# Grassmannian Cluster Varieties and Their Symmetries

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# Overview

1. What's the point?
2. Cluster Varieties and Ensembles
3. Grassmannians
4. The Cluster Modular Group
5. Special foldings of Grassmannian cluster structures

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# What's the point?

- A Cluster Variety is an algebraic variety along with a toric atlas generated by some extra combinatorial data.
- The cluster structure also encodes many new/unexpected coordinate functions, and automorphisms of the cluster structure encode exotic automorphisms of the variety.

## Examples

- The Grassmannian (really some open set in/affine cone of) $\mathrm{Gr}(k, n)$ has such a structure, which contains all of the Plücker coordinates as coordinate functions in its cluster atlas.
- The cluster structure gives a convenient way of packaging up all of the Plücker relations, but also gives surprising new symmetries.
- Cluster structure encodes Polylogarithm relations and Scattering amplitude symbol alphabets in interesting ways

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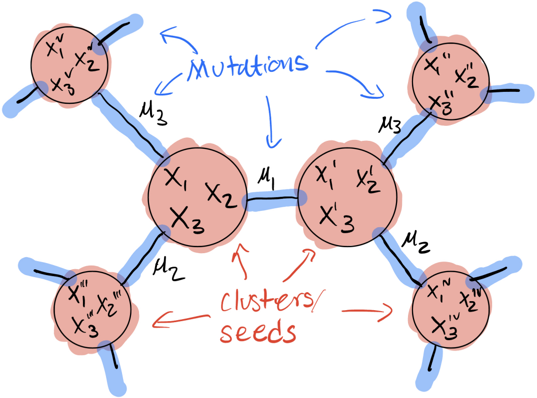
# Cluster Varieties (in brief)

Following Fock-Goncharov

- A Cluster Variety is an Algebraic variety built out of seed tori glued along birational morphisms called mutations.
- Each seed comes with some extra combinatorial data which encodes all of the mutation maps from that seed.
- Each mutation produces a new seed, so the entire cluster structure is generated by just one seed.

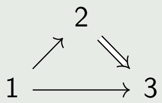


# Cluster Varieties of Geometric Type

- We will only consider cluster varieties of Geometric type meaning that we use quivers to track the combinatorial data underlying mutation maps.
- Quivers allowed for seeds cannot contain self loops or 2-cycles.
- Quivers are graphical representations of the exchange matrix $M = [\epsilon_{ij}]$ with entries equal to the number of arrows from node $i$ to node $j$.

## Examples

$Q =$




$$\longleftrightarrow M = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

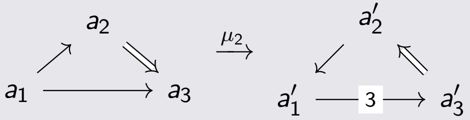
# Cluster Ensembles of Geometric Type

- Associate variables to the nodes of a quiver to make a Seed; $\mathcal{A}_S$ and $\mathcal{X}_S$.
- To Mutate a seed at the (non frozen node/variable) $i$ we
  1. Mutate the quiver at node $i$
  2. Replace all the variables according to the $\mathcal{A}$ or $\mathcal{X}$ mutation rule

## Mutation of quivers/Seeds

1. For every pair of arrows $(k \to i), (i \to j)$ add a arrow from node $k$ to node $j$, cancelling any 2-cycles between nodes $k$ and $j$.
2. Reverse every arrow incident to $i$.

$$A_S \xrightarrow{\mu_2} A_{\mu_2(S)} :$$



# The $\mathcal{A}$ and $\mathcal{X}$ Cluster Varieties

**$\mathcal{A}$-type mutation rule**

$$\mu_i^*(a_j') = \begin{cases} \frac{1}{a_i}\left(\prod_{\epsilon_{ik}>0} a_k{}^{\epsilon_{ik}} + \prod_{\epsilon_{ik}<0} a_k{}^{-\epsilon_{ik}}\right) & i = j \\ a_j & i \neq j \end{cases}$$

**$\mathcal{X}$-type mutation rule**

$$\mu_i^*(x_j') = \begin{cases} x_i^{-1} & i = j \\ x_j(1 + x_i^{-\operatorname{sgn}\epsilon_{ji}})^{-\epsilon_{ji}} & i \neq j \end{cases}.$$

**Definitions**

The Cluster Ensemble associated with $Q$ is the pair of cluster varieties $(\mathcal{A}_Q, \mathcal{X}_Q)$ along with the map $\rho : \mathcal{A}_Q \to \mathcal{X}_Q$ defined on coordinates on the same seed by

$$\rho^*(x_i) = \prod_j a_j^{\epsilon_{ij}}$$

There are lots of interesting aspects of these spaces, e.g. Poisson structure, symplectic structure etc. but we wont really need them

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# Grassmannian Cluster Ensembles

## $\mathcal{A}$-Space

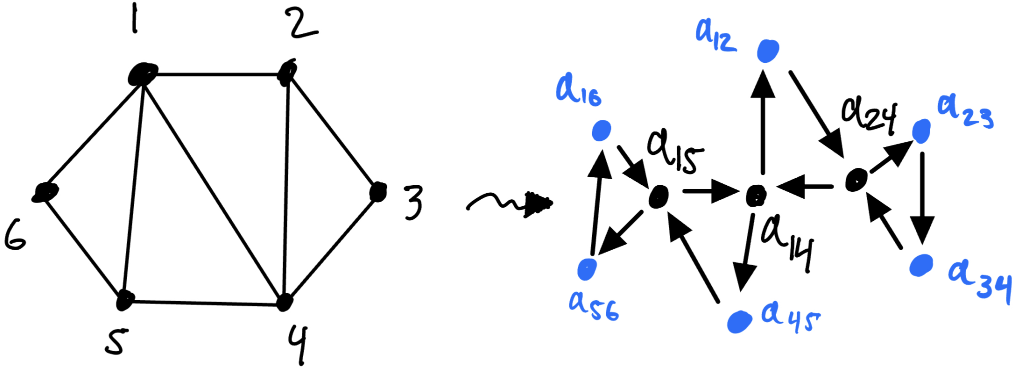
- Open subset of Affine cone over Gr$(k, n)$ where Plücker Corrdinates are non-zero.
- Cluster variables are Plücker Corrdinates + "Exotic" Variables
- Exotics variables are all polynomials in Plückers
- $\mathcal{A}-$variables on Gr$(4, n)$ give "Symbol Alphabet" for $n < 8(10?)$ particle amplitudes $\mathcal{N} = 4$ SYM Theory

## $\mathcal{X}$-Space

- Open Subset of $Conf_n(\mathbb{P}^{k-1})$ (Generic configurations)
- Cluster variables are Cross ratios + "Exotic" ratios (e.g. triple ratio)
- $\mathcal{X}-$Variables are arguments to "Cluster Polylogarithms" and their relations.

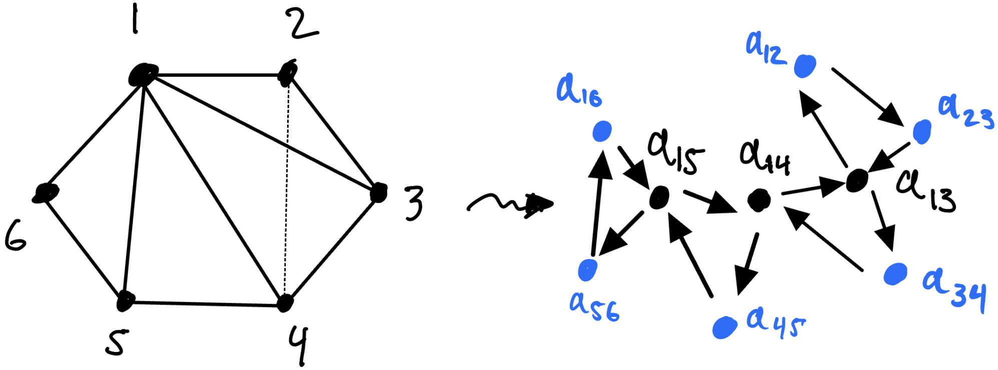
What are the Seeds? What is the combinatorics of these cluster ensembles?

# Example: Gr(2,n)




Seed for Gr(2, 6)

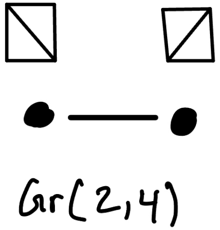
# Example: Gr(2,n)

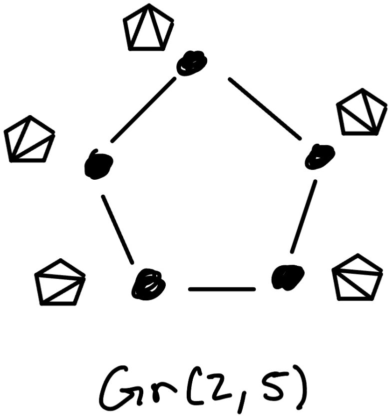


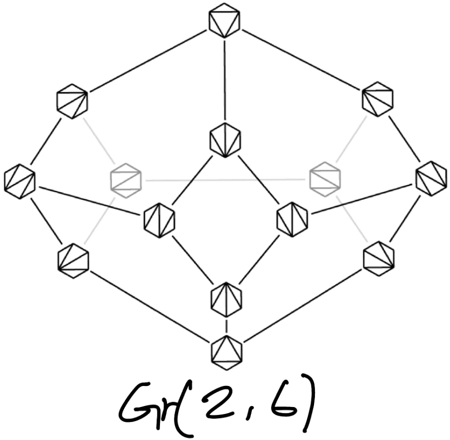

- Exchange Relation($\mathcal{A}$): $a_{13}a_{24} = a_{12}a_{34} + a_{14}a_{23}$
- Exchanges Relation ($\mathcal{X}$): $x_{1345} = x_{1245}(1 + x_{1234}^{-1})^{-1}$

# Exchange Complexes

Exchange Complexes for Gr$(2, n)$ are the Associahedra:



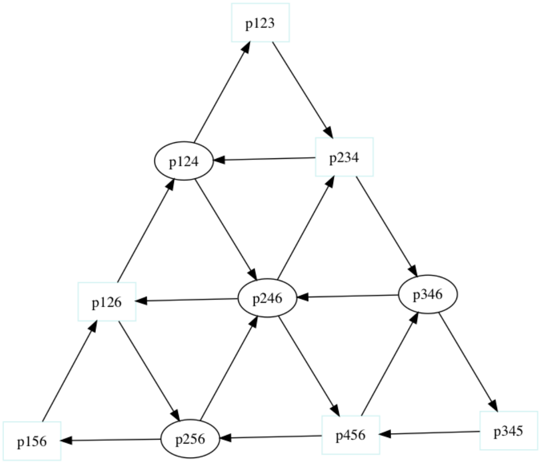



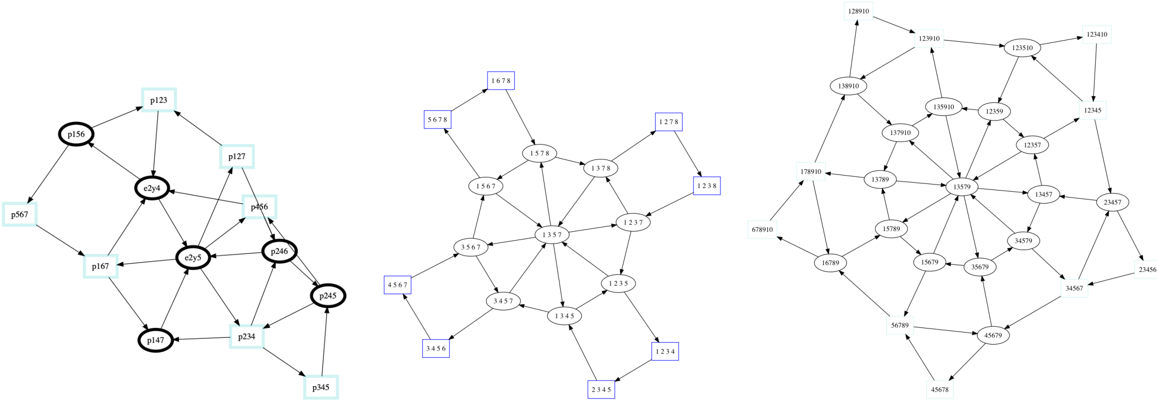
# Example: Gr(3, 6)

New things happen starting at Gr(3, 6):

- Not all mutations produce Plücker coordinates
- Exchange Relation: $a_{246}A = a_{124}a_{346}a_{256} + a_{126}a_{234}a_{456}$
- $A = det(v_1 \times v_2, v_3 \times v_4, v_5 \times v_6)$
- $\mathcal{X}$ coordinate at center is Goncharov's triple ratio.



# Grassmannian Seeds




(a) Gr(3, 7) (b) Gr(4, 8) (c) Gr(5, 10)

# Seeds for General Gr(k,n)

Mutatble portion of seed for $\mathrm{Gr}(p, p+q)$:

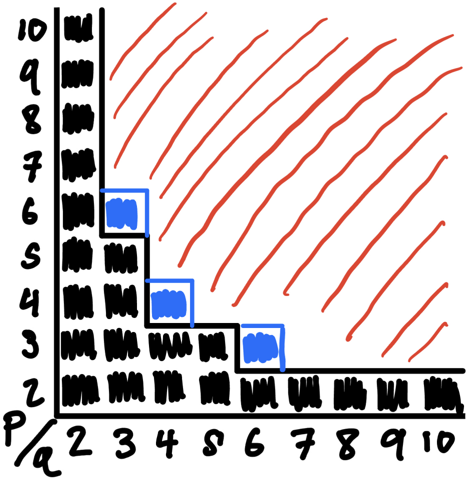
$$\begin{array}{ccccccc}
a_{11} & \longleftarrow & a_{12} & \longrightarrow & \ldots & \longleftarrow & a_{1(p-1)} \\
\downarrow & & \uparrow & & \downarrow & & \uparrow \\
\ldots & \longrightarrow & & \longleftarrow & & \longrightarrow & \\
\uparrow & & \downarrow & & \uparrow & & \downarrow \\
a_{(q-1)1} & \longleftarrow & & \longrightarrow & & \longleftarrow & a_{(q-1)(p-1)}
\end{array}$$

Use B. Kellers Mutation applet `https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/`

# Grassmannian Cluster Types

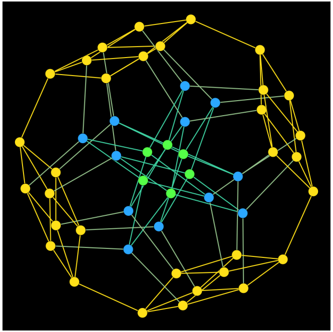
Cluster Type of $\mathrm{Gr}(p, p+q)$ :

- $(p-2)(q-2) < 4 \rightarrow$ finitely many seeds **(Finite type)**
- $(p-2)(q-2) = 4 \rightarrow$ infinitely many seeds, but finite quivers (Mutation Finite)
- $(p-2)(q-2) > 4 \rightarrow$ infinitely many seeds and quivers (Infinite Type)

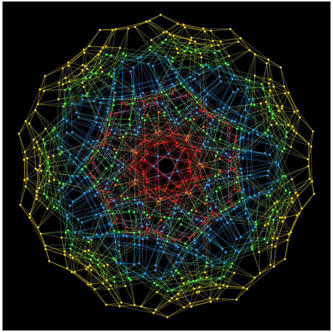


# Finite Type Grassmannians

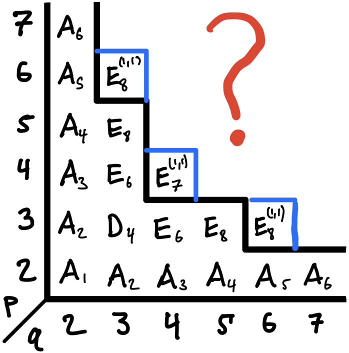
Finite type Cluster structures are classified by by the ADE Dynkin diagrams



(a) $\mathrm{Gr}(3,6) = D_4$



(b) $\mathrm{Gr}(3,7) = E_6$




(c) Dynkin Types

# Mutation Finite Grassmannians

- Gr(4, 8) and Gr(3, 9) are only "Mutation Finite"
- Their Dynkin types are best described by "Elliptic root systems", defined by Satio.
- Have a Dynkin Diagram given by $T_{pq2}$ Quiver
- $\mathrm{Gr}(4,8) = E_7^{(1,1)} = T_{442}$,
  $\mathrm{Gr}(3,9) = E_8^{(1,1)} = T_{632}$

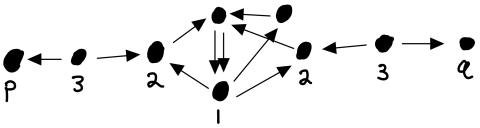
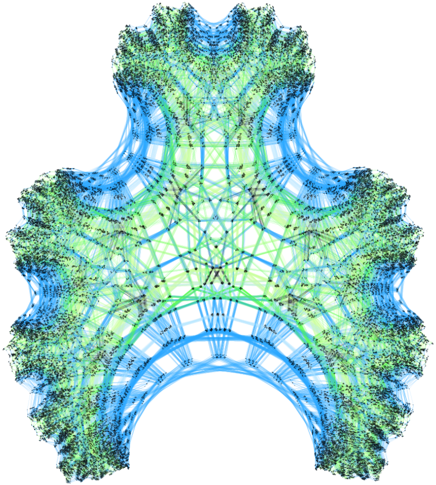



Figure: $T_{pq2}$



# Infinite Type Grassmannians

- Singularity theory of Complex Surface: $x^p + y^q + (z^2) = 0$
- Can be classified by "Modality" (For small $p, q$) (V.I Arnold)
- Can be given Dynkin Diagrams; Each is a $T_{pq2}$ plus $(p-2)(q-2)-4$ extra nodes.
- Connection through "Cluster Categories" (Jensen, King, Su)

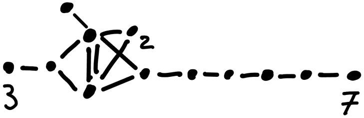
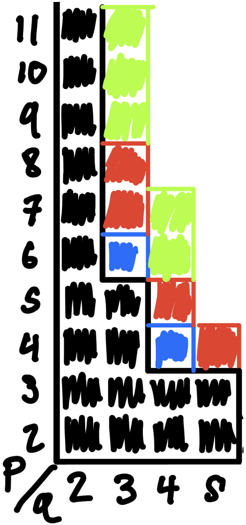



Figure: $\mathrm{Gr}(3,10), x^3 + y^7 + z^2 = 0$

Singularity Types:

- **Simple**
- Elliptic
- Unimodal
- Bimodal



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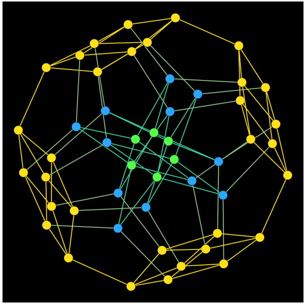
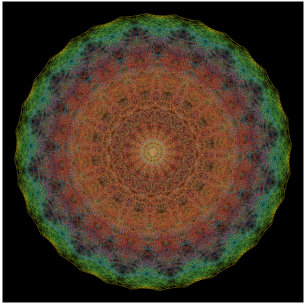
# The Cluster Modular Group

The Cluster Modular Group, $\Gamma$, is the group of automorphisms of a cluster structure. We can think of it as a generalization of a mapping class group.

1. The group of automorphisms of the Exchange Complex of the cluster structure.
2. The group consisting of pairs $\{P, \sigma\}$ of mutation sequences and quiver isomorphisms, up to those that do not change the cluster variables.



(a) $\mathrm{Gr}(3,6) = D_4, \Gamma = S_3 \times \mathbb{Z}_4$



(b) $\mathrm{Gr}(3,8) = E_8, \Gamma = \mathbb{Z}_{16}$

# The Cluster Modular Group

Lets explain this last description.

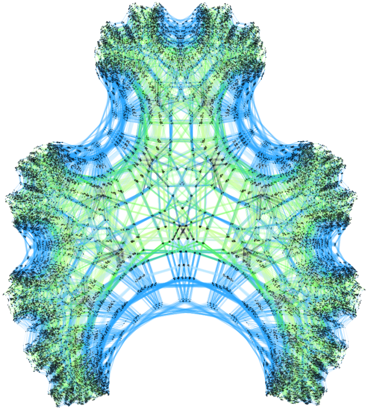
- $P$ is a path of mutations, $P = \{\mu_{P_1}, \mu_{P_2}, \dots \mu_{P_k}\}$ and $\sigma$ is a quiver isomorphism $\sigma : Q \to P(Q) = \mu_{P_k} \circ \cdots \circ \mu_{P_1}(Q)$.
- Labeling the nodes of $Q$ by 1 up to $n$, we can write $P$ as a list of nodes to mutate at and $\sigma$ as an element of the symmetric group $S_n$.
- Pairs $\{P, \sigma\}$ can be composed by viewing them as elements of the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{*n} \rtimes S_n$.
- Explicitly, we have

$$\{P_1, \sigma_1\} \cdot \{P_2, \sigma_2\} = \{P_1 \sigma_1(P_2), \sigma_1 \sigma_2\}$$

- $\sigma$ gives a map between the cluster variables on $S$ to those on $P(S)$, and pairs for which this map is the identity are the identity in $\Gamma$.

# Cluster Modular Groups of Grassmannians

- C. Fraser gives action of "Affine Braid Group" on $d = gcd(n, k)$ strands on $\mathrm{Gr}(n, k)$ by cluster automorphisms.
- This is not faithfull; if $n = 2k$ this gives an action of the "spherical braid group" on $k$ strands
- In joint work with Z. Greenberg we calculate the cluster modular group of all elliptic cluster algebras (e.g $\mathrm{Gr}(4, 8)$ and $\mathrm{Gr}(3, 9)$)
- We show all elliptic cluster modular groups have $\mathrm{PSL}(2, \mathbb{Z})$ as a quotient by a finite normal subgroup.



$$1 \to N \to \Gamma \to \mathrm{PSL}(2, \mathbb{Z}) \to 1$$

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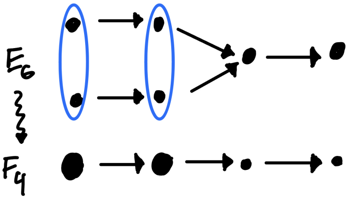
# Folding quivers

Idea: The usual Folding of Dynkin diagrams can be extended to cluster structures.




How to make a folded seed:

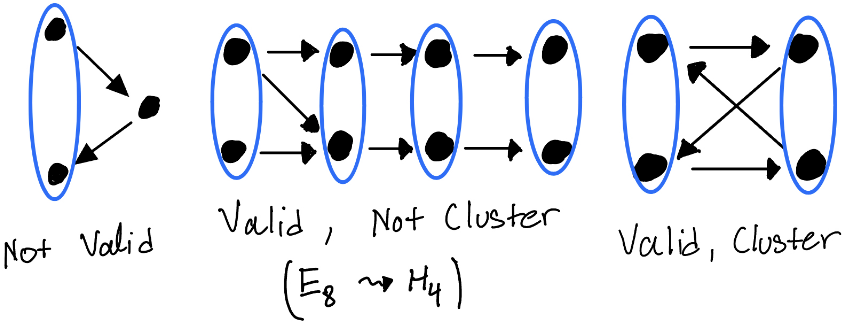
1. Put same cluster variable on each node in a group
2. Mutate each group together - called Group Mutation

# What do we need to fold?

Folded seeds:

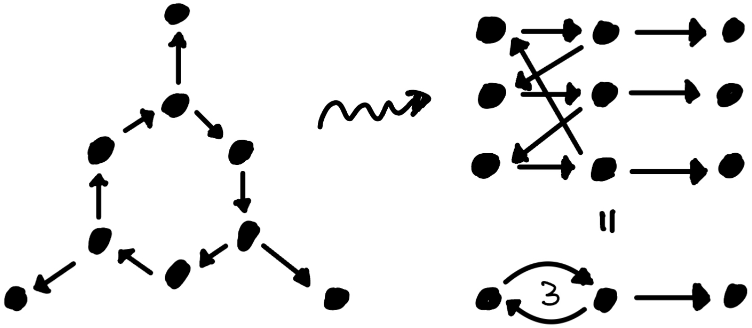
1. Put same cluster variable on each node in a group
2. Mutate each group together

- A folding is valid if there is a way to do group mutations e.g if the nodes in each group remain disconnected after mutations
- A folding is Cluster if each seed produced by group mutation is a folded seed.



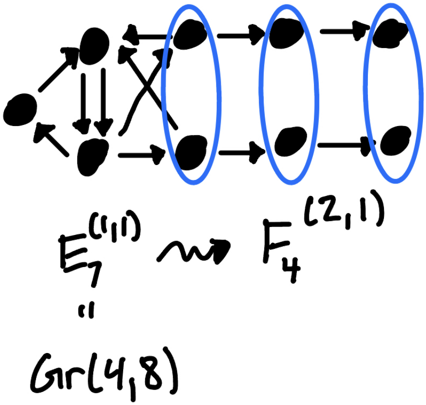
# Cyclic Folding

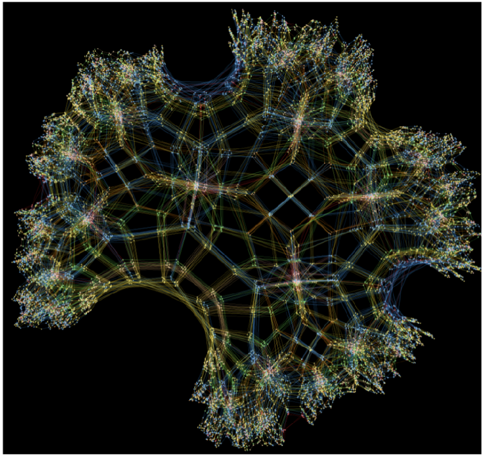
Folding By a Cyclic quiver automorphism is often valid and cluster



# Cyclic folding of Grassmannian seeds

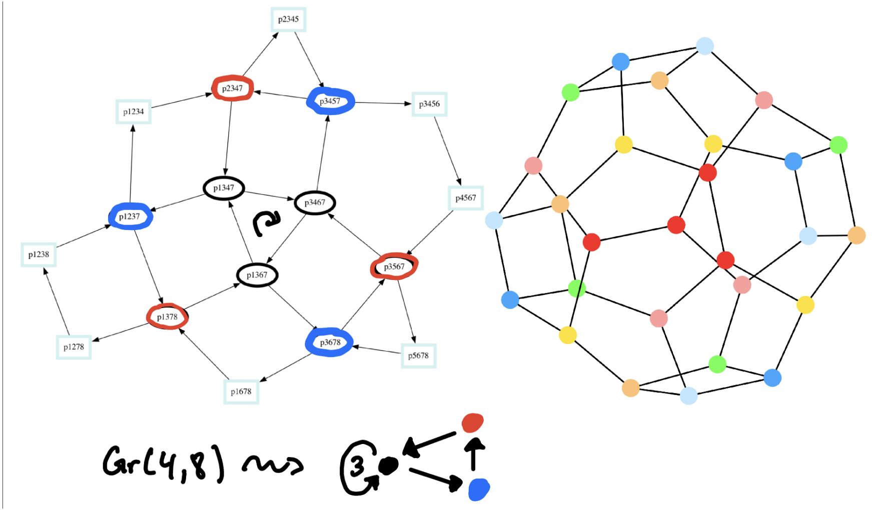
Many Grassmannians admit special cyclic foldings. Folding that minimics Dynkin Folding does not seem to reduce the complexity:



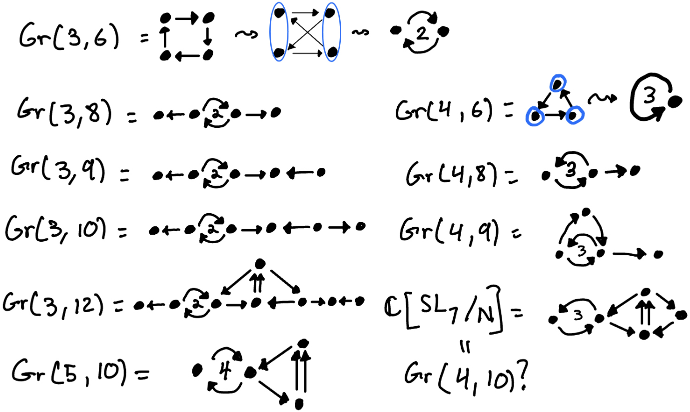



# Cyclic folding of Grassmannian seeds

Special Folding can actually reduce! the complexity of the algebra!



# Special $q-1$ Cyclic foldings of $\mathrm{Gr}(p, p+q)$




# Examples of special foldings

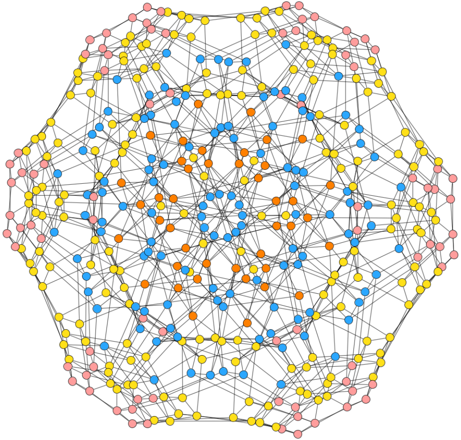


Figure: Gr(3,9)

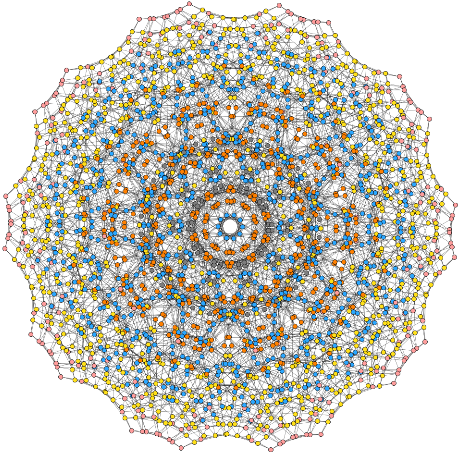


Figure: Gr(3,10)

# Examples of special foldings

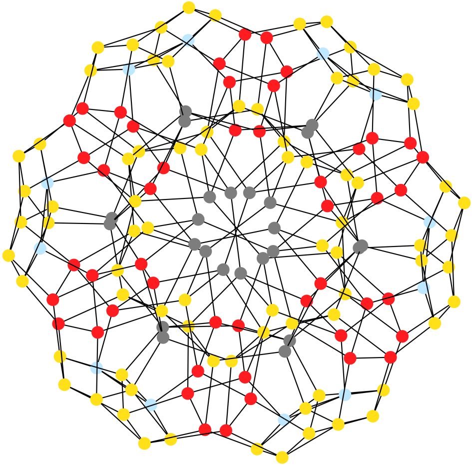


Figure: Gr(4,9)

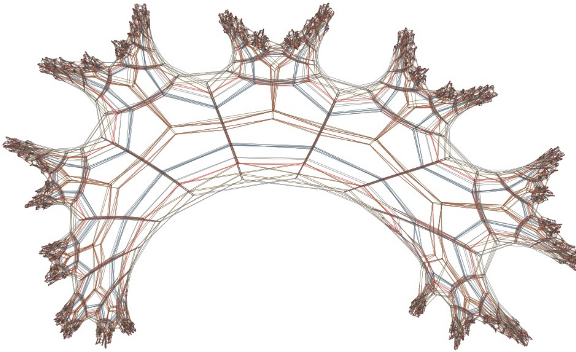


Figure: $\mathbb{C}[SL_7/N]$