Variational Approach to Multiple Particles Interaction in finite volume

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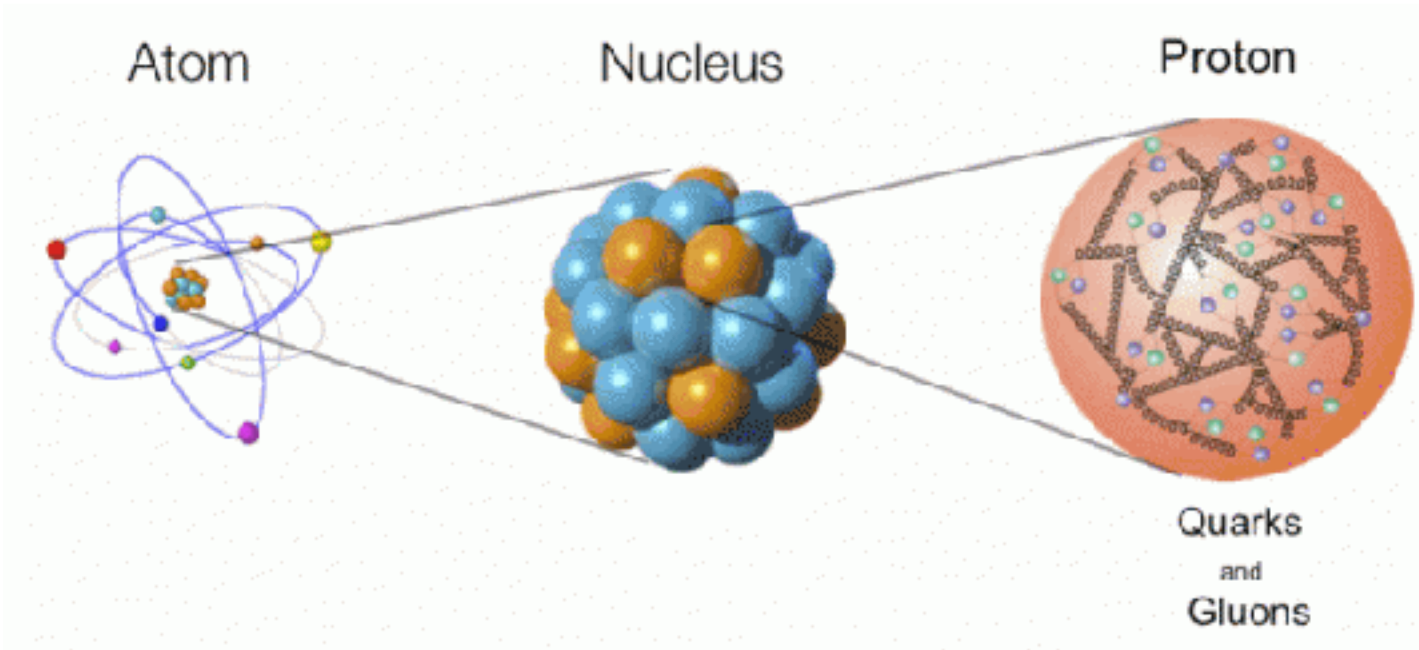
Bethe Forum, Bonn Univ, Sept.10, 2019

Hadron Spectroscopy and QCD: Exploring the nature of matter



- Understanding the role of gluons and confinement: exotic hadrons and gluonic excitations

- Understanding the dynamics of physical states and formation of resonances and bound states

Hadron Spectroscopy and QCD: Exploring the nature of matter

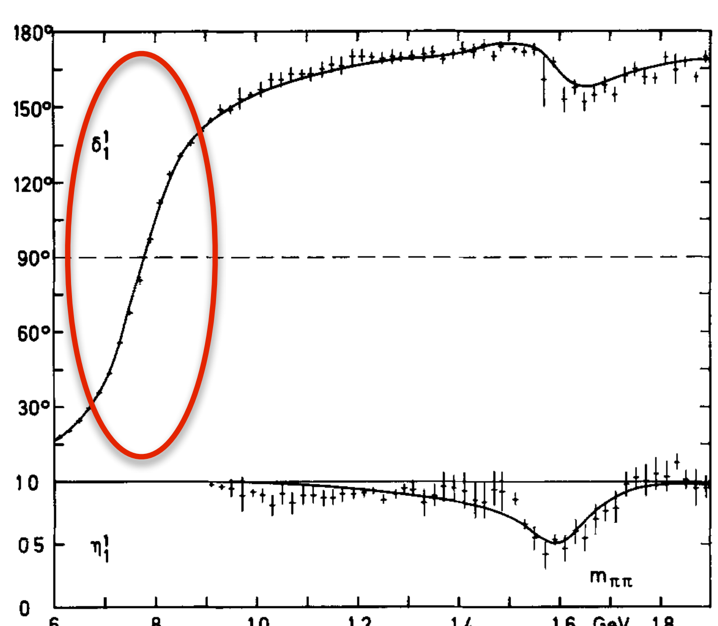
- Establishing connections between experimental observables and theoretical predictions at heart of matter

long distance behavior: decays, scattering

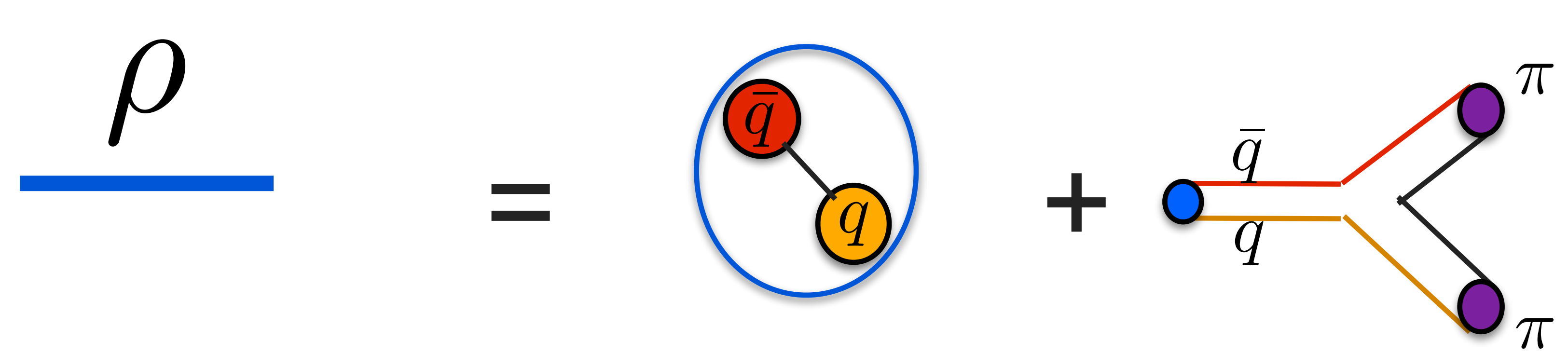
short distance nature: quarks and gluons

Experiments

Theories



B.Hyams *et al.* Nucl.Phys.B64(1973) 134

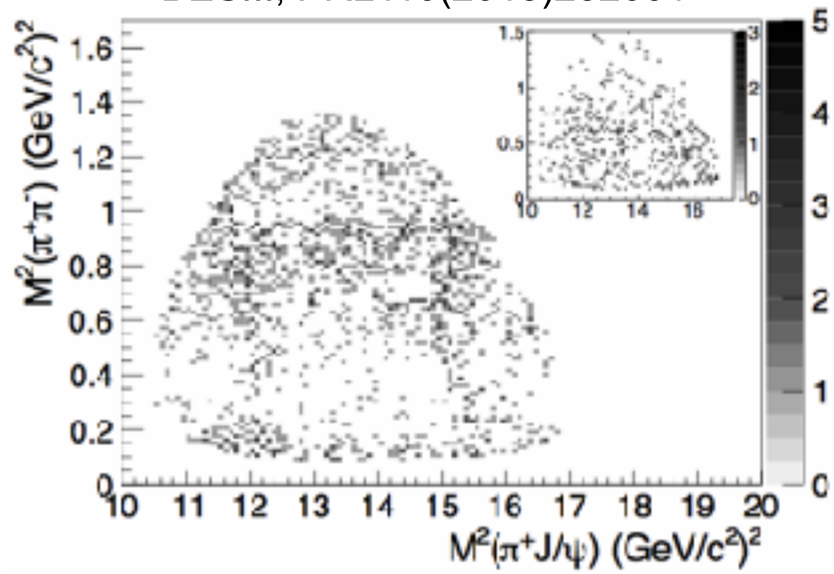


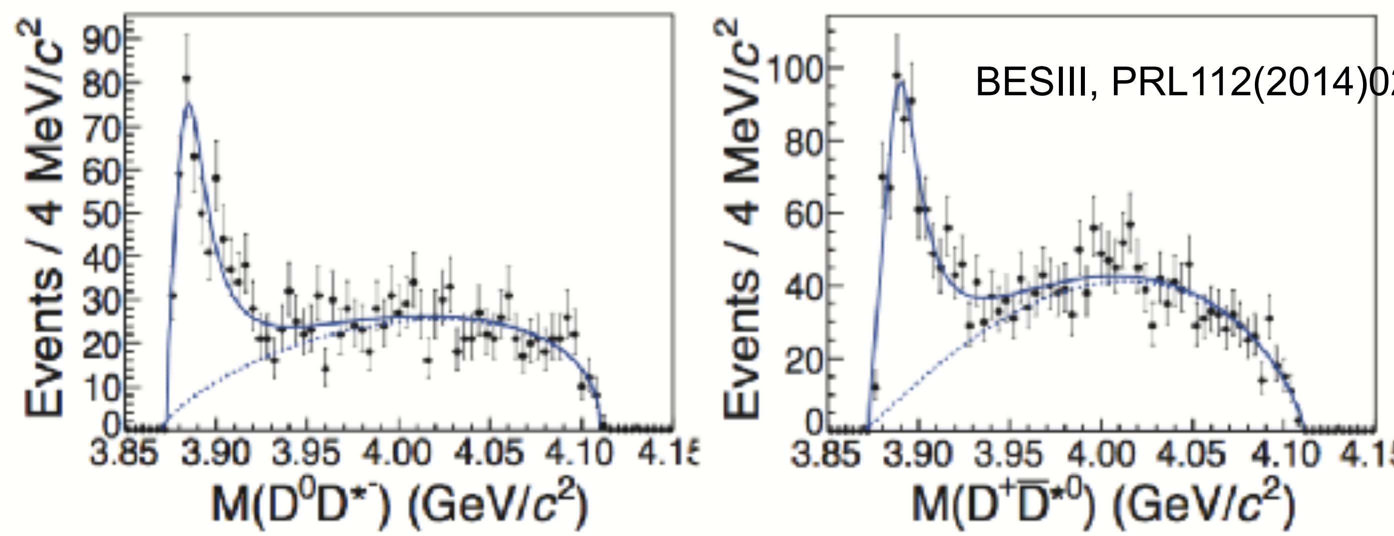
One of Challenges: few-body dynamics and strongly coupled systems

- Exotic hadrons often emerge through few-body dynamics

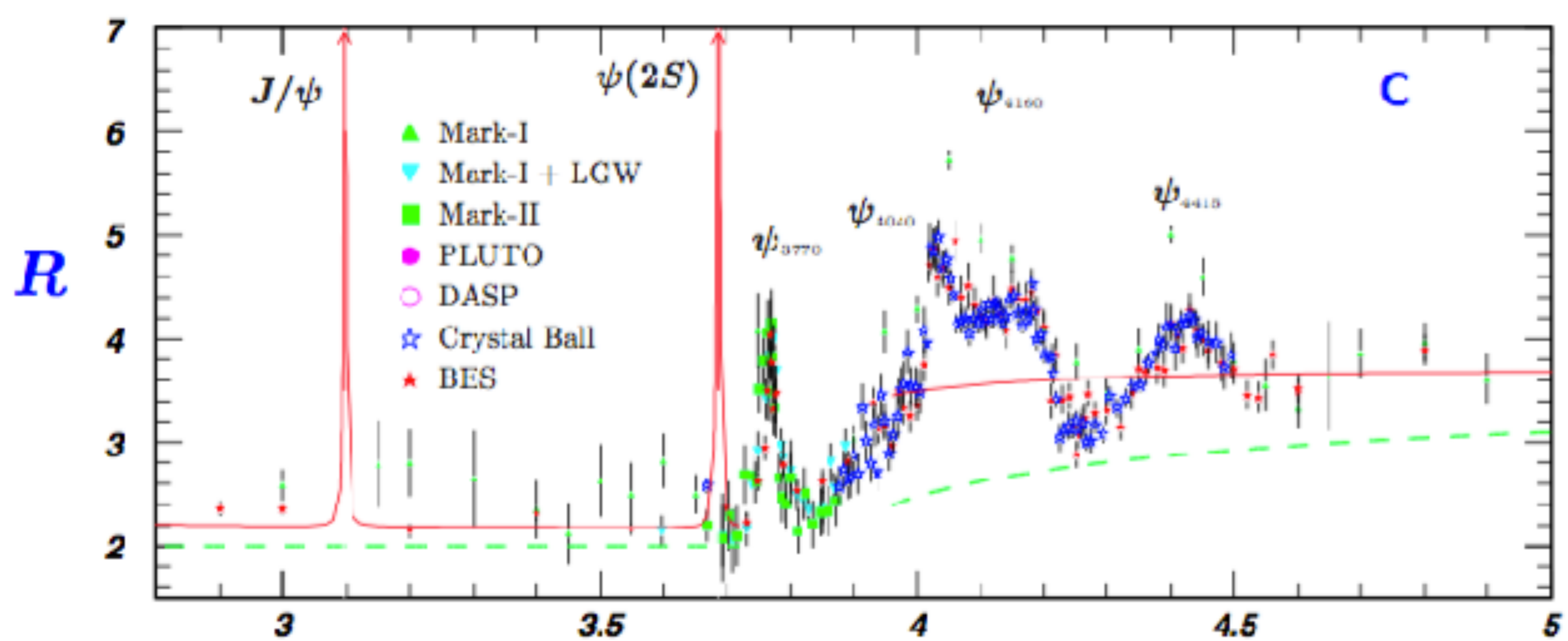
$$Z_c(3900)\ in\ Y(4260) \to \pi^+(\pi^- J/\Psi)\ and\ Y(4260) \to \pi^+(D\bar{D}^*)^-$$

BESIII, PRL110(2013)252001



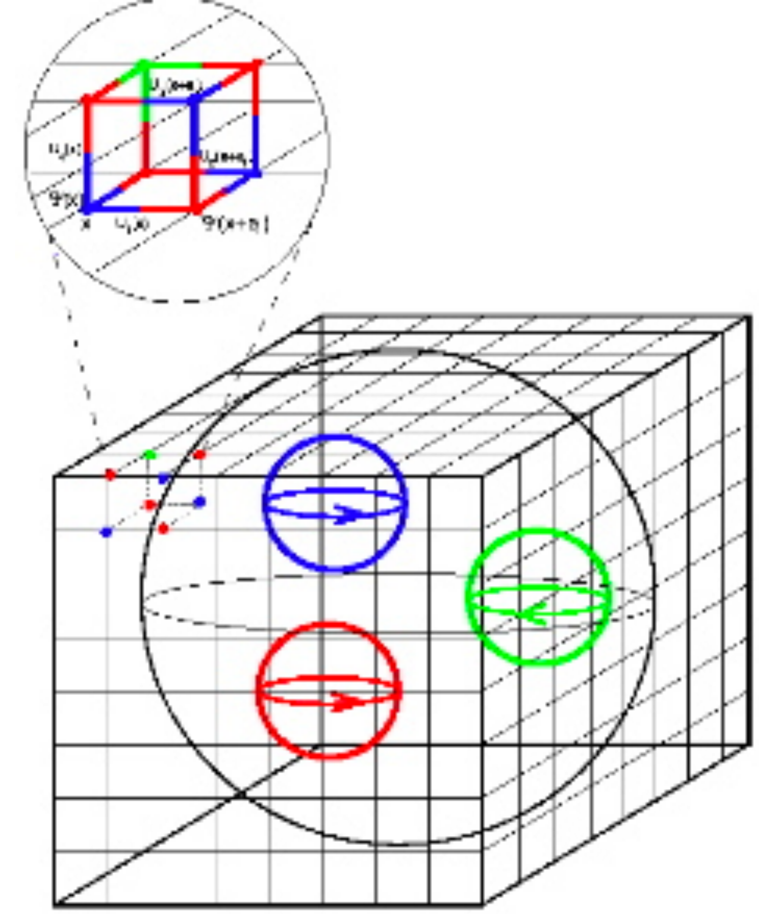


- QCD is a strongly coupled system



Mission

Few-body in Lattice QCD



- Forced to deal with few-body dynamics and coupled systems

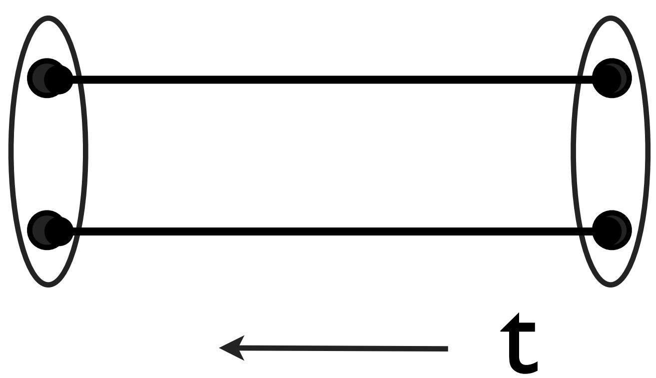
- Additional challenges:

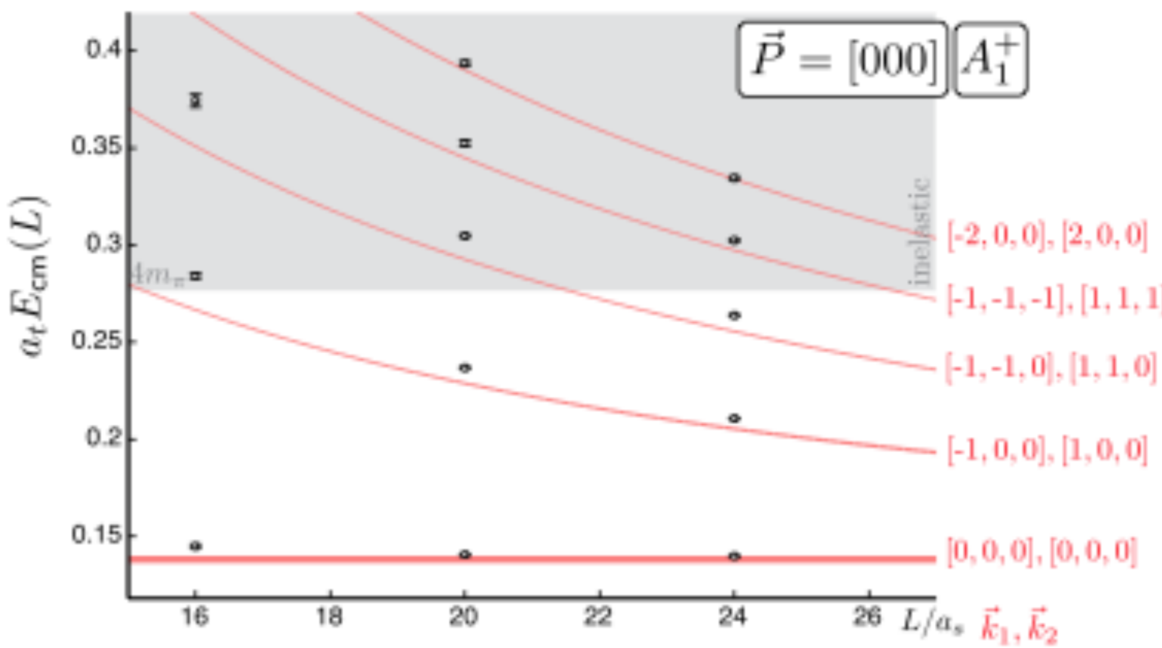
  - Euclidean space computation -> no easy access to scattering amplitudes

  - Computation in a finite box with periodic boundary condition -> finite volume effects

LQCD: multiple energy levels

$$C_{ij}(t) = \langle 0|\mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)|0\rangle$$





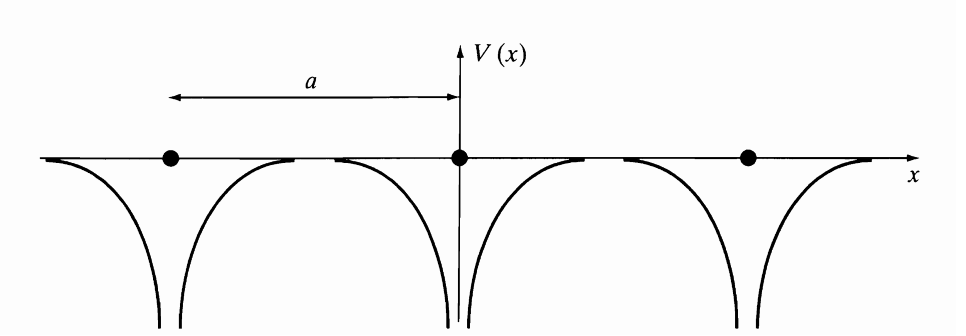
J. J. Dudek *et al.* (Hadron Spectrum Collaboration) Phys. Rev. D **86**, 034031 (2012).

Periodic boundary condition: discrete energies

- Taking advantage of periodic boundary condition

↓

Luescher's formula: relation between periodic lattice and two-body scattering amplitude



M. Luscher, Nucl. Phys. **B364**, 237 (1991).

$$\det\left[\delta_{JM,J'M'}\cot\delta_J(k) - \mathcal{M}_{JM,J'M'}(k)\right] = 0$$

$$k = \frac{\sqrt{E^2 - 4m_\pi^2}}{2}$$

- Taking advantage of periodic boundary condition

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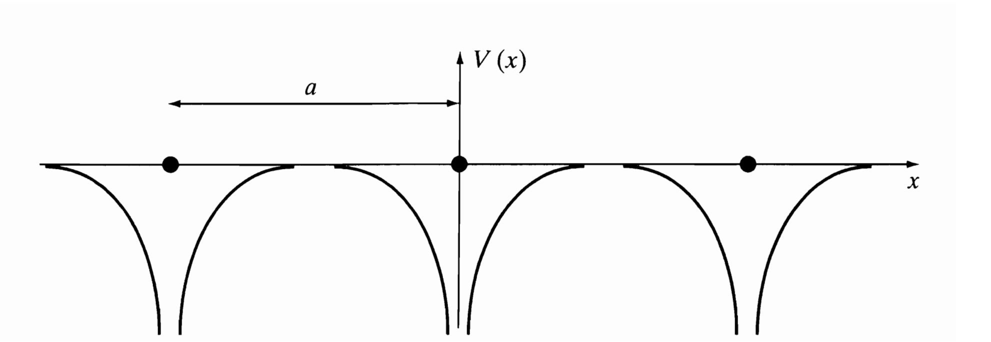

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Short-range Dynamics

- Taking advantage of periodic boundary condition

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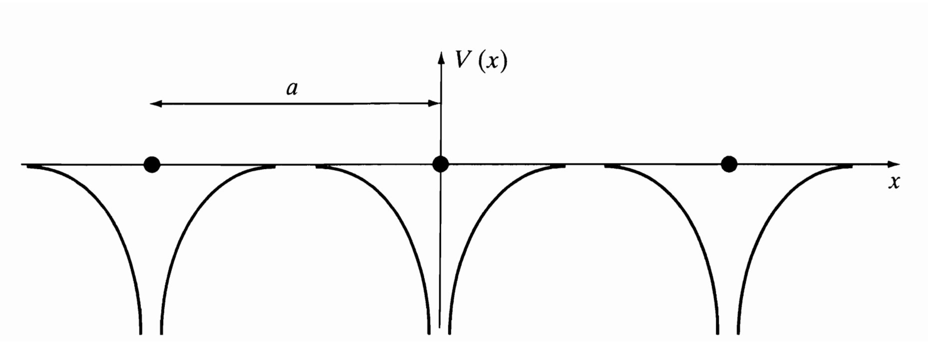

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Short-range Dynamics

Long-range correlation in periodic lattice structure

Mission

Few-body in Lattice QCD

A ideal framework for Few-body in LQCD?

- ☑ Describe relativistic strongly coupled few-body systems
- ☑ Satisfy periodic boundary condition -> discrete energy spectra
- ☑ Map out infinite volume dynamics -> resonance, etc.
- ☑ Mathematically transparent -> user friendly

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Recent development of Variational approach to particles interaction in finite volume

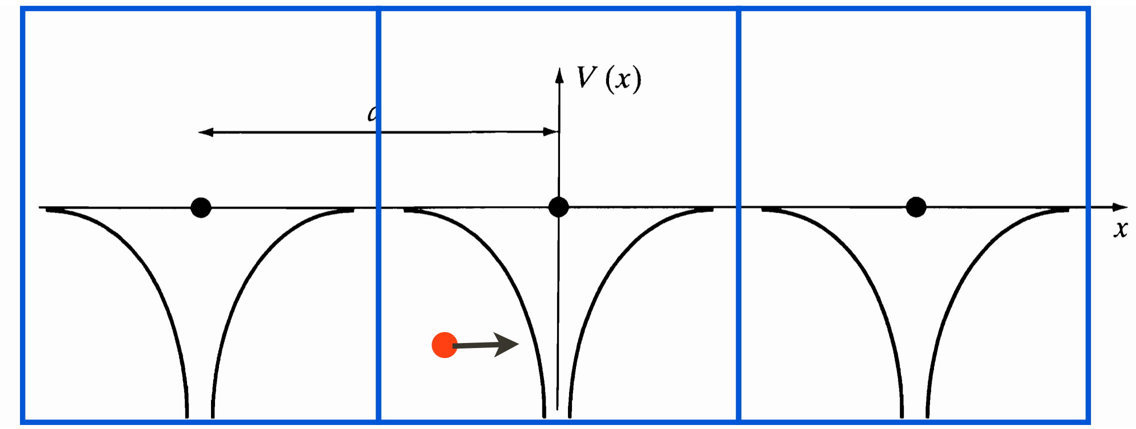
P.Guo, M.Doering and A.Szczepaniak, PRD98(2018)094502
P.Guo, arXiv:1908.08081[hep-lat]

Variational approach in 1D

- QM in 1D: particle propagating through a periodic potential

$$\left(E+\frac{\nabla^2}{2m}\right)\phi(x,E)=V(x)\phi(x,E),\quad x\in[-\frac{L}{2},\frac{L}{2}];$$

$$\phi(-\frac{L}{2},E)=e^{-iQL}\phi(\frac{L}{2},E),\;\; \phi'(-\frac{L}{2},E)=e^{-iQL}\phi'(\frac{L}{2},E)$$

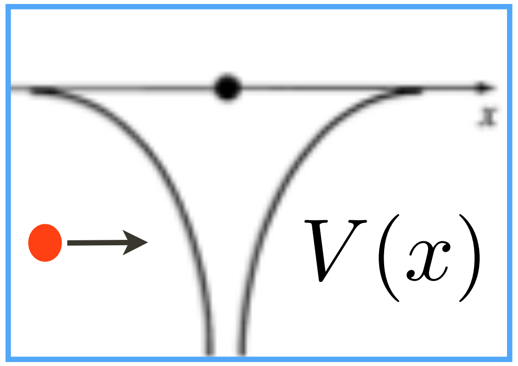


Variational approach in 1D

- QM in 1D: particle propagating through a periodic potential

  - Independent solutions with open boundary condition

$$\left(E + \frac{\nabla^2}{2m}\right)\psi_J(x,E) = V(x)\psi_J(x,E), \quad x \in [-\infty, \infty],$$



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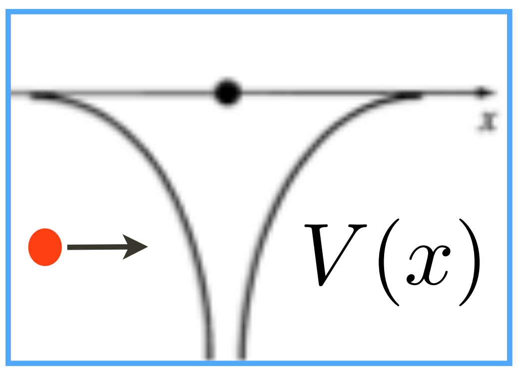

- Imposing BC to find out allowed energy levels and coefficients

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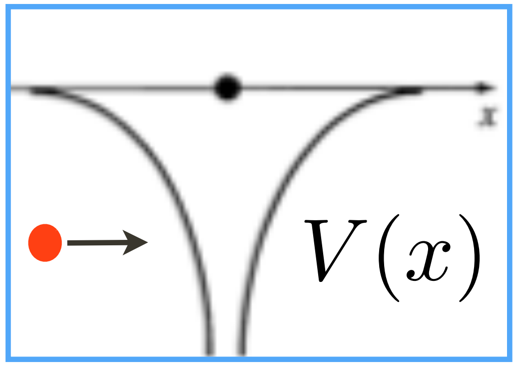

  - Imposing BC to find out allowed energy levels and coefficients

$$\phi(x,E) = \sum_J c_J \psi_J(x,E)$$

Variational approach in 1D

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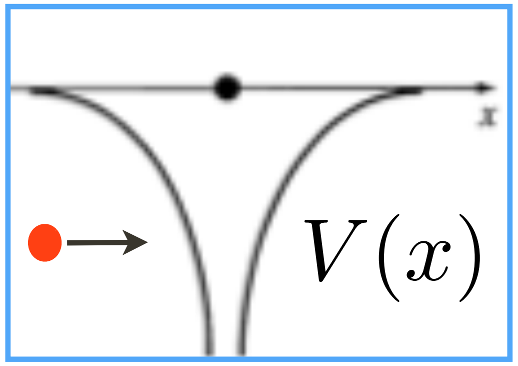
  - Two independent solutions with open boundary condition



$$\psi_\pm(x,E) \overset{|x|\sim\frac{L}{2}}{\longrightarrow} \psi^{(0)}_\pm(x,E)+it_\pm(k)e^{ik|x|}Y_\pm(x)$$

$$\phi(x,E)=\sum_J c_J\psi_J(x,E)$$

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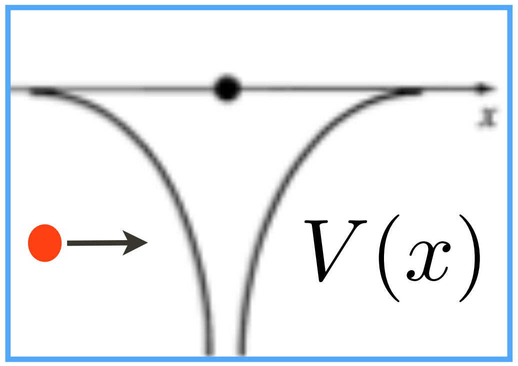

$$\psi_{\pm}(x,E) \overset{|x|\sim\frac{L}{2}}{\longrightarrow} \psi_{\pm}^{(0)}(x,E) + i t_{\pm}(k) e^{ik|x|} Y_{\pm}(x)$$

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- Secular equation -> Quantization condition and ratio of coefficients

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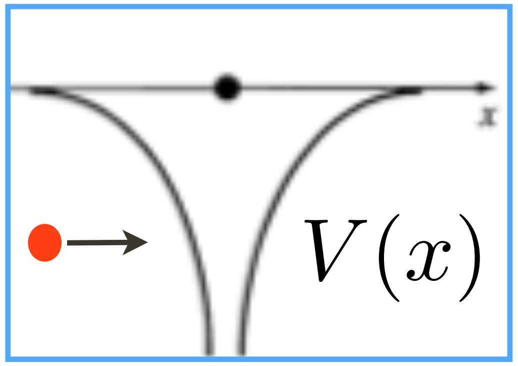

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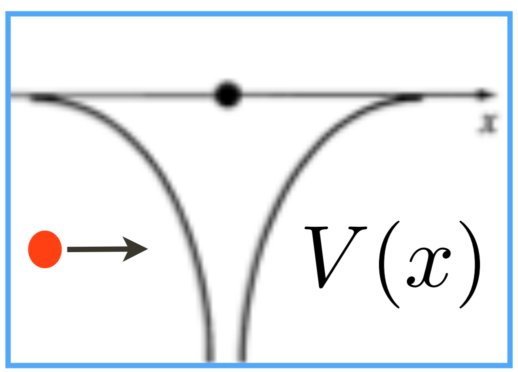
- Secular equation -> Quantization condition and ratio of coefficients

$$\begin{bmatrix} (1-e^{-iQL})\left(\cos\frac{kL}{2} + it_+ e^{i\frac{kL}{2}}\right) & -(1+e^{-iQL})\left(i\sin\frac{kL}{2} + it_- e^{i\frac{kL}{2}}\right) \\ (1+e^{-iQL})\left(i\sin\frac{kL}{2} + it_+ e^{i\frac{kL}{2}}\right) & -(1-e^{-iQL})\left(\cos\frac{kL}{2} + it_- e^{i\frac{kL}{2}}\right) \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = 0$$

Variational approach in 1D

- QM in 1D: particle propagating through a periodic potential
  - Replacing differential equation + boundary conditions by integral equation

$$\phi(x,E) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' G_L^{(Q)}(x-x',E)V(x')\phi(x',E)$$



$$G_L^{(Q)}(x,E) = \frac{1}{L}\sum_{n\in\mathbb{Z}}^{p=\frac{2\pi n}{L}+Q} \frac{e^{ipx}}{E-\frac{p^2}{2m}}.$$

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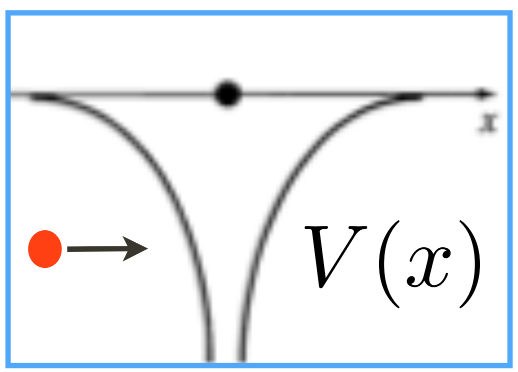

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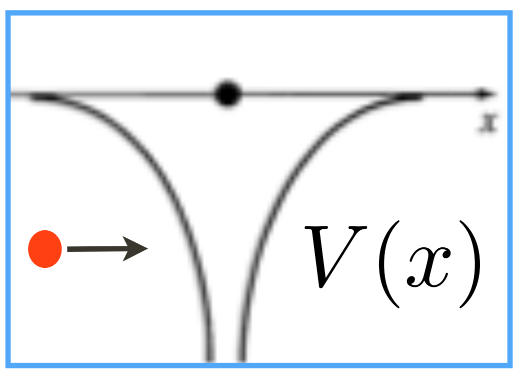

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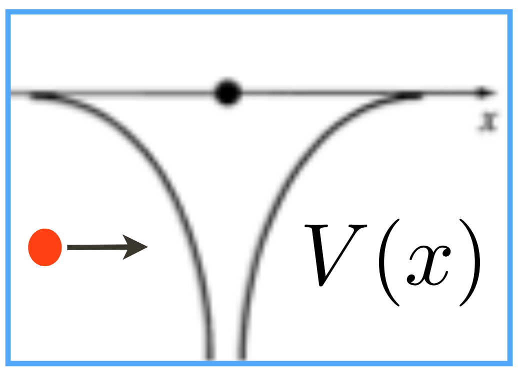

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  - Quantization condition and ratio of coefficients

Variational approach in 1D

- QM in 1D: particle propagating through a periodic potential

  - **Secular equations: a numerical framework**

$$\sum_J \left[S_{J',J}(E) - H^{(Q)}_{J',J}(E)\right] c_J = 0,$$

$$S_{J',J}(E) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \psi^*_{J'}(x,E)\psi_J(x,E)$$

$$H^{(Q)}_{J',J} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx dx' \psi^*_{J'}(x,E) G^{(Q)}_L(x-x',E) V(x') \psi_J(x',E)$$

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$$\det \left[S_{J',J}(E) - H^{(Q)}_{J',J}(E) \right] = 0,$$

Variational approach in 1D

- QM in 1D: particle propagating through a periodic potential

  - Solving finite volume LS eq directly

$$\phi(x,E) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' G_L^{(Q)}(x-x',E)V(x')\phi(x',E)$$

$$\downarrow$$

$$\sum_j \left[\delta_{i,j} - G_L^{(Q)}(x_i - x_j, E)V(x_j)\right]\phi_j = 0,$$

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Variational approach to few-body problem

- Faddeev method in infinite volume

$$\boxed{\Psi_J = \Psi_J^{(0)} + G_0(v_1 + v_2 + v_3 + v_4)\Psi_J}$$

non-compact kernel

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Fredholm-type Faddeev eq:

$$T_{\alpha,J} + t_\alpha G_0 \sum_{\beta \neq \alpha} T_{\beta,J} = t_\alpha \Psi_J^{(0)}$$

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Fredholm-type Faddeev eq:

$$T_{\alpha,J} + t_\alpha G_0 \sum_{\beta \neq \alpha} T_{\beta,J} = t_\alpha \Psi_J^{(0)}$$

- Solutions in infinite volume: $\boxed{T_{\alpha,J} = [\delta_{\alpha,\beta}(\mathbb{I} - t_\alpha G_0) + t_\alpha G_0]^{-1} t_\alpha \Psi_J^{(0)}}$

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$$\downarrow$$

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$$T_{\alpha}^{(L)} = -v_{\alpha}\Phi$$

$$\Downarrow$$

$$T_{\alpha}^{(L)} = v_{\alpha} G_L \sum_{\beta} T_{\beta}^{(L)}$$

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$$t_\alpha^{(L)} = -(\mathbb{I} - v_\alpha G_L)^{-1} v_\alpha$$

Homogeneous eq: $\boxed{T_\alpha^{(L)} + t_\alpha^{(L)} G_L \sum_{\beta \neq \alpha} T_\beta^{(L)} = 0}$

Variational approach to few-body problem

- Few-body Secular equations:

$$T_\alpha^{(L)} = \sum_J c_J T_{\alpha,J}$$

$$\sum_J \left[\sum_{\alpha,\beta} \langle T_{\alpha,J'} | \delta_{\alpha,\beta} (\mathbb{I} - t_\alpha^{(L)} G_L) + t_\alpha^{(L)} G_L | T_{\beta,J} \rangle \right] c_J = 0$$

Variational approach to few-body problem

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- Quantization condition and ratio of coefficients

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Variational approach to few-body problem

- Solving finite volume homogenous Faddeev equations directly:

$$\boxed{T_\alpha^{(L)} + t_\alpha^{(L)} G_L \sum_{\beta \neq \alpha} T_\beta^{(L)} = 0}$$

$$\sum_{\beta, j} \left[\delta_{\alpha,\beta} \delta_{i,j} - (v_\alpha G_L)_{i,j} \right] T_{\beta,j}^{(L)} = 0,$$

Variational approach to few-body problem

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Short Summary of variational approach

P.Guo, M.Doering and A.Szczepaniak, PRD98(2018)094502
P.Guo, arXiv:1908.08081[hep-lat]

- Variational approach is a general framework to few-body problem in finite volume

- Mapping out both infinite volume dynamics and allowed energy levels are not easy

- Solving finite volume homogeneous equation directly might be more practical for lattice data fitting

$$\det\left[\delta_{\alpha,\beta}\delta_{i,j} - (v_\alpha G_L)_{i,j}\right] = 0.$$

Lattice toy model check

- Heavy-light few-body system

$$S = \int d^2x \Big[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{g_{\phi\phi}}{4!}\phi^4 + g_{\sigma\phi}\delta_{x_1,0}\phi^2(x_0,0)\sigma^2(x_0,0) \Big],$$

Preliminary

Light bosons

Static heavy boson

Lattice toy model check

- Heavy-light few-body system

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Preliminary

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- Dynamics of Heavy-light three-body system:

Lattice toy model check

- Heavy-light few-body system

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Preliminary

Light bosons

Static heavy boson

- Dynamics of Heavy-light three-body system:

$$\left[\sigma^2 + \hat{T} - U_L(x_1) - U_L(x_2) - V_L(r) \right] \Phi(x_1, x_2) = 0,$$

Lattice toy model check

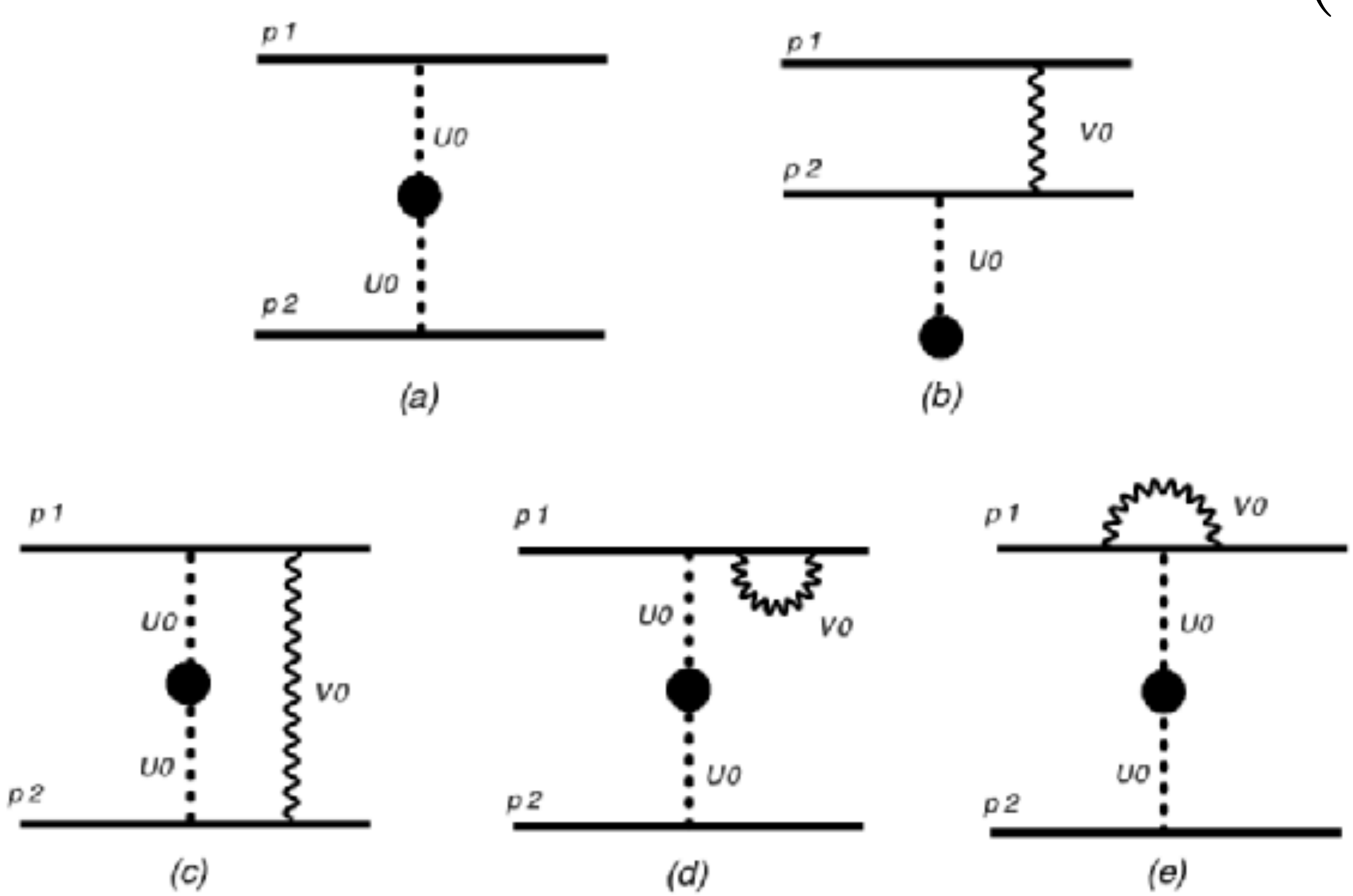
Preliminary

- Dynamics of Heavy-light three-body system:

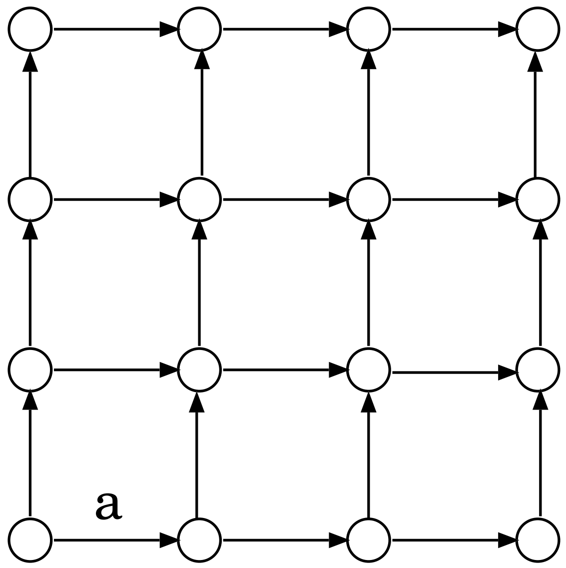
$$\left[\sigma^2 + \hat{T} - U_L(x_1) - U_L(x_2) - V_L(r)\right]\Phi(x_1, x_2) = 0,$$

$$U(x) = U_0\delta(x)$$

$$V(x) = V_0\delta(x)$$



Lattice toy model check

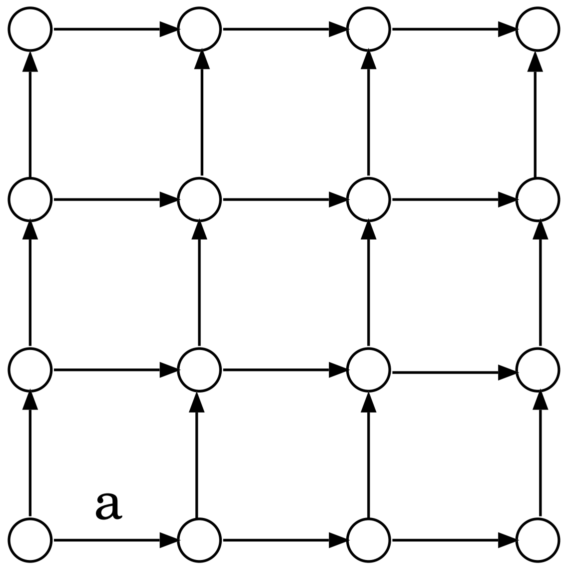


- Lattice model simulation:

$$S_{lat} = \sum_x \left[-2\kappa \sum_{\hat{n}} \phi(x)\phi(x+\hat{n}) + (1-2\lambda_{\phi\phi})\phi^2(x) \right]$$
$$+ \sum_x \left[\lambda_{\phi\phi}\phi^4(x) + \lambda_{\sigma\phi}\delta_{x_1,0}\phi^2(x_0,0)\sigma^2(x_0,0) \right].$$

Preliminary

Lattice toy model check



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* Three parameters in the model: mass of light boson, coupling strengths: U0 and V0

Preliminary

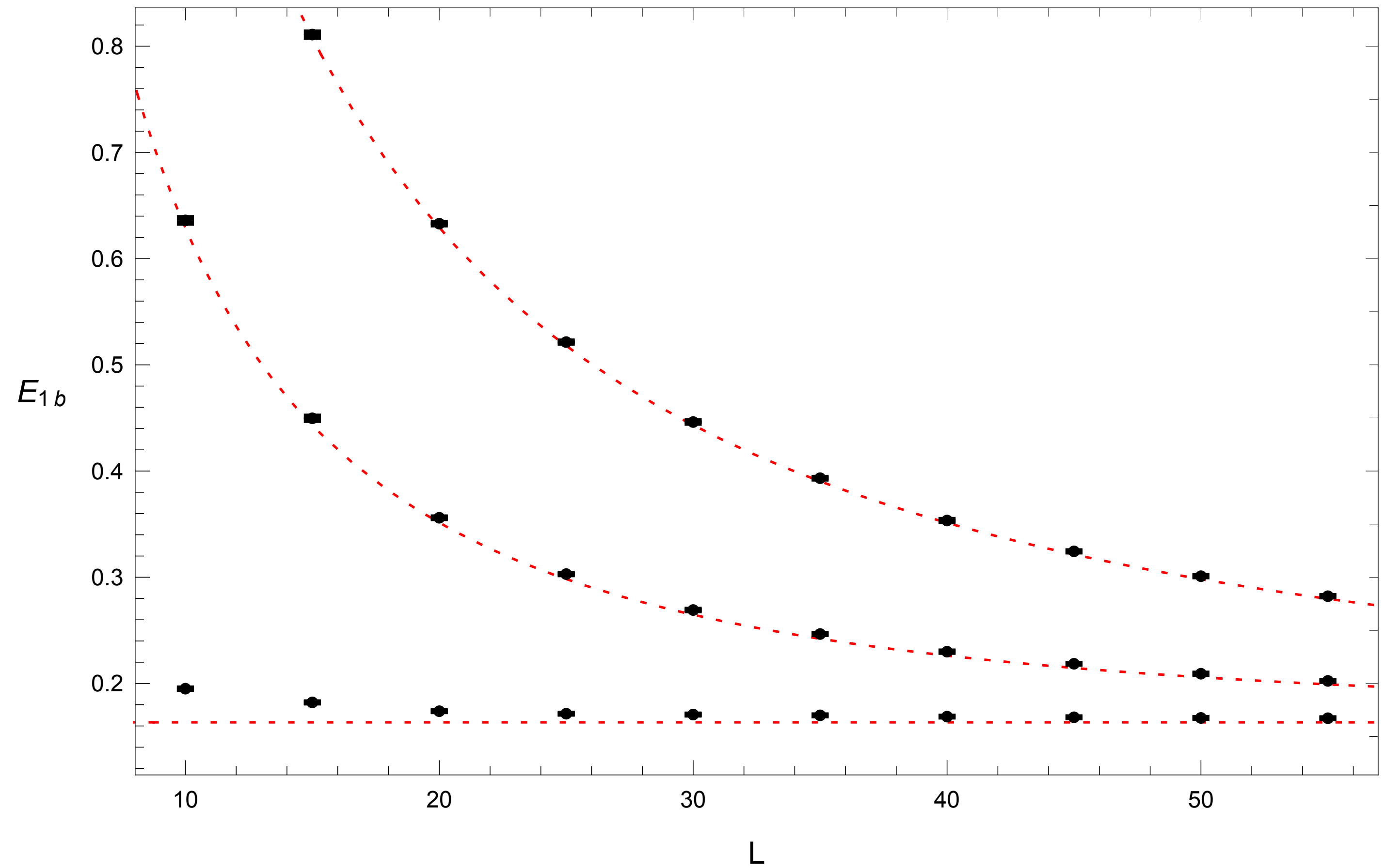
Lattice toy model check

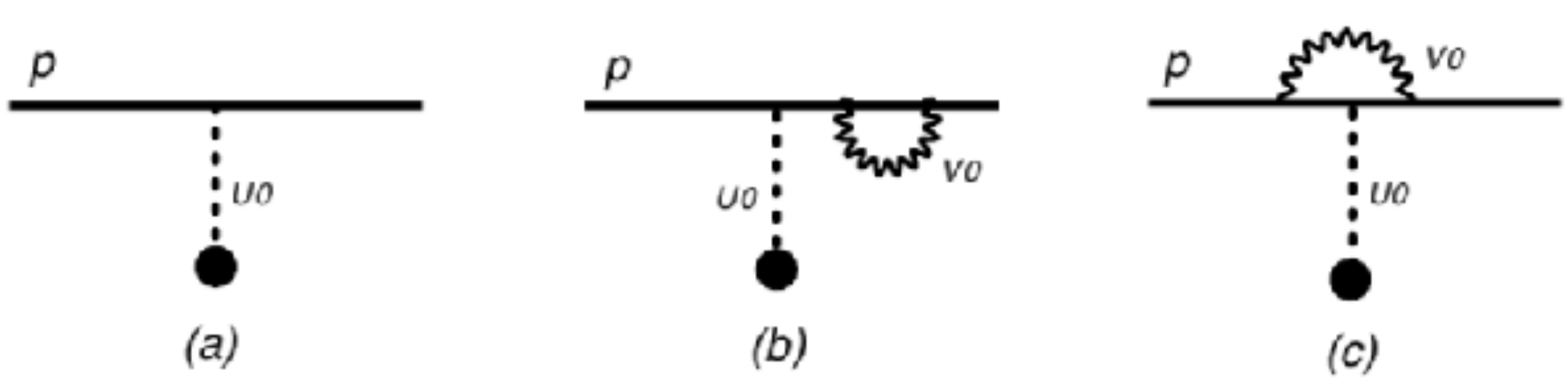
Preliminary

☑ One light particle spectra -> mass of light boson + U0

$$C_{1b,n}(x_0) = \langle \widetilde{\phi}^*_n(x_0)\widetilde{\phi}_n(0)\rangle \propto e^{-E_{1b,n}x_0},$$

$$\widetilde{\phi}_n(x_0) = \frac{1}{L}\sum_{x_1}\phi(x)e^{ix_1\frac{2\pi}{L}n}, \quad n \in \mathbb{Z}.$$





Lattice toy model check

Preliminary

- Two light particles spectra -> V0

$$C^{(d)}_{2b,(i,j)}(x_0) = \langle \left[O^{(d)*}_{2b,i}(x_0) - \delta_{d,0} O^{(d)*}_{2b,i}(x_0+1) \right] O^{(d)}_{2b,j}(0) \rangle,$$

$$O^{(0)}_{2b}(x_0) = \widetilde{\phi}_n(x_0)\widetilde{\phi}_{-n}(x_0), \quad n = 0,1,2,3,$$

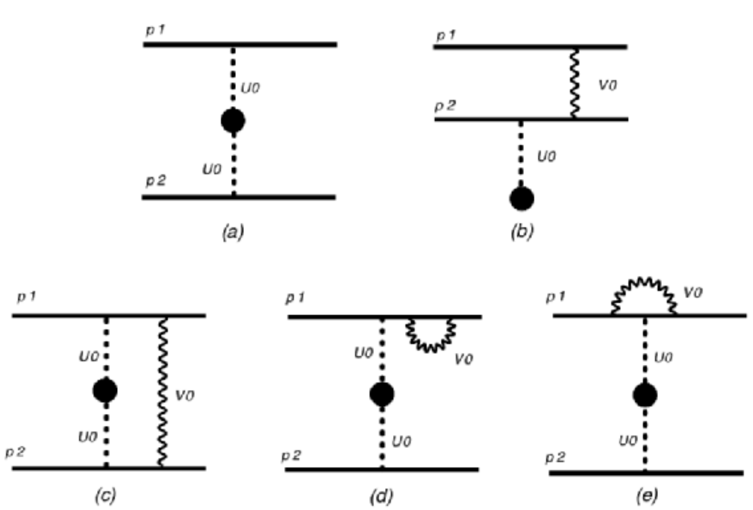
$$O^{(1)}_{2b}(x_0) = \widetilde{\phi}_n(x_0)\widetilde{\phi}_{1-n}(x_0), \quad n = 1,2,3,$$

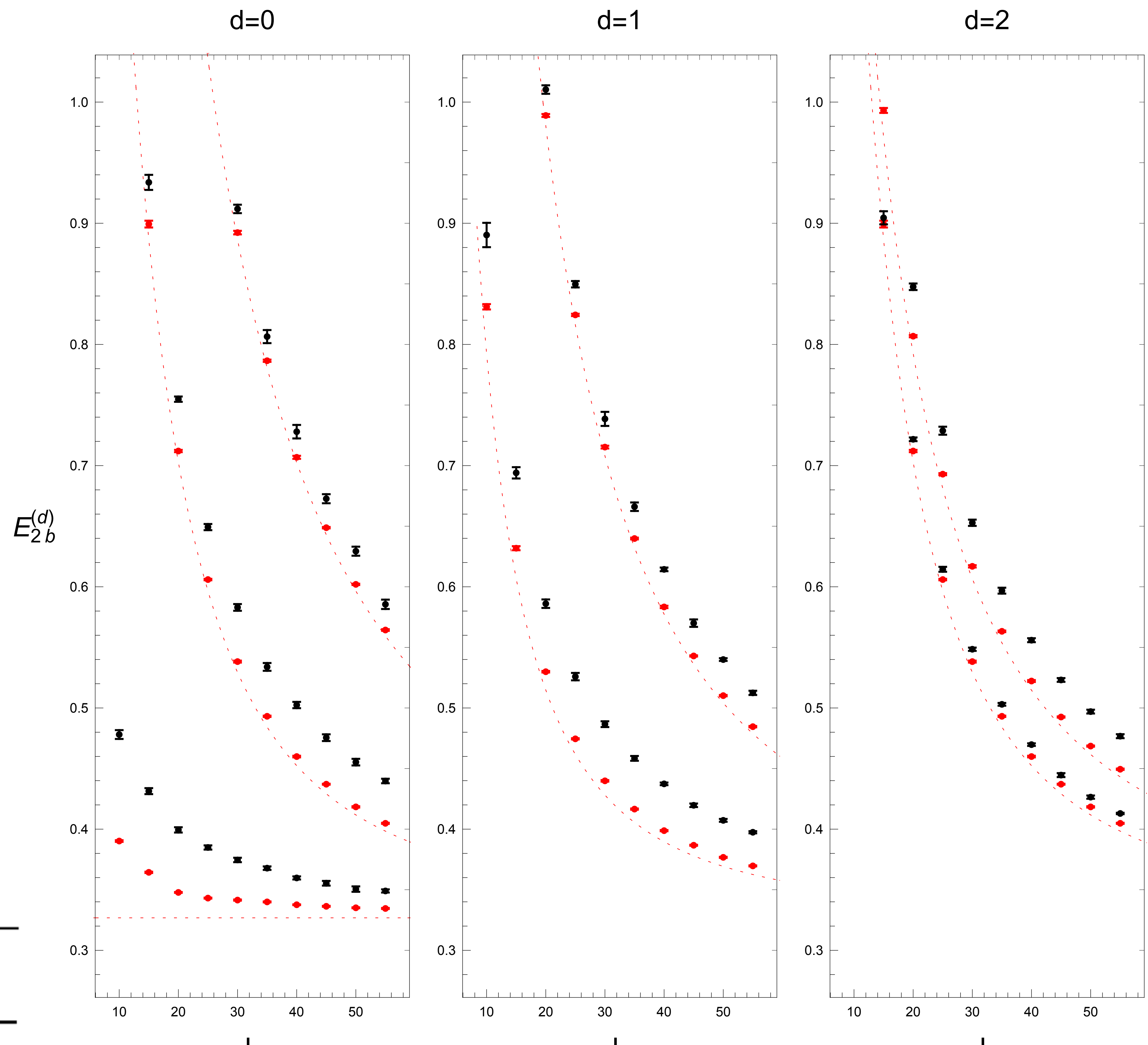
$$O^{(2)}_{2b}(x_0) = \widetilde{\phi}_1(x_0)\widetilde{\phi}_1(x_0), \;\; \widetilde{\phi}_n(x_0)\widetilde{\phi}_{2-n}(x_0), \quad n = 2,3.$$

Lattice toy model check

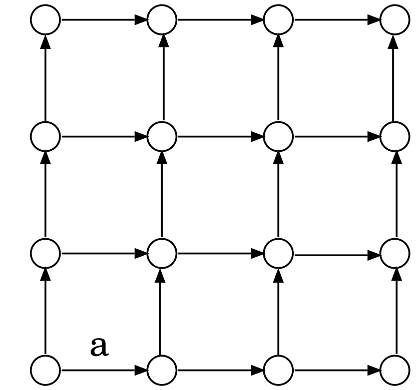
☑ **Two light particles spectra -> V0**

Preliminary





Lattice toy model check

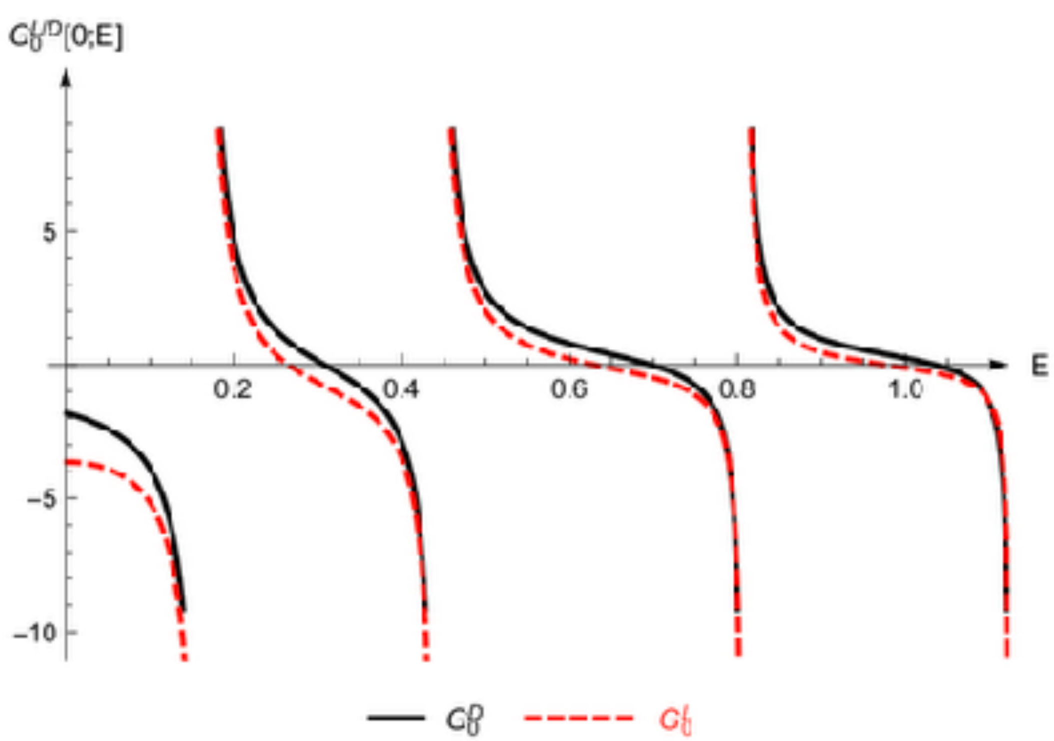


☑ One light particle Dynamics on lattice with finite lattice spacing (a=1):

$$\Phi^D(x) = \sum_{x' \in [0,L-1]} G_0^D(x-x';E) U_0 \delta_{x',0} \Phi^D(x'),$$

Preliminary

$$\cosh E \;=\; \cosh m + 1 - \cos p$$



$$G_0^D(x;E) = \frac{1}{L} \sum_{n \in [0,L-1]}^{p=\frac{2\pi}{L}n} \frac{1}{4\sin\frac{E_p}{2}} \frac{e^{ipx}}{2\sinh\frac{E}{2} - 2\sin\frac{E_p}{2}},$$

As lattice spacing a ->0

$$G_0^D(x;E) \to \frac{1}{L} \sum_{n \in \mathbb{Z}}^{p=\frac{2\pi}{L}n} \frac{1}{2\sqrt{p^2+m^2}} \frac{e^{ipx}}{E - \sqrt{p^2+m^2}}.$$

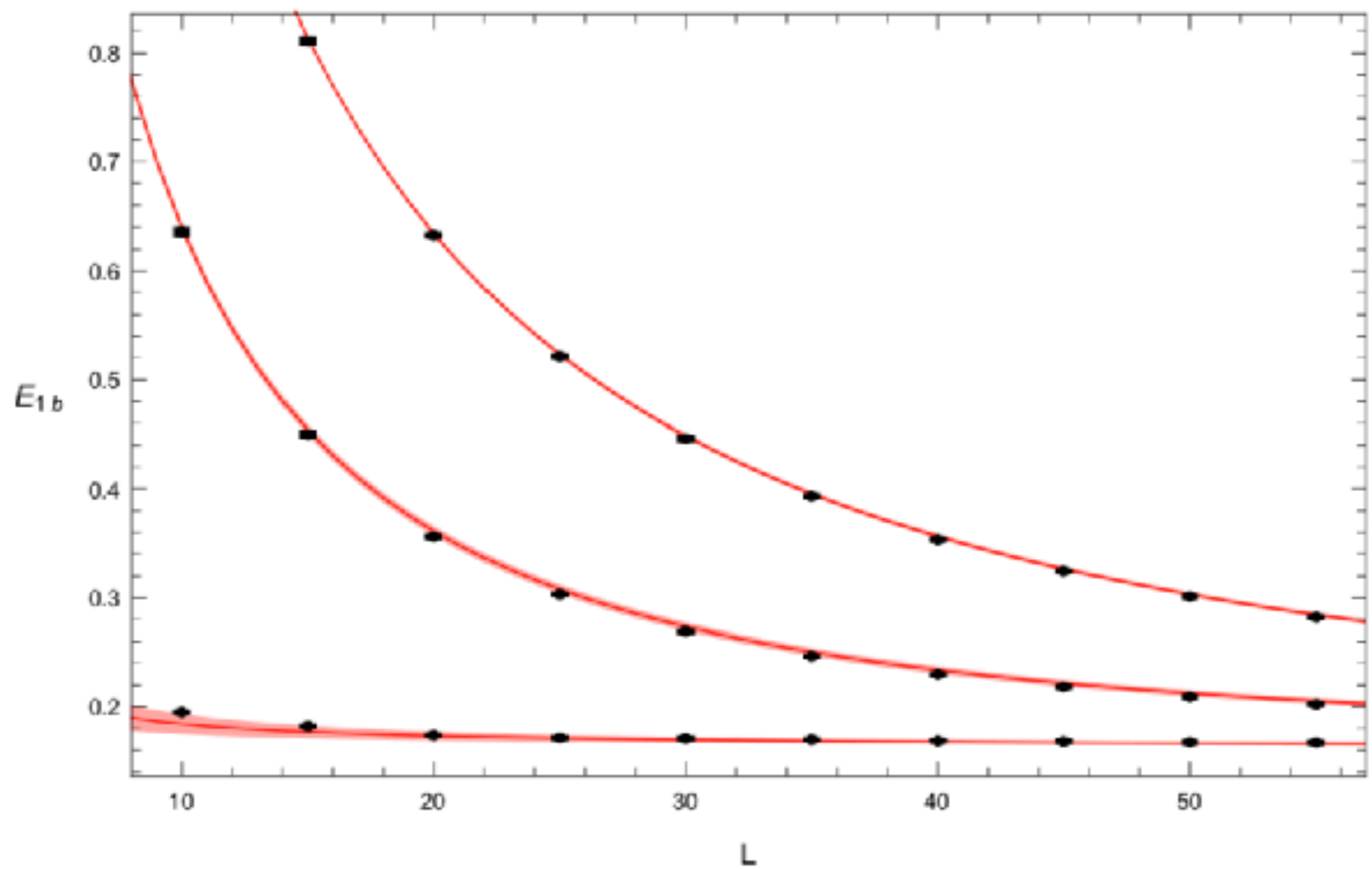
Lattice toy model check

Preliminary

☑ One light particle spectra -> mass of light boson + U0

$$m = 0.163 \pm 0.001$$

$$U_0 = 0.07 \pm 0.03$$

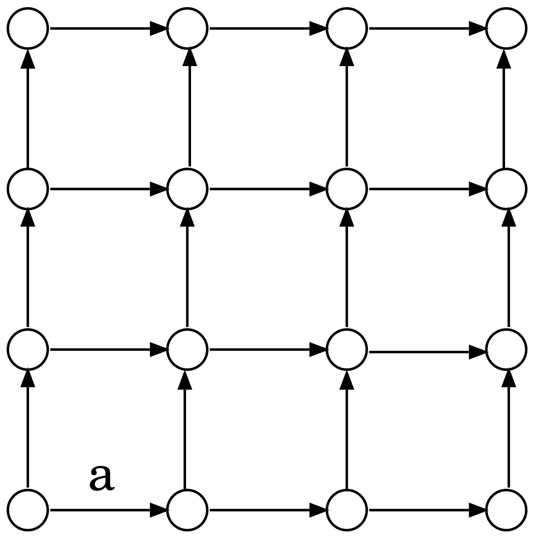


Lattice toy model check

Preliminary

- **Two light particle Dynamics on lattice with finite lattice spacing (a=1):**

$$\Phi^D(x_1,x_2) = \sum_{x_1',x_2' \in [0,L-1]} G_0^D(x_1-x_1', x_2-x_2'; E) \times \left(U_0\delta_{x_1',0} + U_0\delta_{x_2',0} + V_0\delta_{x_1',x_2'}\right)\Phi^D(x_1',x_2')$$



$$G_0^D(x_1,x_2;E) = \frac{1}{L^2}\sum_{n_1,n_2\in[0,L-1]}^{p_i=\frac{2\pi}{L}n_i} \frac{4\sinh\frac{m}{2}}{4\sinh\frac{E_1}{2}4\sinh\frac{E_2}{2}} \times \frac{e^{i(p_1x_1+p_2x_2)}}{2\sinh\frac{E}{2}-(2\sinh\frac{E_1}{2}+2\sinh\frac{E_2}{2})}$$

Lattice toy model check

Preliminary

☑ **Two light particles Dynamics on lattice with finite lattice spacing (a=1):**

$$\begin{bmatrix} \Phi^D(x,x) \\ \Phi^D(x,0) \end{bmatrix} = \sum_{x' \in [0,L-1]} \mathcal{G}^D(x,x';E) \begin{bmatrix} \Phi^D(x',x') \\ \Phi^D(x',0) \end{bmatrix}, \quad x \in [0,L-1],$$

Lattice toy model check

Preliminary

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☑ Quantization condition of three-body interaction:

Lattice toy model check

Preliminary

Two light particles Dynamics on lattice with finite lattice spacing (a=1):

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Quantization condition of three-body interaction:

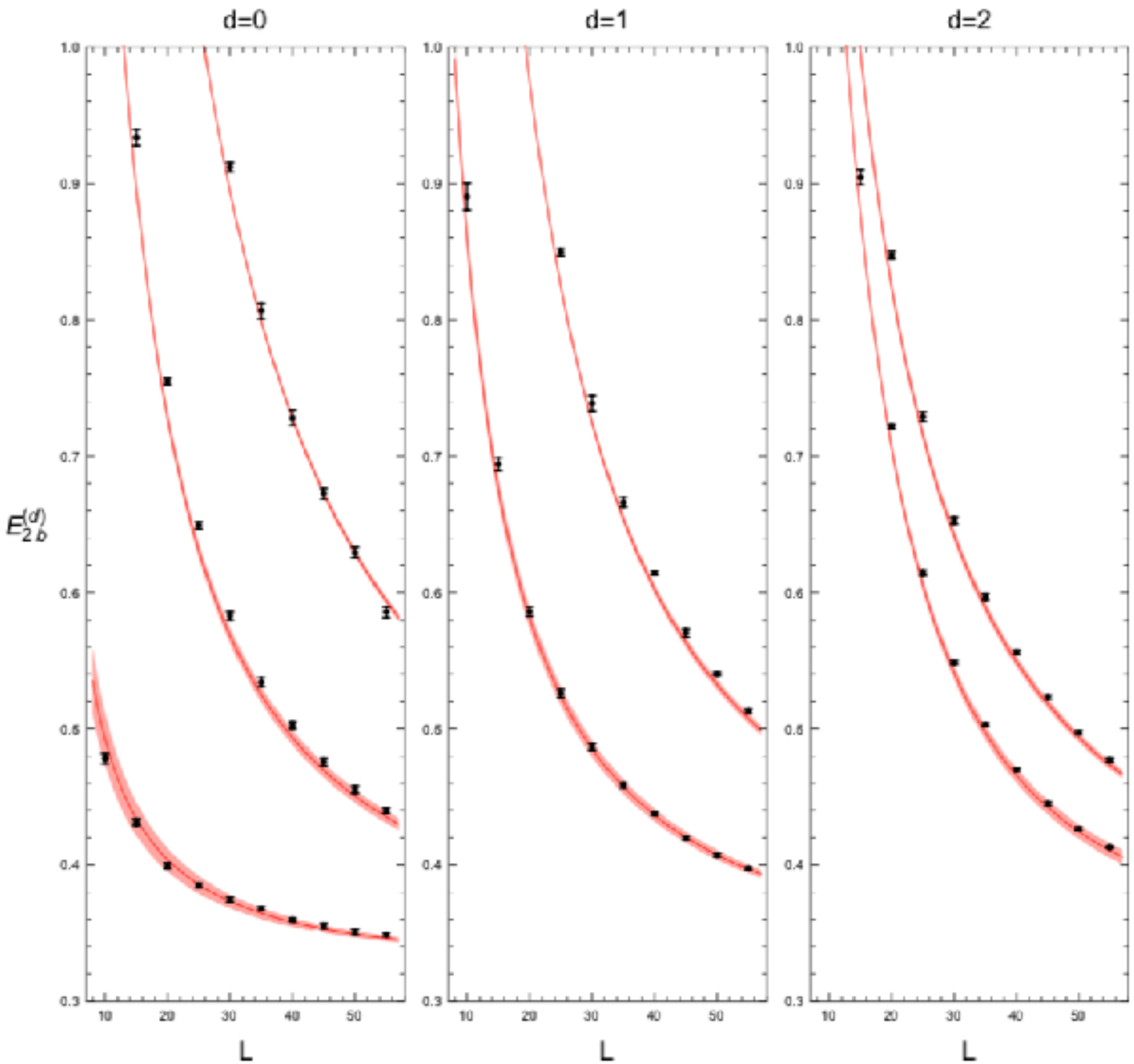
$$\det\left[\delta_{\alpha,\beta}\delta_{i,j} - \mathcal{G}^D_{\alpha,\beta}(x_i,x_j;\sigma)\right] = 0,$$
$$(x_i,x_j)\in[0,\cdots,L-1].$$

Lattice toy model check

Preliminary

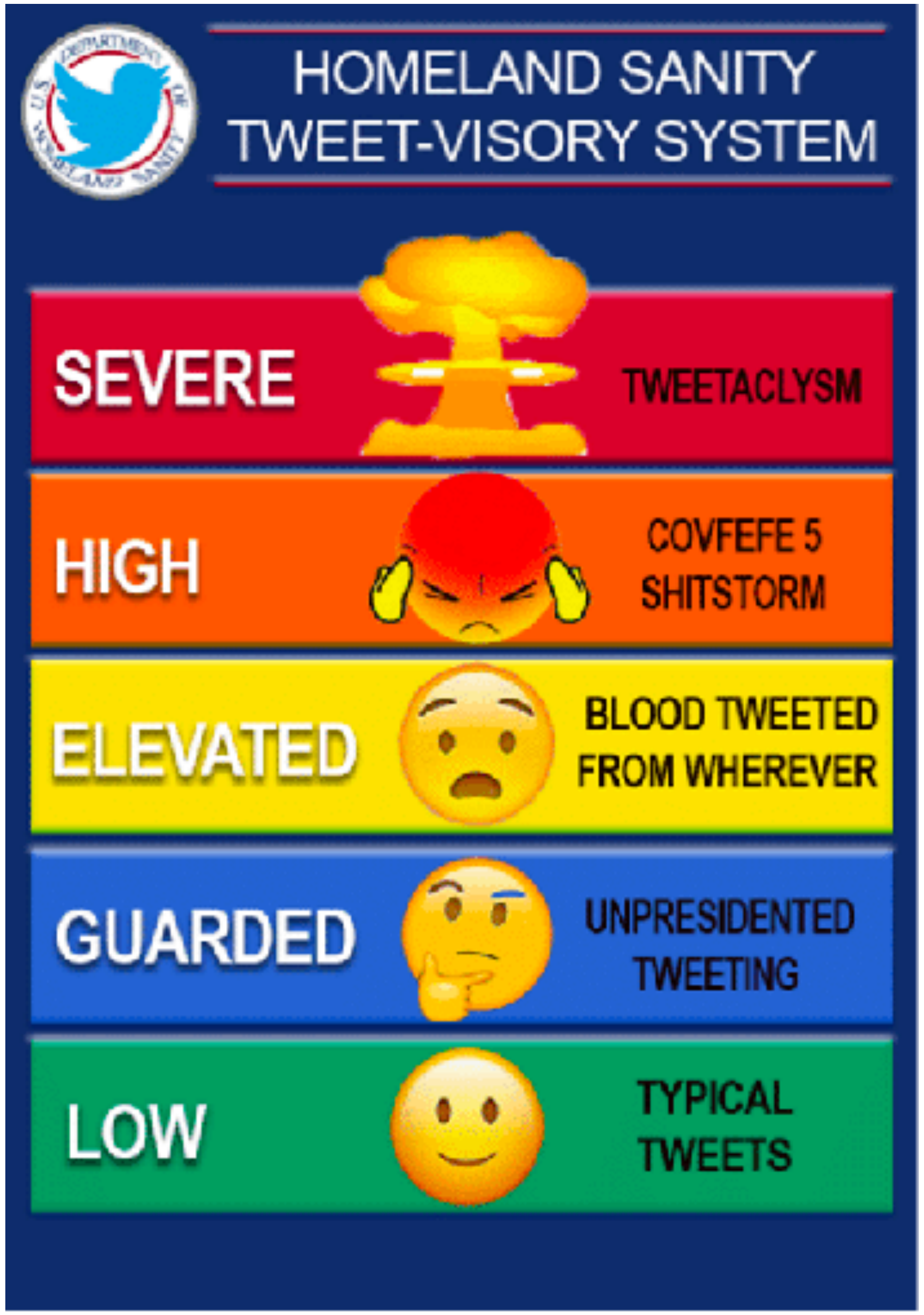
☑ Two light particles Dynamics on lattice with finite lattice spacing (a=1):

$$V_0 = 0.43 \pm 0.03$$



Summary and outlook

- ☑ **Diagnosis of sanity level:**



Summary and outlook

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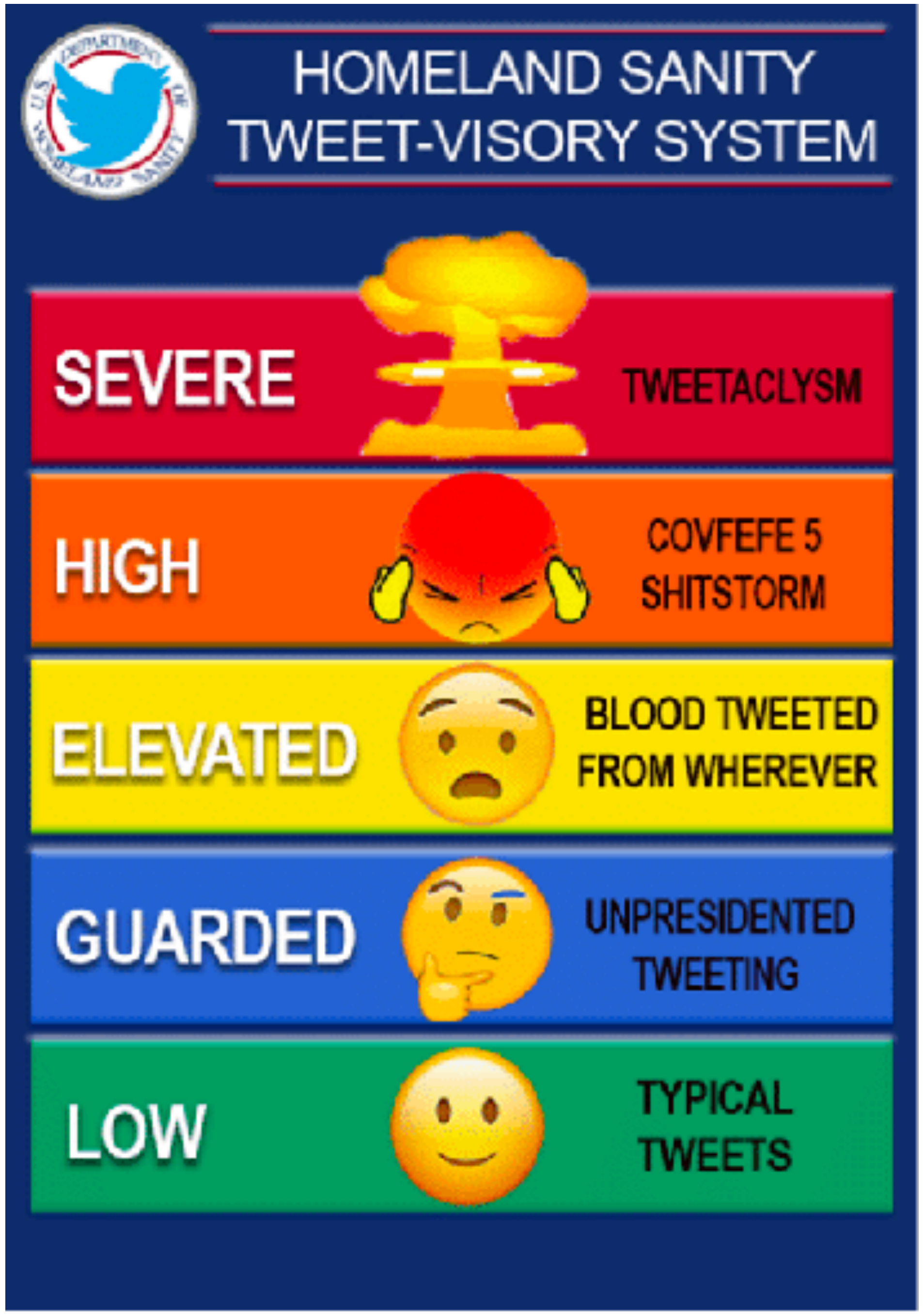

Summary and outlook

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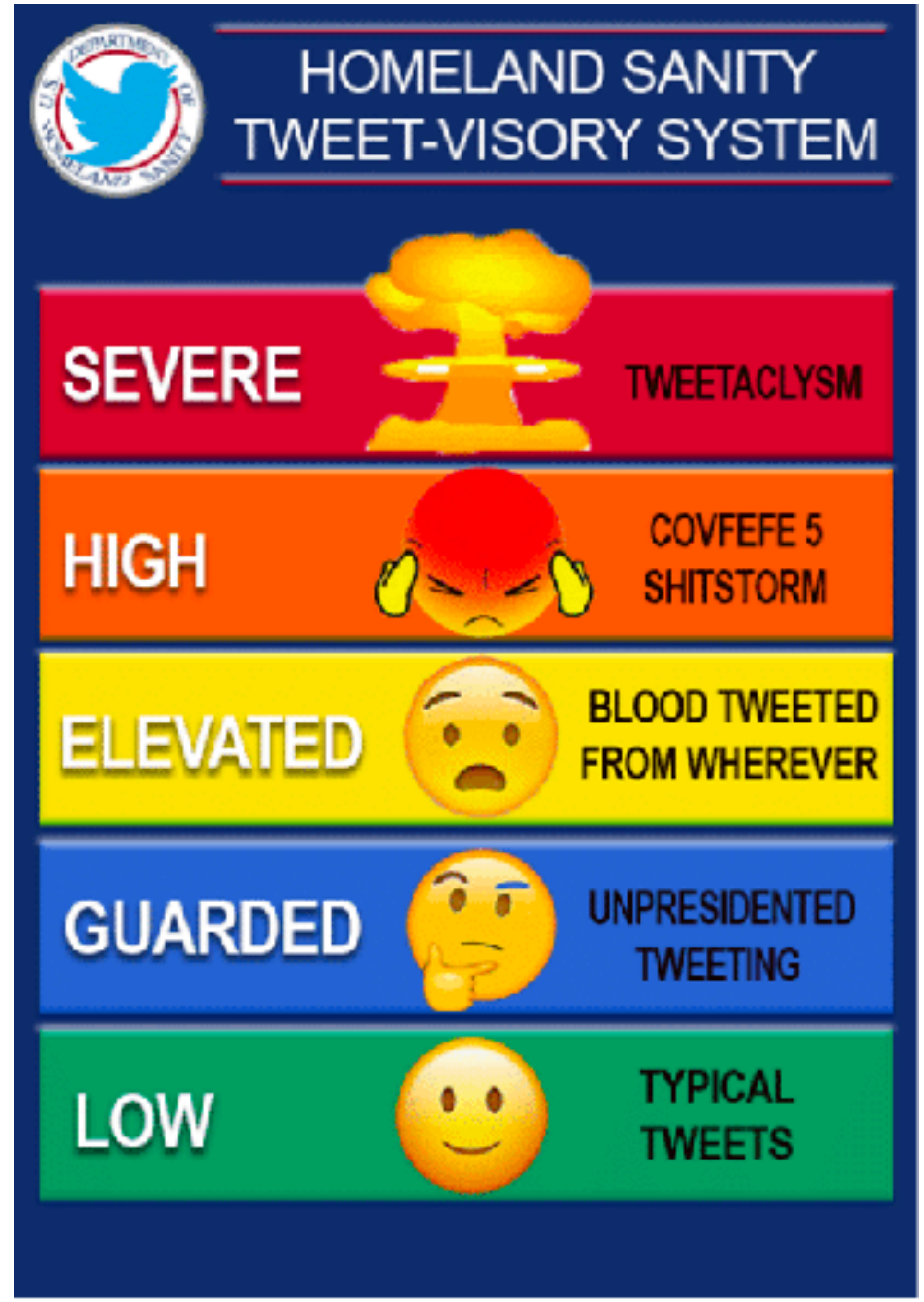

- More sanity check: Extension to include coupled channels and resonances