Scattering amplitudes from finite-volume spectral functions

John Bulava

University of Southern Denmark
CP3-Origins

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Observables from lattice QCD: Euclidean correlation functions

- Large time separation: ground state saturation

$$C_{ij}(\tau) = \langle \mathcal{O}_i(\tau)\bar{\mathcal{O}}_j(0)\rangle_U$$

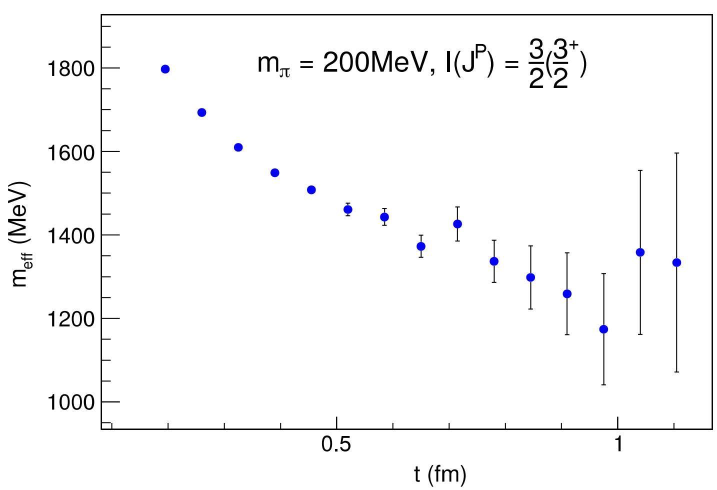
$$\lim_{\tau\to\infty} C_{ij}(\tau) = \langle 0|\hat{O}_i|E_1\rangle\langle E_1|\hat{O}_j^\dagger|0\rangle\, \mathrm{e}^{-E_1\tau}\left\{1 + O(\mathrm{e}^{-(E_2-E_1)\tau})\right\}$$

- Generalized Eigenvalue methods provide a few excited states:

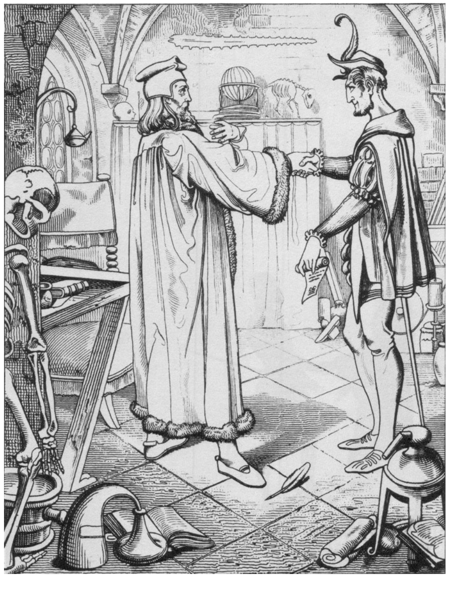
$$C(\tau)v_n(\tau) = \lambda_n(\tau)C(\tau_0)v_n(\tau) \qquad \lim_{\tau\to\infty}\lambda_n(\tau) = \mathrm{e}^{-E_n\tau}$$

- Signal-to-noise problem
  => ‘Teufelspakt’

$$m_{\mathrm{eff}}(\tau) = \log\left[\frac{C(\tau)}{C(\tau+1)}\right]$$



=>



Plot courtesy of C. W. Andersen, apologies to J. W. von Goethe

Scattering amplitudes: finite volume formalism

- In Euclidean time, asymptotic limit of $C(\tau_1, \ldots, \tau_n)$ contains no info about on-shell amplitudes (in general). L. Maiani, M. Testa, *Phys. Lett.* **B245** (1990) 585

- Finite volume method: below $n \geq 3$ hadron thresholds:

$$\det[K^{-1}(E_{\rm cm}) - B(L\mathbf{q}_{\rm cm})] + \mathrm{O}(\mathrm{e}^{-ML}) = 0$$

$$S = (1 - iK)^{-1}(1 + iK)$$

M. Lüscher, *Nucl. Phys.* **B354** (1991) 531

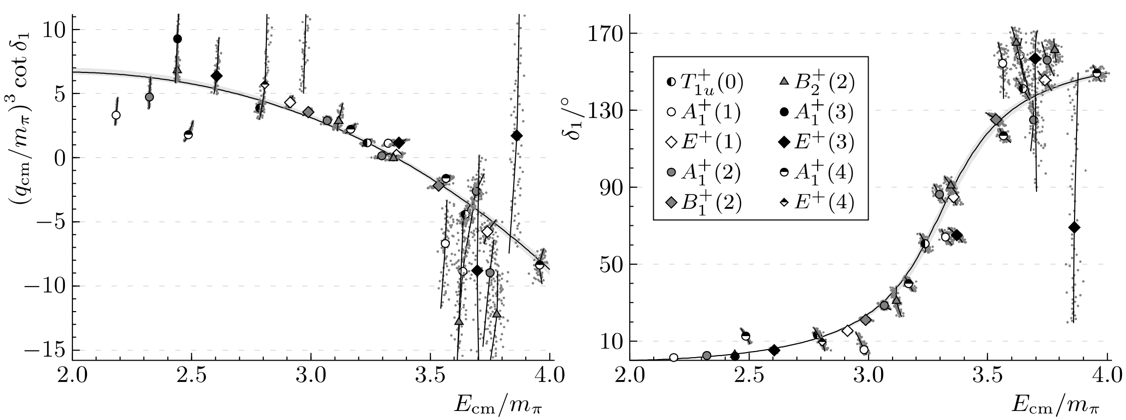
- Determinant over total angular momentum, channel, and total spin

- Block-diagonal in finite-volume irreps.

Finite volume approach: disadvantages

- Model-dependent determination of coupled channels
- Cumbersome in large volume:
  - Recent pi-pi scattering: 43 energies

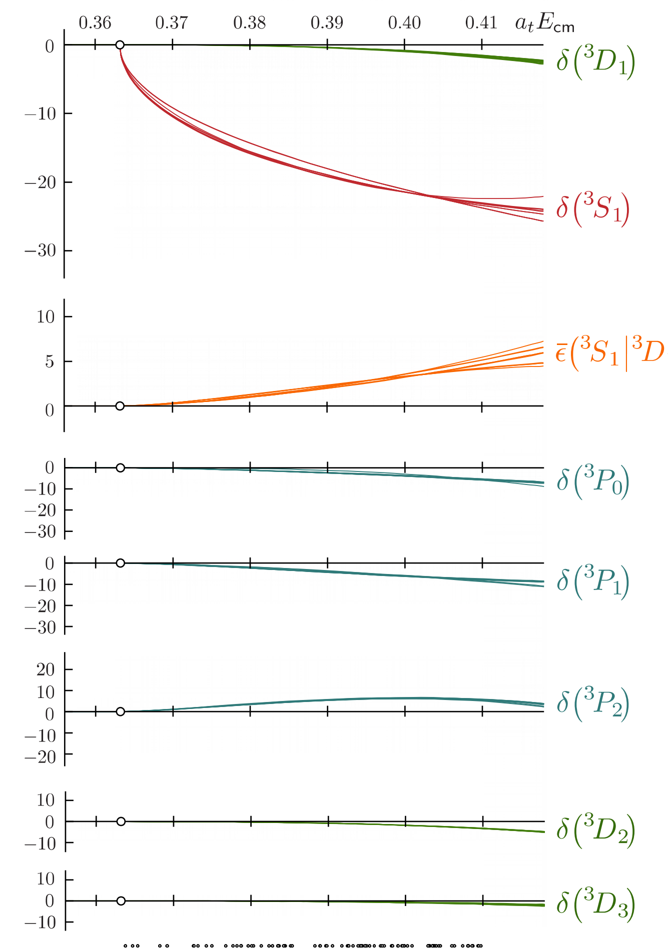
$$m_\pi = 220\text{MeV},\ m_\pi L = 6.1$$



C. Andersen, JB, B. Hörz, C. Morningstar
Nucl. Phys. B939 (2019) 145173

  - Recent rho-pi scattering: 141 energies

$$m_\pi = 700\text{MeV},\ m_\pi L = 10.3,\ 12.4$$



A. Woss, C. Thomas, J. Dudek, R. Edwards,
D. Wilson, JHEP 07 (2018)

Finite volume approach: disadvantages

- Currently applicable below three (four) hadron thresholds. Problem for:
  - Hidden charm exotics
  - Nucleon-nucleon scattering
  - 1-3GeV hadron resonances
  - $D \to \pi\pi$
  - ...
- Three particle formalism adds additional complication, requires two-to-two amplitude
- Higher partial waves must be truncated/difficult to access
- Inclusive rates? $p + p \to \mathrm{hadrons}$

An alternative approach to real-time physics: spectral functions

- Spectral function independent of metric signature:

$$C(\boldsymbol{p}, \tau) = \int_0^\infty dE\, \rho(E)\, \mathrm{e}^{-E\tau}$$

(finite volume is a nuisance; not a tool!)

- Recent development: solve (ill-posed) inverse problem using smearing

$$\rho_\epsilon(E) = \int_0^\infty d\omega\, \hat{\delta}_\epsilon(E, \omega)\, \rho(\omega)$$

Backus, Gilbert '68-'70; H. Meyer, M. T. Hansen, D. Robaina '17; M. R. Hansen, A. Lupo, N. Tantalo, '19; JB, M. T. Hansen, '19

$$\hat{\delta}_\epsilon(E - \omega) = \frac{i}{E - \omega + i\epsilon} = \frac{\epsilon}{(E-\omega)^2 + \epsilon^2} + i\frac{E - \omega}{(E-\omega)^2 + \epsilon^2}$$

- Natural smearing kernel:
  - $\rho_\epsilon(E)$ analytic near $\epsilon = 0$
  - Implements standard $i\epsilon$-prescription
  - Same smearing for Quark-Hadron duality

Poggio, Quinn, Weinberg '76

HLT algorithm for spectral reconstruction

M. R. Hansen, A. Lupo, N. Tantalo, Phys.Rev. D99 (2019) 094508
Backus, Gilbert `68, `70

- Linear ansatz between input and output:

$$\hat{\rho}^{\epsilon}(E) = \sum_{i=1}^{N_\tau} q_i(E) C(\tau_i)$$

- Estimator for smearing kernel:

$$\hat{\delta}^{\epsilon}(E,\omega) = \sum_{i=1}^{N_\tau} q_i(E)\, \mathrm{e}^{-\omega\tau_i} \equiv \boldsymbol{q}(E)\cdot\boldsymbol{s}(\omega)$$

HLT algorithm for spectral reconstruction

- How to choose $\{q_i(E)\}$? Two competing considerations:
  - Resolution: $\hat{\delta}_\epsilon(E,\omega)$ close to $\delta_\epsilon(E,\omega)$
  - Stability: wild $\{q_i(E)\}$ => large errors on $\hat{\rho}^\epsilon(E)$

- $\{q_i(E)\}$ chosen to minimize $G_\lambda[q] = (1-\lambda)A[q] + \lambda B[q]$

$$A[q(E)] = \int_0^\infty \frac{d\omega}{\pi} |\hat{\delta}_\epsilon(E,\omega) - \delta_\epsilon(E-\omega)|^2$$

$$B[q(E)] = \sum_{i,j=1}^{N_\tau} q_i(E) q_j(E)\, \mathrm{Cov}\{C(\tau_i), C(\tau_j)\}$$

BG algorithm for spectral reconstruction

- How to choose $\{q_i(E)\}$? Two competing considerations:
  - Resolution: $\hat{\delta}_\epsilon(E,\omega)$ narrowly peaked
  - Stability: wild $\{q_i(E)\}$ => large errors on $\hat{\rho}^\epsilon(E)$

- $\{q_i(E)\}$ chosen to minimize $G_\lambda[q] = (1-\lambda)A[q] + \lambda B[q]$

$$A[q(E)] = \int_0^\infty \frac{d\omega}{\pi}\,(E-\omega)^2\,\hat{\delta}_\epsilon^2(E,\omega)$$

$$B[q(E)] = \sum_{i,j=1}^{N_\tau} q_i(E)q_j(E)\,\mathrm{Cov}\{C(\tau_i),C(\tau_j)\}$$

HLT algorithm for spectral reconstruction

- The $\{q_i(E)\}$ obtained by extremizing $G_\lambda[q]$

$$\boldsymbol{q}(E) = (1-\lambda) M_\lambda^{-1} \boldsymbol{v}$$

$$v_j = \int_0^\infty \frac{d\omega}{\pi}\, \delta_\epsilon(E-\omega)\, \mathrm{e}^{-\omega \tau_j}$$

$$\{M_\lambda\}_{ij} = (1-\lambda) w_{ij} + \lambda \mathrm{Cov}_{ij}, \quad w_{ij} = \int_0^\infty \frac{d\omega}{\pi} \mathrm{e}^{-\omega(\tau_i+\tau_j)}$$

- M_λ is ill-conditioned. Extended precision required.

HLT algorithm for spectral reconstruction

Advantages:

- Optimized $\{q_i(E)\}$ determined analytically. Model independent.
- $\lambda \to 0$ limit is stable, corresponds to no $B[q(E)]$.
- Deviation $|\delta_\epsilon(E-\omega) - \hat{\delta}_\epsilon(E,\omega)|$ a measure of systematic error.
- Constraints on optimization are possible:
  - Fixed area: $\int_0^\infty d\omega\, \hat{\delta}_\epsilon(E,\omega) = c$
  - Fixed at point: $\hat{\delta}_\epsilon(E,\omega^*) = c$
  - Fixed integral: $\int_0^\infty d\omega\, \frac{\hat{\delta}_\epsilon(E,\omega) - \hat{\delta}_\epsilon(E,-\omega)}{E} = c$

First application: pion two-point function

JB, M. W. Hansen, M. R. Hansen, M. T. Hansen, A. Lupo, A. Patella, N. Tantalo
Preliminary

- Lattice details:

$$N_{\rm f} = 2+1,\ a_s = 3.5 a_\tau,\ m_\pi = 240 {\rm MeV},\ 32^3 \times 256$$

JB, B. Fahy, B. Hoerz, K. J. Juge, C. Morningstar, C. H. Wong '16

- $\rho^L(E)$ has a single pole at m_π, gap until $3m_\pi$

$$C_\pi(\tau) = \frac{Z_\pi}{2m_\pi} {\rm e}^{-m_\pi \tau} + {\rm O}({\rm e}^{-3m_\pi\tau}) = \int_0^\infty \frac{dE}{\pi} \rho^L(E)\, {\rm e}^{-E\tau}$$

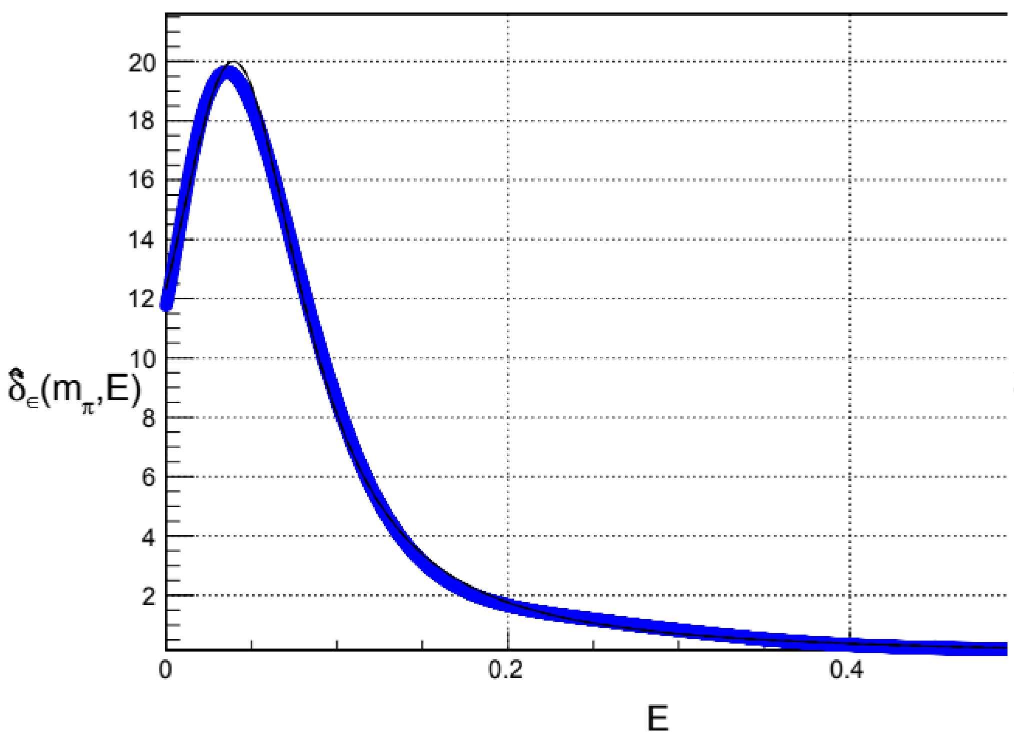
- Simple test of 'amputation' procedure: $\dfrac{2m_\pi}{Z_\pi} \lim_{\epsilon \to 0} \epsilon\, \rho^{\epsilon, L}(m_\pi) = 1$

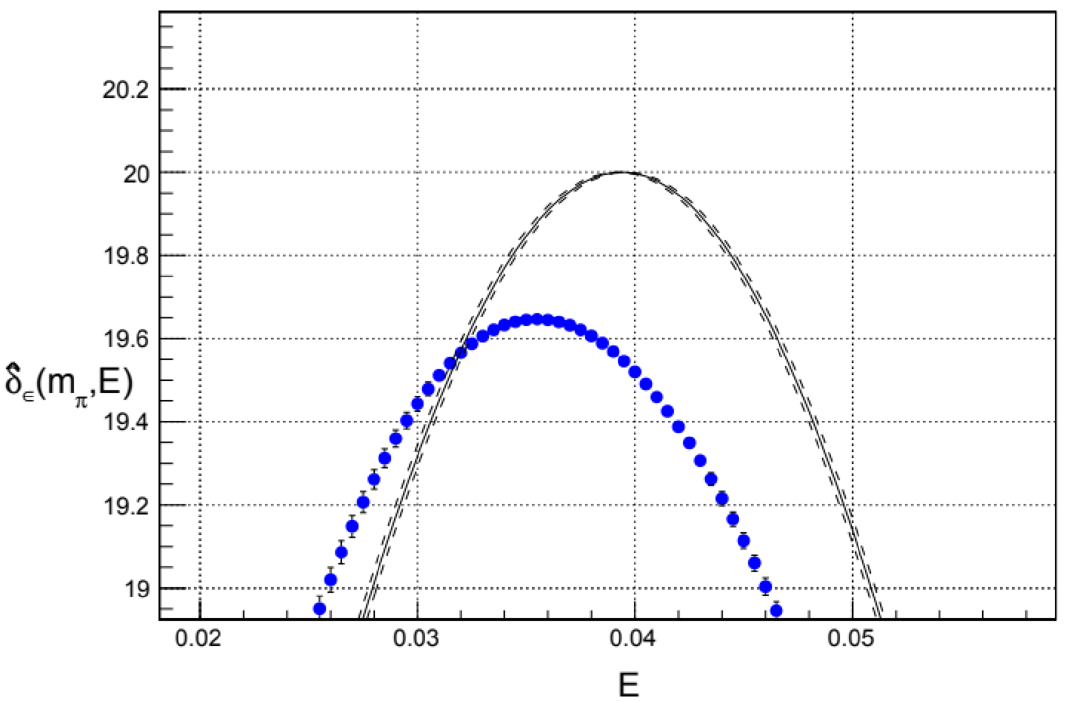
$$\rho^{\epsilon, L}(m_\pi) = \int_0^\infty \frac{dE}{\pi} \frac{\epsilon}{(E-m_\pi)^2 + \epsilon^2} \rho^L(E)$$

First application: pion two-point function

- Reproduction of real kernel: (no constraints)

$$\epsilon = 0.05 = 1.28 m_\pi,\; N_\tau = 36,\; \lambda = 10^{-7}$$

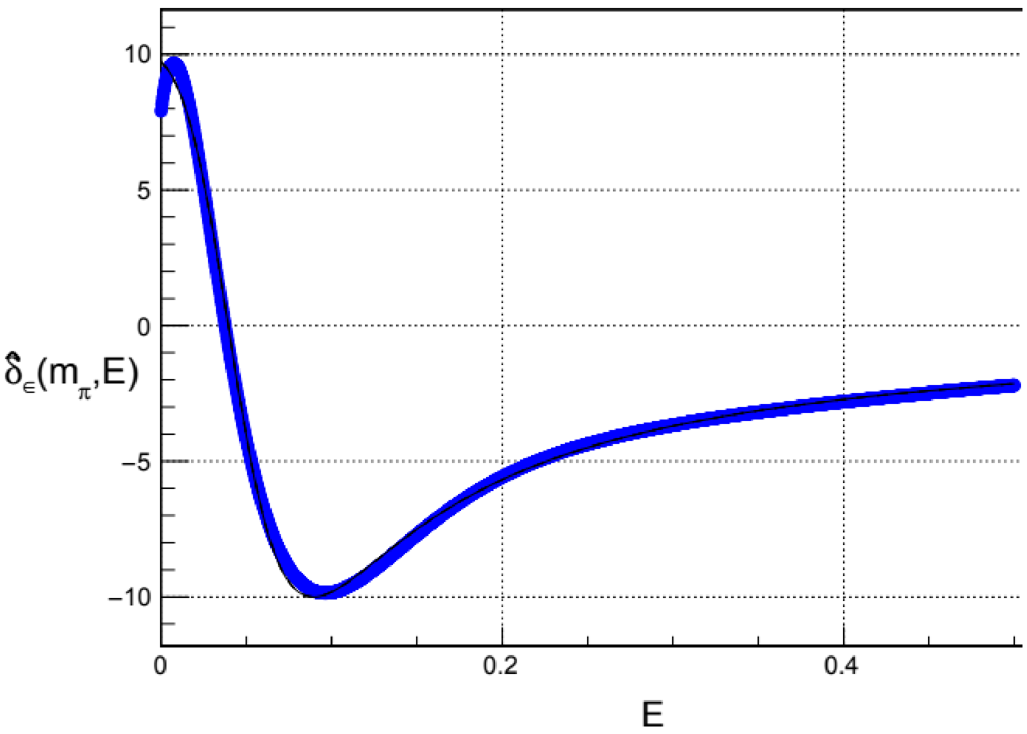


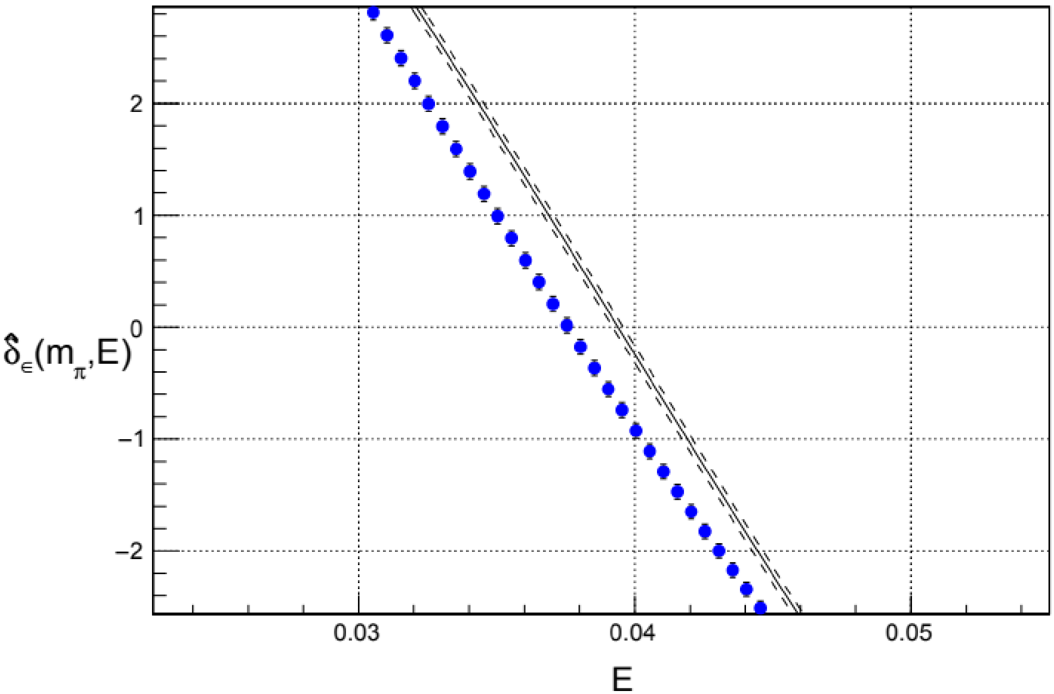


First application: pion two-point function

- Reproduction of imaginary kernel: (no constraints)

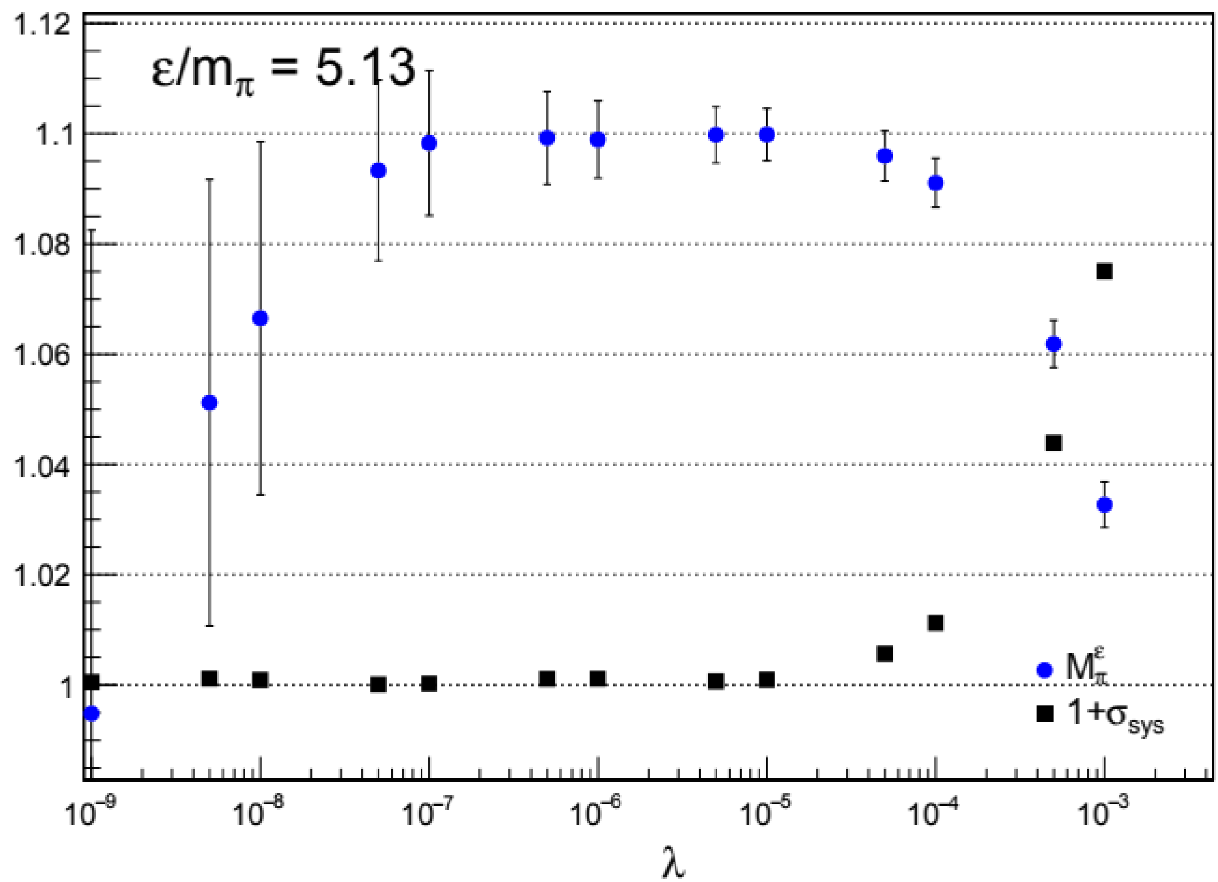
$$\delta_\epsilon(m_\pi, E) = \frac{m_\pi - E}{(m_\pi - E)^2 + \epsilon^2}$$





First application: pion two-point function

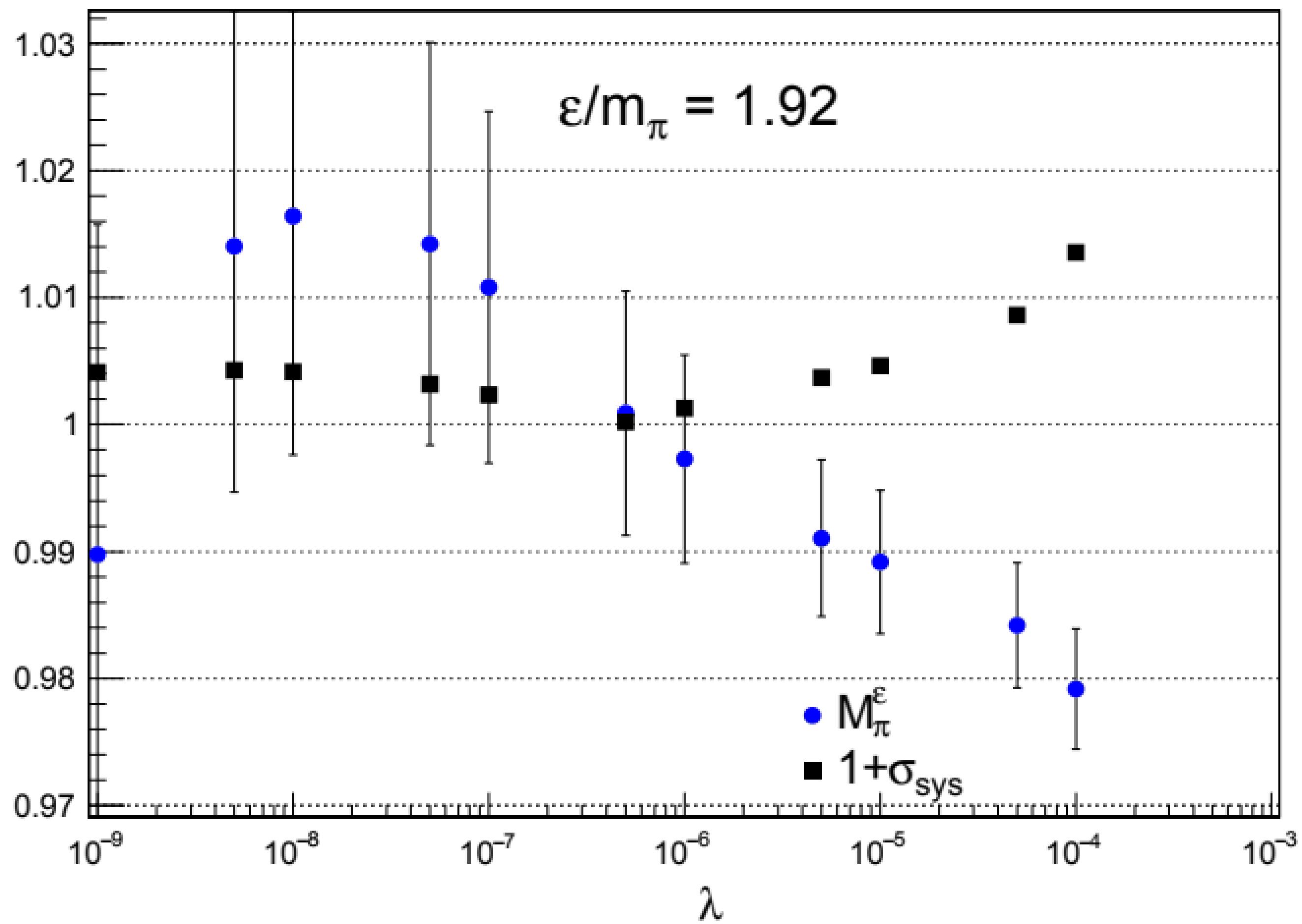
Study of systematic errors: $\sigma_{\rm sys} = |1 - \epsilon\hat{\delta}_\epsilon(m_\pi, m_\pi)|$



Large λ => small statistical error, Small λ => small systematic error

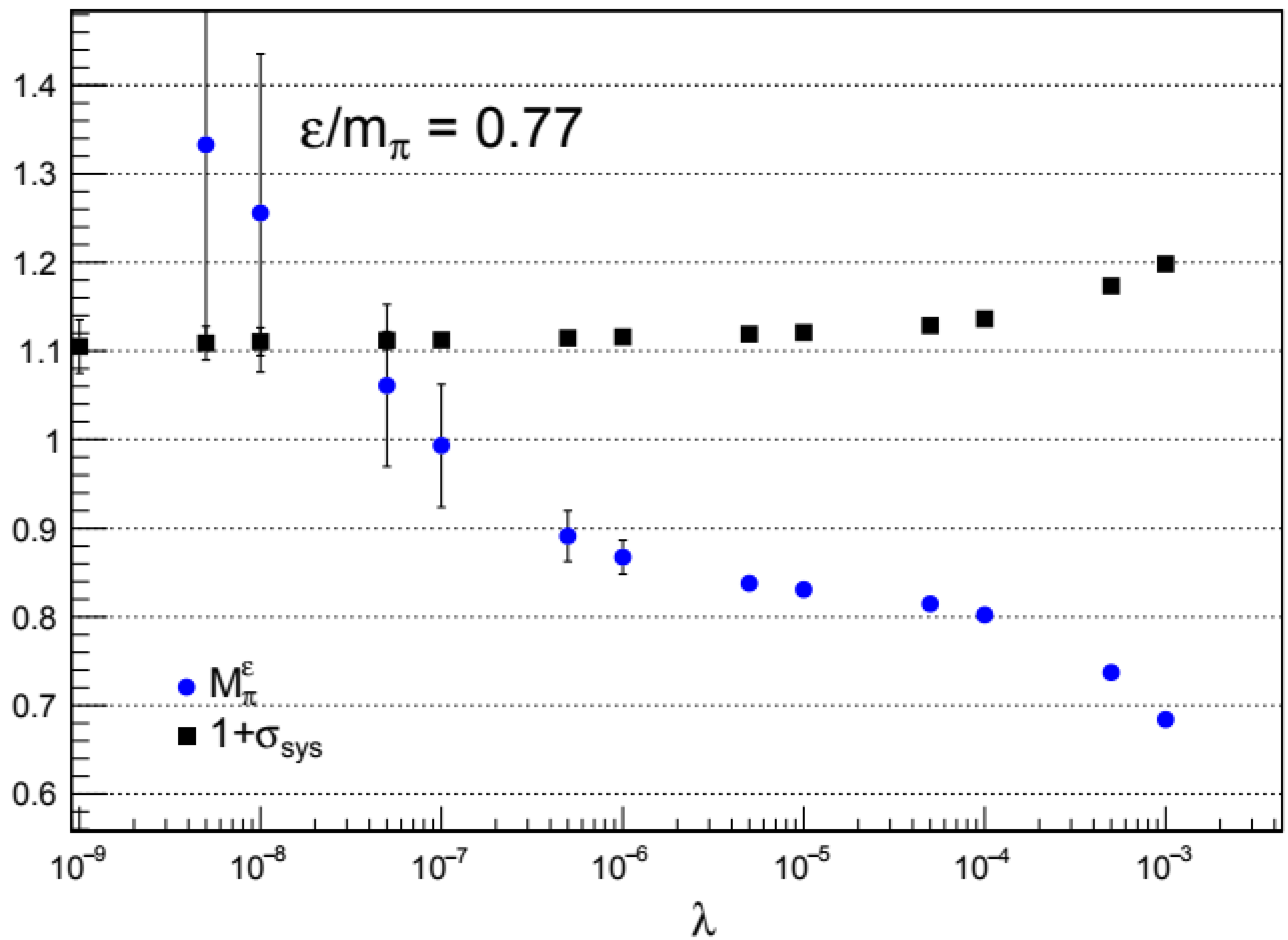
First application: pion two-point function

- Study of systematic errors: $\sigma_{\rm sys} = |1 - \epsilon\hat{\delta}_\epsilon(m_\pi, m_\pi)|$



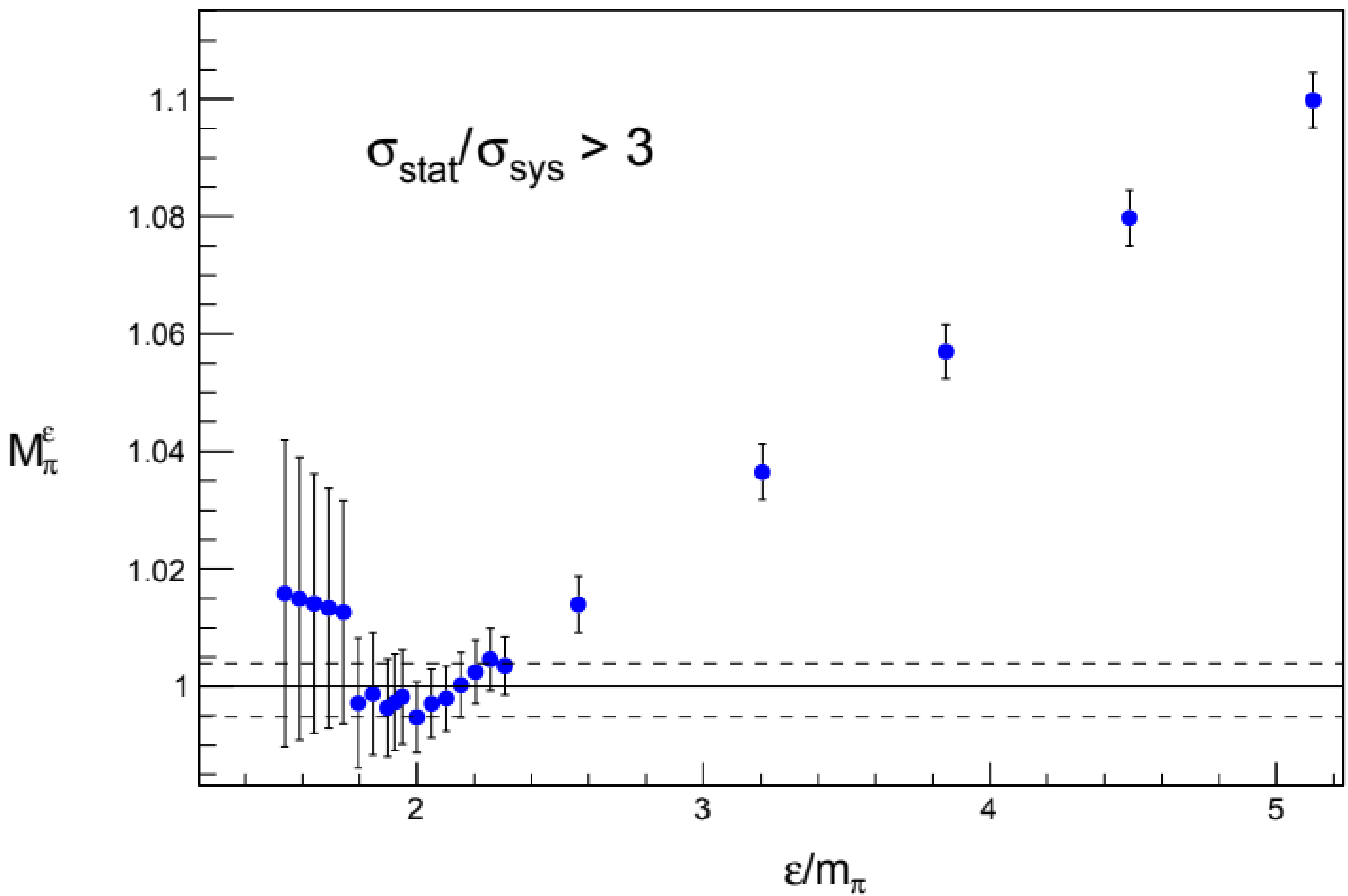
First application: pion two-point function

- Study of systematic errors: $\sigma_{\text{sys}} = |1 - \epsilon\hat{\delta}_\epsilon(m_\pi, m_\pi)|$



First application: pion two-point function

- Extrapolation in ϵ at fixed *L:*



- Dotted lines: single-exponential fit

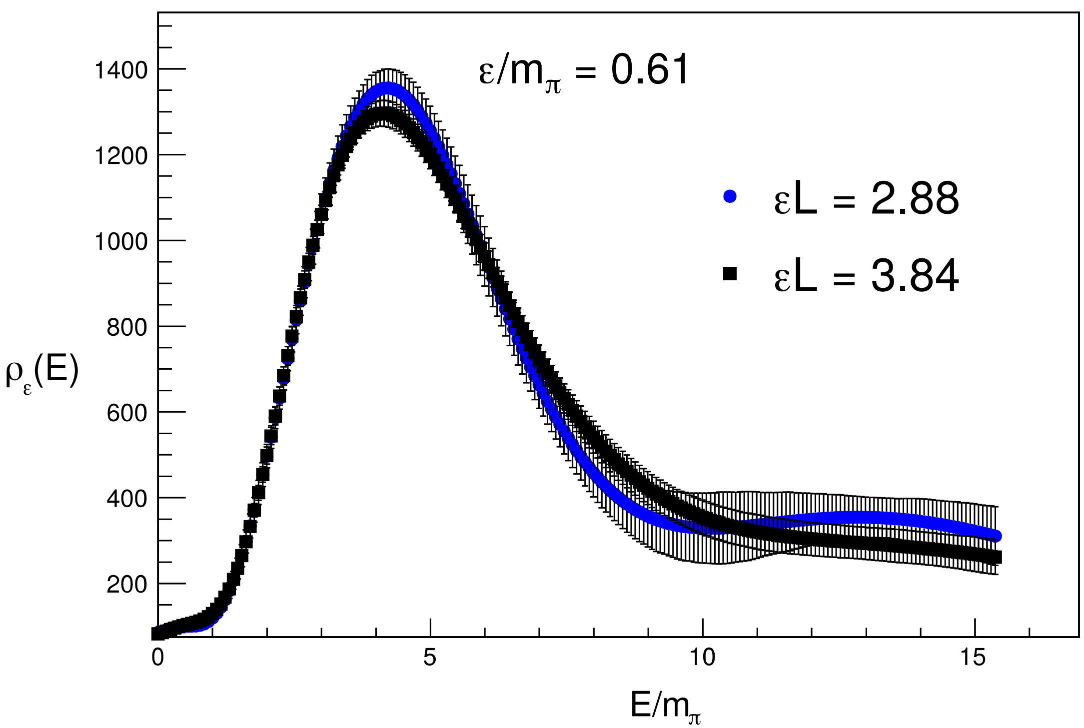
Crude example: Inclusive rates from hadron 'probes':

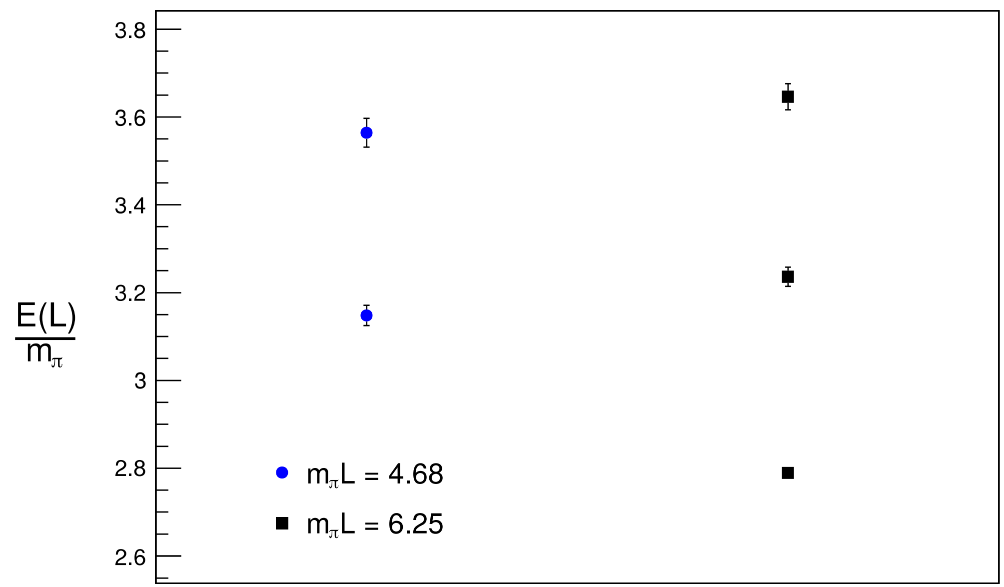
- Fictitious inclusive process: $\hat{\mathcal{O}}_{\rm had} \to \text{hadrons}$

$$\sigma_{\rm tot}^{\mathcal{O}_{\rm had}\to X}(E) = \lim_{\epsilon\to 0}\lim_{L\to\infty}\rho_\epsilon(E)$$

H. Meyer, M. T. Hansen, D. Robaina '17; D. Agadjanov, M. Doering, M. Mai, U. Meissner, A. Rusetsky '16

- First example: smeared rho-meson probe $\hat{\mathcal{O}}_{\rm had} = \bar{\tilde{u}}\gamma_i\tilde{d}(\boldsymbol{p}=0)$





$m_\pi = 220\text{MeV},\ a = 0.086\text{fm}$

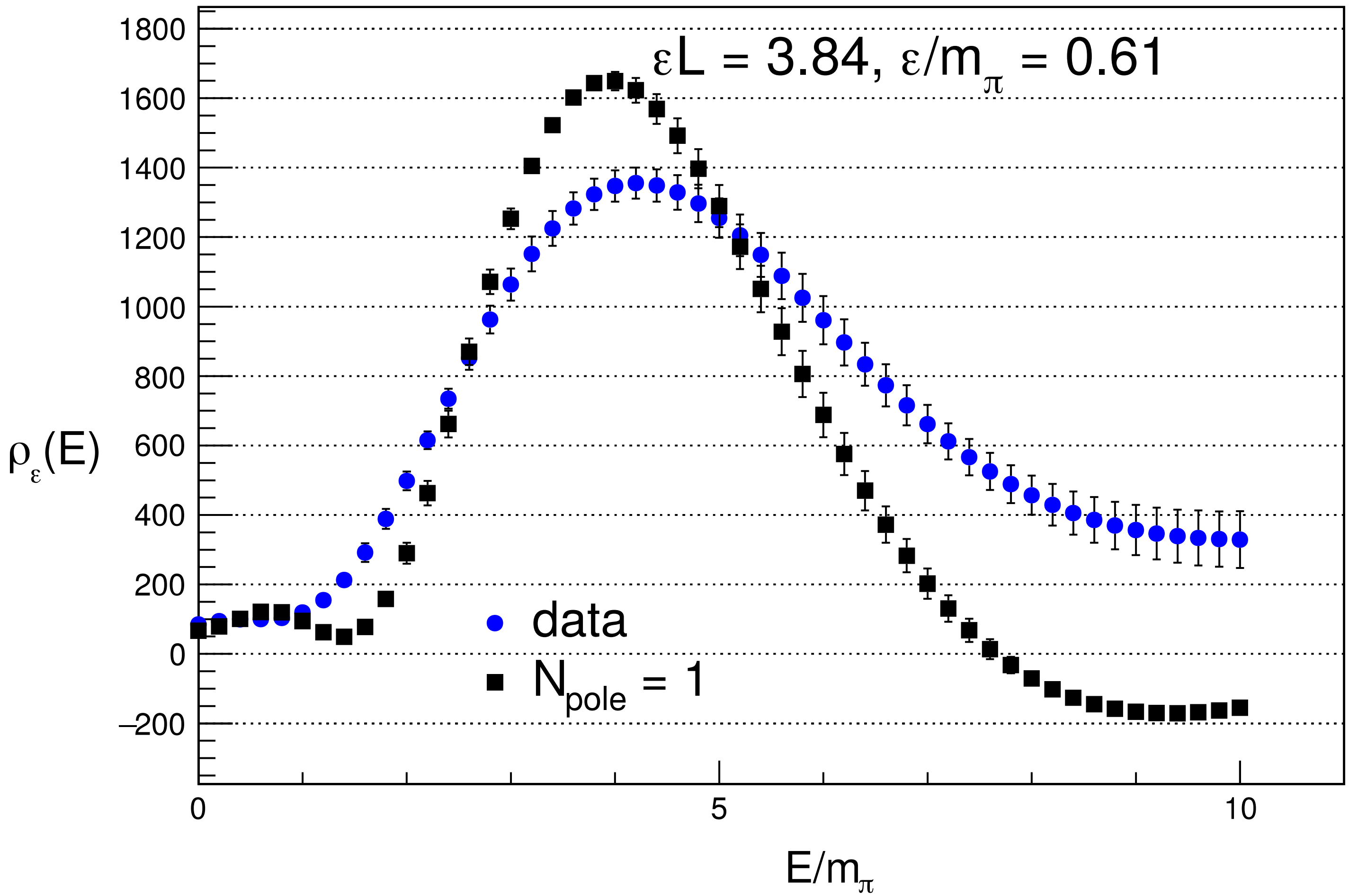
Data from: C. Andersen, JB, B. Hoerz, C. Morningstar '19

- WARNING: hadron probes not renormalized!

Inclusive rates from hadron probes: fitting the data

- Simplest (infinite-volume) Ansatz:

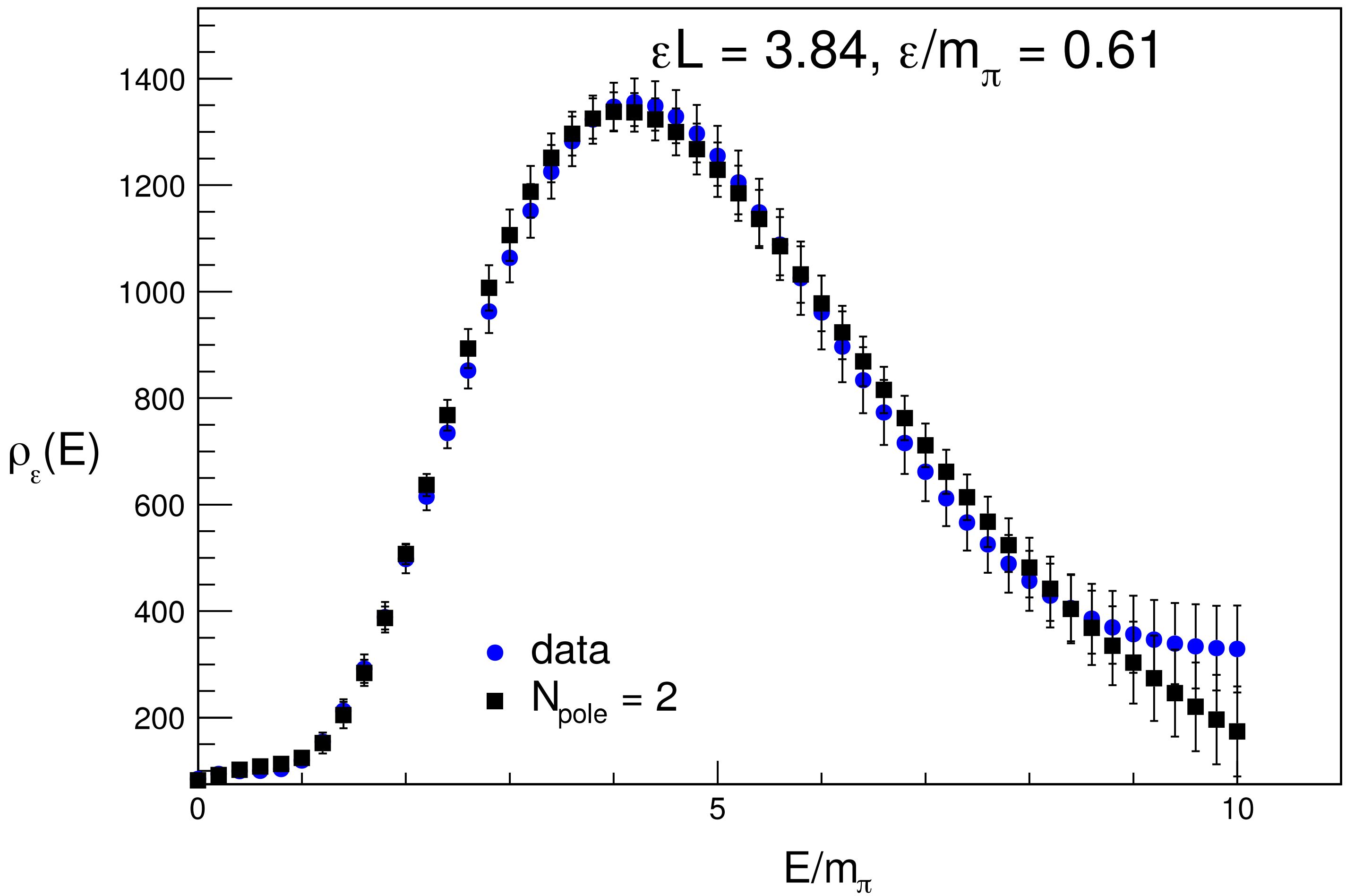
$$\rho(E) = \sum_{n=1}^{N_{\rm pole}} A_n\, \delta(E - m_n)$$



$$m_1 = 890(15)\text{MeV}, \quad \chi^2/\text{d.o.f.} = 42.8$$

Inclusive rates from hadron probes: fitting the data

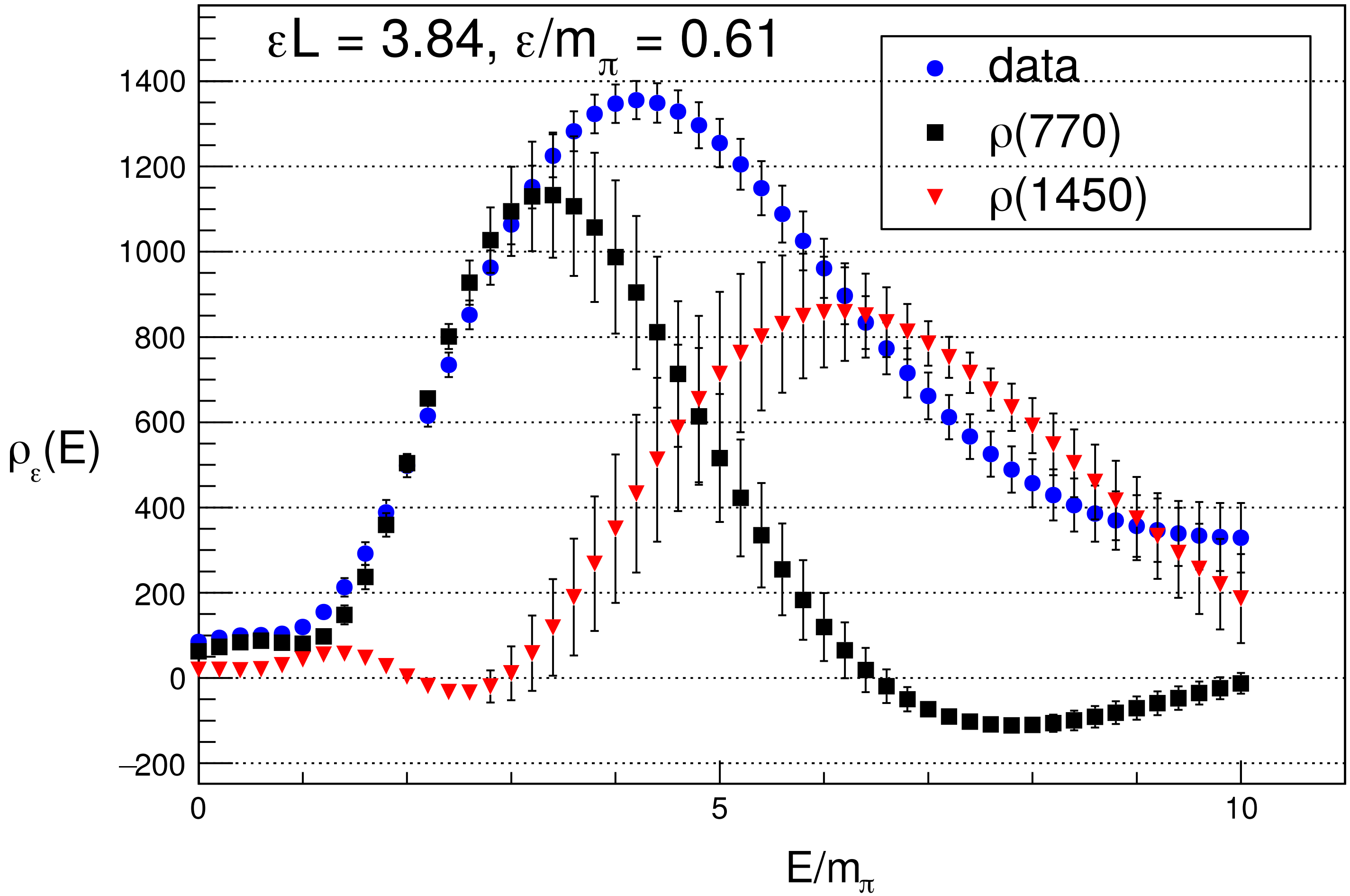
- Two poles provide a much better description



$$m_1 = 740(30)\mathrm{MeV}, \quad m_2 = 1394(109)\mathrm{MeV}, \quad \chi^2/\mathrm{d.o.f.} = 0.65$$

Inclusive rates from hadron probes: fitting the data

- Two poles provide a much better description



$$m_1 = 740(30)\text{MeV}, \quad m_2 = 1394(109)\text{MeV}, \quad \chi^2/\text{d.o.f.} = 0.65$$

- Position of rho(770) consistent with finite-volume analysis.
- Position of rho(1450) stable under variation of L and pion mass.

Inclusive rates from hadronic probes: summary

- Well-defined infinite-volume limit at finite ϵ.

- Small finite-volume effects if $\epsilon L \gtrsim 3 - 4$. Smaller ϵ possible with larger volume.

- Well-defined continuum limit if probes are renormalized. (NOT DONE YET)

- Ansatz for smeared total rate => (crude) identification of resonances.

- Very different from finite-volume spectroscopy:

  1) Rates are probe dependent, possibly interesting.

  2) Interpolators for all lower states not needed! => hidden charm

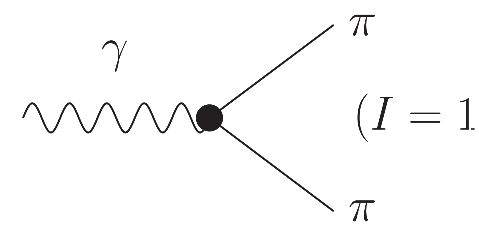
  3) Rethinking the 'Teufelspakt'

LSZ approach with spectral functions

JB, M. T. Hansen '19

$$J_{\rm ew} \to \pi(\boldsymbol{p}_1) + \pi(\boldsymbol{p}_2)$$

- Continuum, infinite-volume, real time.



- Endcap functions: single time ordering,

$$\widetilde{F}(q_2) = \int d^4x\, \mathrm{e}^{-iq_2\cdot x}\theta(x^0)\,\langle \pi(\boldsymbol{p_1})|\hat{\mathcal{O}}_\pi(x)\hat{J}_{\rm ew}(0)|0\rangle_{\rm c}$$

- LSZ reduction: on-shell pole at $q_2^0 = E_\pi(\boldsymbol{p}_2)$

$$\widetilde{F}(q_2) = \frac{Z_\pi^{1/2}(\boldsymbol{p}_2)}{2E_\pi(\boldsymbol{p}_2)}\,\mathcal{M}_{\rm c}(p_2 p_1)\,\frac{i}{q_2^0 - E_\pi(\boldsymbol{p}_2) + i\epsilon} + \,\ldots$$

LSZ approach with spectral functions

- Express as smeared spectral function:

$$\widetilde{F}(q_2) = \int_0^\infty \frac{dE}{\pi} \frac{i}{q_2^0 + E(\boldsymbol{p}_1) - E + i\epsilon} \rho_{\boldsymbol{p}_1}(E, \boldsymbol{p}_2)$$

- Spectral function independent of metric signature:

$$\rho_{\boldsymbol{p}_1}(E, \boldsymbol{p}_2) = \sum_n \pi\, \delta(E - E_n) \langle \pi(\boldsymbol{p}_1) | \hat{O}_\pi(\boldsymbol{p}_2) | n \rangle \langle n | \hat{J}_{\mathrm{w}}(0) | 0 \rangle_{\mathrm{c}}$$

LSZ approach with spectral functions

- Express as smeared spectral function:

$$\widetilde{F}(q_2) = \int_0^\infty \frac{dE}{\pi} \, \frac{i}{q_2^0 + E(\boldsymbol{p}_1) - E + i\epsilon} \, \rho_{\boldsymbol{p_1}}(E, \boldsymbol{p}_2)$$

- Particular complex smearing kernel required:

$$\delta_\epsilon(E-\omega) = \frac{i}{E-\omega+i\epsilon} = \frac{\epsilon}{(E-\omega)^2+\epsilon^2} + i\,\frac{E-\omega}{(E-\omega)^2+\epsilon^2}$$

LSZ approach with spectral functions

- Finite-volume Euclidean endcap function:

$$\langle \pi(\boldsymbol{p}_1) | \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau} \, \hat{J}_{\mathrm{ew}}(0) | 0 \rangle_{L,\mathrm{c}} = \frac{Z_\pi^{1/2}(\boldsymbol{p}_1)}{2E_\pi(\boldsymbol{p}_1)} \lim_{\tau_1 \to \infty} \frac{C_{3\mathrm{pt}}(\tau_1, \tau)}{C_{2\mathrm{pt}}^\pi(\boldsymbol{p}_1, \tau_1)}$$

- Finite-volume spectral function:

$$\langle \pi(\boldsymbol{p}_1) | \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau} \, \hat{J}_{\mathrm{ew}}(0) | 0 \rangle_{L,\mathrm{c}} = \int_0^\infty \frac{dE}{\pi} \, \mathrm{e}^{-E\tau} \, \rho^L_{\boldsymbol{p}_1}(E, \boldsymbol{p}_2)$$

LSZ approach with spectral functions

- Finite-volume smeared spectral function:

$$\tilde{\rho}^{L,\epsilon}_{\boldsymbol{p}_1}(E_\pi(\boldsymbol{p}_2),\boldsymbol{p}_2) = \int_0^\infty \frac{dE}{\pi} \frac{i}{E_\pi(\boldsymbol{p}_1)+E_\pi(\boldsymbol{p}_2)-E+i\epsilon} \times \rho^L_{\boldsymbol{p}_1}(E,\boldsymbol{p}_2)$$

- AMPLITUDE FROM ORDERED DOUBLE LIMIT:

$$\mathcal{M}_{\mathrm{c}}(p_2 p_1) = \frac{2E_\pi(\boldsymbol{p}_2)}{Z_\pi^{1/2}(\boldsymbol{p}_2)} \lim_{\epsilon\to 0} \lim_{L\to\infty} \epsilon \tilde{\rho}^{L,\epsilon}_{\boldsymbol{p}_1}(E_\pi(\boldsymbol{p}_2),\boldsymbol{p}_2)$$

Comparison with finite-volume approach

- Does not rely on finite volume 😁, requires large volume 😒
- Works for energies above arbitrary thresholds 😁
- Each particle isolated separately => unambiguous channels, partial waves 😁
- More particles straightforward 😁, need higher n-point functions 😒
- Difficult inverse problem to determine spectral functions 😒

Scattering test: O(3) model

- 1+1 Dimensional O(3) model:

Lüscher, Wolff, NPB 339 (1990)

$$S[\sigma] = -\beta \sum_{\langle i,j \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

- Updated with Metropolis + Microcanonical sweeps.
- I=2 elastic scattering amplitude known analytically:

$$\sigma(d_1) + \sigma(d_2) \to \sigma(d_3) + \sigma(d_4)$$

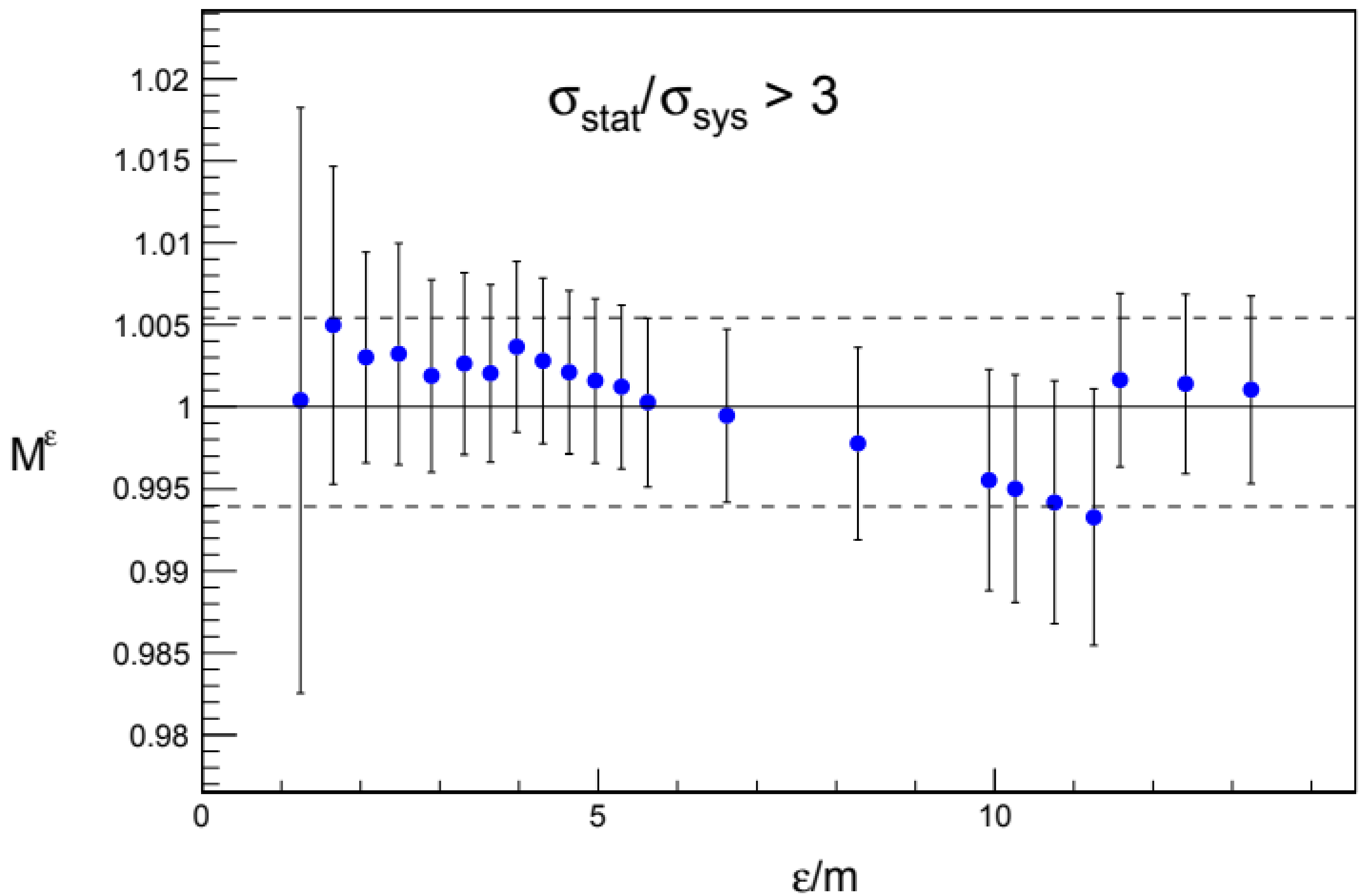
$$\mathrm{e}^{2i\delta_2(k)} = \frac{\theta - i\pi}{\theta + i\pi}, \quad k = m \sinh\frac{\theta}{2}$$

A. B. Zamolodchikov, A. B. Zamolodchikov, NPB 133(1978)

Two-point function

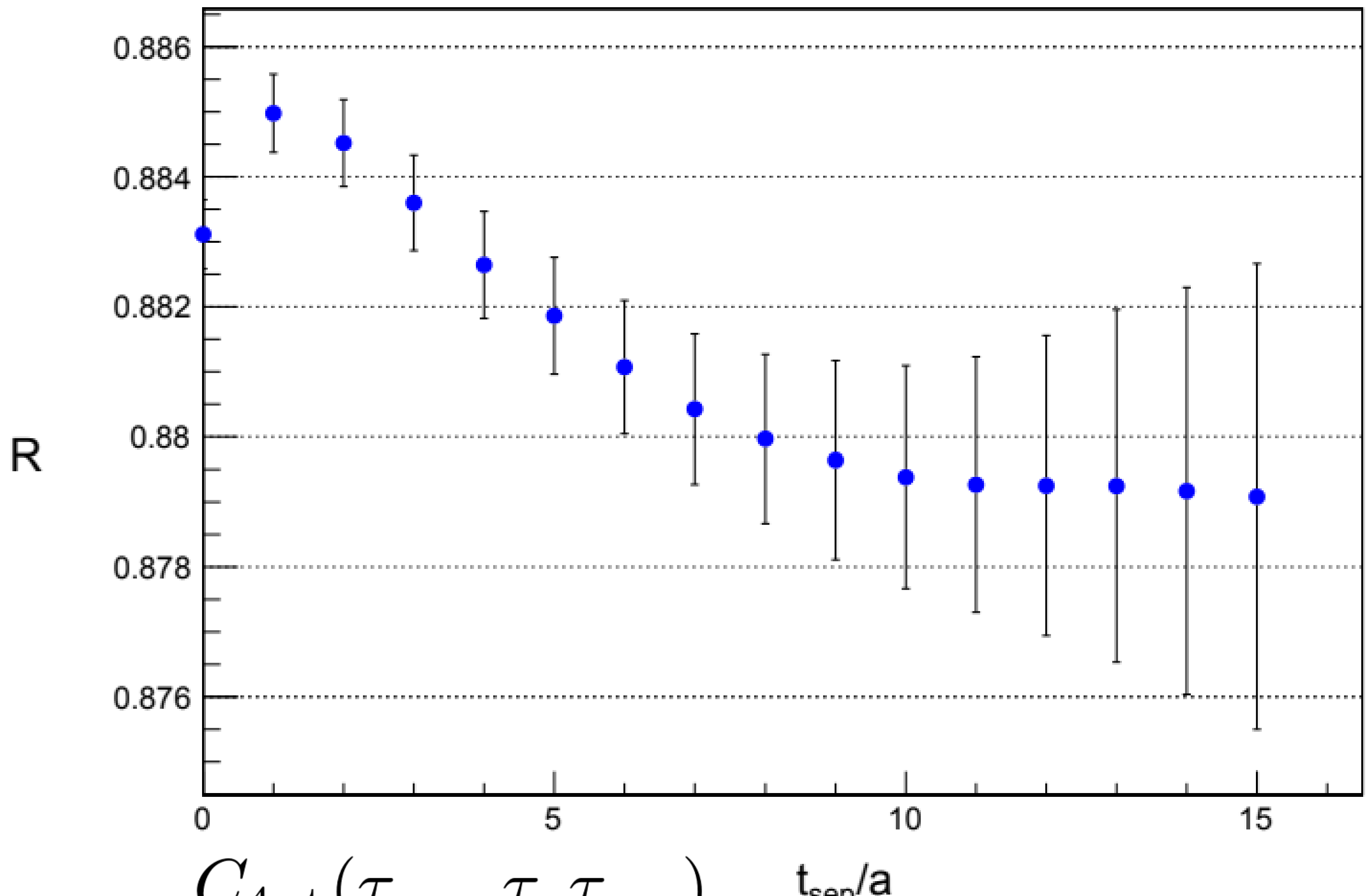
- Parameters: (no constraints)

$$160 \times 320,\ mL = 9.7,\ \beta = 1.5763,\ N_\tau = 50$$



Four-point function

- Test endcap saturation: $\tau_1 = \tau_2 = \tau_{\rm sep} = 0$ is a small correction



$$R(\tau_{\rm sep}) = \frac{C_{\rm 4pt}(\tau_{\rm sep}, \tau, \tau_{\rm sep})}{C^2_{\rm 2pt}(\tau_{\rm sep})}$$

$$d_1 = d_2 = d_3 = d_4 = 0$$

Four-point function: spectral reconstruction

Scattering amplitude: Preliminary!

- Consider total zero momentum. Three kinematics:

$$d_1 = -d_2 = d_3 = -d_4 = 0, 1, 2$$

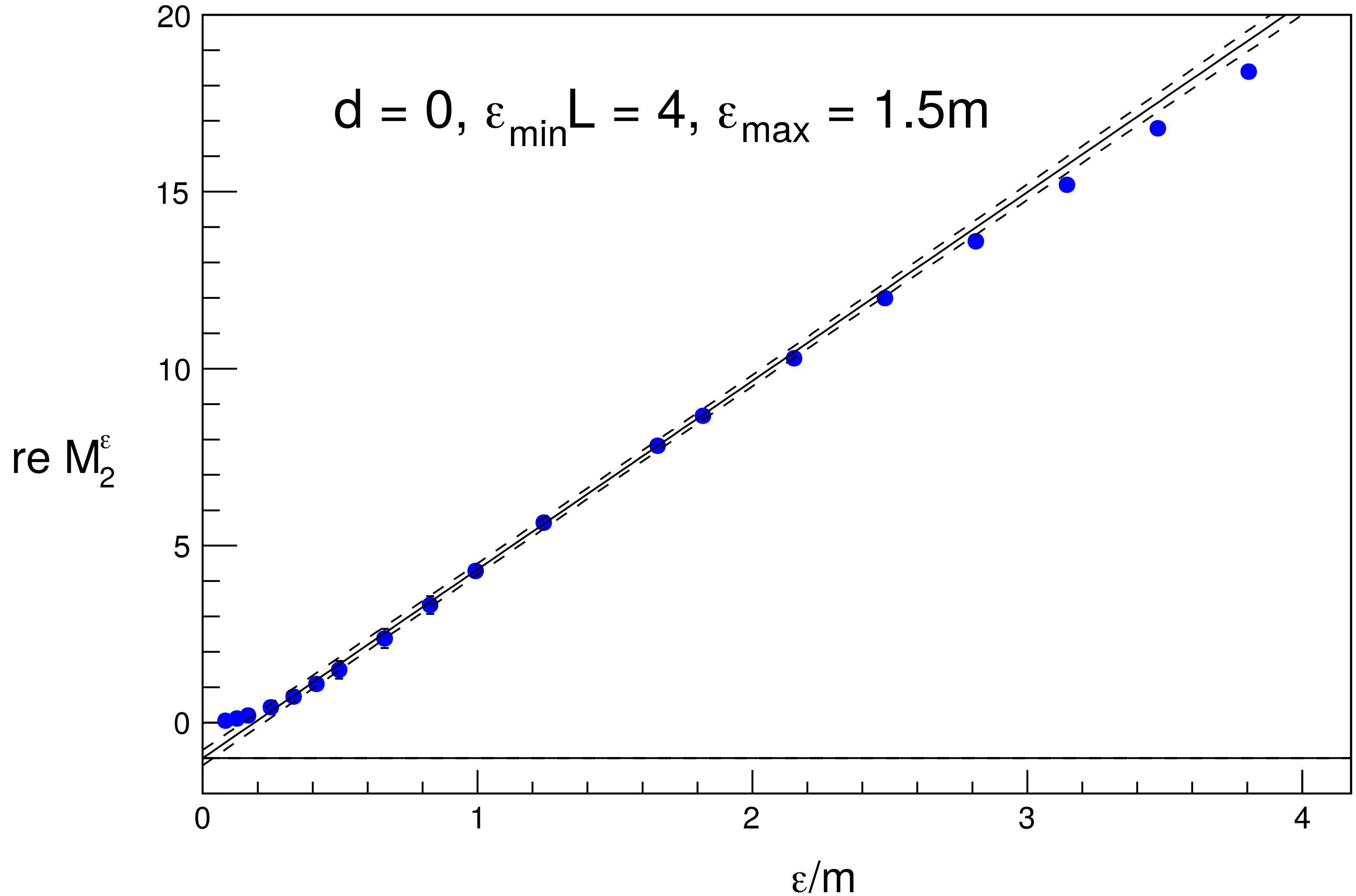
- On-shell point (in general) above ground state

- Consider the real part only:

$$\mathrm{Re}\, M(k) = \cos 2\delta_2(k) = \frac{\theta^2 - \pi^2}{\theta^2 + \pi^2}$$

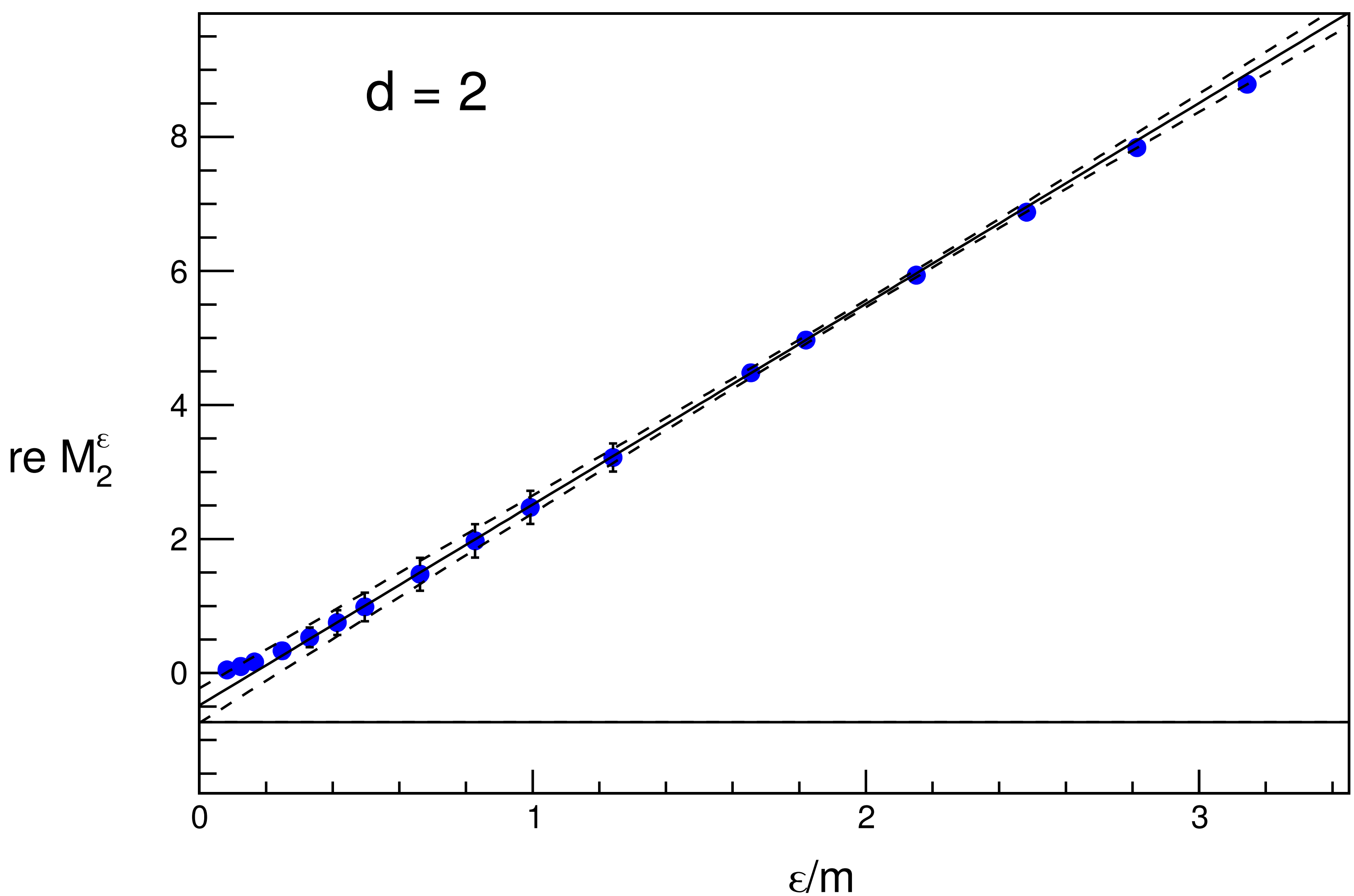
Four-point function: spectral reconstruction

Momentum $mL \approx 19,\ d_1 = -d_2 = d_3 = -d_4 = 0$



Four-point function: spectral reconstruction

Momentum $mL \approx 19, d_1 = -d_2 = d_3 = -d_4 = 2$



Four-point function: spectral reconstruction

Ignored systematic errors:

- Dependence on λ
- Outer separations: $\tau_{\rm sep}$
- Finite-volume effects, $L \to \infty$
- Extrapolation $\epsilon \to 0$

Possible improvements:

- Momentum smearing
- GEVP reconstruction
- Better estimate of sys. err.
- Much more statistics!

Conclusions

- Spectral reconstruction: controlled determination of real-time physics.
- Smearing is crucial: bridge between finite and infinite volume. $\epsilon L \gtrsim 4$ is enough.
- LSZ approach theoretically provides an alternative to the finite-volume formalism. Many advantages/difficulties
- Main question: Is this a viable alternative to finite-volume approach?
  - Larger volume simulations
  - Calculation of higher n-point functions
  - Theoretical advances: alternatives to LSZ?

LSZ approach for three hadrons

JB, M. T. Hansen '19

$$J_{\mathrm{ew}} \to \pi(\boldsymbol{p}_1) + \pi(\boldsymbol{p}_2) + \pi(\boldsymbol{p}_3)$$

- Endcap functions: single time ordering,

$$\widetilde{F}(q_2, q_3) = \int d^4x \, \mathrm{e}^{-i(q_2 \cdot x_2 + q_3 \cdot x_3)} \theta(x_2^0 - x_3^0) \, \theta(x_3^0) \times$$
$$\langle \pi(\boldsymbol{p_1}) | \hat{\mathcal{O}}_\pi(x_2) \hat{\mathcal{O}}_\pi(x_3) \hat{J}_{\mathrm{ew}}(0) | 0 \rangle_{\mathrm{c}}$$

- LSZ reduction: on-shell pole at $q_2^0 = E_\pi(\boldsymbol{p}_2), \; q_3^0 = E_\pi(\boldsymbol{p}_3)$

$$\widetilde{F}(q_2) = \frac{Z_\pi^{1/2}(\boldsymbol{p}_2)}{2E_\pi(\boldsymbol{p}_2)} \frac{Z_\pi^{1/2}(\boldsymbol{p}_3)}{2E_\pi(\boldsymbol{p}_3)} \mathcal{M}_{\mathrm{c}}(p_3 p_2 p_1) \times$$
$$\frac{i}{q_2^0 - E_\pi(\boldsymbol{p}_2) + i\epsilon} \frac{i}{q_3^0 - E_\pi(\boldsymbol{p}_3) + i\epsilon} + \ldots$$

LSZ approach for three hadrons

- Smeared spectral function:

$$\widetilde{F}(q_2, q_3) = \int_0^\infty \frac{dE_1}{\pi}\frac{dE_2}{\pi}\,\frac{i}{q_2^0 + E(\boldsymbol{p}_1) - E_1 + i\epsilon} \times \frac{i}{q_2^0 + q_3^0 + E(\boldsymbol{p}_1) - E_2 + i\epsilon}\,\rho_{\boldsymbol{p}_1}(E_1, E_2, \boldsymbol{p}_2, \boldsymbol{p}_3)$$

- Unsmeared spectral function:

$$\rho_{\boldsymbol{p}_1}(E_1, E_2, \boldsymbol{p}_2, \boldsymbol{p}_3) = \sum_{n_1, n_2} \pi\,\delta(E - E_{n_1})\,\pi\,\delta(E - E_{n_2}) \times \langle \pi(\boldsymbol{p}_1)|\hat{O}_\pi(\boldsymbol{p}_2)|n_2\rangle\,\langle n_2|\hat{O}_\pi(\boldsymbol{p}_3)|n_1\rangle\,\langle n_1|\hat{J}_{\mathrm{w}}(0)|0\rangle_{\mathrm{c}}$$

LSZ approach for three hadrons

- Finite-volume Euclidean endcap function:

$$\langle \pi(\boldsymbol{p}_1) | \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau_1} \, \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau_2} \, \hat{J}_{\mathrm{ew}}(0) | 0 \rangle_{L,\mathrm{c}} = \frac{Z_\pi^{1/2}(\boldsymbol{p}_1)}{2E_\pi(\boldsymbol{p}_1)} \lim_{\tau \to \infty} \frac{C_{4\mathrm{pt}}(\tau, \tau_1, \tau_2)}{C_{2\mathrm{pt}}^{\pi}(\boldsymbol{p}_1, \tau)}$$

- Finite-volume spectral function:

$$\langle \pi(\boldsymbol{p}_1) | \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau_1} \, \hat{O}_\pi(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}\tau_2} \, \hat{J}_{\mathrm{ew}}(0) | 0 \rangle_{L,\mathrm{c}} = \int_0^\infty \frac{dE_1}{\pi} \frac{dE_2}{\pi} \, \mathrm{e}^{-E_1\tau_1 - E_2\tau_2} \, \rho^L_{\boldsymbol{p}_1}(E_1, E_2, \boldsymbol{p}_2, \boldsymbol{p}_3)$$

LSZ approach with spectral functions

- Finite-volume smeared spectral function:

$$\widetilde{\rho}^{L,\epsilon}_{\boldsymbol{p}_1}(E_\pi(\boldsymbol{p}_2), E_\pi(\boldsymbol{p}_3), \boldsymbol{p_2}, \boldsymbol{p}_3) = \int_0^\infty \frac{dE_1}{\pi}\frac{dE_2}{\pi} \frac{i}{E(\boldsymbol{p}_2)+E(\boldsymbol{p}_1)-E_1+i\epsilon} \times \frac{i}{E(\boldsymbol{p}_2)+E(\boldsymbol{p}_3)+E(\boldsymbol{p}_1)-E_2+i\epsilon}\,\rho_{\boldsymbol{p}_1}(E_1, E_2, \boldsymbol{p}_2, \boldsymbol{p}_3)$$

- AMPLITUDE FROM ORDERED DOUBLE LIMIT:

$$\mathcal{M}_{\rm c}(p_3 p_2 p_1) = \frac{2E_\pi(\boldsymbol{p}_2)}{Z_\pi^{1/2}(\boldsymbol{p}_2)}\frac{2E_\pi(\boldsymbol{p}_3)}{Z_\pi^{1/2}(\boldsymbol{p}_3)} \times \lim_{\epsilon\to 0}\lim_{L\to\infty} \epsilon^2 \tilde{\rho}^{L,\epsilon}_{\boldsymbol{p}_1}(E_\pi(\boldsymbol{p}_2), E_\pi(\boldsymbol{p}_3), \boldsymbol{p}_2, \boldsymbol{p}_3)$$