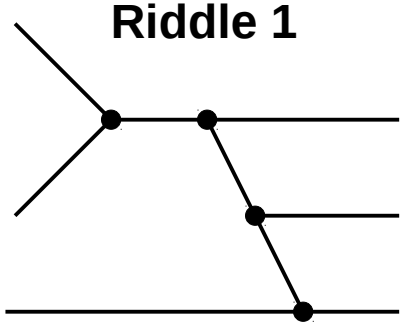
provide full imaginary part of all possible $3 \to 3$ transitions?

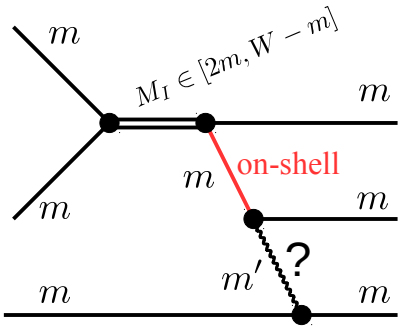
The Power of Unitarity

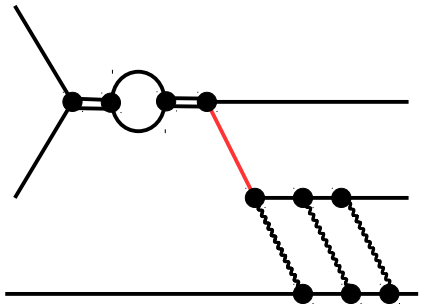
Question: Does 

provide full imaginary part of all possible 3→ 3 transitions?

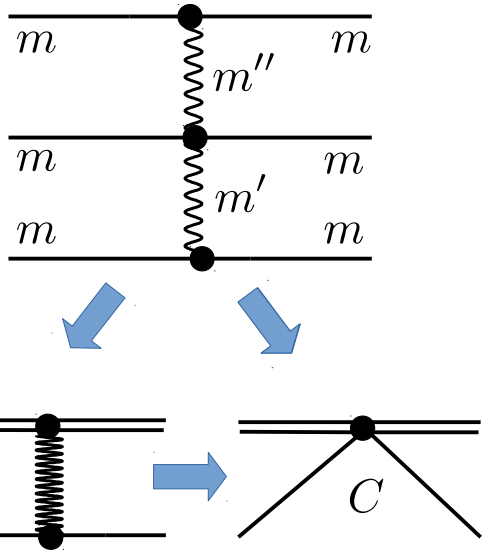
Riddle 1

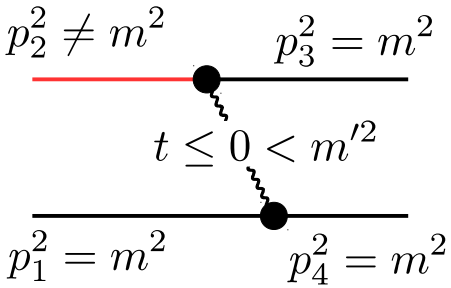


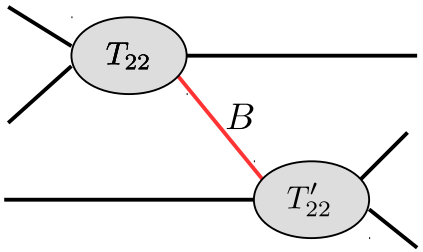




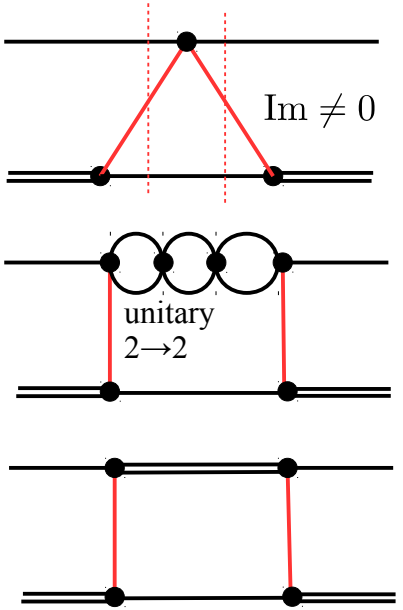
Riddle 2



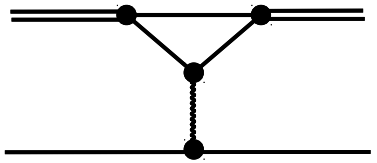




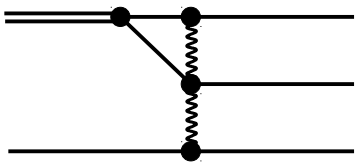
Riddle 3



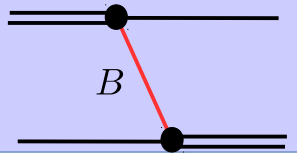
Riddle 4



Riddle 5

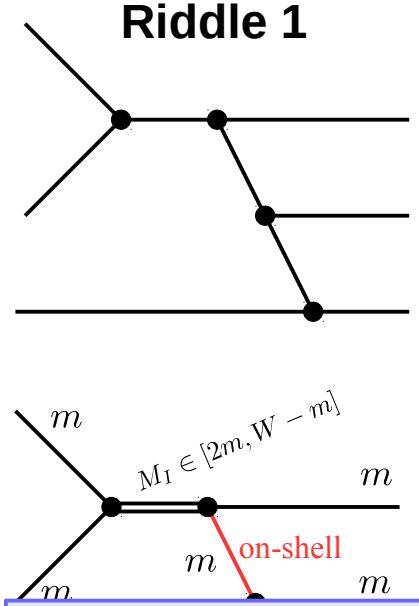


The Power of Unitarity

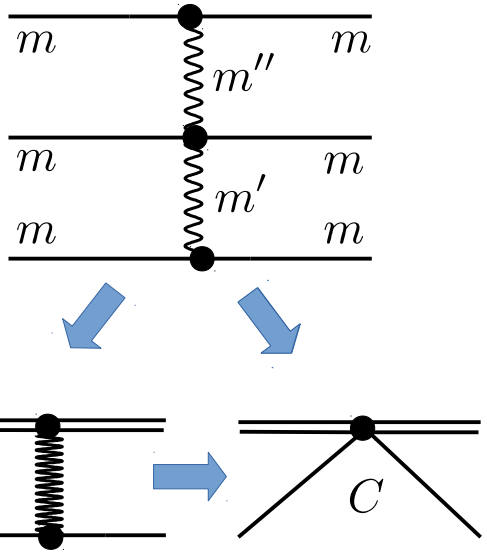
Question: Does 

provide full imaginary part of all possible 3→ 3 transitions?

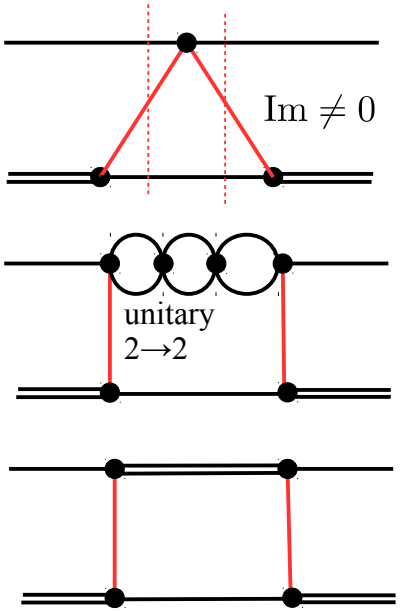
Riddle 1

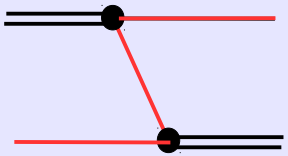
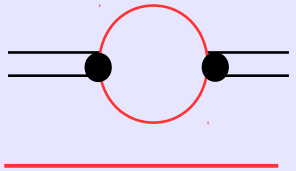


Riddle 2



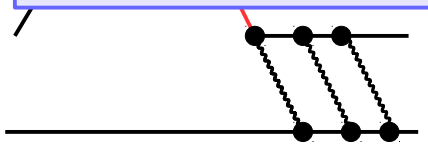
Riddle 3

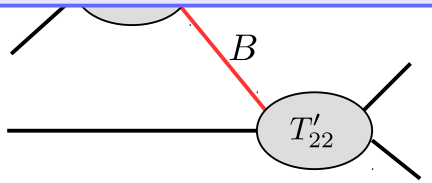


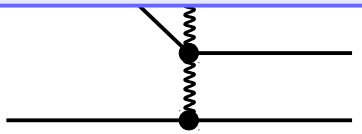
Answer: Yes.  and 

are the only on-shell configurations in physical region. Three-body unitarity avoids many artificial complications of diagrammatic expansions.

e 4





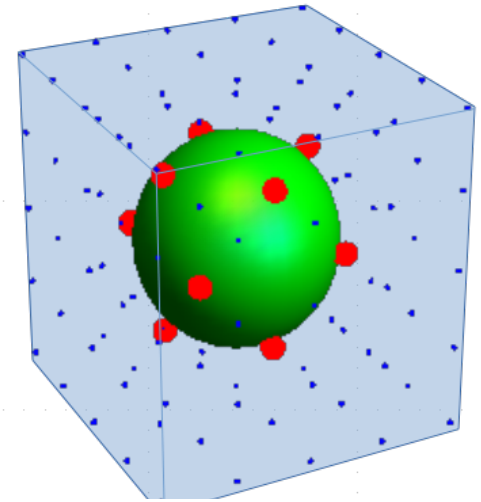


Riddle 5

Projection to irreps

[M.D.,Hammer,Mai,Pang,Rusetsky,Wu(2018)]

- **Lüscher formalism relies on regular 2→ 2 potentials**
  - Now: manifestly singular interactions
  - Find generalization that projects also the interactions to the irreps of cubic symmetry, not only propagation
- **Separation of variables**
  - shells = sets of points related by $\boldsymbol{O_h}$
  - Analogous to radial coordinate in infinite volume
- **Find the orthonormal basis for arbitrary functions defined on each point of a given shell.**



$$q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1,2,3$$

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_s \sum_{i=1}^{\vartheta(s)}$$

- **J (inf. volume) → irreps (finite volume):** $\Gamma \in \{A_1^\pm, A_2^\pm, E^\pm, T_1^\pm, T_2^\pm\}$
- **<u>Partial wave projection</u> (inf. Volume) → <u>Irrep. projection</u> (fin.)**

$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{p}) f_{\ell m}(p)$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{p}) f(\mathbf{p})$$

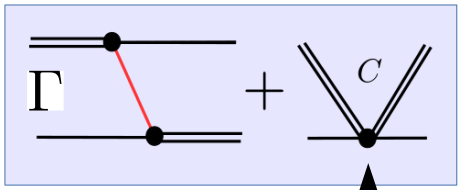
$$f^s(\hat{p}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_a f_a^{\Gamma\alpha s} \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

$$f_a^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{p}_j) \chi_a^{\Gamma\alpha s}(\hat{p}_j)$$

(a is index u in quantization condition; Quantization condition has projection in incoming AND outgoing basis states with indices u, u')

Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{ss'}\delta_{uu'}\right) = \mathbf{0}$$



W – total energy

s/s' - shell index

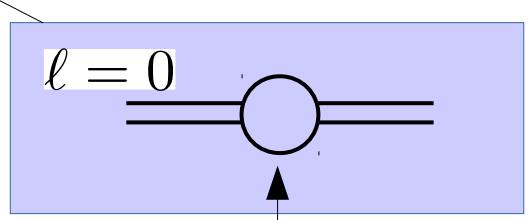
u/u' - basis index

ϑ – multiplicity

L – lattice volume

E_s – spect. energy

Determinant of (s,u) x (s',u') matrix at fixed W, Γ, L



Fix to 3→3 data

Fix to 2→2 data:

$T_{22} = v\,\tau\,v$

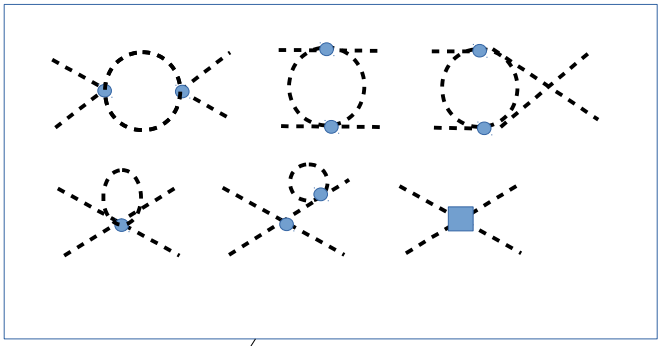
- Not a Lüscher-like equation ("left": infinite volume, "right": finite volume)
- Instead: Fix parameters to lattice eigenvalues
- With parameters fixed, evaluate infinite-volume amplitude
- Same workflow as in many 2-body coupled-channel fits (see, e.g., M.D., Meißner, Oset, Rusetsky, EPJA (2012))

Finite-volume & chiral extrapolations

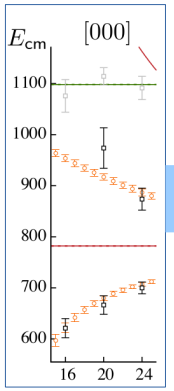
QCD calculations in finite volume

- unphysical pion mass
- (periodic) boundary conditions

→ <u>discrete momenta</u> & <u>discrete spectrum</u>

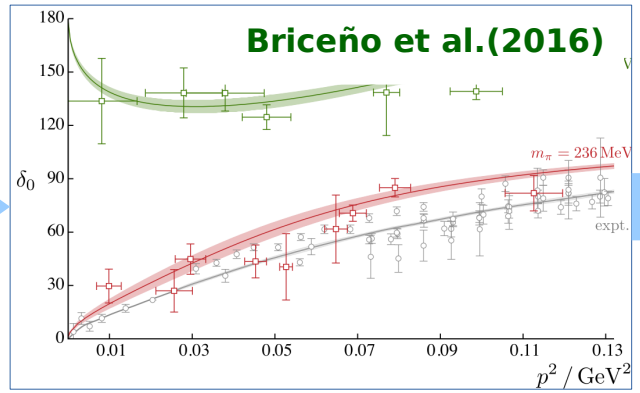


Recipe for 2 → 2 scattering (e.g. *I=J=0* ππ scattering)



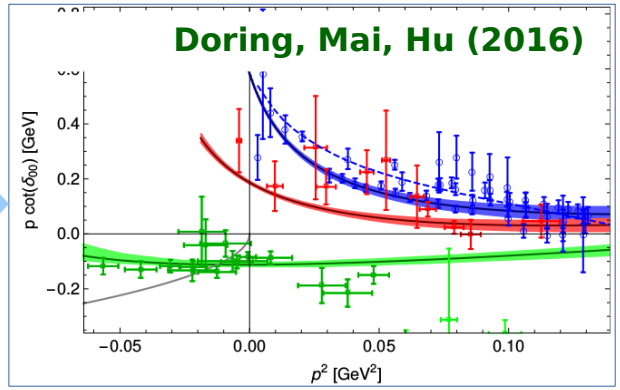
HSC(2016)

step 1



(This step can be skipped)

step 2



<u>LÜSCHER(1986)</u>

- 1 eigenenergy ↔ 1 phase-shift in infinite volume
- also with coupled channels

He et al. (2005)

Doring, Prelovsek, HSC

<u>CHIRAL EXTRAPOLATIONS</u>

- M_π dependence from NLO ChPT (IAM)

Gasser, Leutwyler(1981)

Dobado, Pelaez (1997)

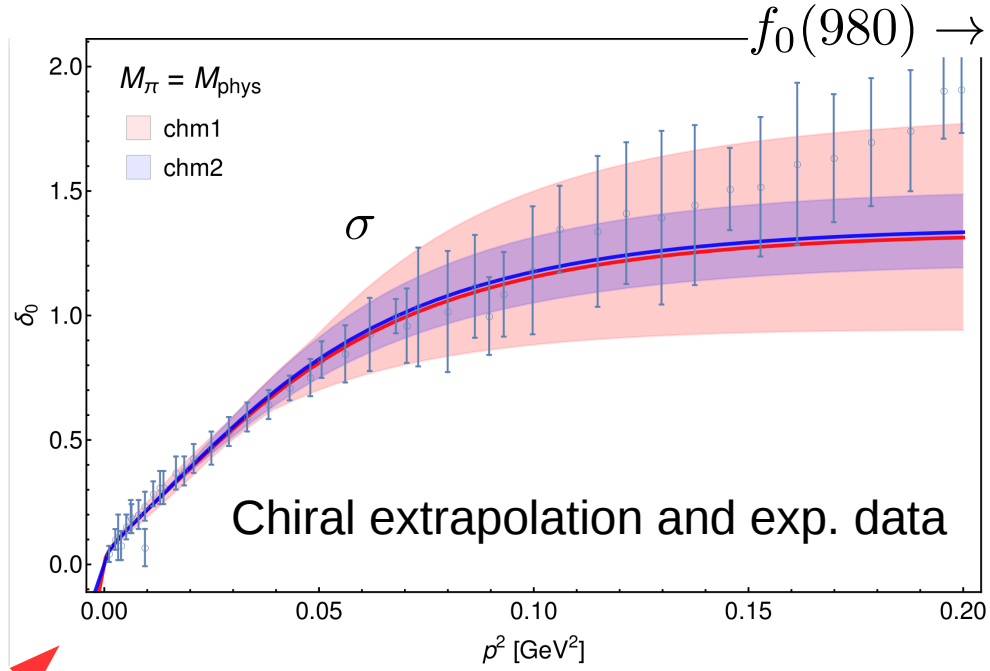
- Extrapolation in flavor

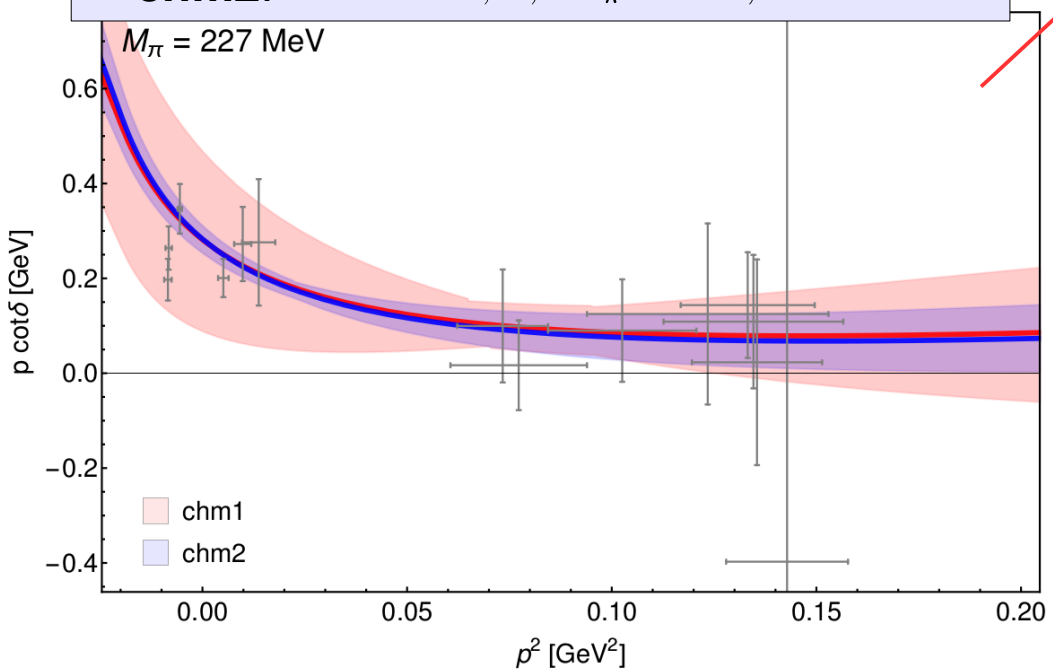
B. Hu, MD, R. Molina M. Mai et al. (2016)

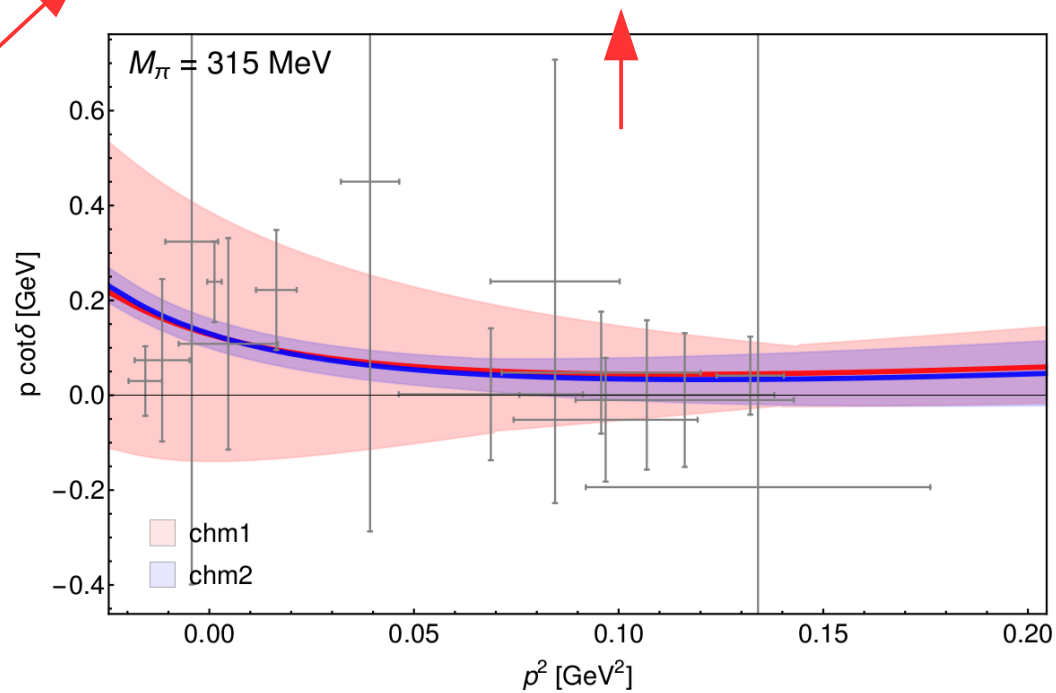
GWU lattice group: the isoscalar sector

[Guo, Alexandru, Molina, M.D., M. Mai, PRD (2018)]

- nHYP-smeared clover fermions with mass-degenerate quark flavors (N_f = 2)
- M_π=227 MeV and 315 MeV
- 3 elongated boxes
- Large variational basis including several meson-meson operators
- Moving frames
- Conformal mapping for σ pole extraction
- Unitarized Chiral Perturbation Theory fits for chiral extrapolation:
  chm1: $I = L = 0,\ M_\pi = 227, 315\,\mathrm{MeV}$
  chm2: $I = L = 0,\ 1,\ M_\pi = 227, 315\,\mathrm{MeV}$







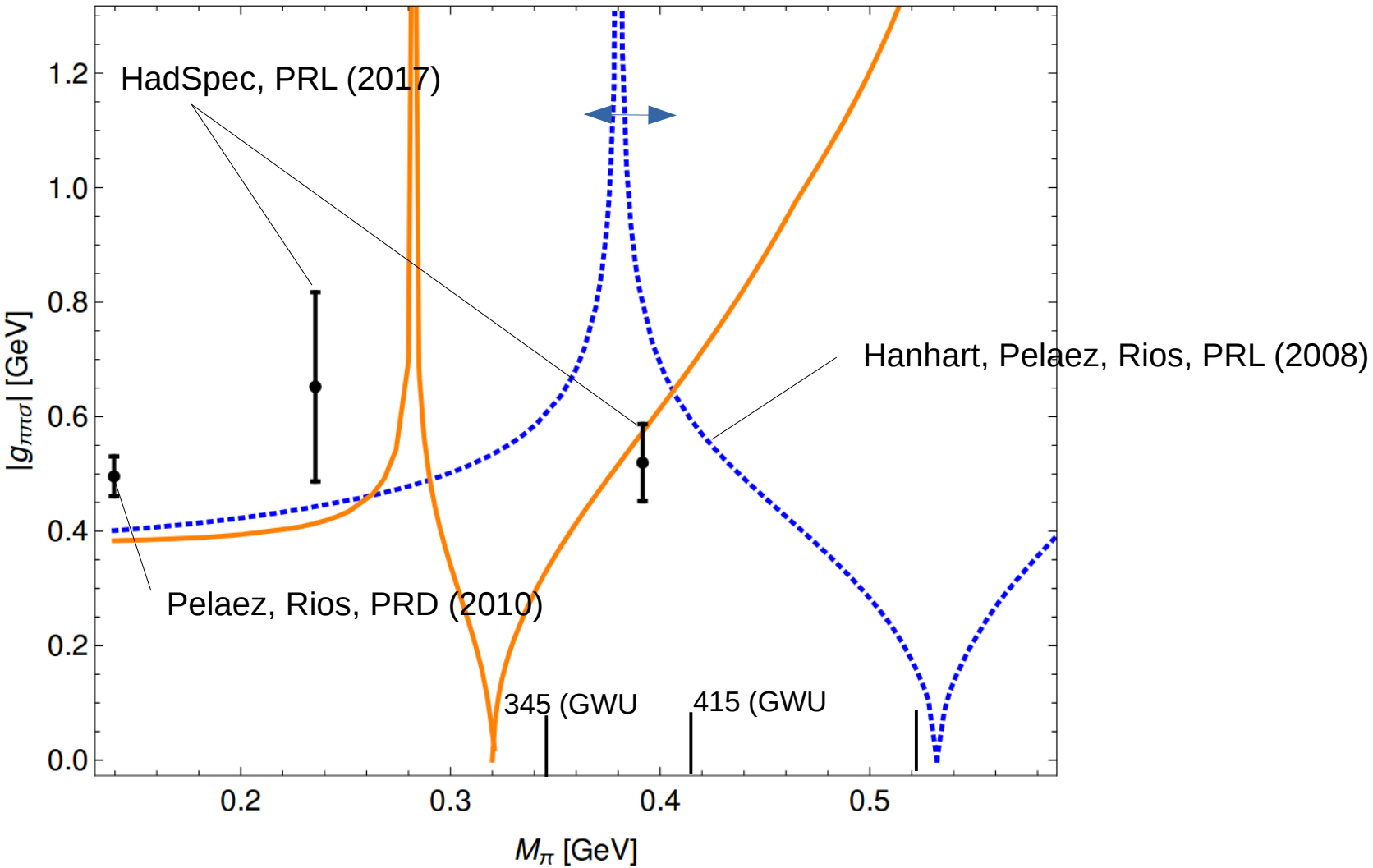
Chiral extrapolation of σ pole

		$M_\pi = 138$ MeV		
Parametrization	Fitted data	$\mathrm{Re}\, z^*$	$-\mathrm{Im}\, z^*$	$\lvert g \rvert$
chm1	$\sigma_{227,315}$	440^{+60}_{-90}	240^{+20}_{-50}	$3.0^{+0.2}_{-0.6}$
chm2	$\sigma_{227}\ \rho_{227}$	430^{+20}_{-30}	250^{+30}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	$\sigma_{315}\ \rho_{315}$	460^{+10}_{-15}	210^{+40}_{-30}	$3.0^{+0.1}_{-0.1}$
chm2	$\sigma_{227,315}\ \rho_{227,315}$	440^{+10}_{-16}	240^{+20}_{-20}	$3.0^{+0.0}_{-0.0}$
Ref. [1]	experimental	449^{+22}_{-16}	275^{+12}_{-12}	$3.5^{+0.3}_{-0.2}$

[1] J. R. Pelaez, Phys. Rept. **658**, 1 (2016), arXiv:1510.00653 [hep-ph].

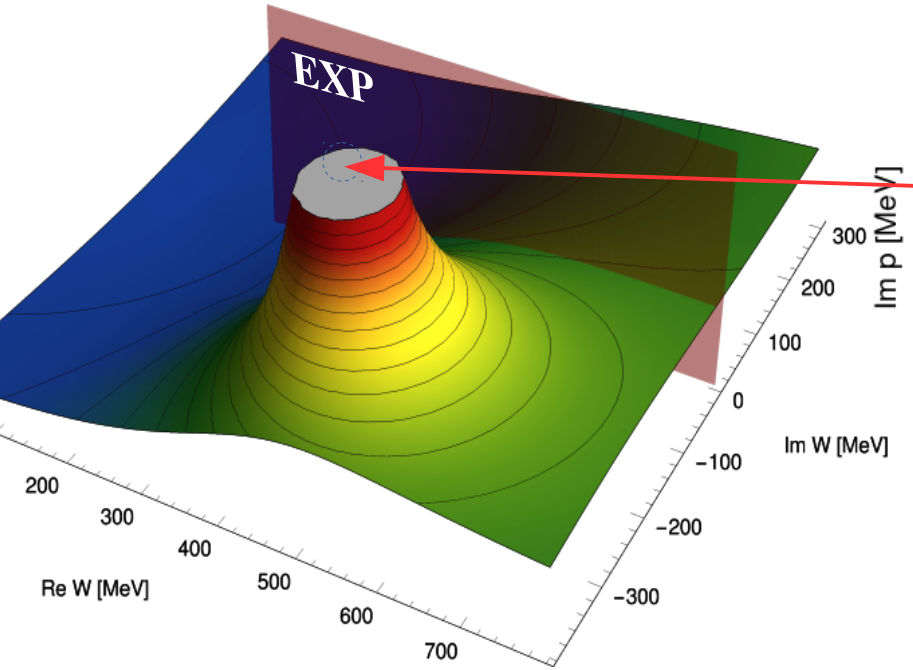
[Consistent with conformal-mapping amplitude parametrization (model-independent, not shown)]

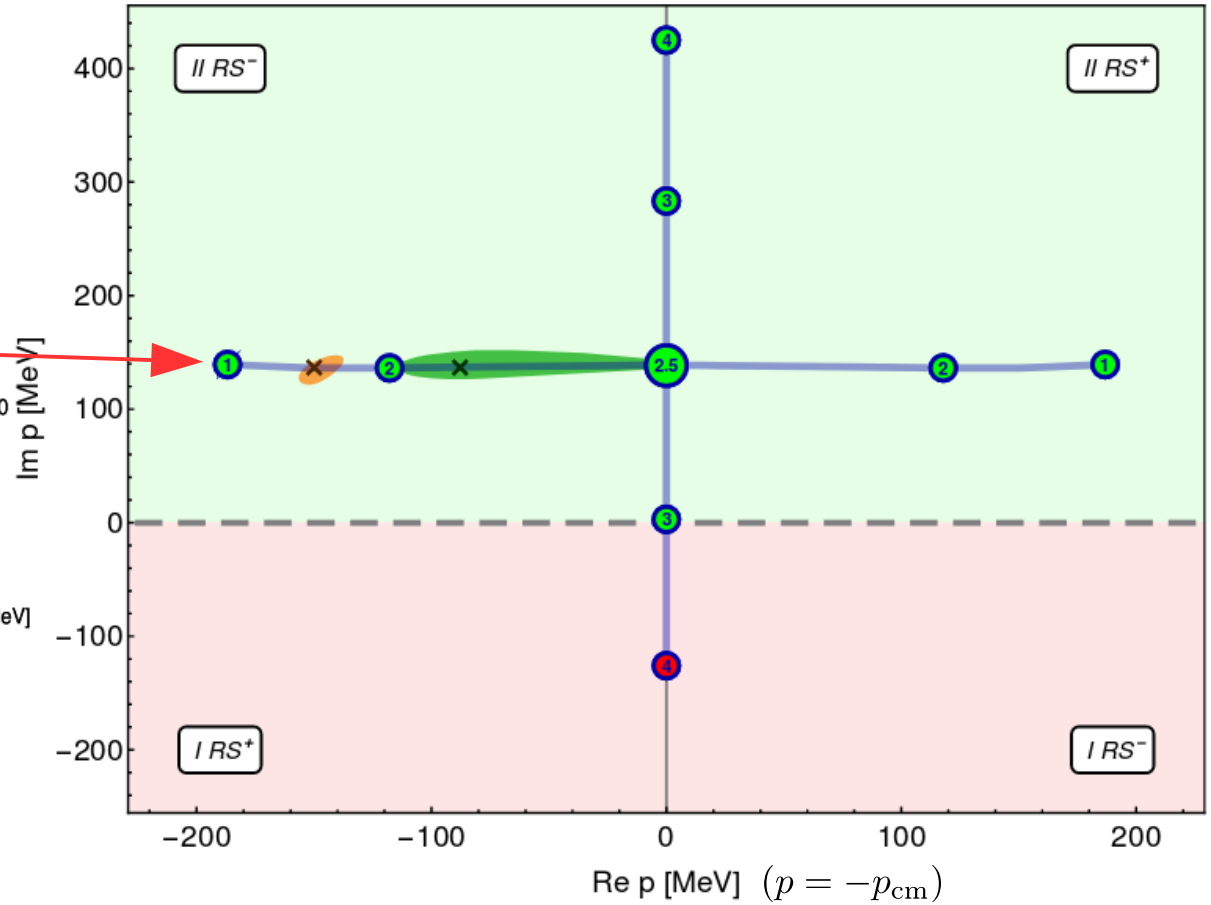
Residues


HadSpec, PRL (2017)
Hanhart, Pelaez, Rios, PRL (2008)
Pelaez, Rios, PRD (2010)
345 (GWU
415 (GWU
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.3
0.4
0.5
$|g_{\pi\pi\sigma}|$ [GeV]
M_π [GeV]

Pole trajectory

First prediction: Hanhart, Pealez, Rios, PRL (2008)





→ σ becomes a (virtual) bound state @ *M_π = (345) 415 MeV*