Motivic Galois theory for Feynman integrals via twisted cohomology

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Bethe Forum
“Geometries and Special Functions for Physics and Mathematics”
Bethe Center for Theoretical Physics
Bonn, 20-24 March 2023

Theorem (Brown–D.–Fresán–Tapušković)

The space of Laurent expansions of Feynman integrals in dimensional regularization is closed under the action of the motivic Galois group / closed under the motivic coaction.

- "Cosmic Galois theory" (Cartier).
- Conjectured and checked by Abreu–Britto–Duhr–Gardi–Matthew.
- Is an application of a general theorem for *algebraic Mellin transforms*.
- Main tool: a new view on *twisted cohomology*.

Algebraic Mellin transforms

The classical Mellin transform (Mellin, 1897)

$$\varphi : (0,\infty) \to \mathbb{C} \qquad \rightsquigarrow \qquad (\mathcal{M}\varphi)(s) = \int_0^\infty x^s \varphi(x) \frac{dx}{x}.$$

Algebraic Mellin transforms (Aomoto, 1974)

$$I(s) = \int_\sigma f^s \omega.$$

- X an (affine, smooth) algebraic variety over a field $k \subset \mathbb{C}$.
- $f : X \to \mathbb{G}_m$ an invertible function on X.
- ω an algebraic differential form on X, σ a topological cycle on X.

More generally, for $f = (f_1, \ldots, f_N) : X \to \mathbb{G}_m^N$, consider multivariate versions:

$$I(s_1, \ldots, s_N) = \int_\sigma f_1^{s_1} \cdots f_N^{s_N} \omega.$$

Examples of algebraic Mellin transforms

- Any function $z^s \times$ (period).
- The beta function

$$\mathrm{B}(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = \int_0^1 x^s (1-x)^t \frac{dx}{x(1-x)} .$$

- String theory amplitudes in genus zero

$$\int_{0=t_0<t_1<\cdots<t_n<t_{n+1}=1} \prod_{i<j} (t_j - t_i)^{s_{i,j}} \, \omega .$$

- The classical hypergeometric function

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \text{where } (t)_n = t(t+1)\cdots(t+n-1).$$

$$\mathrm{B}(b, c-b)\, {}_2F_1(a,b;c;z) = \int_0^1 x^b (1-x)^{c-b} (1-zx)^{-a} \frac{dx}{x(1-x)} .$$

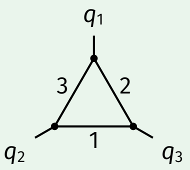
- Dimensional regularization: work in space-time dimension $D = D_0 - 2\varepsilon$.

$$I_\Gamma(\varepsilon) = \int_{\mathbb{P}^{n-1}(\mathbb{R}_+)} \frac{\Psi_\Gamma^{n-(h+1)D/2}}{\Xi_\Gamma^{n-hD/2}}\,\Omega = \int_{\mathbb{P}^{n-1}(\mathbb{R}_+)} \left(\frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma^{h}}\right)^{\varepsilon} \omega_\Gamma.$$

- It is an algebraic Mellin transform for

$$X = \mathbb{P}^{n-1} \setminus \{\Psi_\Gamma \Xi_\Gamma = 0\} \quad \text{and} \quad f = \frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma^{h}} : X \longrightarrow \mathbb{G}_m.$$

Example: the massless triangle graph ($D_0 = 4$)



$$I_\Gamma(\varepsilon) = \iint_{(0,\infty)^2} \left(\frac{(x+y+1)^2}{q_1^2 x + q_2^2 y + q_3^2 xy}\right)^{\varepsilon} \frac{dxdy}{(x+y+1)(q_1^2 x + q_2^2 y + q_3^2 xy)}$$

Structure of algebraic Mellin transforms

(Not in this talk) Systems of finite difference equations

$$I_i(s+1) = \sum_{j=1}^{N} f_{i,j}(s)\, I_j(s) \quad \text{with } f_{i,j}(s) \in k(s).$$

- Example: $\mathrm{B}(s+1,t) = \frac{s}{s+t}\,\mathrm{B}(s,t)\ ,\ \mathrm{B}(s,t+1) = \frac{t}{s+t}\,\mathrm{B}(s,t).$

(Not in this talk) Systems of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z} I_i(s;z) = \sum_{j=1}^{N} g_{i,j}(s;z)\, I_j(s;z) \quad \text{with } g_{i,j}(s;z) \in k(s,z).$$

- Example: differential equation for $F(z) = {}_2F_1(a,b,c;z)$

$$z(1-z)\,F''(z) + (c-(a+b+1)z)\,F'(z) - ab\,F(z) = 0.$$

Algebraic structure

They are both controlled by *twisted cohomology groups*.

(Not in this talk) Values at $s \in \mathbb{Q}$

For $s \in \mathbb{Q}$, $I(s)$ is a period of a cyclic cover of X.

- Example: $\mathrm{B}(\frac{k}{n}, \frac{l}{n})$ is a period of an open Fermat curve $\{x^n + y^n = 1\}$.

(In this talk) Laurent expansion at $s = 0$

$$I(s) = \sum_{n \gg -\infty} \alpha_n s^n \quad \text{where the } \alpha_n \text{ are periods.}$$

- Example: $$\mathrm{B}(s,t) = \frac{s+t}{st}\left(1 - \sum_{m,n \geqslant 1} (-s)^m (-t)^n \zeta(\underbrace{1, \ldots, 1}_{n-1}, m+1)\right).$$

What this talk is about...

- We are interested in the *motivic Galois theory / coaction* of the α_n.
- It is also controlled by a *twisted cohomology group*!

Slogan

Galois theory of algebraic numbers *should* extend to a Galois theory for periods, where the Galois groups are *algebraic groups* over $\mathbb{Q}$.

- Periods arise as coefficients of the perfect pairing

$$\int : \mathrm{H}_n^{\mathrm{B}}(X) \times \mathrm{H}_{\mathrm{dR}}^n(X) \longrightarrow \mathbb{C} \quad , \quad (\sigma, \omega) \mapsto \int_\sigma \omega$$

for algebraic varieties X, or pairs (X, Y), defined over $\mathbb{Q}$.

- Assuming Grothendieck's *period conjecture*, the *motivic Galois group* G acts on the algebra of periods:

$$\text{for } g \in G, \quad g \, . \int_\sigma \omega := \int_\sigma g.\omega$$

- Unconditional: Galois theory for *motivic periods*.
- Computable: (motivic) *coaction*

$$\rho\,(\text{period}) = \sum (\text{period}) \otimes (\text{function on } G).$$

- The "symbol" of a hyperlogarithm is a byproduct of the coaction.

The key example: the beta function

$$\mathrm{B}(s,t) = \frac{s+t}{st} \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n)\,(s^n + t^n - (s+t)^n)\right).$$

- Galois theory for zeta values: for $g \in G$,

$$g\,.\,\zeta(n) = \zeta(n) + a_g^{(n)} \quad \text{with} \quad a_g^{(n)} \in \mathbb{Q}.$$

  Or equivalently, for the motivic coaction:

$$\rho(\zeta(n)) = \zeta(n) \otimes 1 + 1 \otimes a^{(n)}.$$

- Gives rise to a Galois theory for the beta function:

$$g\,.\,\mathrm{B}(s,t) = A_g(s,t)\,\mathrm{B}(s,t) \quad \text{with} \quad A_g(s,t) \in \mathbb{Q}((s,t))^{\times}.$$

  Or equivalently, for the motivic coaction:

$$\rho(\mathrm{B}(s,t)) = \mathrm{B}(s,t) \otimes A(s,t).$$

Theorem (Brown–D.–Fresán–Tapušković)

The motivic Galois group acts on Taylor expansions of algebraic Mellin transforms via power series, i.e., for g in the motivic Galois group G:

$$g.\int_\sigma f^s\omega = \sum_{i=1}^{N} A_g^{(i)}(s)\int_\sigma f^s\omega_i$$

where the $A_g^{(i)}(s)$ are in $k((s))$. Equivalently, for the motivic coaction:

$$\rho\left(\int_\sigma f^s\omega\right) = \sum_{i=1}^{N}\left(\int_\sigma f^s\omega_i\right)\otimes A^{(i)}(s).$$

- This is a *finite* formula which computes the Galois theory of *infinitely many* periods.

Proof of concept

- A two-term example:

$$L(s;z) = \frac{1}{s}\left({}_2F_1(s,1,s+1;z) - 1\right) = \int_0^1 x^s \frac{z\,dx}{1-zx} = \sum_{n=0}^{\infty} (-s)^n \operatorname{Li}_{n+1}(z).$$

  Motivic coaction for classical polylogarithms:

$$\rho(\operatorname{Li}_{n+1}(z)) = \sum_{k=0}^{n} \operatorname{Li}_{n+1-k}(z) \otimes \frac{\lambda(z)^k}{k!} + 1 \otimes b_n(z)$$

  Gives rise to a two-term formula (already noticed by Goncharov):

$$\rho(L(s;z)) = L(s;z) \otimes A(s;z) + 1 \otimes B(s;z).$$

- A family of examples (Brown–D. '23): Lauricella hypergeometric functions

$$\int_0^{\sigma_i} x^{s_0}(1 - x\sigma_1^{-1})^{s_1} \cdots (1 - x\sigma_n^{-1})^{s_n} \frac{dx}{x - \sigma_j}.$$

Twisted cohomology, traditional version

On this slide, *s is a fixed complex number*.

Twisted (de Rham) cohomology, traditional version

$$\mathrm{H}^i_{\mathrm{dR}}(X, f) := \mathrm{H}^i(\Omega^\bullet(X), \nabla_s) = \frac{\ker(\nabla_s : \Omega^i(X) \to \Omega^{i+1}(X))}{\operatorname{Im}(\nabla_s : \Omega^{i-1}(X) \to \Omega^i(X))}$$

$$\text{where} \quad \nabla_s : \omega \mapsto \mathrm{d}\omega + s\frac{\mathrm{d}f}{f} \wedge \omega \quad (\text{so that } \mathrm{d}(f^s\omega) = f^s\nabla_s(\omega)).$$

- This is where the *integrands* of algebraic Mellin transforms live.
- The relations $\nabla_s(\omega) = 0$ are the “IBP relations”.
- $\mathrm{H}^i_{\mathrm{dR}}(X, f)$ is a finite dimensional k-vector space, whose dimension depends on s.
- The case when s is generic is easier: generic vanishing, intersection pairing.
- Basis for s generic : “master integrands”.

Twisted cohomology, local version

How motivic is twisted cohomology?

- $\mathrm{H}^\bullet(X, f)$ *is not motivic* (does not come from geometry) if $s \notin \mathbb{Q}$.
- A *formal generic* version of $\mathrm{H}^\bullet(X, f)$ *is motivic* (comes from geometry).

Twisted (de Rham) cohomology, local version

$$\mathrm{M}^i_{\mathrm{dR}}(X, f) := \mathrm{H}^i(\Omega^\bullet(X)(\!(s)\!), \nabla)$$

$$\text{where} \qquad \nabla : \omega \mapsto \mathrm{d}\omega + s\frac{\mathrm{d}f}{f} \wedge \omega \ .$$

- It is a finite dimensional $k(\!(s)\!)$-vector space, whose dimension is the generic dimension of “traditional” twisted cohomology.

- Key remark: $\mathrm{H}^i(\Omega^\bullet(X)[s]/(s^{n+1}), \nabla)$ can be interpreted in terms of the *motivic fundamental group* of $\mathbb{G}_m$.

Back to Feynman integrals

Theorem (Brown–D.–Fresán–Tapušković)

The space of Laurent expansions of Feynman integrals in dimensional regularization is closed under the action of the motivic Galois group:

$$g.I_\Gamma(\varepsilon) = \sum_{i=1}^{N} A_g^{(i)}(\varepsilon)\, I_{\Gamma_i}(\varepsilon).$$

Or equivalently, for the motivic coaction:

$$\rho\left(I_\Gamma(\varepsilon)\right) = \sum_{i=1}^{N} I_{\Gamma_i}(\varepsilon) \otimes A^{(i)}(\varepsilon).$$

- Still difficult to make explicit. Problem: how to make sense of the functions $A^{(i)}(\varepsilon)$?
- No “diagrammatic coaction” yet (Abreu–Britto–Duhr–Gardi–Matthew).