Mould theory and the elliptic associator

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Geometries and Special Functions for Physics and Mathematics

Bethe Institute, Bonn, March 20-24, 2023

Part I. Elliptic associators and elliptic multizeta values

Let θ be the (odd) Jacobi theta function,

$$\begin{aligned}\theta(z;\tau) &= \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} q^{(\frac{n+1}{2})^2} e^{(2n+1)iz} \\ &= 2q^{\frac{1}{4}}\sin(z) - 2q^{\frac{9}{4}}\sin(3z) + 2q^{\frac{25}{4}}\sin(5z) + \cdots\end{aligned}$$

with $q = e^{i\pi\tau}$, and τ runs over the Poincaré upper half-plane. Let

$$F_\tau(u,v) = \frac{\theta(u+v;\tau)}{\theta(u;\tau)\theta(v;\tau)}$$

denote the Kronecker function.

The starting point in the construction of the elliptic associator by B. Enriquez is the pair of iterated integrals (for each $r \geq 1$) over simplices on the boundary components $[0;1]$ and $[0;\tau]$ of the fundamental parallelogram of the elliptic curve associated to $\tau \in \mathcal{H}$:

$$I^{A_\tau}(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<1} F_\tau(u_1,v_1)\cdots F_\tau(u_r,v_r)\,dv_r\cdots dv_1$$

$$I^{B_\tau}(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<\tau} F_\tau(u_1,v_1)\cdots F_\tau(u_r,v_r)\,dv_r\cdots dv_1,$$

Fix a field k and a k-algebra R. A *mould* is a family

$$M = (M_r)_{r \geq 0}$$

of functions defined over R, such that $M(\emptyset) \in k$ and $M_r(u_1, \ldots, u_r)$ is a function of r commutative variables. In this talk we consider only rational functions and polynomials. The vector space of moulds with constant term 0 is denoted ARI, the set of moulds with constant term 1 is denoted GARI.

The functions I^{A_τ} and I^{B_τ} are thus *moulds*, with constant term 1.

Proposition. (B. Enriquez) *(i) For each $r \geq 1$, define moulds A_τ and B_τ by setting*

$$\begin{cases} A_\tau(u_1, \ldots, u_r) = u_1 \cdots u_r I^{A_\tau}(u_1, \ldots, u_r) \\ B_\tau(u_1, \ldots, u_r) = u_1 \cdots u_r I^{B_\tau}(u_1, \ldots, u_r) \end{cases}$$

for each $r \geq 1$. Then A_τ and B_τ are both polynomial moulds.

Definition. The pair (A_τ, B_τ) is known as the *elliptic associator*.

Polynomial moulds

Let $\mathrm{Lie}[a,b]$ denote the degree-completed free Lie algebra on two generators and $\mathbb{Q}\langle\langle a,b\rangle\rangle$ its universal enveloping algebra. We can write

$$\mathbb{Q}\langle\langle a,b\rangle\rangle \supset \mathbb{Q}a \oplus \mathbb{Q}\langle\langle c_1,c_2,\ldots\rangle\rangle$$

where for $i \geq 0$ we set

$$c_i = ad(a)^{i-1}(b).$$

The subspace $\mathbb{Q}\langle\langle c_1,c_2,\ldots\rangle\rangle$ contains all elements in the kernel of the derivation $partial_x$ defined by $\partial_x(x)=1$, $\partial_x(y)=0$ and contains all the series we will see (Lie-like, group-like etc.)

There is an an isomorphism

$$\mathbb{Q}\langle\langle c_1,c_2,\ldots\rangle\rangle \xrightarrow{\sim} \mathrm{ARI}^{pol}$$

given by linearly extending the map on monomials

$$c_{a_1}\cdots c_{a_r} \mapsto u_1^{a_1-1}\cdots u_r^{a_r-1}.$$

By a slight abuse of notation, we write A_τ and B_τ also for the power series in a,b associated to the moulds. Enriquez shows that these power series are *group-like*, i.e. they lie in the group

$$\exp(\mathrm{Lie}[a,b]) \subset \mathbb{Q}\langle\langle a,b\rangle\rangle.$$

Goal: Show how the elliptic associator breaks into two parts, an arithmetic part and a geometric part; show how the arithmetic part comes from the Drinfeld associator in genus zero; show how any genus zero associator gives rise to an elliptic associator.

Caveat: we work mod $2\pi i$, see more below.

Quick reminder on multizeta values and the Drinfeld associator.

For each sequence $(k_1, \ldots, k_r)$ of strictly positive integers, $k_1 \geq 2$, the **multiple zeta value** is defined by the convergent series

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

These real numbers have been studied since Euler (1775).

They form a $\mathbb{Q}$-algebra, the *multizeta algebra* $\mathcal{Z}$.

Multiplication of multizeta values

It is easy to see that we have the iterated integral form for multizeta values

$$\zeta(k_1,\ldots,k_r) = (-1)^r \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n-\epsilon_n} \cdots \frac{dt_2}{t_2-\epsilon_2}\frac{dt_1}{t_1-\epsilon_1}$$

where

$$(\epsilon_1,\ldots,\epsilon_n) = (\underbrace{0,\ldots,0}_{k_1-1},1,\underbrace{0,\ldots,0}_{k_2-1},1,\ldots,\underbrace{0,\ldots,0}_{k_r-1},1).$$

The product of two simplices is a union of simplices, giving an expression for the product of two multizeta values as a sum of multizeta values. This is the **shuffle product**.

Example. We have

$$\zeta(2) = \int_0^1 \int_0^{t_1} \frac{dt_2}{1-t_2}\frac{dt_1}{t_1}$$

$$\zeta(2,2) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4}\frac{dt_3}{t_3}\frac{dt_2}{1-t_2}\frac{dt_1}{t_1}$$

$$\zeta(3,1) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{t_1}$$

and

$$\zeta(2)^2 = 2\,\zeta(2,2) + 4\,\zeta(3,1).$$

Convergent and non-convergent words

A **convergent word** $w \in \mathbb{Q}\langle x, y\rangle$ is a word $w = xvy$.

The reason for this notation is that it gives a bijection

$$\begin{aligned}\{\text{tuples with } k_1 \geq 2\} &\leftrightarrow \{\text{convergent words}\}\\ (k_1, \ldots, k_r) &\leftrightarrow x^{k_1-1}y \cdots x^{k_r-1}y.\end{aligned}$$

As a notation, we use this to write

$$\zeta(k_1, \ldots, k_r) = \zeta(x^{k_1-1}y \cdots x^{k_r-1}y).$$

We extend the definition to $\zeta(w)$ for any word $w = y^a u x^b$ with u convergent:

$$\zeta(w) = \sum_{r=0}^{a}\sum_{s=0}^{b}(-1)^{r+s}\zeta\big(sh(y^r, y^{a-r}ux^{b-s}, x^s)\big).$$

Proposition. *The $\zeta(w)$ for all words w satisfy the shuffle relations*

$$\zeta(w)\zeta(u) = \zeta\big(sh(w, u)\big).$$

The *depth* of a word w the number of y's and the *weight* is the degree; correspondingly, the *depth* of $\zeta(k_1, \ldots, k_r)$ is r and the *weight* is $k_1+\cdots+k_r$.

The Drinfel'd associator

Definition. The *Drinfel'd associator* is the power series given by

$$\Phi_{KZ}(x,y) = 1 + \sum_{w \in \mathbb{Q}\langle x,y\rangle} (-1)^{d_w} \zeta(w) w$$

where d_w is the number of y's in the word w. It is a generating series for multizeta values.

• It can be obtained as monodromy of the KZB equation

$$\frac{d}{dz} G(z) = \Big(\frac{x}{v} + \frac{y}{1-v}\Big) G(z);$$

more specifically $\Phi_{KZ}(x,y) = G_1(z)^{-1} G_0(z)$, where G_0 (resp. G_1) is the solution to the KZ equation that tends to z^x as $z \to 0$ (resp. to $(1-z)^y$ as $z \to 1$).

• If Φ^r_{KZ} denotes the depth r part of Φ_{KZ}, then Φ^r_{KZ} is given by the iterated integral

$$\Phi^r_{KZ}(x,y) = \int_{0<v_r<\cdots<v_1<1} \Big(\frac{x}{v_1} + \frac{y}{1-v_1}\Big) \cdots \Big(\frac{x}{v_r} + \frac{y}{1-v_r}\Big)\, dv_r \cdots dv_1.$$

Geometric part of the elliptic associator

For $i \geq 0$, let ϵ_{2k} denote the derivation of $\mathrm{Lie}[a,b]$ defined by

$$\epsilon_{2k}(a) = ad(a)^{2k}(b), \quad \epsilon_{2k}([a,b]) = 0,$$

and let $\mathfrak{u}$ denote the Lie subalgebra of $Der^0(\mathrm{Lie}[a,b])$ generated by these.

The Lie algebra $\mathrm{Lie}[\epsilon_0, \epsilon_2, \epsilon_4, \ldots]$ is far from free. There are many interesting relations among the derivations ϵ_{2i}, closely related to period polynomials associated to cusp forms on $\mathrm{SL}_2(\mathbb{Z})$.

Example. $[\epsilon_4, \epsilon_{10}] - 3[\epsilon_6, \epsilon_8] = 0$ in weight 14 corresponds to the period polynomial

$$(X^8 - X^2) - 3(X^6 - X^4)$$

associated to the Ramanujan Δ-function. These relations were first investigated by A. Pollack.

Let g_τ denote the power series in $\mathcal{U}\mathfrak{u}$

$$g_\tau = id + \sum_{n>0} \sum_{(k_1,\dots,k_n)} \mathcal{G}_{(2k_1,\dots,2k_n)}(\tau)\epsilon_{2k_1} \circ \cdots \circ \epsilon_{2k_n}$$

where all $k_i \geq 0$ and $\mathcal{G}_{(2k_1,\dots,2k_n)}(\tau)$ denotes (a regularization of) *the iterated integral of the Eisenstein series G_{2k} from τ to $i\infty$.* This g_τ satisfies the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau} g_\tau = -\Big(\sum_{k\geq 0} G_{2k}(\tau)\epsilon_{2k}\Big) g_\tau,$$

so g_τ is group-like, so it gives an automorphism of $\mathbb{Q}\langle\langle a,b\rangle\rangle$.

Let $\mathcal{E}$ denote the $\mathbb{Q}$-algebra generated inside $\mathcal{O}(\mathcal{H})$ by the coefficients of g_τ written in any basis of the subspace generated inside $\mathcal{U}\mathfrak{u}$ by the monomials $\epsilon_{2k_1} \circ \cdots \circ \epsilon_{2k_n}$. These coefficients are "sufficiently independent" in the sense that we have:

Proposition.

$$\mathcal{E} \simeq \mathcal{U}\mathfrak{u}^*.$$

Elliptic multiple zeta algebra

In analogy with the $\mathbb{Q}$-algebra $\mathcal{Z}$ of multizeta values arising as the coefficients of the Drinfeld associator, we write $\mathcal{EZ}$ for the $\mathbb{Q}$-algebra of *elliptic multiple zeta values*, generated by the coefficients of A_τ and B_τ (essentially just A_τ). Note that these form a $\mathbb{Q}$-algebra thanks to the fact that A_τ is group-like.

Let

$$t_{01} = \frac{ad(b)}{\exp(b)-1}(-a), \quad t_{02} = \frac{ad(-b)}{\exp(-b)-1}(a), \quad t_{12} = [a,b]$$

lie inside the free Lie algebra $\mathrm{Lie}[a,b]$. They satisfy $t_{01}+t_{02}+t_{12}=0$.

We can view this as the image of a map

$$\begin{aligned} \mathrm{Lie}[x,y] &\to \mathrm{Lie}[a,b] \\ x,y,z &\mapsto t_{12}, t_{01}, t_{02} \end{aligned}$$

which is the Lie algebra (or pro-unipotent) "translation" of the homomorphism of topological π_1s of the thrice-punctured sphere to the once-punctured torus coming from joining two of the punctures.

Theorem. (Enriquez) *Let Φ_{KZ} be the Drinfeld associator, and set*

$$A = \Phi_{KZ}(t_{01}, t_{12})^{-1} e^{2\pi i t_{01}} \Phi_{KZ}(t_{01}, t_{12}).$$

Then

$$A_\tau = g_\tau(A)$$

where

This theorem shows that the coefficients of A_τ are polynomial combinations of multiple zeta values and $2\pi i$ and elements of $\mathcal{E}$ (all viewed inside $\mathcal{O}(\mathcal{H})$).

Theorem. *We have the isomorphism*

$$\mathcal{E}\mathcal{Z} \simeq \mathcal{Z}[2\pi i] \otimes_{\mathbb{Q}} \mathcal{E}.$$

Modulo the ideal generated by $2\pi i$, we have

$$\overline{\mathcal{E}\mathcal{Z}} \simeq \overline{\mathcal{Z}} \otimes_{\mathbb{Q}} \mathcal{E}.$$

If we replace the multiple zeta values by their motivic versions (or make the conjecture that the motivic multizeta value algebra is isomorphic to the real one), we then have:

Theorem. (F. Brown) $\overline{\mathcal{Z}}$ *is a Hopf algebra dual to the universal enveloping algebra of a free Lie algebra with one generator in each odd rank* ≥ 3*:*

$$\overline{\mathcal{Z}}^{\vee} \simeq \mathcal{U}\mathrm{Lie}[\sigma_3, \sigma_5, \ldots].$$

Recall that we had

$$\mathcal{E}^{\vee} = \mathcal{U}\mathfrak{u}$$

where $\mathfrak{u} \subset Der^0(\mathrm{Lie}[a,b])$ is genearted by the ϵ_{2k}.

$$\overline{\mathcal{E}\mathcal{Z}}^{\vee} = \mathcal{U}\big(\mathfrak{u} \rtimes \mathrm{Lie}[\sigma_3, \sigma_5, \ldots]\big).$$

Alternatively, if $\overline{\mathbf{ez}}$ denotes the quotient of the Hopf algebra $\overline{\mathcal{E}\mathcal{Z}}$ by products, then

$$\overline{\mathbf{ez}}^{\vee} \simeq \mathfrak{u} \rtimes \mathrm{Lie}[\sigma_3, \sigma_5, \ldots].$$

Brief motivic remark

The Lie algebra $\mathrm{Lie}[\sigma_3, \sigma_5, \ldots]$ is the fundamental Lie algebra (Lie algebra of the pro-unipotent radical of the Tannakian fundamental group) of the Tannakian category MTM of mixed Tate motives over $\mathbb{Z}$, and as we saw above, its dual is the Hopf algebra of motivic multizeta values mod products.

In the elliptic situation, Hain and Matsumoto constructed the Tannakian category MEM of mixed elliptic motives and showed that the fundamental Lie algebra is isomorphic to

$$\mathfrak{u} \rtimes \mathrm{Lie}[\sigma_3, \sigma_5, \ldots].$$

As we just saw, the universal enveloping algebra is dual to the Hopf algebra of "elliptic motivic multizetas", i.e. elliptic multizetas in which the "real multizetas" are replaced by the motivic ones.

Part II. How mould theory helps

There is much interplay between genus zero and elliptic associators is reflected for example in fact that $\mathcal{Z}$ is contained in $\mathcal{E}\mathcal{Z}$, or in the identity

$$A_\tau = g_\tau\big(\Phi_{KZ}(t_{01},t_{12})^{-1}e^{2\pi i t_{01}}\Phi_{KZ}(t_{01},t_{12})\big)$$

(where $t_{01} = Ber_b(-a)$, $t_{12} = [a,b]$), or in the interplay between the genus zero multizeta values in $\mathcal{Z}$ and the geometric elements in $\mathcal{E}$, or in the dual situation in which the elliptic fundamental Lie algebra takes the form

$$\mathfrak{u}\rtimes\mathrm{Lie}[\sigma_3,\sigma_5,\ldots].$$

Mould theory is particularly helpful in understanding this interplay.

To every element $f \in \mathrm{Lie}[x,y]$, associate a derivation $D_f \in \mathrm{Der}(\mathrm{Lie}[x,y])$ by

$$D_f(x) = 0, \quad D_f(y) = [y, f].$$

Define a Lie algebra $\mathbb{L}$ by the underlying vector space of $\mathrm{Lie}[x,y]$ and the *Ihara bracket* defined by

$$\{f, g\} = D_f(g) - D_g(f) + [f, g]$$

or equivalently, by

$$[D_f, D_g] = D_{\{f,g\}}.$$

There is an exponential map from $\mathbb{L}$ to $\exp(\mathbb{L})$, a Campbell-Hausdorff law defining the multiplication in the group $\exp(\mathbb{L})$ and and an adjoint action of $\exp(\mathbb{L})$ on $\mathbb{L}$, denoted $ad_{\{,\}}(F)$ for $F \in \exp(\mathbb{L})$.

All of these definitions were extended by Écalle to all of ARI by explicit formulas. The bracket $\{,\}$ extends to a Lie bracket called the *ari*-bracket on ARI, written $ari(A,B)$. There is an exponential map from ARI equipped with the *ari*-bracket to GARI, which gives a multiplication law on GARI, written $gari(P,Q)$, and an an adjoint action of GARI on ARI, denote $ad_{ari}(P)$ for $P \in$ GARI. Let *invpal* denote the inverse of *pal* for the GARI multiplication.

Let Δ be the map from ARI to ARI defined by

$$\Delta(A)(u_1,\ldots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)A(u_1,\ldots,u_r),$$

and let Δ^* be its "group version" that makes the diagram

$$\begin{array}{ccc} \mathrm{GARI} & \xrightarrow{\Delta^*} & \mathrm{GARI} \\ \uparrow & & \uparrow \\ \mathrm{ARI} & \xrightarrow{\Delta} & \mathrm{ARI} \end{array}$$

commute.

Écalle's magic mould *pal*

Let *dupal* be the "Bernoulli map mould"

$$dupal(u_1,\ldots,u_r) = \frac{B_r}{r!}\frac{1}{u_1\cdots u_r}\left(\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}u_{j+1}\right).$$

Let *dur* be the mould operator defined by

$$dur(P)(u_1,\ldots,u_r) = (u_1+\cdots+u_r)P(u_1,\ldots,u_r).$$

Define the mould *pal* recursively by $pal(\emptyset)=1$ and

$$dur(pal) = pal\cdot dupal.$$

In low weights for instance, We have

$$\begin{cases} pal(u_1) = -\frac{1}{2u_1} \\ pal(u_1,u_2) = \frac{u_1+2u_2}{12u_1u_2(u_1+u_2)} \end{cases}$$

We write *invpal* for the inverse of *pal* in GARI.

Construction Theorem for the elliptic associatior. *Let $\overline{\Phi}_{KZ}$ denote the Drinfeld associator mod $\zeta(2)$. Let P_{KZ} denote the associated polynomial-valued mould. Let*

$$C = \Delta^*\big(gari(invpal, P_{KZ}, pal)\big)$$

and set

$$C_\tau = g_\tau(C).$$

Then there exists an automorphism of

$$\exp\big(\mathrm{Lie}[a,b] \otimes_{\mathbb{Q}} \mathcal{O}(\mathcal{H})\big)$$

mapping $e^a \mapsto C_\tau$ and fixing $[a,b]$. This automorphism also maps

$$e^{t_{01}} \mapsto A_\tau, \quad e^b \mapsto B_\tau.$$

Another application of mould theory is to make the semi-direct product

$$\mathfrak{u} \rtimes \mathrm{Lie}[\sigma_3, \sigma_5, \ldots]$$

explicit.

Theorem. For each $n \geq 1$, there exists a (non-unique) "genus zero Lie associator" (satisfying Lie versions of the relations satisfied by the Drinfeld associator) f_{2n+1} of degree $2n+1$ starting with the Lie term $ad(x)^{n-1}(y)$. Let F_n be the associated polynomial mould. Then the mould

$$\Delta \circ ad_{ari}(invpal)(F_n)$$

is a polynomial-valued mould. Let $\Sigma_n \in \mathrm{Lie}[a, b]$ be the associated power series. Then letting $\sigma_{2n+1} \in \mathrm{Der}(\mathrm{Lie}[a, b])$ be defined by

$$\sigma_{2n+1}(a) = \Sigma_n, \quad \sigma_{2n+1}([a, b]) = 0$$

gives a map from

$$\mathrm{Lie}[\sigma_3, \sigma_5, \ldots] \rightarrow \mathrm{Der}(\mathrm{Lie}[a, b])$$

that extends to a Lie morphism

$$\mathfrak{u} \rtimes \mathrm{Lie}[\sigma_3, \sigma_5, \ldots] \rightarrow \mathrm{Der}(\mathrm{Lie}[a, b]).$$

Thanks to the explicit knowledge of σ_{2n+1} as derivations of Lie$[x,y]$, we can compute their brackets with the elements of $\mathfrak{u}$ and thus explicitly determine the semi-direct product structure.

Example. For $n = 3$ there is a unique choice for the genus zero Lie associator,

$$f_3 = [x,[x,y]] - [[x,y],y] = c_3 - [c_2, c_1]$$

with $c_i = ad(x)^{i-1}(y)$. The associated mould F_3 is given by

$$F_3(\emptyset) = 0, \quad F_3(u_1) = u_2^2, \quad F_3(u_1,u_2) = u_1 - u_2.$$

We have

$\Delta \circ ad_{ari}(invpal)(F_3)(\emptyset) = 0$

$\Delta \circ ad_{ari}(invpal)(F_3)(u_1) = u_1^4$

$\Delta \circ ad_{ari}(invpal)(F_3)(u_1,u_2) = 0$

$\Delta \circ ad_{ari}(invpal)(F_3)(u_1,u_2,u_3) = -3u_1^3u_2 + 3u_1^3u_3 - 3u_1^2u_2^2$
$-u_1^2u_2u_3 + 6u_1^2u_3^2 + 2u_1u_2^2u_3 - u_1u_2u_3^2 + 3u_1u_3^3 - 3u_2^2u_3^2 - 3u_2u_3^3.$

Converting back to a, b we find

$$\Sigma_3 = c_5 - 3c_4c_2c_1 + 3c_4c_1c_2 - 3c_3c_3c_1 - c_3c_2c_2 + 6c_3c_1c_3 + 2c_2c_3c_2$$
$$-c_2c_2c_3 + 3c_2c_1c_4 - 3c_1c_3c_3 - 3c_1c_2c_4 + \cdots$$

$$= c_5 + 3[c_4, [c_1, c_2]] - 3[c_3, [c_3, c_1]] - [c_2, [c_2, c_3]] + \cdots$$

Thus σ_{2n+1} can be identified with the derivation of $\mathrm{Lie}[a, b]$ mapping a to Σ_3 and annihilating $[a, b]$, which determines it completely, and its action on $\mathfrak{u}$ can then be determined explicitly.