Talk at Bethe Center, wed. 22-3-2023 ①
(workshop Geometries & Special function
for Physics & Mathematics).

Calabi-Yau Periods.
- geometry & arithmetic -

thank coauthors; Almkvist, Candelas, de la Ossa
Golyshev, GdT.

<u>CY-operator</u>: MUM point, self-dual, fuchsian
+ integrality conditions.

⇝ families on CY varieties.

AESZ-list. for 4^{th} order.

<u>Examples</u>:

LEGENDRE $y^2 = x(x-1)(x-t)$; $\omega = \frac{dx}{y}$

$$\int_0^1 \frac{dx}{y} = \pi\, {}_2F_1(1/2\ 1/2\ 1; t)$$
$$= \pi\left(1 + \left(\tfrac{1}{2}\right)^2 t + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 t^2 + \dots\right)$$
$$= \pi \sum \binom{2n}{n}^2 \left(\frac{t}{16}\right)^n$$

$$\phi_L(t) = \sum \binom{2n}{n}^2 t^n \in \mathbb{Z}[[t]]$$
annihilated by $\theta^2 - 16t(\theta + 1/2)^2$.

BANANIANA 

$$B^*_{r,n} = \sum_{n_1+\dots+n_r = n} \left(\frac{n!}{n_1! \cdots n_r!}\right)^2$$

$$\phi_{B_r} = \sum B^*_{r,n} t^n$$

$r=2$ $B^*_{2,n} = \binom{2n}{n}$ $\phi_{B_2^*} = \frac{1}{\sqrt{1-4t}}$

$r=3$, $r=4$, $r=5$: AESZ 34 Verrill.

Laurent Polynomial

$$F_r = (x_1 + \dots + x_r)\left(\frac{1}{x_1} + \dots \frac{1}{x_r}\right)$$

$$B_{r,n} = [F_r^n]_0 \quad \text{constant term}$$

Geometry $X_t \supset \{1 - t F = 0\} \subset (\mathbb{C}^*)^r$

$$\phi(t) = \frac{1}{(2\pi i)^r} \oint \frac{1}{1-tF} \frac{dx_1}{x_1} \dots \frac{dx_r}{x_r}$$

APERY: $A_n = \sum \binom{n}{k}^2 \binom{n+k}{k}$; 1, 3, 19, 147

$$\phi_A = \sum A_n t^n$$

$$\Theta^2 - t(11\Theta^2 + 11\Theta + 3) - t^2(\Theta+1)^2$$

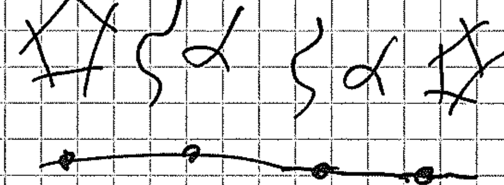
Geometry $A_n = [F_A^n]_0 \quad F_A = \frac{(1+x+y)(1+x)(1+y)}{xy}$

$E_{A,t} \supset \cancel{E^0_{A,t}} \{1 - tF_A = 0\} \subset (\mathbb{C}^*)^2$

E_A

$\downarrow$

$\mathbb{P}^1$



§2 Hadamard Product

$$\phi = \sum a_n t^n, \quad \psi = \sum b_n t^n$$

$$\phi * \psi(t) = \sum a_n b_n t^n = \frac{1}{2\pi i} \oint \phi(u)\, \psi\left(\frac{t}{u}\right) du.$$

$$\frac{1}{1-\alpha t} * \frac{1}{1-\beta t} = \frac{1}{1-\alpha\beta t}$$

$$\frac{1}{\sqrt{1-4t}} * \frac{1}{\sqrt{1-4t}} = \phi_L(t)$$

ϕ, ψ P-F $\longrightarrow$ $\phi * \psi$ P-F

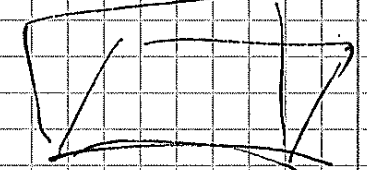
$$\phi_L * \phi_A = \sum_{n=0}^{\infty} \binom{2n}{n} A_n t^n$$

$$\Theta^4 - 4t(2\Theta+1)^2(11\Theta^2 + 11\Theta + 3) - 16t^2 (2\Theta+1)^2(2\Theta+3)^2$$

$$\begin{cases}\phi & 1-tF(x)\\ \psi & 1-tG(y)\end{cases} \Rightarrow \phi*\psi : \quad 1-tF(x)F(y)$$

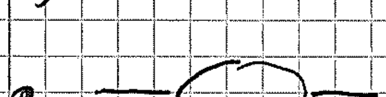
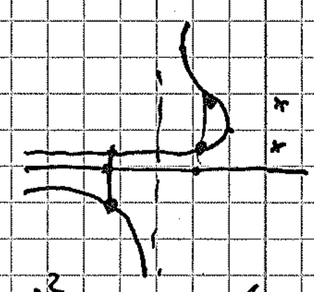
$\Rightarrow$ Geometry for $\phi_L * \phi_A$: fibre product of two elliptic surfaces.

$$\cancel{X}Y_t = E_A \tilde{\times}_t E_L$$



<u>Thm:</u> Y_t CY, $h^{11}=1,\ h^{12}=61$ mirror dual to $2,2,1 \subset G(2,5)$.

§3. Frobenius.

-  $X_t : 1 = t\dfrac{(1+y)^2}{y}$: 

$$\# X_t(\mathbb{F}_p) = \#\ y - t(1+y)^2 \bmod p = \begin{cases}0\\1\\2\end{cases}$$

$$\phi_B^{(p-1)}(t) := \sum_{n=0}^{p-1}\binom{2n}{n} t^n \bmod p.$$

$$\phi_B^{(p-1)}(t) \bmod p = \# X_t(\mathbb{F}_p).$$

- $\# E_{A,t} = 1+p-a_p(t) \qquad |a_p(t)| \le 2\sqrt{p}$

$$a_p(t) = \phi_A^{(p-1)} \bmod p$$

Extension to other CY-operators

p-adic cohomology DWORK

BEUKERS-VLASENKO.

CdOvS: L CY-operator

$\rightsquigarrow \mathrm{Frob}_p(t) \in \mathrm{Mat}(N\times N, R)$

$$\mathbb{Z}_p[t, \tfrac{1}{\Delta}] \subset R \subset \mathbb{Z}_p[[t]].$$

Euler factor $Q_{p,t} = \det(1 - T\, U_p(t))$

$$U(t) = E^{-1}(t^p)\, U(0)\, E(t)$$

$$\begin{pmatrix} \phi_0 & \vartheta\phi_0 & \cdots \\ \phi_1 & \cdot & \\ \phi_2 & \cdot & \\ \phi_3 & \cdot & \end{pmatrix} = E(t).$$

$$U(0) = \begin{pmatrix} 1 & & & 0 \\ & p & & \\ & 0 & p^2 & \\ * & & & p^3 \end{pmatrix} \qquad * = r \cdot \zeta_p(3)$$

$r \in \mathbb{Q}$.

Conjecture: $U_p(t) = \mathrm{Frob}_p(t)$

§ 4. What to do with it?

① Attractorpoints of rank 2.

$$H^3(X_{t_0}) = A \oplus B \qquad Q = Q_A\, Q_B.$$

$1\;1\;1\;1 \qquad 1001 + 0110.$

special pts in moduli G. Moore

Found in AESZ 34 Verrill-Family.

$t_0 = -1/7, \quad t_0 = 33 \pm 8\sqrt{17}$

CEdOvS.

Periods expressible in modul forms

eg. $\varphi_{B_5}(-1/7) =$

$$\varphi(33 - 8\sqrt{17}) = \frac{119 + 29\sqrt{17}}{16\pi^2} L(f_4, 2)$$

$f_4 = q - 2q^2 + 2i\, q^3 + 4q^4 + \ldots$

34.4.b.a in LMFDB

Other exples: Kilian Bönisch

Project: cases more parameters.

(2) Paramodular quest (5)

$E/\mathbb{Q}$ elliptic curve $\quad H^1E$ 2-dim

conductor $N \qquad Q = 1 - a_pT + pT^2$

<u>Wiles</u>: "E is modular."

$$1 + a_p$$

$$\# E(\mathbb{F}_p) = 1 + p - a_p$$

a_p Four. coeff of modular form.

$$L(H^1E, s) = L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$f = \sum a_n q^n$$

$$f \in S_2(\Gamma_0(N))^{new}_{\mathbb{Q}}$$

Smallest conductor:

$$N = 11; \quad y^2 + y = x^3 - x^2$$

$$f = q \prod (1-q^n)^2 (1-q^{11n})^2$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + \dots$$

$X/\mathbb{Q}$ CY 3-fold $\quad K^{12} = 1$

$H^3X \quad$ 4-dimensional.

$$Q = 1 + \alpha_p T + p\beta_p T^2 + p^3\alpha_p T^3 + p^6 T^4$$

Quest.. $L(H^3X, s) = L(F, s)$

$F \in S_3(K(N))^{new}_{\mathbb{Q}} \quad$ "Paramodular form."

$\mathbb{H}_2 \ni Z \quad$ 2×2 symm

$\quad \operatorname{Im} > 0.$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B)(CZ+D)^{-1}$$

$$K(N) = \begin{pmatrix} \cdot & N & \cdot & /N \\ \cdot & N & \cdot & \cdot \\ N & N & N & \cdot \end{pmatrix} \cdot \in \mathbb{Z}$$

$\mathbb{H}_2/K(N)$: 1:N polarised moduli.

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list of paramodular forms

$N = 61, 73, \underline{\underline{79}}, \ldots \quad \sim 1000.$

Golyshev – vS: $Y_{79} := E_L \times_{-1} E_A$

$\phi_L * \phi_A$

$L(H^3 Y_{79}, s) \overset{?!}{=} L(F_{79}, s) = 1 - \frac{5}{2^s} - \frac{5}{3^s} + \frac{11}{\cancel{7}^s} + \ldots$

first few hundred coefficients fit.

Fibred structure of Y_{79}:

congruence property mod 5

to Hilbert modular form.