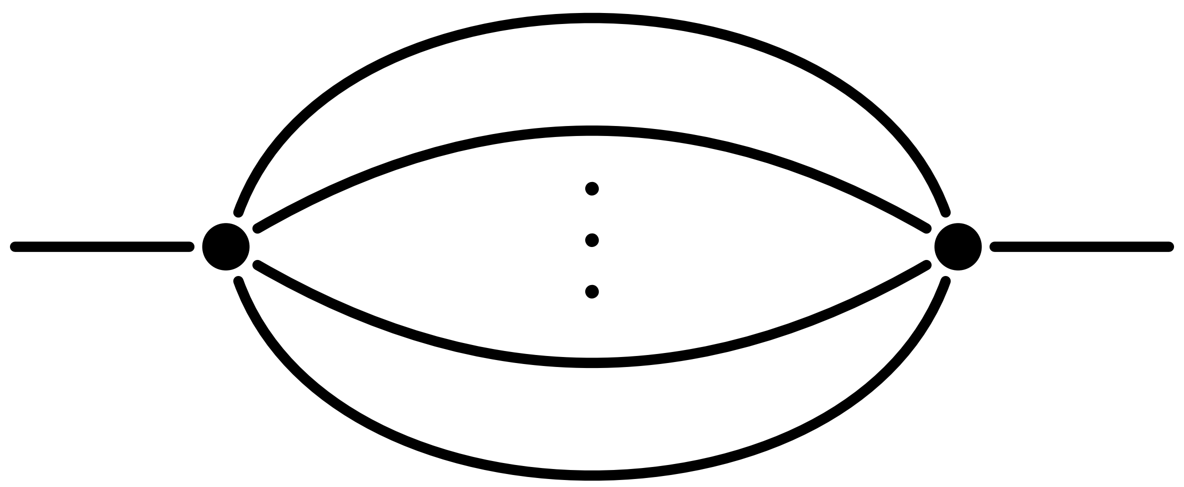
Automorphic forms for Calabi—Yau Integrals?

My Favourite Problem

Sebastian Pögel, University of Mainz
Geometries and Special Functions for Physics and Mathematics, Bonn
21st March 2023

Based on work in collaboration with Xing Wang and Stefan Weinzierl

`2207.12893` (JHEP 09 (2022) 062)
`2211.04292` (PRL 130 (2023) 10, 101601)
`2212.08908`



Closely related to Albrecht's and Christoph's talks

What is the generalisation of modular forms to higher-dimensional Calabi—Yaus?

Modular Forms

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : H \to H, \ \gamma\tau \to \frac{a\tau + b}{c\tau + d} \qquad \tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$\gamma \in \Gamma \quad \Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ Eg. Congruence subgroup of modular group

Modular Forms: Transform “nicely” under modular transformations

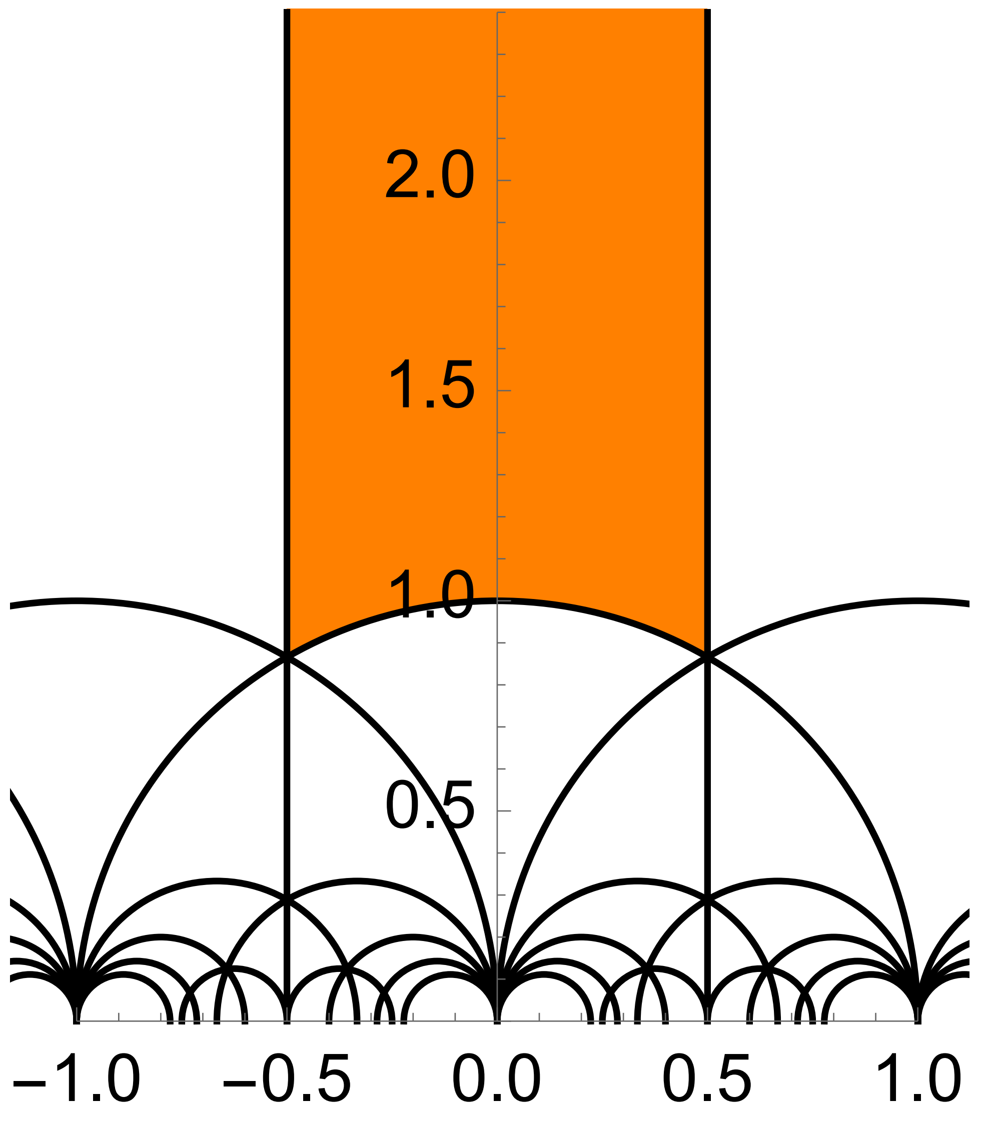
$$\omega_k(\gamma\tau) = (c\tau + d)^k \omega_k(\tau)$$

Modular Forms in Physics

Define function space of certain Feynman integrals
Iterated integrals of modular forms

Convenient for physics:
Use modular transformation to map to "more convergent point"
[Weinzierl]

With suitable redefinition of integrals

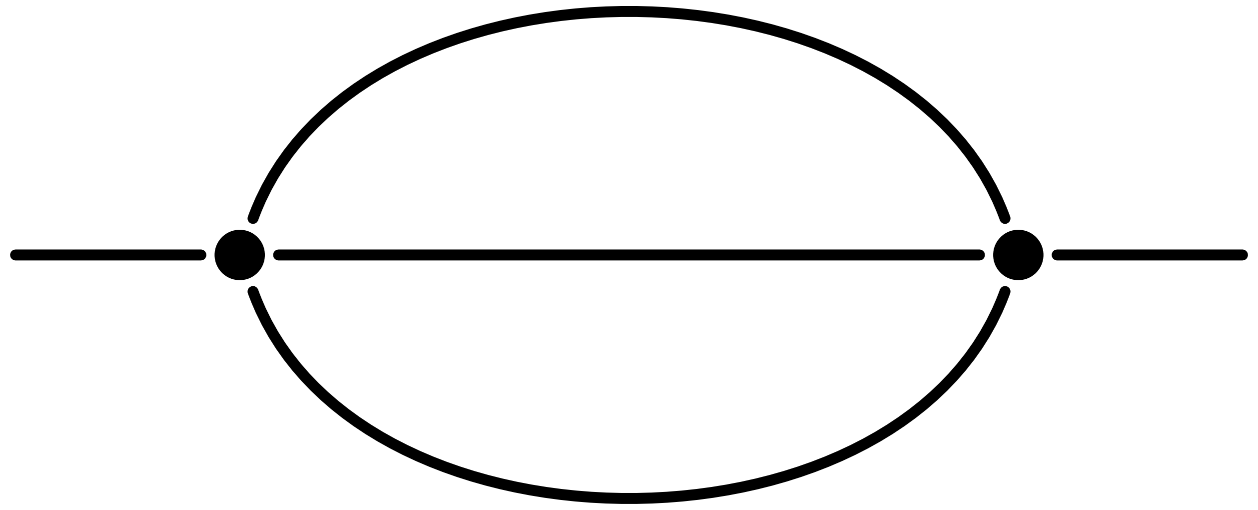


For all τ, can find $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, such that $|q'| \leq \exp(-\pi\sqrt{3}) \sim 0.004$ for $\tau' = \gamma\tau$

Quickly convergent expansions in new nome q'

Modular Forms for Elliptic Integrals

ε-Factorization of Sunrise



$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

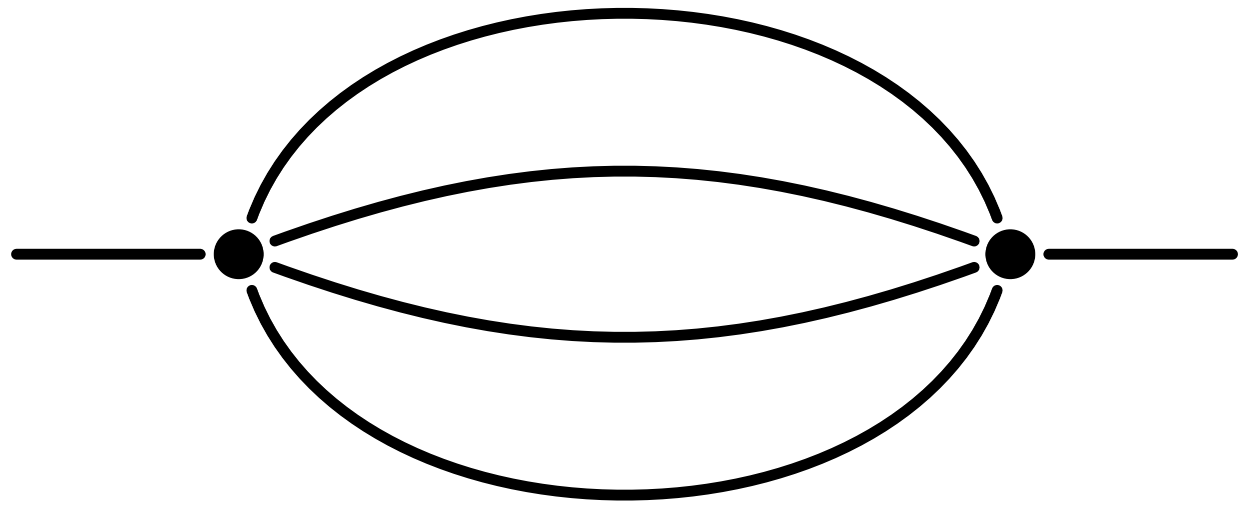
$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_2^{(2)} & 1 \\ \omega_3^{(2)} & \omega_4^{(2)} & \omega_2^{(2)} \end{pmatrix} I^{(2)}$$

Organised by modular weight

$\omega_k^{(2)}$: Modular forms of $\Gamma_1(6)$ of weight k

Modular Forms for K3 Integrals

ε-Factorization of Three-loop Banana



Degree 3 Picard—Fuchs operator is symmetric square:

Function space still elliptic!

$$
\begin{aligned}
I_1 &= \varepsilon^3 I_{1110}, \\
I_2 &= \varepsilon^3 \frac{1}{\omega} I_{1111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.
\end{aligned}
$$

$$
\mathrm{d}I^{(3)} = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \omega_2^{(3)} & 1 & \\ 0 & \omega_4^{(3)} & \omega_2^{(3)} & 1 \\ \omega_4^{(3)} & \omega_6^{(3)} & \omega_4^{(3)} & \omega_2^{(3)} \end{pmatrix} I^{(3)}
$$

Still organised by modular weight

τ: still ratio of elliptic periods

$\omega_k^{(3)}$: Meromorphic modular forms of $\Gamma_1(6)$ of weight k +

One special function F_2 New object mentioned in Christoph's talk

F_2 has to satisfy

$$\frac{d^2F_2}{dx^2} + \left[\frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)} + 2\left(\frac{d\ln\omega_0}{dx}\right)\right]\frac{dF_2}{dx} = \frac{\left(x-8\right)\left(x+8\right)^3}{6x\left(x-4\right)^{5/2}\left(x-16\right)^{5/2}}\left(\frac{\omega_0}{\pi}\right)^2$$

Solution: Iterated integral of meromorphic modular form of weight 6!

$$F_2 = \left(2\pi i\right)^2 \int_{i\infty}^{\tau} d\tau_1 \int_{i\infty}^{\tau_1} d\tau_2 \overbrace{\frac{x\left(x-8\right)\left(x+8\right)^3}{864\left(4-x\right)^{\frac{3}{2}}\left(16-x\right)^{\frac{3}{2}}}\left(\frac{\omega_0}{\pi}\right)^6}^{g_6}$$
$$= -2q + 26q^2 - 254q^3 + 1882q^4 - 12252q^5 + \dots$$

Some special properties:

- q expansion of g_6 has only integer coefficients
- q^n coefficient of g_6 divisible by n^2

F_2: Not a Modular Form

Iterated integral → non-trivial transformation under $\Gamma_1(6)$

Path decomposition gives us

$$(F_2|_2\gamma)(\tau) = F_2(\tau)$$

$$-6\frac{c}{c\tau+d}\frac{1}{2\pi i}I\left(1,1,g_6;\tau\right)+18\left(\frac{c}{c\tau+d}\right)^2\frac{1}{(2\pi i)^2}I\left(1,1,1,g_6;\tau\right)$$

$$-24\left(\frac{c}{c\tau+d}\right)^3\frac{1}{(2\pi i)^3}I\left(1,1,1,1,g_6;\tau\right)$$

$$+\frac{C_{1,6}}{(c\tau+d)^2}-\frac{2\pi i C_6}{c\,(c\tau+d)^3}$$

Defining "Quasi-Eichler" of weight k, depth p:

$$(f|_k\gamma)(\tau) = f(\tau)+\sum_{j=1}^{p}\left(\frac{c}{c\tau+d}\right)^j f_j(\tau)+\frac{P_\gamma(\tau)}{(c\tau+d)^p}$$

Constants:

$$C_{1,6} = I\left(1,g_6;i\infty,\frac{a}{c}\right)$$
$$C_6 = I\left(g_6;i\infty,\frac{a}{c}\right)$$

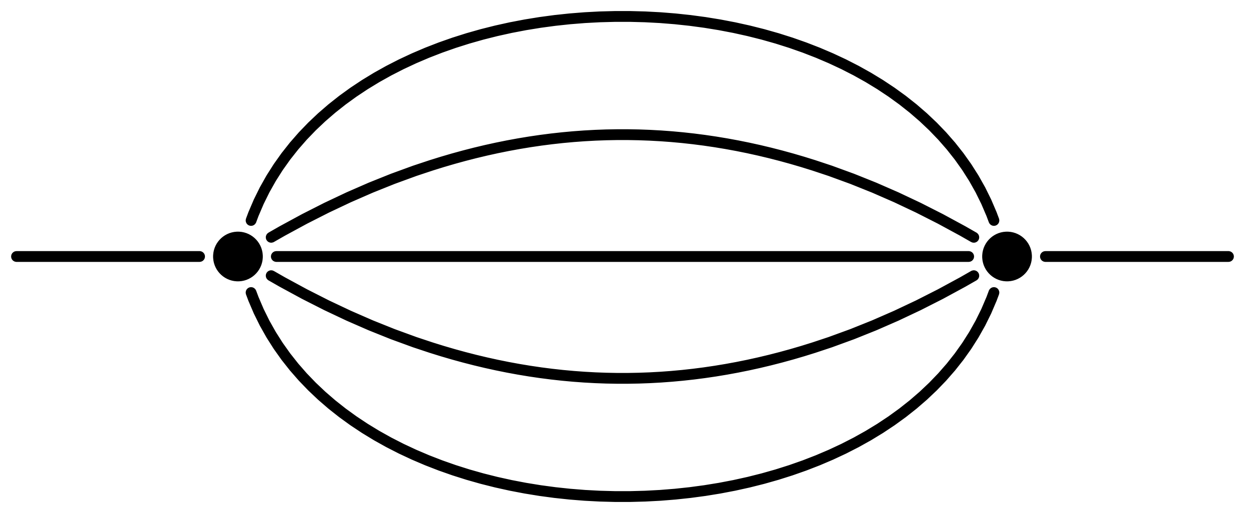
Integration between singular points obstructs simple evaluation. From numerics:

E.g. $a/c = 1/6$: $C_{1,6}=5$, $C_6 = \frac{1620\zeta_3}{\pi^4} - i\frac{42}{\pi}$

F_2 transforms "Quasi-Eichler" of modular weight 2 and depth 3

Modular Forms for n-fold CY Integrals

ε-Factorization of $(n+1)$-loop Banana



Example: $n=3$

Function space no longer elliptic!

Special Coordinate:
$q = \exp(2\pi i\tau)$,
$\tau = \varpi_1/\varpi_0$

See Christoph's talk

$$\mathrm{d}I^{(4)} = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2^{(4)} & 1 & 0 & 0 \\ 0 & \omega_4^{(4)} & \omega_2^{(4)} & Y & 0 \\ 0 & \omega_6^{(4)} & \omega_4^{(4)} & \omega_2^{(4)} & 1 \\ \omega_5^{(4)} & \omega_8^{(3)} & \omega_6^{(3)} & \omega_4^{(3)} & \omega_2^{(4)} \end{pmatrix} I^{(4)}$$

Natural extension of modular weight

ϖ_0: weight n

$\frac{\mathrm{d}\tau}{\mathrm{d}y}$: weight 2

Y: weight 0

$\omega_k^{(4)}$: Automorphic forms of "weight k"?

All $\omega_k^{(4)}$ have integral q-expansions (up to algebraic prefactors):

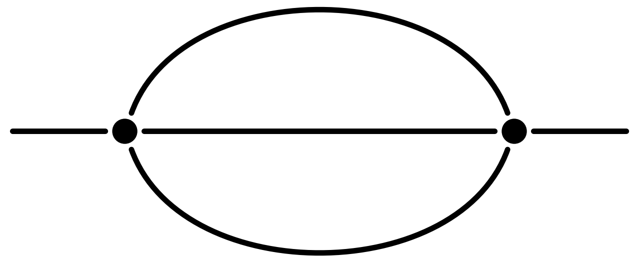
$$\omega_k^{(4)}(\tau+1) = \omega_k^{(4)}(\tau)$$

Monodromy Group of Periods

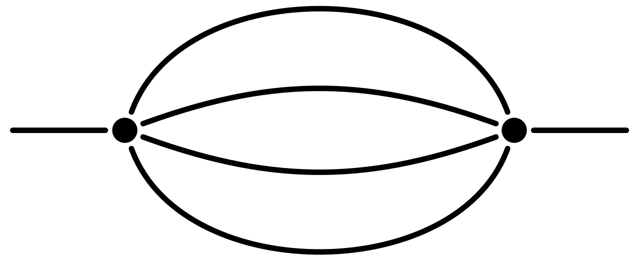
$$\Sigma = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}$$

Monodromy group Γ is subgroup of $O(\Sigma, \mathbb{Z}) = \{T \in \mathrm{GL}(r, \mathbb{Z}) | T^t \Sigma T = \Sigma\}$

[Bönisch, Duhr, Fischbach, Klemm, Nega]



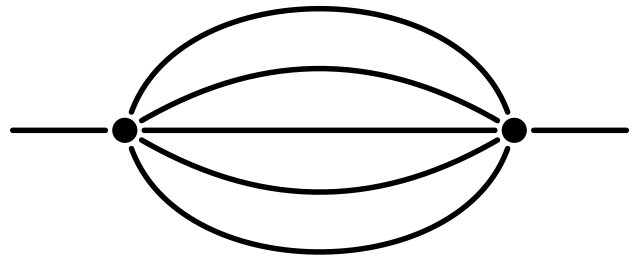
$\Gamma = \Gamma_1(6)$



$\Gamma = \Gamma_0(6)^{+}3 \supset \Gamma_1(6)$

[Verrill]

$$\Gamma_0(6)^{+}3 = \Gamma_0(6) \cup \left\{ \sqrt{3} \begin{pmatrix} a & b/3 \\ 2c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) | a, b, c, d \in \mathbb{Z} \right\}$$



$\Gamma =$ see [Bönisch, Duhr, Fischbach, Klemm, Nega]

“Automorphy” w.r.t. monodromy group?

Backup

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

3-loop banana in $d = 2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2\,(x-4)} + \frac{1}{2\,(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x\,(x-4)\,(x-16)}$$

Sunrise in disguise

with solutions $\psi_1,\ \psi_2,\ \mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

$$\omega_i \in \langle \psi_1^2,\ \psi_1\psi_2,\ \psi_2^2 \rangle$$