Symmetries and twisted cohomology

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Section 1

Introduction

Motivation

- We are interested in a countable set of integrals ($\nu_j \in \mathbb{Z}$)

$$I_{\nu_1 \dots \nu_N}$$

- For Feynman integral reductions, computer programs like `Reduze`, `Fire`, `Kira` are widely used.
- These programs take into account two concepts:
  - **Integration-by-parts identities**
  - **Symmetries**
- The mathematical setting for integration-by-parts is twisted cohomology.
- We want to study how twisted cohomology interacts with symmetries.

View integrals $I = \langle \alpha | \mathcal{C} \rangle$ as pairings between differential forms α and chains $\mathcal{C}$.
Study equivalence classes modulo Stokes' formula and changes of variables, which do not change the class of integrals.

Motivation

We carefully distinguish between **master integrands** and **master integrals**:

- If we take only integration-by-parts into account (i.e. if we look at twisted cohomology) the dimension of the relevant vector space is given by the number of master integrands.
- If we take integration-by-parts and symmetries into account (i.e. by running `Reduze`, `Fire`, `Kira`) the dimension of the relevant vector space is the number of master integrals.

In general:

$$N_{\text{master integrals}} \quad \leq \quad N_{\text{master integrands}}$$

Example

A simple example is given by the one-loop two-point function with equal masses:

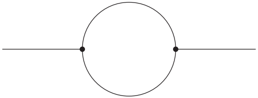
- There are **3 master integrands**, which can be taken as the integrands of

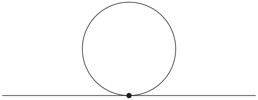
$$I_{11}, I_{10}, I_{01}$$

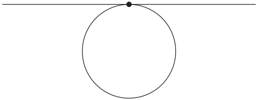
- There are **2 master integrals**, for example

$$I_{11}, I_{10}$$

as we have the symmetry $I_{\nu_1 \nu_2} = I_{\nu_2 \nu_1}$.







Example

An even simpler example: Let

$$\begin{aligned} \omega_1 &= z_1 \, dz_1 \wedge dz_2 \\ \omega_2 &= z_2 \, dz_1 \wedge dz_2 \end{aligned}$$

- At the level of **integrands**

$$\omega_1 \neq \omega_2$$

- At the level of **integrals**

$$\int\limits_{[0,1]^2} \omega_1 = \int\limits_{[0,1]^2} \omega_2$$

Integrals

Examples of integrals we are interested in ($\nu_j \in \mathbb{Z}$ or $\nu_j \in \mathbb{N}_0$):

- Feynman integrals

$$I_{\nu_1 \nu_2 \dots \nu_N} = \int \frac{d^D k_1}{(2\pi)^D} \dots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{N} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}}$$

- Lattice integrals

$$I_{\nu_1 \nu_2 \dots \nu_N} = \int d^N \phi \left(\prod_{k=1}^{N} \phi_{x_k}^{\nu_k} \right) \exp\left(-S\right)$$

Section 2

Quantum field theory on a lattice

Lattice field theory

Our main interest are scattering amplitudes and correlation functions in (continuum) quantum field theory.

- At **small coupling** we may use perturbation theory and compute Feynman integrals.
- At **finite** (non-small) **coupling** lattice field theory is the method of choice:
  - Space-time is discretised by a lattice.
  - Correlation functions are (usually) computed numerically with Monte Carlo methods.
    To avoid oscillatory integrands one works with Euclidean signature.
  - We may also consider lattice correlation functions analytically.
    S.W. '20, Gasparotto, Rapakoulias, S.W., '22

Lattice field theory

- Consider a lattice Λ with **lattice spacing a** in $D \in \mathbb{N}$ space-time dimensions.
  For simplicity we assume that the lattice consists of **L points in any direction**, hence the lattice has $N = L^D$ points.
- We assume **periodic boundary conditions**.
- Notation:
  - We label the lattice points by $x_1, \dots, x_N$.
  - We denote by b_j the unit vector in the j-th space-time direction.
  - We denote the field at a lattice point x by ϕ_x and the field at the next lattice point in the (positive) j-direction modulo L by ϕ_{x+ab_j}.

Lattice field theory

As an example consider ϕ^4-theory.

- The continuum Lagrange density reads

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4$$

- On a lattice with **Euclidean** metric the action reads

$$S_E = \sum_{x \in \Lambda} \left[- \sum_{j=0}^{D-1} \phi_x \phi_{x+a b_j} + \left(D + \frac{m^2}{2} \right) \phi_x^2 + \frac{1}{4!} \lambda \phi_x^4 \right]$$

- On a lattice with **Minkowskian** metric the action reads

$$S_M = i \sum_{x \in \Lambda} \left[\phi_x \phi_{x+a b_0} - \sum_{j=1}^{D-1} \phi_x \phi_{x+a b_j} + \left(D + \frac{m^2}{2} - 2 \right) \phi_x^2 + \frac{1}{4!} \lambda \phi_x^4 \right]$$

Lattice field theory

We are interested in the **lattice integrals**

$$I_{\nu_1 \nu_2 \dots \nu_N} = \int\limits_{\mathbb{R}^N} d^N\phi \left(\prod_{k=1}^{N} \phi_{x_k}^{\nu_k} \right) \exp\left(-S\right)$$

The correlation functions are then given by

$$G_{\nu_1 \nu_2 \dots \nu_N} = \frac{I_{\nu_1 \nu_2 \dots \nu_N}}{I_{00\dots0}}$$

Section 3

Twisted cohomology

Stokes' theorem

Let M be a N-dimensional complex manifold and $\mathcal{C}$ an N-chain.

The starting point for integration-by-parts identities is Stokes' theorem:

$$\int\limits_{\mathcal{C}} d\eta \;=\; \int\limits_{\partial\mathcal{C}} \eta$$

If η vanishes on $\partial\mathcal{C}$:

$$\int\limits_{\mathcal{C}} d\eta \;=\; 0$$

and hence

$$\int\limits_{\mathcal{C}} (\alpha + d\eta) \;=\; \int\limits_{\mathcal{C}} \alpha$$

Cohomology

We are interested in the case, where α is a $(N,0)$-form, depending only on the holomorphic coordinates $z_1, \dots, z_N$, but not on the anti-holomorphic coordinates $\bar{z}_1, \dots, \bar{z}_N$.

In this case

$$d\alpha = 0$$

and since we may always add an exact form, **α represents a cohomology class $[\alpha]$**.

Twisted cohomology

- In mathematics, the case where α is a rational N-form is well studied.

  Example: A **numerical period** is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.
  (Kontsevich, Zagier)

- In physics, we often are in the situation where

$$\alpha = \underbrace{u}_{\text{possibly multivalued function}} \cdot \underbrace{\Phi}_{\text{rational } \mathbf{N}-\text{form}}$$

Examples

Example (Feynman integrals in the Lee-Pomeransky representation)

$$u = \mathcal{G}^{-\frac{D}{2}}$$

$$\Phi = \left(\prod_{k=1}^{N} z_k^{\nu_k - 1} \right) d^N z$$

Example (Lattice integrals)

$$u = \exp\left(-S\right)$$

$$\Phi = \left(\prod_{k=1}^{N} \phi_{x_k}^{\nu_k} \right) d^N \phi$$

Twisted cohomology

Question:

How do integration-by-parts identities look in terms of Φ instead of α?

- Set

$$\omega = d \ln u \qquad \nabla_\omega = d + \omega$$

- If $u\Xi$ vanishes on $\partial\mathcal{C}$

$$\int_{\mathcal{C}} u \left(\Phi + \nabla_\omega \Xi \right) = \int_{\mathcal{C}} u \, \Phi$$

- The function u is called the **twist** and determines the **connection** ω and the **covariant derivative** ∇_ω.

Twisted cohomology

Two N-forms Φ' and Φ are called **equivalent**, if

$$\Phi' \sim \Phi \quad \Leftrightarrow \quad \Phi' = \Phi + \nabla_\omega \Xi$$

Denote equivalence classes by $\langle \Phi |$. Each Φ is trivially closed:

$$\nabla_\omega \Phi = 0$$

The equivalence classes define the **twisted cohomology group** H_ω^N:

$$\langle \Phi | \in H_\omega^N = \frac{\nabla_\omega\text{−closed } N\text{−forms}}{\nabla_\omega\text{−exact } N\text{−forms}}$$

Dual twisted cohomology

We may also consider the dual twisted cohomology group. Equivalence classes are now defined by

$$\left|\Phi'\right\rangle = \left|\Phi\right\rangle \quad \Leftrightarrow \quad \Phi' \; = \; \Phi + \nabla_{-\omega}\Xi$$

Equivalence classes $\left|\Phi\right\rangle$ are elements of the dual twisted cohomology group

$$\left(H^N_\omega\right)^* \quad = \quad H^N_{-\omega}.$$

Key properties

With (mild) assumptions:

1. The cohomology groups H^N_ω and $(H^N_\omega)^*$ are **finite-dimensional**.
2. There is a non-degenerate inner product between H^N_ω and $(H^N_\omega)^*$, called the **intersection number** and denoted by

$$\langle \Phi_L | \Phi_R \rangle, \qquad \langle \Phi_L | \in H^n_\omega, \qquad | \Phi_R \rangle \in (H^n_\omega)^*.$$

Aomoto '75, Matsumoto '94, Cho, Matsumoto, '95, Aomoto, Kita, '94 (jap.), '11 (engl.), Yoshida '97

Application 1: Finiteness

A priori we have countable many integrals ($\nu_j \in \mathbb{Z}$ or $\nu_j \in \mathbb{N}_0$)

$$I_{\nu_1 \dots \nu_N}.$$

As H^N_ω is finite-dimensional we may express any integral as a **linear combination of a finite set** of them.

Aomoto, Kita, '94 (jap.), '11 (engl.), Smirnov and Petukhov, '10

Application 2: Reduction

Let $\langle e_1|, \langle e_2|, \ldots$ be a basis of H^N_ω and $|d_1\rangle, |d_2\rangle, \ldots$ the dual basis of $(H^N_\omega)^*$ such that

$$\langle e_i | d_j \rangle = \delta_{ij}.$$

Consider an arbitrary element $\langle \Phi| \in H^N_\omega$. We may **express $\langle \Phi|$ in terms of the basis**:

$$\langle \Phi| = c_1 \langle e_1| + c_2 \langle e_2| + \ldots$$

The **coefficients** are **given by the intersection numbers**

$$c_j = \langle \Phi | d_j \rangle$$

This is exactly what we need for integral reduction!

Mastrolia, Mizera, '18

Application 3: Differential equations

Set $N_F = \dim H_\omega^N$ and let $\vec{I} = (I_1, I_2, \dots, I_{N_F})^T$ a vector of integrals such that the **integrands form a basis of H_ω^N**.

Let x be a parameter these integrals depend on.

We then get the **differential equation**

$$\frac{d}{dx}\vec{I} = A\vec{I}$$

with an $(N_F \times N_F)$-matrix A.

Section 4

Symmetries

Motivation

Let's consider the lattice integrals for ϕ^4-theory. We take $L = 2$ lattice points in each direction.

What is the size of the system of differential equations as a function of the space-time dimension?

D	1	2	3	4
N_F	9	81	6561	43046721

The size of the system grows exponentially with the number of lattice points:

$$N_F = 3^{L^D}$$

Symmetries

Assume for simplicity $\mathcal{C} = \mathbb{R}^N$. Let $\vec{z} = (z_1, \dots, z_N)^T$ and $g \in \mathrm{SL}_N(\mathbb{R})$. Set

$$\vec{z}\,' = g \cdot \vec{z}$$

Note that due to the requirement $g \in \mathrm{SL}_N(\mathbb{R})$ the Jacobian of the transformation is equal to one and therefore

$$d^N z' = d^N z$$

Let $G \subseteq \mathrm{SL}_N(\mathbb{R})$ be the subgroup, which leaves the **twist invariant**:

$$u(g \cdot \vec{z}) = u(\vec{z}) \quad \forall g \in G$$

Symmetries

We write

$$I = \int_{\mathcal{C}} u \, \Phi = \langle \Phi | \mathcal{C} \rangle \quad \text{and} \quad \Phi'(\vec{z}) = g \cdot \Phi(\vec{z}) = \Phi\left(g^{-1} \cdot \vec{z}\right).$$

Let $g \in G$ and let $\langle \Phi |$ be an integrand on which g acts non-trivially:

$$\left\langle \Phi' \right| \neq \left\langle \Phi \right|$$

We then have

$$\left\langle \Phi' | \mathcal{C} \right\rangle = \left\langle \Phi | \mathcal{C} \right\rangle$$

We call this a **symmetry relation**.

Proof

Change of variables and invariance of the twist function:

$$\begin{aligned}
\left\langle \Phi' \middle| \mathcal{C} \right\rangle &= \int\limits_{\mathcal{C}} \Phi'\left(\vec{z}\,'\right) u\left(\vec{z}\,'\right) = \int\limits_{\mathcal{C}} \Phi\left(g^{-1}\cdot\vec{z}\,'\right) u\left(\vec{z}\,'\right) \\
&\overset{\text{change of variables}}{=} \int\limits_{\mathcal{C}} \Phi\left(\vec{z}\right) u\left(g\cdot\vec{z}\right) \\
&\overset{\text{invariance}}{=} \int\limits_{\mathcal{C}} \Phi\left(\vec{z}\right) u\left(\vec{z}\right) \\
&= \left\langle \Phi \middle| \mathcal{C} \right\rangle
\end{aligned}$$

Symmetries

Example

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\Phi = z_1^2 d^2 z, \quad \Phi' = z_2^2 d^2 z, \quad u = e^{-z_1^4 - z_2^4}$$

Then

$$\left\langle \Phi' \right| \neq \left\langle \Phi \right| \quad \text{but} \quad \int\limits_{\mathbb{R}^2} u \, \Phi' = \int\limits_{\mathbb{R}^2} u \, \Phi$$

Symmetries

Symmetry relations allow us to reduce the number of elements in the spanning set of integrals.

- Suppose that $\langle \Phi | \mathcal{C} \rangle$ and $\langle \Phi' | \mathcal{C} \rangle$ are in the spanning set. Then it is sufficient to **just keep one element**, say

$$\frac{1}{2} \left\langle \left(\Phi + \Phi' \right) | \mathcal{C} \right\rangle,$$

  as $(\Phi - \Phi')$ integrates to zero.

- Suppose $\langle \Phi | \mathcal{C} \rangle$ is in the spanning set and $\Phi' = c\Phi$ with $c \neq 1$. Then Φ integrates to zero and we may **eliminate this integral** from the spanning set.

Theorem

Let $e_1, e_2, \dots, e_{N_F}$ be a basis of H^N_ω and consider the orbits

$$o_j = \frac{1}{|G|} \sum_{g \in G} g \cdot e_j$$

Suppose that there are N_O non-zero orbits and suppose we label $e_1, e_2, \dots, e_{N_O}, \dots, e_{N_F}$ sucht that the first N_O elements are in distinct non-zero orbits. Then we may replace the spanning set of integrals by the smaller set

$$\langle o_1 | \mathcal{C} \rangle, \dots, \langle o_{N_O} | \mathcal{C} \rangle .$$

Remark: This also works for a subgroup $G' \subset G$.

Symmetries of lattice integrals

The lattice integrals of ϕ^4-theory with L lattice points in each direction have the symmetries:

- A **global $\mathbb{Z}_2$-symmetry** $\phi'_x = -\phi_x$.
- The **remnants of Poincaré symmetry** on a discrete lattice:
  - Translations by one lattice spacing
  - Spatial rotations by 90° in the ij-pane
  - Boosts (rotations by 90° in the $0j$-plane for the Euclidean action, for the Minkowskian action with some additional sign flips)
  - Time reversal and spatial reversal

The order of the symmetry group as a function of the space-time dimension for $L = 2$:

D	1	2	3	4
$\lvert G' \rvert$	4	16	96	768

Symmetries

We may now compare the dimension $N_F = \dim H_\omega^N$ of the twisted cohomology group to the number of non-zero orbits N_O for a lattice with $L = 2$ points in each direction:

D	1	2	3	4
N_F	9	81	6561	43046721
N_O	4	13	147	66524

This **reduces the size of the system of differential equations** from $(N_F \times N_F)$ to $(N_O \times N_O)$.

Remark 1

Premature optimization is the root of all evil.

D. Knuth

The connection in the differential equation

$$d\vec{I} = A\vec{I}$$

is flat:

$$dA = A \wedge A.$$

Keeping the size system at size $(N_F \times N_F)$, but using some of the symmetries may destroy the integrability condition.

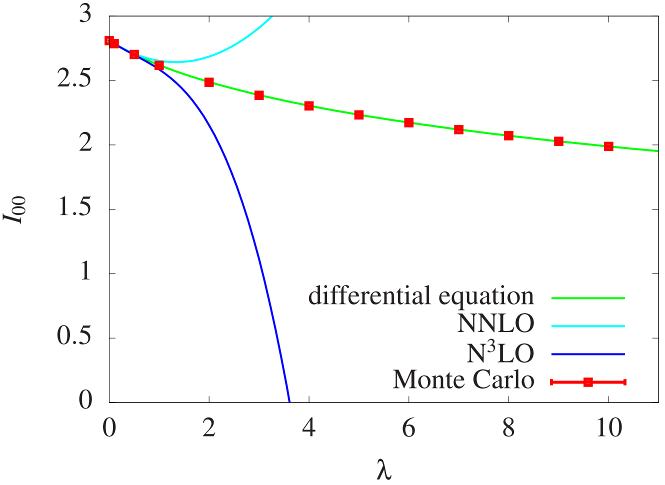
Remark 2

- Very often we are actually not interested in specific integrals, but in equivalence classes of integrals modulo relations induced by **linearity** in α and $\mathcal{C}$, **changes of variables** and **Stokes' formula**.

  Example: Is the period map from effective periods to numerical periods injective?

- For the integrals of the type "twist times rational N-form":
  - Twisted cohomology takes care of **Stokes' formula**.
  - Symmetries correspond to a **change of variables**.

Euclidean vacuum-to-vacuum integral I_{00} for $D = 1$ space-time dimensions and $L = 2$ lattice points as a function of the coupling λ:



Conclusions

- Integrands in physics are often of the type “twist times rational N-form”.
- Integration-by-parts reduces integrands to linear combinations of a basis of H^N_ω.
- The method of differential equations allows us to compute these integrals.
- Symmetries of the twist function allow us to reduces the size of the system from $\dim H^N_\omega$ to the number of non-zero orbits.