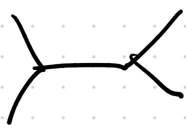
LEADING SINGULARITIES BEYOND MPLS

LORENZO TANCREDI

- We compute Feynman Integrals in order to compute SCATTERING AMPLITUDES

- Amplitudes in "physically relevant theories" (gauge theories) have special behaviours motivated by general considerations in QFT

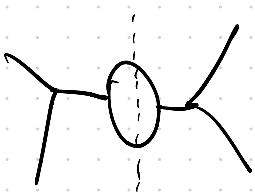
  * poles when single particles go on-shell



$$\sim \frac{1}{s-m^2}$$

  * branch cuts when multiple particles go on shell

$D=2-2\varepsilon$



$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}-\sqrt{s}}{\sqrt{s-4m^2}+\sqrt{s}}\right)$$

problem for $s>4m^2$

  * Logarithmic singularities (IR, etc)

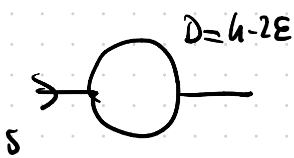
QUESTION 1. Can we choose integrals to "decompose" these amplitudes, which make these properties manifest?

ANSWER 1. if amplitudes can be written in terms of multiple polylogs (geometry Riemann sphere) we do!

$\Longrightarrow$ what we call CANONICAL INTEGRALS

"pure" ints of UNIFORM TRANSC. WEIGHT

EXAMPLE (NOT CANONICAL)

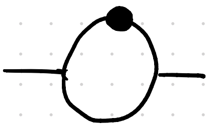


$$= \# \left[\frac{1}{\varepsilon} + \underbrace{2 - \sqrt{\frac{s-4m^2}{s}} \ln\left(\frac{-\sqrt{s} - \sqrt{s-4m^2}}{\sqrt{s} - \sqrt{s-4m^2}} \right)} \right]$$

$\frac{1}{\varepsilon} \times$ weight "0"

$\varepsilon^0 \times \{$ weight 0 & 1 with algebraic prefactor $\}$

EXAMPLE (CANONICAL)

 $= \frac{1}{\sqrt{s(s-4m^2)}} \underbrace{\ln\left(\frac{-\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}-\sqrt{s-4m^2}}\right)} + O(\epsilon)$

$\frac{1}{\epsilon}$ not there but 0 is a number of weight "0"

ϵ^0 is just a log (weight 1) with one single algebraic prefactor

Cubsmes

by selecting these type of integrals we achieve:

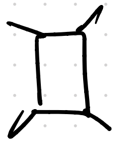
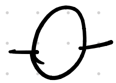
1. Ints contain "branch cuts" (including square roots)
2. Overall prefactors give the <u>poles</u> (rational functions) <u>If any</u>

⇒ these integrals are also particularly simple to compute because they satisfy simple diff eqs in DIM REG

IMPORTANTLY, we know how to FIND them without having to compute them!

this is where leading singularities enter -

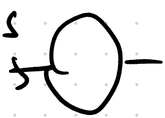
QUESTION

say I write an integral  or  etc

what is the simplest way to know if it is a PURE INTEGRAL

ANSWER is Look at residues of integrand!

$p^2 = s$



$$= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-m^2)((k-p)^2-m^2)} = \text{BAIKOV}$$

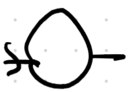
$$= \# \oint \frac{dz_1\, dz_2}{z_1 z_2} \left[\frac{P(s,m,z_1,z_2)}{s} \right]^{\frac{D-2}{2}} \frac{1}{\sqrt{P(s,m^2,z_1,z_2)}}$$

props as int variables

$$P(s, m^2, z_1, z_2) = (4m^2 - s)s + 2s\left[z_1 + z_2\right] - (z_1 - z_2)^2$$

second order polynomial

For simplicity, do analysis in $D = 2 - 2\varepsilon$ instead of $D = 4 - 2\varepsilon$ (two results can be related)



$$\sim \oint dz_1 dz_2 \frac{1}{z_1 z_2 \sqrt{P(s, m^2, z_1, z_1)}} \left[\quad\right]^{\varepsilon}$$

CONJECTURE : look at all variables iteratively and take all residues in $\varepsilon = 0$ - IF :

1. there are only simple poles & branch cuts
2. All iterated residues are EQUAL

$\Longrightarrow$ the integral is CANONICAL

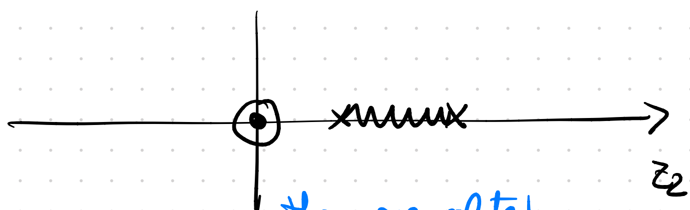
$\left[\quad\right]^{\varepsilon}$ when expanded produces only PURE LOGARITHMS, so we can neglect it

$\varepsilon = 0$

$$-\bigcirc- = \oint \frac{dz_1}{z_1} \oint \frac{dz_2}{z_2} \frac{1}{\sqrt{P(s, m^2, z_1, z_2)}}$$

residue of $z_2 = 0$

Quadratic residue at infinity + branch cut



they are related by global res theorem

limit to $z_2 = 0$

$$\operatorname*{Res}_{z_2=0}\left(-\bigcirc-\right) = \oint \frac{dz_1}{z_1} \frac{1}{\underbrace{\sqrt{4m^2 s - (s - z_1)^2}}_{\text{same structure in } z_1}}$$

$$\operatorname*{Res}_{z_1=0} \operatorname*{Res}_{z_2=0} -\bigcirc- = \frac{1}{\sqrt{s(s-4m^2)}}$$

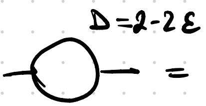
$\Rightarrow$ indeed, only single poles and iterated residues are all equal to $\dfrac{1}{\sqrt{s(s-4m^2)}}$

 in $D \sim 2-2\varepsilon$ IS CANONICAL

in fact, it turns out

so we have "proven" that  is a canonical integral in $D = 4-2\varepsilon$

this can be reinterpreted as



$$= \# \oint \frac{dz_1}{z_1} \frac{1}{\sqrt{4m^2 s - (s-z_1)^2}} \oint \frac{d \log\left[f(s,m^2,z_1,z_2)\right]}{dz_2} dz_2$$

$$= \frac{\#}{\sqrt{s(s-4m^2)}} \oint \frac{d \log(g(s,m^2,z_1))}{dz_1} dz_1 \oint \frac{d \log(f(s,m^2,z_1,z_2))}{dz_2} dz_2$$

× other logs from $()^{\varepsilon}$

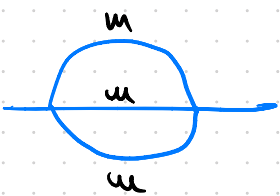
$\Rightarrow$ we say it's in d-log FORM

Conjecture verified in multitude of problems

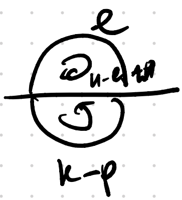
Problem: how do we extend this when non-polylog geometries are involved?

What is the generalization of this statement?

in Elliptic case, we could hope answer is forms of $\mathcal{E}_4$ or $\tilde{\Gamma}$ etc



two-loop sunrise

$=$ 

$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{\underbrace{(k-p)^2-m^2}_{z_2}} \mathrm{Bub}^{(1L)}(\underbrace{k^2}_{z_1})$$

$D = 2-2\varepsilon$

$$= \int \frac{dz_1 dz_2}{z_2} \cancel{\left[\;\right]^{\varepsilon}} \frac{1}{\sqrt{P_2(s,m^2,z_1,z_2)}} \mathrm{Bub}^{(1L)}(z_1)$$

neglect

↑ quadratic pol

$$= \oint dz_1 \, \mathrm{Bub}_{1L}(z_1) \oint \frac{dz_2}{z_2} \underbrace{\frac{1}{\sqrt{P_2(s,m^2,z_1,z_2)}}}_{\text{same as at 1L in } z_2}$$

$$= \oint dz_1 \frac{\mathrm{Bub}_{1L}(z_1)}{\sqrt{(z_1-a_1)(z_1-a_2)}} \oint d\log\big[\underset{dz_2}{\underline{f(s,m^2,z_1,z_2)}}\big]_{dz_2}$$

with $a_1 = (\sqrt{s}-m)^2 \qquad a_2 = (\sqrt{s}+m)^2$

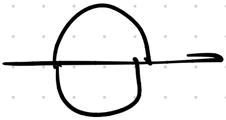
moreover we know that

$$\mathrm{Bub}_{1L}(z_1) = \frac{1}{\sqrt{z_1(z_1-4m^2)}} \oint d\log \oint d\log$$

so

$$= \# \oint \underbrace{\frac{dz_1}{\sqrt{z_1(z_1-4m^2)(z_1-a_1)(z_1-a_2)}}}_{\text{this is NOT a d-log}} \underbrace{\oint d\log}_{\text{from 2L part}} \underbrace{\oint d\log \oint d\log}_{\text{from 1L Bub}}$$

$y = \sqrt{P_4(z)}$ elliptic curve

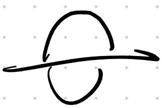
 $= \underbrace{\oint \frac{dz}{y}} \oint d\log \oint d\log \oint d\log$

this is exactly one of the differential forms that define "PURE" elliptic polylogs $\underline{\mathcal{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix}; z\right)}$

UP TO ALGEBRAIC PREFACTOR

$$\int \frac{dz}{y} \sim \omega_1 \, dz \quad \text{on TORUS}$$

so this analysis tells us that we expect

 to be "CANONICAL" in $D = 2 - 2\varepsilon$

and in fact it turns out to be writable, at every order in ε, in terms of $\underline{\text{pure } \mathcal{E}_4 \text{ or } \tilde{\Gamma} \text{ functions}}$

PROBLEM $\Rightarrow$ we need a second candidate integral, this family requires two master integrals $\Rightarrow$ 

this is the candidate

we usually start from (see talks of Albrecht, Christoph, Sebastian)

According to analysis before, namely that candidate would not work $\Rightarrow$ it has an apparent DOUBLE POLE

- so how do we choose one?
- what about more complicated geometries (CYn etc...)

?