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Calabi-Yau operators and p-adic zeta function

joint with Frits Beukers

(I) Differential operators with nice arithmetic properties

Mirror symmetry

A-side $\quad X$	X' $\quad$ B-side
enumerative geometry	differential equation for period integrals
n_d = # of rational curves of degree d on X	$Y(q) = 1 + \sum_{d \ge 1} n_d \cdot d^3 \cdot \frac{q^d}{1-q^d}$ (n_d: instanton #s)
Gromov-Witten invariants	

Mirror Theorem (Givental, Lian-Liu-Yau, ≈ 1995)

$$n_d(A) = n_d(B)$$

key example $\quad X \subset \mathbb{P}^4$ generic quintic threefold

$n_1(A) = 2875$ Schubert 1886

$n_2(A) = 609250$ S. Katz 1986

$$F_0(t) = \frac{1}{(2\pi i)^4} \oint\oint\oint\oint \frac{1}{1 - t\left(x_1+x_2+x_3+x_4+\frac{1}{x_1x_2x_3x_4}\right)} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4}$$

$$= \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} t^{5n}$$

period integral annihilated by

$$L = \theta^4 - (5t)^5(\theta+1)(\theta+2)(\theta+3)(\theta+4) \qquad \theta = t\frac{d}{dt}$$

1991 Candelas, de la Ossa, Green, Parkes

solutions to $L(y(t)) = 0$ near $t = 0$

$$y_0(t) = F_0(t) \in \mathbb{Z}[\![t]\!]$$

$$y_1(t) = F_0(t)\log t + F_1(t) \qquad F_1 \in t\,\mathbb{Q}[\![t]\!]$$

but $\quad q(t) = \exp\left(\frac{y_1}{y_0}\right) = t\exp\left(\frac{F_1}{F_0}\right) \in t\,\mathbb{Z}[\![t]\!]$

proved by Lian, Yau in 1996

$q(t)$ is called canonical coordinate

inverse series $t(q)$ is mirror map

$$y_2(t) = F_0(t)\frac{(\log t)^2}{2!} + F_1(t)\log t + F_2(t) \qquad F_2 \in t\,\mathbb{Q}[\![t]\!]$$

$$\frac{y_1}{y_0} = \log q \qquad \frac{y_2}{y_0} = \frac{1}{2}(\log q)^2 + \text{a power series in } q$$

Yukawa coupling $\quad Y(q) = \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + \ldots$

$$= \sum_{d \geq 0} N_d \frac{q^d}{1-q^d} \qquad \text{Lambert expansion}$$

instanton #s $\qquad n_d = N_d / d^3 \qquad d \geq 1$

<u>Theorem 1</u> (V - Beukers 2020) For the quintic case above denominators of n_d can only have prime divisors 2, 3, 5

$\Leftrightarrow$ n_d are p-integral for all $p > 5$

> 2018 Ionel, Parker BPS #s $\in \mathbb{Z}$
>
> our proof stays on B - side

$$L = \theta^n + a_1(t)\theta^{n-1} + \ldots + a_n(t) \qquad a_j \in \mathbb{Q}(t)$$

MUM at $t=0$ $\quad$ <u>m</u>aximally <u>u</u>nipotent local <u>m</u>onodromy $\Leftrightarrow$ $a_j(0) = 0$, $j = 1, \ldots, n$

$\Leftrightarrow$ has basis of solutions of the form

$$y_i(t) = F_0\frac{(\log t)^i}{i!} + F_1\frac{(\log t)^{i-1}}{(i-1)!} + \ldots + F_i \qquad i = 0, \ldots, n-1$$

$F_i \in \mathbb{Q}[\![t]\!] \qquad F_0(0) = 1 \qquad F_i(0) = 0 \quad i > 0$

A MUM-type diff. operator

Def L of order $n=4$ is a Calabi-Yau differential operator if it is selfdual, has regular singularities and

* $y_0(t) \in \mathbb{Z}_p[[t]]$
* $q(t) = \exp(y_1/y_0) \in t\,\mathbb{Z}_p[[t]]$
* $n_d = N_d/d^3 \in \mathbb{Z}_p$, $d = 1, 2, 3, \ldots$

for all but finitely many primes p

↑

tables by Almkvist, van Enckevort, van Straten, Zudilin

< 500 cases found experimentally

Theorem 2 (V- Beukers)

If L has a p-adic Frobenius structure then $y_0(t) \in \mathbb{Z}_p[[t]]$ and $q(t) \in \mathbb{Z}_p[[t]]$.

If this Frobenius structure has $\alpha_1 = 0$ then $N_d/d^2 \in \mathbb{Z}_p$, $d = 1, 2, 3, \ldots$

Moreover, if L is of order $n=4$ and self-dual then $N_d/d^3 \in \mathbb{Z}_p$, $d = 1, 2, 3, \ldots$

II p-adic Frobenius structure

<u>Definition</u> Diff. operator $L = \theta^n + a_1(t)\theta^{n-1} + \dots$ has a p-adic Frobenius str-re over a ring $R \subset \mathbb{Z}_p[\![t]\!]$ if there exists diff. operator

$$\mathcal{A} = \sum_{j=0}^{n-1} A_j(t)\,\theta^j \quad \text{with } A_j \in R,\ A_0(0) = 1$$

such that for every solution $y(t)$ to $L(y(t)) = 0$ function $\tilde{y}(t) = \mathcal{A}(y(t^p))$ is again a solution, i.e. $L(\tilde{y}(t)) = 0$.

↑

Very strong property, only expected from diff. operators arising from Gauss-Manin connection in algebraic geometry

<u>Remarks</u> i) If $R \subset E_p$ then such operator $\mathcal{A}$ is unique, when it exists (Dwork)

E_p = p-adic completion of $\mathbb{Q}(t)$

ii) in the standard basis of solutions one has

$$\mathcal{A}(y_i(t^p)) = p^i \sum_{j=0}^{i} \alpha_{i-j}\, y_j(t) \qquad (*)$$

where $\alpha_j = A_j(0) \in \mathbb{Z}_p$, $j = 0, \dots, n-1$.

Moreover, any set of p-adic constants $\alpha_0 = 1, \dots, \alpha_{n-1}$ determines uniquely an operator $\mathcal{A}$ with $A_j \in \mathbb{Q}_p[\![t]\!]$ such that $(*)$ holds. For some unique α_j's one has $A_j \in \mathbb{Z}_p[\![t]\!]$!

<u>Conjecture</u> (Candelas, de la Ossa, van Straten 2017... 2021)

For Calabi-Yau diff. operators of order $n = 4$ one has $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \lambda \cdot \zeta_p(3)$ where $\lambda \in \mathbb{Q}$ is independent of p. Moreover,

$$\lambda = \frac{\chi(X)}{Y}$$

$\chi(X)$ ← Euler char of the mirror manifold X

Y ← Large complex str-re value of the Yukawa coupling

Theorem (V-Beukers 2022)

Operator

$$L = \theta^n - ((n+1)t)^{n+1}(\theta+1)\dots(\theta+n)$$

has a p-adic Frobenius str-re for each $p > n+1$. This Frob. str-re is defined over

$$R = p\text{-adic completion of } \mathbb{Z}\left[t, \frac{1}{1-((n+1)t)^{n+1}}\right]$$

and the respective constants are given by

$$\alpha_j = \text{coeff. of } X^j \text{ in } \frac{\Gamma_p(X)}{\Gamma_p\left(\frac{X}{n+1}\right)^{n+1}}$$

where $\Gamma_p(x)$ is ~~the~~ ~~Mor~~ ~~eta~~ a p-adic gamma function.

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$$

$$\Gamma(1+n) = n!$$

$$\log \Gamma(1+x) = \Gamma'(1)x + \sum_{m\geq 2} \frac{\zeta(m)}{m}(-x)^m$$

Riemann zeta function

$\Gamma_p(x)$ = p-adic interpolation of $n \mapsto n!$

$$\log \Gamma_p(x) = \Gamma_p'(0) + \sum_{m\geq 2} \frac{\zeta_p(m)}{m} x^m$$

p-adic zeta function

$$\zeta(1-n) = (-1)^{n-1}\frac{B_n}{n}$$

Bernoulli #s

$\longleftarrow\!\!\!| \; n$ admits p-adic interpolation!

(Kummer's congruences)

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}$$

the result is the p-adic ~~zeta~~

E.g. $n=4$

$$\frac{\Gamma_p(x)}{\Gamma_p(x/5)^5} = 1 - \frac{8}{25}\zeta_p(3)X^3 + O(X^4)$$