Geometries and Special Functions

for Physics and Mathematics

Single-valued polylogarithms and modular forms from zeta generators

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based on 2209.06772 & work in progress with D. Dorigoni, M. Doroudiani, J. Drewitt, M. Hidding, A. Kleinschmidt, N. Matthes, B. Verbeek

in progress with H. Frost, M. Hidding, D. Kamlesh, C. Rodriguez, B. Verbeek

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Outline

I. Genus zero: single-valued polylogarithms

[Brown 2004; Broedel, Sprenger, Torres Orjuela 1606.08411; Del Duca, Druc, . . .

Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 1606.08807]

[Frost, Hidding, Kamlesh, Rodriguez, OS, Verbeek: work in progress]

II. Genus one: non-holomorphic modular forms

[Brown 1707.01230, 1708.03354; Dorigoni, Doroudiani, Drewitt, Hidding, . . .

. . ., Kleinschmidt, Matthes, OS, Verbeek (DDDHKMSV) 2209.06772 & in progress]

III. Summary and outlook

I. Genus zero: single-valued polylogarithms

I. 1 Definitions and basics

How to construct single-valued versions of mero' polylogs ($a_j, z \in \mathbb{C}$)

$$G(a_1, a_2, \ldots, a_\ell; z) = \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2, \ldots, a_\ell; t) \, , \qquad G(\emptyset; z) = 1$$

- one variable $\leftrightarrow a_j \in \{0, 1\}$: [Brown 2004]

- ≥ 2 variables $z, y, \ldots \leftrightarrow a_j \in \{0, 1, y, \ldots\}$

[Broedel, Sprenger, Torres Orjuela 1606.08411 & Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 1606.08807]

Use meromorphic generating series with non-commutative variables e_0, e_1

$$I(e_0, e_1; z) = \sum_{\ell=0}^{\infty} \sum_{a_1, a_2, \ldots, a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} G(a_\ell, \ldots, a_2, a_1; z)$$

$$= 1 + e_0 G(0; z) + e_1 G(1; z) + \sum_{a_1, a_2 \in \{0,1\}} e_{a_1} e_{a_2} G(a_2, a_1; z) + \mathcal{O}(e_j^3)$$

I. 2 Reformulating the 1-variable construction

Decorate mero' generating series (non-commutative variables e_0, e_1) ...

$$I(e_0, e_1; z) = \sum_{\ell=0}^{\infty} \sum_{a_1,a_2,\ldots,a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} G(a_\ell, \ldots, a_2, a_1; z)$$

... with complex conjugate polylogs and multiple zeta values (MZVs)

$$I^{\rm SV}(e_0, e_1; z) = \sum_{\ell=0}^{\infty} \sum_{a_1,a_2,\ldots,a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} \underbrace{G^{\rm SV}(a_\ell, \ldots, a_2, a_1; z)}_{\substack{\text{single-valued polylogs} \\ \text{no monodromies around } z=0,1}}$$

$$= \big(\mathbb{M}^{\rm SV}(\sigma_k)\big)^{-1} \overline{I(e_0, e_1; z)^t} \, \mathbb{M}^{\rm SV}(\sigma_k) \, I(e_0, e_1; z)$$

Reversal $(e_i e_j)^t = e_j e_i$, so coefficient of $e_{a_1} e_{a_2} \ldots e_{a_\ell}$ is

$$G^{\rm SV}(a_\ell, \ldots, a_2, a_1; z) = \sum_{j=0}^{\ell} G(a_\ell, \ldots, a_{j+2}, a_{j+1}; z) \overline{G(a_1, a_2, \ldots, a_j; z)} + \text{MZVs}$$

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$$I^{\rm SV}(e_0,e_1;z) \ = \sum_{\ell=0}^{\infty} \sum_{a_1,a_2,\ldots,a_\ell \in \{0,1\}} e_{a_1} e_{a_2} \ldots e_{a_\ell} \ \underbrace{G^{\rm SV}(a_\ell,\ldots,a_2,a_1;z)}_{\substack{\text{single-valued polylogs} \\ \text{no monodromies around } z=0,1}}$$

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Reversal $(e_i e_j)^t = e_j e_i$ and series $\mathbb{M}^{\rm SV}$ in MZVs & zeta generators σ_k

$$\mathbb{M}^{\rm SV}(\sigma_k) = \ 1 + \sum_{k \in 2\mathbb{N}+1} 2\,\zeta_k \, \sigma_k + \sum_{k_1,k_2 \in 2\mathbb{N}+1} 2\,\zeta_{k_1} \zeta_{k_2} \, \sigma_{k_1} \sigma_{k_2} + \mathcal{O}(\sigma_k^3)$$

I. 2 Reformulating the 1-variable construction

$$I^{\rm sv}(e_0,e_1;z) \ = \sum_{\ell=0}^{\infty} \ \sum_{a_1,a_2,\ldots,a_\ell\in\{0,1\}} e_{a_1}e_{a_2}\ldots e_{a_\ell}\, G^{\rm sv}(a_\ell,\ldots,a_2,a_1;z)$$

$$= \ \big(\mathbb{M}^{\rm sv}(\sigma_k)\big)^{-1}\, \overline{I(e_0,e_1;z)^t}\, \mathbb{M}^{\rm sv}(\sigma_k)\, I(e_0,e_1;z)$$

Only single-valued MZVs enter $\mathbb{M}^{\rm sv}$

$$\mathbb{M}^{\rm sv}(\sigma_k) \ = \ 1 + \sum_{k\in 2\mathbb{N}+1} \underbrace{\phi^{-1}({\rm sv}\ f_k)}_{2\zeta_k}\sigma_k + \sum_{k_1,k_2\in 2\mathbb{N}+1} \underbrace{\phi^{-1}({\rm sv}\ f_{k_1}f_{k_2})}_{2\zeta_{k_1}\zeta_{k_2}}\sigma_{k_1}\sigma_{k_2} + \mathcal{O}(\sigma_k^3)$$

$$= \sum_{r=0}^{\infty} \ \sum_{k_1,k_2,\ldots,k_r\in 2\mathbb{N}+1} \underbrace{\phi^{-1}({\rm sv}\ f_{k_1}f_{k_2}\ldots f_{k_r})}_{\text{also } \zeta^{\rm sv}_{3,5,3} \text{ etc.}}\sigma_{k_1}\sigma_{k_2}\ldots\sigma_{k_r}$$

which are most conveniently described in the f-alphabet

$$\phi : \ (\text{motivic) MZVs} \ \to \ \mathbb{Q}[f_2]\otimes_{\mathbb{Q}}\mathbb{Q}\langle \underbrace{f_3,f_5,f_7,\ldots}_{\text{non-commutative}} \rangle$$

I. 3 Brief recap of the f-alphabet

Motivation: mod out by multitude of $\mathbb{Q}$-relations among (motivic) MZVs and obtain simple formulae for coaction and single-valued map!

Isomorphism ϕ: (motivic) MZVs $\rightarrow \mathbb{Q}[f_2] \otimes_{\mathbb{Q}} \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle$ with

- $\phi(\zeta_w) = f_w$ with f_{2k+1} non-commutative and $f_{2k} \in \mathbb{Q} f_2^k$ commutative
- preserving $\sqcup\!\sqcup$ and Δ such that for instance

$$\phi(\zeta_{3,5}) \ = \ -5 f_3 f_5 \, , \qquad \phi(\zeta_{3,5,3}) \ = \ -5 f_3 f_5 f_3 + \frac{299}{2} f_{11}$$

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- $\mathrm{sv}(f_2^N f_{i_1} f_{i_2} \ldots f_{i_r}) = \delta_{N,0} \sum_{j=0}^{r} f_{i_j} \ldots f_{i_2} f_{i_1} ⧢ f_{i_{j+1}} \ldots f_{i_r}$

  such that $\mathrm{sv}(\zeta_{2k+1}) = 2\zeta_{2k+1}$ and $\mathrm{sv}(\zeta_{3,5}) = -10\zeta_3\zeta_5$ etc.

I. 4 Commutators $\left[\text{braid operators } e_0, e_1 \,,\ \text{zeta generators } \sigma_k\right]$

$$I^{\rm SV}(e_0,e_1;z) \ = \ \Big(\mathbb{M}^{\rm SV}(\sigma_k)\Big)^{-1} \overline{I(e_0,e_1;z)^t}\, \mathbb{M}^{\rm SV}(\sigma_k)\, I(e_0,e_1;z)$$

Need to find representation in terms of only e_0, e_1 (i.e. no leftover σ_k) for

$$(\mathbb{M}^{\rm SV})^{-1}\,\overline{I(\ldots)^t}\,\mathbb{M}^{\rm SV} \ = \ \overline{I(\ldots)^t} + \sum_{k\in 2\mathbb{N}+1} 2\zeta_k \left[\overline{I(\ldots)^t}, \sigma_k\right]$$

$$+ \sum_{k_1,k_2\in 2\mathbb{N}+1} 2\zeta_{k_1}\zeta_{k_2} \left[\left[\overline{I(\ldots)^t}, \sigma_{k_1}\right], \sigma_{k_2}\right] + \mathcal{O}(\sigma_k^3)$$

Indeed, both of $[e_0,\sigma_k]$ and $[e_1,\sigma_k]$ boil down to words in e_0, e_1:

$$[e_0,\sigma_k] = 0 \quad \Longrightarrow \quad (\mathbb{M}^{\rm SV})^{-1}\,\overline{I(e_0,e_1)^t}\,\mathbb{M}^{\rm SV} \ = \ \overline{I\big(e_0, (\mathbb{M}^{\rm SV})^{-1} e_1 \mathbb{M}^{\rm SV}\big)^t}$$

$$[e_1,\sigma_k] = \ \Big[\Phi(e_0,e_1)\Big|_{\zeta_k}, e_1\Big] \qquad \underbrace{\text{Drinfeld associator } \Phi(e_0,e_1) = I(e_0,e_1;z{=}1)}_{\text{in } \mathbb{Q}\text{-basis of MZVs}}$$

[**e.g. Ihara 1992; Furushu 0011261**]

I. 5 Unpacking example

Recover $G^{\rm SV}(0,0,1,1;z)$ from ζ_3-correction to $\mathbb{M}^{\rm SV}=1+2\zeta_3\sigma_3+\ldots$

$$G^{\rm SV}(0,0,1,1;z) \;=\; I^{\rm SV}(e_0,e_1;z)\,\big|_{e_1e_1e_0e_0}$$

$$= \;(1-2\zeta_3\sigma_3+\ldots)\,\overline{I(e_0,e_1;z)^t}\,(1+2\zeta_3\sigma_3+\ldots)\,I(e_0,e_1;z)\,\big|_{e_1e_1e_0e_0}$$

$$= \;\overline{I(e_0,e_1;z)^t}\,I(e_0,e_1;z)\,\big|_{e_1e_1e_0e_0}+2\zeta_3[\overline{I(e_0,e_1;z)^t},\sigma_3]\,\big|_{e_1e_1e_0e_0}$$

By $[e_0,\sigma_3]=0$ and $[e_1,\sigma_3]=[[[e_0,e_1],e_0{+}e_1],e_1]$, obtain

$$G^{\rm SV}(0,0,1,1;z) \;=\; G(0,0,1,1;z)+G(0,0,1;z)\overline{G(1;z)}$$

$$+\,G(0,0;z)\overline{G(1,1;z)}+G(0;z)\overline{G(1,1,0;z)}$$

$$+\,\overline{G(1,1,0,0;z)}+2\zeta_3\overline{G(1;z)}$$

I. 6 Comments

- recover single-valued polylogarithms in 1 var of **[Brown 2004]** since

$$I^{\rm sv}(e_0,e_1;z) \ = \ \overline{I(e_0,e_1';z)^t}\, I(e_0,e_1;z)\ , \quad \text{where}$$

$$e_1' \ = \ \big(\mathbb{M}^{\rm sv}(\sigma_k)\big)^{-1} e_1\, \mathbb{M}^{\rm sv}(\sigma_k) \ = \ \big(\Phi^{\rm sv}(e_0,e_1)\big)^{-1} e_1\, \Phi^{\rm sv}(e_0,e_1)$$

[proof by Deepak Kamlesh, in progress]

- in $I^{\rm sv} = (\mathbb{M}^{\rm sv})^{-1}\overline{I^t}\mathbb{M}^{\rm sv} I$, $\left\{ \begin{array}{l} \mathbb{M}^{\rm sv} \text{ factor cancels monodromies of } I, \overline{I^t} \text{ at } z{=}1 \\ (\mathbb{M}^{\rm sv})^{-1} \text{ on the left “drains out” } \sigma_k \end{array}\right.$

- virtue of $\mathbb{M}^{\rm sv}$: depth-one data $\zeta_k \leftrightarrow [e_j,\sigma_k]$ already fixes higher depth:

  coeff's of $\zeta_{k_1}\zeta_{k_2}$ or $\zeta^{\rm sv}_{3,5,3}$, *etc.*, in $G^{\rm sv}(a_1,\ldots,a_\ell;z)$ from iterative $[\sigma_k,\cdot]$

- rewrote genus-zero construction of $G^{\rm sv}$ in this way to

  illustrate parallels to genus-one case (maybe also higher genus?)

I. 7 Generalizations

Conjugation with MZV-series also occurs in motivic coaction

$$\Delta I^{\mathfrak{m}}(e_0,e_1;z) = \big(\mathbb{M}^{\mathfrak{dr}}(\sigma_k)\big)^{-1} I^{\mathfrak{m}}(e_0,e_1;z)\,\mathbb{M}^{\mathfrak{dr}}(\sigma_k)\,I^{\mathfrak{dr}}(e_0,e_1;z)$$

$$\mathbb{M}^{\mathfrak{dr}}(\sigma_k) = \sum_{r=0}^{\infty} \sum_{k_1,k_2,\ldots,k_r \in 2\mathbb{N}+1} \phi^{-1}(f_{k_1} f_{k_2} \ldots f_{k_r})^{\mathfrak{dr}}\, \sigma_{k_1}\sigma_{k_2}\ldots\sigma_{k_r}$$

[see Britto, Mizera, Rodriguez, OS 2102.06206 for matrix representations]

Loosely speaking, $(\ldots)^{\mathfrak{m}}$ and $(\ldots)^{\mathfrak{dr}}$ distinguish 1st and 2nd entry of

$$\zeta^{\mathfrak{dr}}_{n_1,n_2} G^{\mathfrak{m}}(a_1,a_2;z) \rightarrow G(a_1,a_2;z) \otimes \zeta_{n_1,n_2}$$

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Similar construction for polylogs in multiple variables, e.g. 2-var case

$$I^{\rm SV}(e_0,e_1,e_y;z) = \sum_{\ell=0}^{\infty} \sum_{a_1,a_2,\ldots,a_\ell\in\{0,1,y\}} e_{a_1}e_{a_2}\ldots e_{a_\ell}\, G^{\rm SV}(a_\ell,\ldots,a_2,a_1;z)$$

$$= \big(I^{\rm SV}(g_0,g_1;y)\big)^{-1}\big(\mathbb{M}^{\rm SV}(\sigma_k)\big)^{-1}\,\overline{I(e_0,e_1,e_y;z)^t}$$

$$\times\, \mathbb{M}^{\rm SV}(\sigma_k)\, I^{\rm SV}(g_0,g_1;y)\, I(e_0,e_1,e_y;z)$$

where all brackets $[e_j,g_m]$ and $[e_j,\sigma_k]$ are expressible via e_0,e_1,e_y.

II. Genus one: non-holomorphic modular forms

II. 1 Iterated Eisenstein integrals at genus one

Meromorphic targets at genus one: iterated Eisenstein integrals

with holo' Eisenstein series $\mathrm{G}_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k}$ and $\mathcal{E}[\emptyset;\tau] = 1$

[Brown 1407.5167; Broedel, Matthes, OS 1507.02254]

$$\mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix};\tau\right] = (2\pi i)^{1+j_\ell-k_\ell} \int_\tau^{i\infty} \mathrm{d}\tau' \, (\tau')^{j_\ell} \, \mathrm{G}_{k_\ell}(\tau') \, \mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_{\ell-1} \\ k_1 & k_2 & \ldots & k_{\ell-1} \end{smallmatrix};\tau'\right]$$

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generating series: instead of e_0, e_1, non-commutative var's are derivations

$$\epsilon_0 \text{ and } \epsilon_k \longleftrightarrow \mathrm{G}_k \text{ at } k\geq 4 \text{ even with } \epsilon_k^{(j)} = \mathrm{ad}^j_{\epsilon_0}(\epsilon_k) \text{ and } \epsilon_k^{(k-1)} = 0$$

[Tsunogai 1995, Goncharov, Gangl-Kaneko-Zagier, Baumard-Schneps, Pollack]

$$J(\epsilon_k;\tau) = 1 + \sum_{k=4}^{\infty} (k-1) \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \mathcal{E}\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right] \epsilon_k^{(j)}$$

$$+ \sum_{k_1,k_2=4}^{\infty} (k_1-1)(k_2-1) \sum_{j_1=0}^{k_1-2} \sum_{j_2=0}^{k_2-2} \frac{(-1)^{j_1+j_2}}{j_1!\, j_2!} \mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix};\tau\right] \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} + \mathcal{O}\big((\epsilon_k^{(j)})^3\big)$$

II. 2 Towards non-holomorphic modular forms

Similar to $I^{\rm SV}(e_0,e_1) = (\mathbb{M}^{\rm SV})^{-1}\,\overline{I(e_0,e_1)^t}\,\mathbb{M}^{\rm SV}\,I(e_0,e_1)$ at genus $h=0$,

construct non-holomorphic modular forms from coeff's of $\epsilon_{k_1}^{(j_1)}\epsilon_{k_2}^{(j_2)}\ldots$ of

$$\big(\mathbb{M}^{\rm SV}(z_k)\big)^{-1}\,\overline{J(\epsilon_k;\tau)^t}\,\mathbb{M}^{\rm SV}(\sigma_k)\,J(\epsilon_k;\tau)$$

↑ only drain out z_k part of σ_k

However, novel feature of genus $h=1$ is that zeta generators σ_k

have $\epsilon_k^{(j)}$-dependent "geometric" part (besides "non-geometric" z_k)

$$\sigma_k \ = \ z_k - \frac{1}{(k-1)!}\,\epsilon_{k+1}^{(k-1)} + \big(\,\text{infinite series in nested brackets of } \epsilon_k^{(j)}\,\big)$$

for instance

$$\sigma_3 \ = \ z_3 - \tfrac{1}{2}\epsilon_4^{(2)} + \tfrac{1}{480}[\epsilon_4,\epsilon_4^{(1)}] + \tfrac{1}{120960}\big(4[\epsilon_4^{(1)},\epsilon_6] - [\epsilon_4,\epsilon_6^{(1)}]\big) + \tfrac{1}{7257600}[\epsilon_4,\epsilon_8^{(1)}]$$

$$-\tfrac{1}{1209600}[\epsilon_4^{(1)},\epsilon_8] + \tfrac{1}{383201280}\big(8[\epsilon_4^{(1)},\epsilon_{10}] - [\epsilon_4,\epsilon_{10}^{(1)}]\big) - \tfrac{1}{58060800}[\epsilon_4,[\epsilon_4,\epsilon_6]] + \ldots$$

$$\sigma_5 \ = \ z_5 - \tfrac{1}{24}\epsilon_6^{(4)} - \tfrac{5}{48}[\epsilon_4^{(1)},\epsilon_4^{(2)}] + \tfrac{1}{5760}\big([\epsilon_4^{(0)},\epsilon_6^{(3)}] - [\epsilon_4^{(1)},\epsilon_6^{(2)}] + [\epsilon_4^{(2)},\epsilon_6^{(1)}]\big) + \ldots$$

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$$\sigma_k = z_k - \frac{1}{(k{-}1)!}\,\epsilon_{k+1}^{(k-1)} + \Big(\underbrace{\text{infinite series in nested brackets of } \epsilon_k^{(j)}}\Big)$$

see **[Brown 1504.04737]** for explicit form of certain $[\epsilon_{k_1}^{(j_1)},\epsilon_{k_2}^{(j_2)}]$

Note that $[z_m,\epsilon_n]$ is expressible in terms of $\epsilon_k^{(j)}$ in the same way as

$[\sigma_m,e_j]$ at genus $h=0$ boiled down to words in e_j.

II. 3 Non-holo' modular forms from Eisenstein integrals

At fixed $k = 4, 6, 8, \ldots$, the derivations $\{\epsilon_k^{(0)}, \epsilon_k^{(1)}, \ldots, \epsilon_k^{(k-2)}\}$ subject to $\epsilon_k^{(k-1)} = 0$ form $(k{-}1)$-dim SL_2 multiplets with ladder operators

$$[\epsilon_0, \epsilon_k^{(j)}] = \epsilon_k^{(j+1)}\,, \qquad [\epsilon_0^\vee, \epsilon_k^{(j)}] = j(k{-}j{-}1)\epsilon_k^{(j-1)}$$

Need to perform SL_2-transformation

$$U(\tau) = \exp\Big(-\frac{\epsilon_0^\vee}{4\pi\,\mathrm{Im}\,\tau}\Big)\,\exp\big(2\pi i\bar\tau\epsilon_0\big)$$

in order to read off non-holo' modular forms $\beta^{\rm eqv}[\ldots;\tau]$ from $\#(\epsilon_k^{(j)})$

$$J^{\rm eqv}(\epsilon_k;\tau) = U(\tau)\,\big(\mathbb{M}^{\rm sv}(z_k)\big)^{-1}\,\overline{J(\epsilon_k;\tau)^t}\,\mathbb{M}^{\rm sv}(\sigma_k)\,J(\epsilon_k;\tau)\,U^{-1}(\tau)$$

$$= 1 + \sum_{k=4}^{\infty}(k{-}1)\sum_{j=0}^{k-2}\frac{(-1)^j}{j!}\,\beta^{\rm eqv}\big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\big]\,\epsilon_k^{(j)} + \mathcal{O}\big((\epsilon_k^{(j)})^2\big)$$

[Brown 1407.5167, 1707.01230, 1708.03354]

$\beta^{\rm eqv} \supset$ modular graph forms in $h{=}1$ string amplitudes **[DDDHKMSV 2209.06772]**

II. 4 Examples at depth one and two

$$J^{\rm eqv}(\epsilon_k;\tau) \;=\; U(\tau)\,\big(\mathbb{M}^{\rm sv}(z_k)\big)^{-1}\,\overline{J(\epsilon_k;\tau)^t}\,\mathbb{M}^{\rm sv}(\sigma_k)\,J(\epsilon_k;\tau)\,U^{-1}(\tau)$$

$$= 1 + \sum_{k=4}^{\infty}(k-1)\sum_{j=0}^{k-2}\frac{(-1)^j}{j!}\,\beta^{\rm eqv}\big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\big]\,\epsilon_k^{(j)} + \mathcal{O}\big((\epsilon_k^{(j)})^2\big)$$

From $\#(\epsilon_{2k}^{(k-1)})$, extract modular invariant non-holo' Eisenstein series

$$\mathrm{E}_k(\tau) \;=\; \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^k \sum_{(m,n)\neq(0,0)} \frac{1}{|m\tau+n|^{2k}} \;=\; -\frac{\Gamma(2k)}{\Gamma(k)^2}\,\beta^{\rm eqv}\big[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix};\tau\big]$$

$$\sim\; (\pi\mathrm{Im}\,\tau)^{1-k}\left\{\frac{2\zeta_{2k-1}}{2k-1} + \mathrm{Im}\int_{i\infty}^{\tau}\frac{\mathrm{d}\tau'}{\pi}\,(\tau-\tau')^{k-1}(\bar\tau-\tau')^{k-1}\,\mathrm{G}_{2k}(\tau')\right\}$$

Odd zeta values $\sim \zeta_{2k-1}$ from $\sigma_k = -\frac{1}{(k-1)!}\,\epsilon_{k+1}^{(k-1)} + \ldots$

i.e. the simplest term which is not drained out by $\big(\mathbb{M}^{\rm sv}(z_k)\big)^{-1}$.

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$$J^{\rm eqv}(\epsilon_k;\tau) \ = \ U(\tau)\, \big(\mathbb{M}^{\rm sv}(z_k)\big)^{-1} \, \overline{J(\epsilon_k;\tau)^t}\, \mathbb{M}^{\rm sv}(\sigma_k)\, J(\epsilon_k;\tau)\, U^{-1}(\tau)$$

$$= \ 1 + \sum_{k=4}^\infty (k{-}1) \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \, \beta^{\rm eqv}\big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\big]\, \epsilon_k^{(j)}$$

$$+ \sum_{k_1,k_2=4}^\infty (k_1{-}1)(k_2{-}1) \sum_{j_1=0}^{k_1-2}\sum_{j_2=0}^{k_2-2} \frac{(-1)^{j_1+j_2}}{j_1!j_2!} \, \beta^{\rm eqv}\big[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix};\tau\big]\, \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} + \ldots$$

Among $\#(\epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)})$ find higher-depth instances of modular graph forms

[D'Hoker, Green Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

$$\text{e.g.} \ \ C_{2,1,1}(\tau) \ = \ \Big(\frac{ {\rm Im}\, \tau }{\pi} \Big)^4 \sum_{(m_i,n_i)\neq (0,0)} \frac{ \delta(m_1{+}m_2{+}m_3)\, \delta(n_1{+}n_2{+}n_3) }{ |m_1\tau{+}n_1|^4 \ |m_2\tau{+}n_2|^2 \ |m_3\tau{+}n_3|^2 }$$

$$= \ -18\, \beta^{\rm eqv}\big[\begin{smallmatrix} 2 & 0 \\ 4 & 4 \end{smallmatrix};\tau\big] - 126\, \beta^{\rm eqv}\big[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix};\tau\big]$$

[Broedel, OS, Zerbini 1803.00527; Gerken, Kleinschmidt, OS 2004.05156]

II. 5 New effects at depth two

$$J^{\rm eqv}(\epsilon_k;\tau) = U(\tau)\,\big(\mathbb{M}^{\rm sv}(z_k)\big)^{-1}\,\overline{J(\epsilon_k;\tau)^t}\,\mathbb{M}^{\rm sv}(\sigma_k)\,J(\epsilon_k;\tau)\,U^{-1}(\tau)$$
$$= 1 + \sum_{k=4}^{\infty}(k-1)\sum_{j=0}^{k-2}\frac{(-1)^j}{j!}\,\beta^{\rm eqv}\big[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\big]\,\epsilon_k^{(j)} + \mathcal{O}\big((\epsilon_k^{(j)})^2\big)$$

In order to drain out z_k, need to iteratively use commutators

$$[z_{2m-1},\epsilon_0] = 0 = [z_{2m-1},\epsilon_0^\vee]\,, \qquad \text{“the } z_{2m-1} \text{ are SL}_2\text{-invariant”}$$

$$[z_{2m-1},\epsilon_{2n+2}] \sim \sum_{\ell=0}^{2m-2}\frac{(-1)^\ell}{\ell!}(2n+\ell)!\,\big[\epsilon_{2m}^{(\ell)},\epsilon_{2n+2m}^{(2m-2-\ell)}\big] + \mathcal{O}\big((\epsilon_k^{(j)})^3\big)$$

[Pollack master thesis 2009; Hain, Matsumoto 1512.03975]

$$\text{e.g. } [z_3,\epsilon_6] = \frac{1}{1200}\Big([\epsilon_8^{(2)},\epsilon_4] - 5[\epsilon_8^{(1)},\epsilon_4^{(1)}] + 15[\epsilon_8,\epsilon_4^{(2)}] + 63[\epsilon_4,[\epsilon_4^{(1)},\epsilon_4]]\Big)$$

$$\Longrightarrow \quad \beta^{\rm eqv}\big[\begin{smallmatrix} j_1 & j_2 \\ 8 & 4 \end{smallmatrix};\tau\big] \supset \zeta_3 \times \big(\text{antiholo' depth-one integrals of } \tau^j \mathrm{G}_6(\tau)\big)$$

II. 6 Predicting higher-depth MZVs in modular graph forms

Main virtue of the series $\mathbb{M}^{\rm sv}$ in $f_m \sigma_m$: coefficients of ζ_m in $\beta^{\rm eqv}$

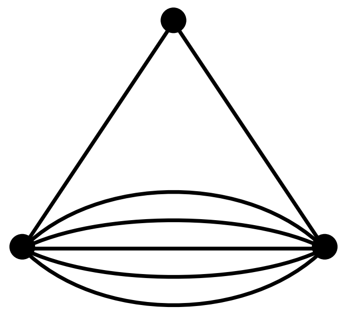
determine appearance of higher-depth $\zeta_a \zeta_b$ and $\zeta^{\rm sv}_{3,5,3}, \zeta^{\rm sv}_{3,7,3}, \zeta^{\rm sv}_{5,3,5}, \ldots$

E.g.: simplest term along with $\phi(\zeta^{\rm sv}_{3,5,3}) = -20(f_3 f_5 f_3 + f_5 f_3 f_3) + 299 f_{11}$

$$\sigma_3 \sigma_5 \sigma_3 = -\frac{1}{96} \epsilon_4^{(2)} \epsilon_6^{(4)} \epsilon_4^{(2)} + \left(\text{contributions to } \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} \ldots \text{ at } \sum_i k_i \geq 16 \right)$$

fixes coefficient of $\zeta^{\rm sv}_{3,5,3}$ in all $\beta^{\rm eqv}\left[\begin{smallmatrix} j_1 & j_2 & j_3 \\ 4 & 6 & 4 \end{smallmatrix}; \tau\right]$ and hence

in all modular graph forms of transcendental weight 7 **[Zerbini 1512.05689]**



$$\longleftrightarrow \quad \mathcal{C}^+\left[\begin{smallmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right] \Big|_{\tau \to i\infty} = \ldots - \frac{9\, \zeta^{\rm sv}_{3,5,3}}{8(\pi {\rm Im}\, \tau)^4} + \ldots$$

II. 7 How to proceed in practice

Since most parts of the $\epsilon_k^{(j)}$-expansion of σ_m are unknown ...

$$\sigma_3 = z_3 - \tfrac{1}{2}\epsilon_4^{(2)} + \tfrac{1}{480}[\epsilon_4, \epsilon_4^{(1)}] + \tfrac{1}{120960}\Big(4[\epsilon_4^{(1)}, \epsilon_6] - [\epsilon_4, \epsilon_6^{(1)}]\Big) + \tfrac{1}{7257600}[\epsilon_4, \epsilon_8^{(1)}]$$

$$- \tfrac{1}{1209600}[\epsilon_4^{(1)}, \epsilon_8] + \tfrac{1}{383201280}\Big(8[\epsilon_4^{(1)}, \epsilon_{10}] - [\epsilon_4, \epsilon_{10}^{(1)}]\Big) - \tfrac{1}{58060800}[\epsilon_4, [\epsilon_4, \epsilon_6]] + \ldots$$

$$\sigma_5 = z_5 - \tfrac{1}{24}\epsilon_6^{(4)} - \tfrac{5}{48}[\epsilon_4^{(1)}, \epsilon_4^{(2)}] + \tfrac{1}{5760}\Big([\epsilon_4^{(0)}, \epsilon_6^{(3)}] - [\epsilon_4^{(1)}, \epsilon_6^{(2)}] + [\epsilon_4^{(2)}, \epsilon_6^{(1)}]\Big)$$

$$- \tfrac{1}{145152}\Big([\epsilon_6^{(0)}, \epsilon_6^{(3)}] - [\epsilon_6^{(1)}, \epsilon_6^{(2)}]\Big) + \tfrac{1}{6912}\Big([\epsilon_4^{(1)}, [\epsilon_4^{(1)}, \epsilon_4^{(0)}]] + 2[\epsilon_4^{(0)}, [\epsilon_4^{(0)}, \epsilon_4^{(2)}]]\Big) + \ldots$$

... make ansatz for new terms $\sigma_m \supset \epsilon_{k_1}^{(j_1)} \ldots \epsilon_{k_r}^{(j_r)}$ at $r+j_1+\ldots+j_r = m$

and determine $\mathbb{Q}$ coeff's of nested brackets by numerically imposing $\mathrm{SL}_2(\mathbb{Z})$

$$\beta^{\rm eqv}\Big[\begin{smallmatrix} j_1 & \ldots & j_r \\ k_1 & \ldots & k_r \end{smallmatrix}; \tfrac{a\tau+b}{c\tau+d}\Big] = \left(\prod_{i=1}^{r} (c\bar{\tau}+d)^{k_i-2-2j_i}\right) \beta^{\rm eqv}\Big[\begin{smallmatrix} j_1 & \ldots & j_r \\ k_1 & \ldots & k_r \end{smallmatrix}; \tau\Big]$$

Moreover, $[z_m, \epsilon_k]$ fixed by σ_m via "inertial relation" **[Brown 1407.5167]**

$$[\sigma_m, N] = 0\,, \quad N = -\epsilon_0 + \sum_{k\geq 2} \frac{(2k-1)\mathrm{B}_{2k}}{(2k)!}\,\epsilon_{2k}$$

II. 8 Ambiguities

While z_3, z_5 are still canonical, $\exists$ ambiguities in the splitting of

$$\sigma_m \;=\; z_m + (\epsilon_k^{(j)}\text{-valued “rest”}) \quad \text{for any } m = 7, 9, \ldots$$

[Brown 1708.03354]

Can for instance redefine $z_7 \rightarrow z_7 + \mathbb{Q}\, \delta z_7$ by SL_2-invariant

$$\begin{aligned}\delta z_7 \;=\; & -[\epsilon_4,[\epsilon_4,\epsilon_6^{(4)}]] + [\epsilon_4,[\epsilon_4^{(1)},\epsilon_6^{(3)}]] - [\epsilon_4,[\epsilon_4^{(2)},\epsilon_6^{(2)}]] \\ & +[\epsilon_4^{(1)},[\epsilon_4,\epsilon_6^{(3)}]] - 2[\epsilon_4^{(1)},[\epsilon_4^{(1)},\epsilon_6^{(2)}]] + 3[\epsilon_4^{(1)},[\epsilon_4^{(2)},\epsilon_6^{(1)}]] \\ & -[\epsilon_4^{(2)},[\epsilon_4,\epsilon_6^{(2)}]] + 3[\epsilon_4^{(2)},[\epsilon_4^{(1)},\epsilon_6^{(1)}]] - 6[\epsilon_4^{(2)},[\epsilon_4^{(2)},\epsilon_6]]\end{aligned}$$

$\Longrightarrow$ left-multiplies J^{eqv} by $(1 + 2\zeta_7\, \delta z_7 + \ldots)$ and

shifts mod. invariant $\beta^{\text{eqv}}\left[\begin{smallmatrix} j_1 & j_2 & j_3 \\ 4 & 6 & 4 \end{smallmatrix}\right]$ at $j_1{+}j_2{+}j_3 = 4$ by $\mathbb{Q}$-multiples of ζ_7

II. 9 Equivariant versus single-valued Eisenstein integrals

Ambiguities in splitting $\sigma_m = z_m + \epsilon' s$ only affects

equivariant iterated Eisenstein integrals $\leftrightarrow$ modular graph forms

$$J^{\text{eqv}}(\epsilon_k;\tau) \;=\; U(\tau)\, \big(\mathbb{M}^{\text{sv}}(z_k)\big)^{-1}\, \overline{J(\epsilon_k;\tau)^t}\, \mathbb{M}^{\text{sv}}(\sigma_k)\, J(\epsilon_k;\tau)\, U^{-1}(\tau)$$

$\exists$ canonically defined "single-valued iterated Eisenstein int's" to all orders

$$J^{\text{sv}}(\epsilon_k;\tau) \;=\; U(\tau)\, \big(\mathbb{M}^{\text{sv}}(\sigma_k)\big)^{-1}\, \overline{J(\epsilon_k;\tau)^t}\, \mathbb{M}^{\text{sv}}(\sigma_k)\, J(\epsilon_k;\tau)\, U^{-1}(\tau)$$

However, components $\#(\epsilon_{k_1}^{(j_1)}\epsilon_{k_2}^{(j_2)}\ldots)$ are no longer modular forms, e.g.

$$J^{\text{sv}}(\epsilon_k;\tau)\Big|_{\epsilon_{2k}^{(k-1)}} \;=\; \mathrm{E}_k(\tau) - \frac{2\,\Gamma(2k{-}1)\,\zeta_{2k-1}}{\Gamma(k)^2\,(4\pi \mathrm{Im}\,\tau)^{k-1}} \quad \text{different mod. weights!}$$

[Brown 1708.03354]

III. Summary and outlook

III. 1 Summary

- revisited construction of single-valued polylogarithms at genus zero: antiholomorphic generating series $\overline{I(e_0, e_1; z)}^t$ of polylogs twisted by

  $\mathbb{M}^{\rm sv}(\sigma_k) \longleftrightarrow$ single-valued MZVs paired with zeta-generators σ_k

- similar construction applies to iterated Eisenstein integrals at genus one: non-holomorphic modular forms in $\overline{J(\epsilon_k; \tau)}^t$ similarly twisted via σ_k

  in both cases: $[\sigma_m, \bullet]$ expressible in terms of $\bullet \rightarrow e_j$ or ϵ_k

- new features at genus one include (non-canonical) splitting $\sigma_k = z_k + \epsilon' s$ and SL_2-multiplet structure of the non-commutative variables $\epsilon_k^{(j)}$

III. 2 Further directions

- by analogy with genus zero, possible new line attack for deRham periods in elliptic coaction formulae [Broedel, Duhr, Dulat, Penante, Tancredi 1803.10256]

$$\Delta I^{\mathfrak{m}}(e_0, e_1; z) = \big(\mathbb{M}^{\mathfrak{dr}}(\sigma_k)\big)^{-1} I^{\mathfrak{m}}(e_0, e_1; z)\, \mathbb{M}^{\mathfrak{dr}}(\sigma_k)\, I^{\mathfrak{dr}}(e_0, e_1; z)$$

$$\Delta J^{\mathfrak{m}}(\epsilon_k; \tau) \stackrel{?}{=} \big(\mathbb{M}^{\mathfrak{dr}}(\sigma_k)\big)^{-1} J^{\mathfrak{m}}(\epsilon_k; \tau)\, \mathbb{M}^{\mathfrak{dr}}(\sigma_k)\, J^{\mathfrak{dr}}(\epsilon_k; \tau)$$

- construct sv elliptic polylogs by conjugating with series in $\epsilon_0, \epsilon_{k\geq 4}$ and additional non-commutative variables $b_{k\geq 2} \longrightarrow$ need brackets with σ_m
[Broedel, Kaderli, OS 2007.03712; Hidding, OS, Verbeek 2208.11116; Sohnle: WIP]

- describe modular graph tensors at genus $h \geq 2$ [D'Hoker, OS 2010.00924] in similar framework, with analogue of ϵ_k for holo' modular tensors

Backup

Contributions to zeta generators from modular graph forms:

$$\text{e.g.}\quad C_{2,1,1}(\tau) = \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^4 \sum_{(m_i,n_i)\neq(0,0)} \frac{\delta(m_1{+}m_2{+}m_3)\,\delta(n_1{+}n_2{+}n_3)}{|m_1\tau{+}n_1|^4\,|m_2\tau{+}n_2|^2\,|m_3\tau{+}n_3|^2}$$

$$= -18\,\beta^{\rm eqv}\big[\begin{smallmatrix}2 & 0\\ 4 & 4\end{smallmatrix};\tau\big] - 126\,\beta^{\rm eqv}\big[\begin{smallmatrix}3\\ 8\end{smallmatrix};\tau\big]$$

Expansion around $(\tau\to i\infty)$ cusp $\Rightarrow$ Laurent polynomial in $y=\pi\mathrm{Im}\,\tau$:

$$C_{2,1,1}(\tau) = \frac{2y^4}{14175} + \frac{y\zeta_3}{45} + \frac{5\zeta_5}{12y} - \frac{\zeta_3^2}{4y^2} + \frac{9\zeta_7}{16y^3} + \mathcal{O}(e^{-2\pi\mathrm{Im}\,\tau})$$

Upon comparison with expansion of $J^{\rm eqv}(\epsilon_k;\tau)$, infer highlighted term of

$$\sigma_5 = z_5 - \tfrac{1}{24}\epsilon_6^{(4)} - \tfrac{5}{48}[\epsilon_4^{(1)},\epsilon_4^{(2)}] + \tfrac{1}{5760}\big([\epsilon_4^{(0)},\epsilon_6^{(3)}] - [\epsilon_4^{(1)},\epsilon_6^{(2)}] + [\epsilon_4^{(2)},\epsilon_6^{(1)}]\big)$$

$$- \tfrac{1}{145152}\big([\epsilon_6^{(0)},\epsilon_6^{(3)}]-[\epsilon_6^{(1)},\epsilon_6^{(2)}]\big) + \tfrac{1}{6912}\big([\epsilon_4^{(1)},[\epsilon_4^{(1)},\epsilon_4^{(0)}]]+2[\epsilon_4^{(0)},[\epsilon_4^{(0)},\epsilon_4^{(2)}]]\big) + \ldots$$