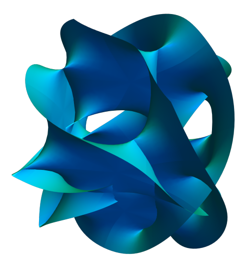
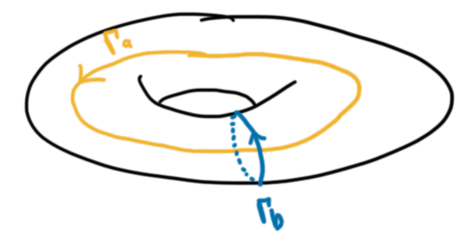
Geometries, Iterated Period Integrals and Feynman Integrals



Christoph Nega



Joint work with:
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"The Ice Cone Family and Iterated Integrals for Calabi-Yau Varieties" [1], "Yangian-Invariant Fishnet Integrals in Two Dimensions as Volumes of Calabi-Yau Varieties" [2], "Feynman Integrals in Dimensional Regularization and Extensions of Calabi-Yau Motives" [3], "Analytic structure of all Banana integrals" [4], "The l-loop Banana amplitude from GKZ systems and relative Calabi-Yau periods" [5]

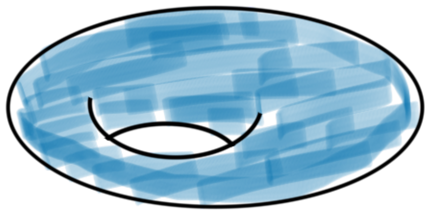
Geometries and Special Functions for Physics and Mathematics
Bonn
March 20, 2023

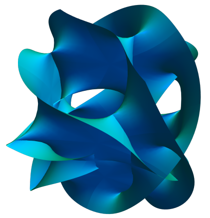
Motivation

- **Feynman integrals** are cornerstone of perturbative QFT and necessary for predictions in collider and gravitational wave experiments.

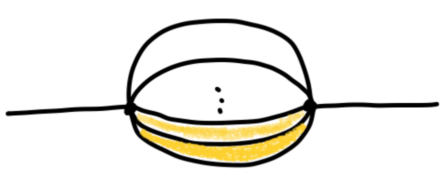
- High precision measurements require **multi-loop** Feynman integral computations.

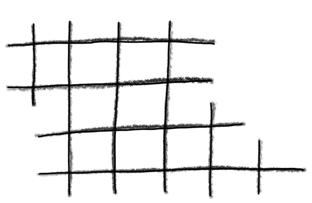
- There are many examples starting at two loops where **elliptic functions** show up. This means that these Feynman integrals have an associated non-trivial geometry.





- At higher loops we have examples where even **more complicated geometries** appear.





- Feynman integrals give us **interesting mathematical structures** (algebraic geometry, number theory, …) we want to understand.

Questions

Q1 **How do we get a geometry behind a Feynman integral?**

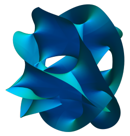
Different approaches depending on particular example

Q2 **How does this geometry help us to compute Feynman integrals?**

Useful insights, for instance **function space & boundary conditions**

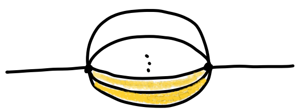
Table of Content

0) Introduction to Calabi-Yau Geometries



[Albrecht Klemm]

1) The Banana Family

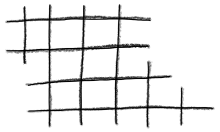


[3-5]

2) The Ice Cone Family

[1]

3) Fishnet Integrals



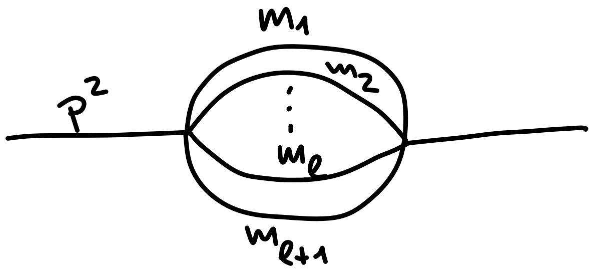
[2]

4) Feynman Integrals in Dimensional Regularization

[new]

5) Conclusion and Remarks

The Banana Family

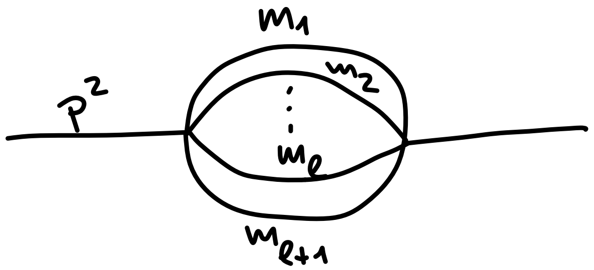


Two dimensions [3-5]

Equal-mass and **generic-mass** case

Q1: Calabi-Yau geometry

The Banana Family



Two dimensions [3-5]

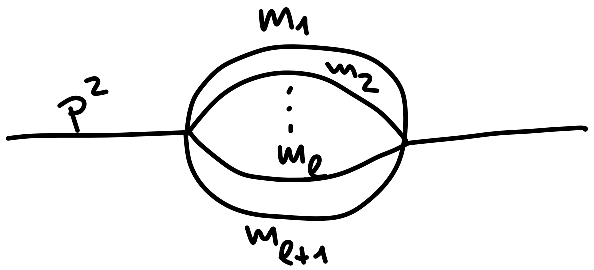
Equal-mass and **generic-mass** case

Q1: Calabi-Yau geometry

- **Second graph polynomial** defines a CY: $\text{“CY} = \{\mathcal{F} = 0\}\text{”}$
  Singular
  - Smooth **Calabi-Yau mirror pairs** from toric resolution and the Batyrev construction:

    $M_{l-1} = \{P_{\Delta_l} = 0 \subset \mathbb{P}_{\Delta_l^\star}\}$ **Mirror Symmetry** $W_{l-1} = \{P_{\Delta_l^\star} = 0 \subset \mathbb{P}_{\Delta_l}\}$
  - Toric resolution introduces **new** parameters: $h^{l-2,1} = l^2$ vs. $l+1$

The Banana Family



Two dimensions [3-5]

Equal-mass and **generic-mass** case

Q1: Calabi-Yau geometry

- **Second graph polynomial** defines a CY: $\text{“CY} = \{\mathcal{F} = 0\}\text{”}$ **Singular**
  - Smooth **Calabi-Yau mirror pairs** from toric resolution and the Batyrev construction:

$$M_{l-1} = \{P_{\Delta_l} = 0 \subset \mathbb{P}_{\Delta_l^\star}\} \qquad \textbf{Mirror Symmetry} \qquad W_{l-1} = \{P_{\Delta_l^\star} = 0 \subset \mathbb{P}_{\Delta_l}\}$$

  - Toric resolution introduces **new** parameters: $h^{l-2,1} = l^2$ vs. $l+1$

- Analysis of "torus period" (holomorphic period): $I_l^{\max} = \int_{T^l} \frac{1}{\mathcal{F}} \mu_l$
  - Associated smooth **complete intersection CY**: $M_{l-1}^{\mathrm{CI}} = \left\{ P_1 = P_2 = 0 \subset F_l \subset \bigtimes_{i=1}^{l+1} \mathrm{P}^1_{(i)} \right\}$ [Kerr, 3]
  - **Correct** number of parameters: $z_i = \frac{m_i^2}{p^2}$ for $i = 1, \ldots, l+1$

The Banana Family

Q2: Calabi-Yau Period Integrals

- The maximal cuts of the banana family are period integrals of the CY M_{l-1}
- One can compute these periods using differential equations: **Picard-Fuchs ideal** or **Gauss-Manin system**
  - **Integration by Parts** identities
  - **Griffiths reduction method** or **GKZ** approach
  - Compute a **single period** and operators via ansatz, e.g. "torus period" $\Pi_0 = \int_{T^n} \Omega$
  - **Combination** of different approaches

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- The **simplex** integration domain of the full banana integral makes it a **relative CY period**:

  → Inhomogeneous diff. eqs.: $\mathcal{D}_r I(\underline{z}) = q_r(\underline{z}, \log(\underline{z}))$

- Full Feynman integral is linear combination of basis solutions $\{\varpi_i\}$ which are the **Calabi-Yau periods** plus additional **special solutions** of the inhomogeneous $\mathcal{D}$-module:

$$I(\underline{z}) = \sum_i \lambda_i \; \varpi_i(\underline{z})$$

The Banana Family

Q2: Iterated Calabi-Yau Period Integrals

- **Griffiths transversality** gives **quadratic relations**: $\mathbf{Z}(z) = \mathbf{W}(z)\Sigma\mathbf{W}(z)^T$ with $\mathbf{W}(z)_{i,j} = \{\partial_z^i \varpi_j\}$

- The **additional special solution** can be interpreted as **iterated Calabi-Yau period**:

$$\underline{I}_{\text{ban},l}(z) \sim \underline{\Pi}_l(z)^T \int_0^z \mathrm{d}z' \, \mathbf{W}_l(z')^{-1} \, \underline{\text{Inhom}}_l(z') + \mathbf{W}_l \underline{\lambda}$$

$$\sim \underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_0^z \frac{\mathrm{d}z'}{z'^2} \, \underline{\Pi}_l(z') + \mathbf{W}_l \underline{\lambda}$$

Quadratic relations to invert Wronskian

Function space banana family	⟷	iterated CY period integrals of M_{l-1}

Generalization of elliptic polylogarithms?

The Banana Family

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Quadratic relations to invert Wronskian

Function space banana family ⟷ iterated CY period integrals of M_{l-1}

Generalization of elliptic polylogarithms?

- With the **mirror map** (canonical variable) we can also express the CY periods as iterated integrals:

$$I_{\mathrm{ban},l} \sim \varpi_0(q) \left(\sum_{k=1}^{l} \lambda_k I(1, Y_1, \ldots, Y_{l-k-1}; q) + I(1, Y_1, \ldots, Y_1, 1, g_{\mathrm{ban}}; q) \right)$$

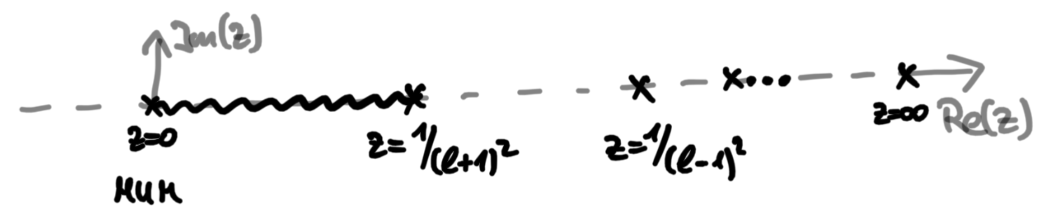
Y-invariants

Pure function of weight l?

The Banana Family

Q2: Special Calabi-Yau Monodromies

- There are plenty of different approaches to get the **boundary condition**:
    - In **certain limits** maybe some integrals are known (standard).
    - The banana integrals have **special monodromies**:



- **Generating series**: $\sum_{l=0}^{\infty} \frac{\lambda_0^{(l)}}{(l+1)!} t^l = -\frac{\Gamma(1-t)}{\Gamma(1+t)} e^{-2\gamma t - i\pi t}$ and $\lambda_k^{(l)} = (-1)^k \binom{l+1}{k} \lambda_0^{(l-k)}$
- $\hat{\Gamma}$**-class** from CY geometry: [Iritani]

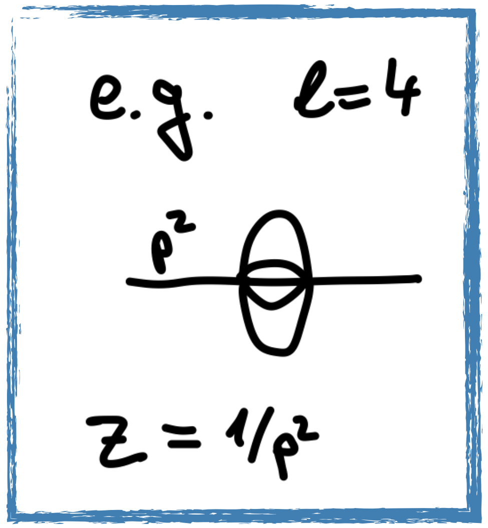
$$\mathrm{Im}(\lambda): \quad \mathrm{Im}(I(T)) = \int_{W_{l-1}} \mathrm{e}^{\omega T} \widehat{\Gamma}(T W_{l-1}) + \mathcal{O}(\mathrm{e}^T)$$

$$\mathrm{Re}(\lambda): \quad \mathrm{Re}(I(T)) = \int_{F_l} \mathrm{e}^{\omega T} \frac{\Gamma(1-c_1)}{\Gamma(1+c_1)} \cos(\pi c_1) + \mathcal{O}(\mathrm{e}^T)$$

- Combination of different approaches

4L Equal-Mass Banana Integral

Equal Mass Case:



1) PF equation:

$$\mathcal{L}_4 = 1 - 5z + (-4 + 28z)\theta + (6 - 63z + 26z^2 - 225z^3)\theta^2 + (-4 + 70z - 450z^3)\theta^3 + (1-z)(1-9z)(1-25z)\theta^4$$

AESZ 34 [Almquist, Enckefort, van Straten and Zudilin]

$$\mathcal{L}_4 I_4(z) = -5!z \qquad \text{or} \qquad (\theta - 1)\mathcal{L}_4 I_4(z) = 0$$

First extension of CY operator

2) Frobenius basis:

$$\varpi_k = \sum_{j=0}^{k} \binom{k}{j} \log(z)^j \, \Sigma_{k-j} \qquad \text{for } k = 1, \dots, 4-1$$

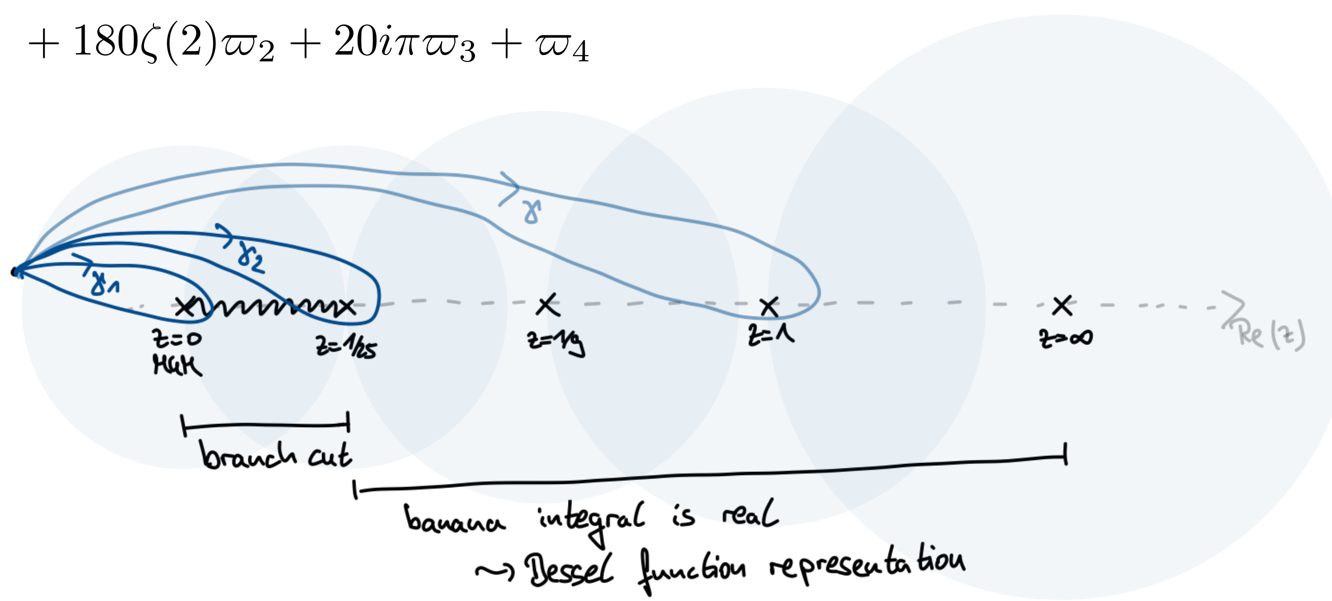
$$\varpi_l = (-1)^{l+1}(l+1) \sum_{j=0}^{l} \binom{l}{j} \log(z)^j \, \Sigma_{l-j}$$

$$\varpi_0 = z + 5z^2 + +45z^3 + 545z^4 + 7885z^5 + \cdots$$
$$\Sigma_1 = 8z^2 + 100z^3 + \frac{4148}{3}z^4 + \frac{64\,198}{3}z^5 + \cdots$$
$$\Sigma_2 = 2z^2 + \frac{197}{2}z^3 + \frac{33\,637}{18}z^4 + \frac{2\,402\,477}{72}z^5 + \cdots$$
$$\Sigma_3 = -12z^2 - \frac{267}{2}z^3 - \frac{19\,295}{18}z^4 - \frac{933\,155}{144}z^5 + \cdots$$
$$\Sigma_4 = 1830z^3 + \frac{112\,720}{3}z^4 + \frac{47\,200\,115}{72}z^5 + \cdots$$

3) Linear combination from $\hat{\Gamma}$-conjecture:

$$I_4(z) = (-450\zeta(4) - 80\zeta(3)i\pi)\varpi_0 + (80\zeta(3) - 120\zeta(2)i\pi)\varpi_1 + 180\zeta(2)\varpi_2 + 20i\pi\varpi_3 + \varpi_4$$

4) Analytic structure:



Monodromy action: $I_4 \xrightarrow{\gamma_1,\gamma_2} M_{1,2}\, I_4$

$I_4 \xrightarrow{\gamma} I_4$

as predicted by the optical theorem

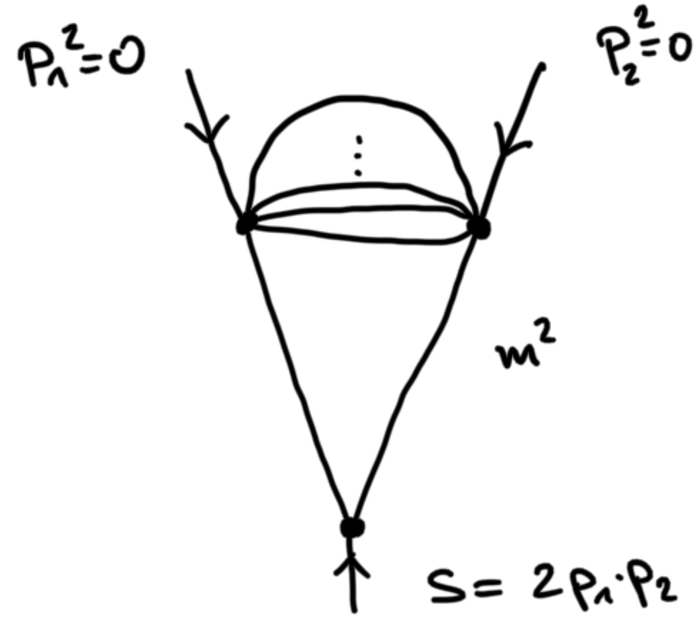
The Ice Cone Family

- Now we consider the family of **ice cone integrals**: [1]

external parameters: p_1 and p_2 with $p_1^2 = p_2^2 = 0$
so we have only $s = 2p_1 \cdot p_2$

internal masses: all equal to m

dimension: two



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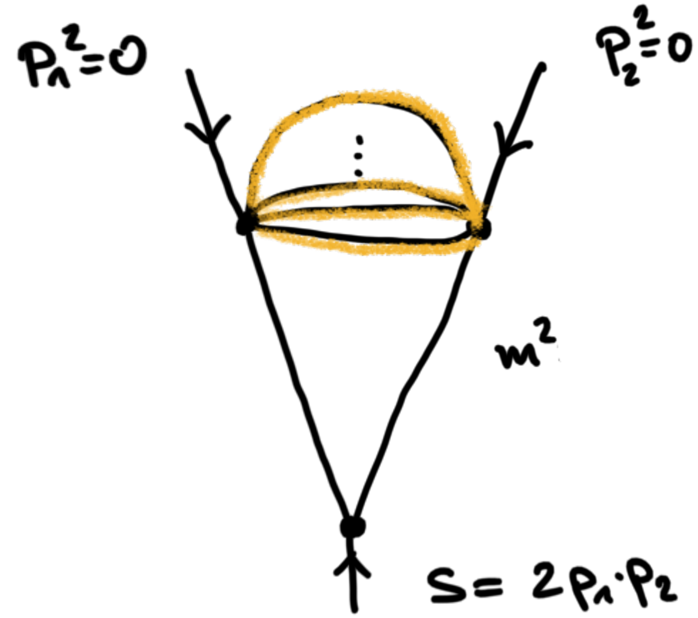

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Q1: Calabi-Yau geometry

- Naively, we expect that the **banana integrals** and therefore a **CY geometry** play a prominent role for ice cone integrals since they explicitly appear in their diagrams.

- We will not find the CY from the graph polynomials here. [Doran, Harder, Vanhove]

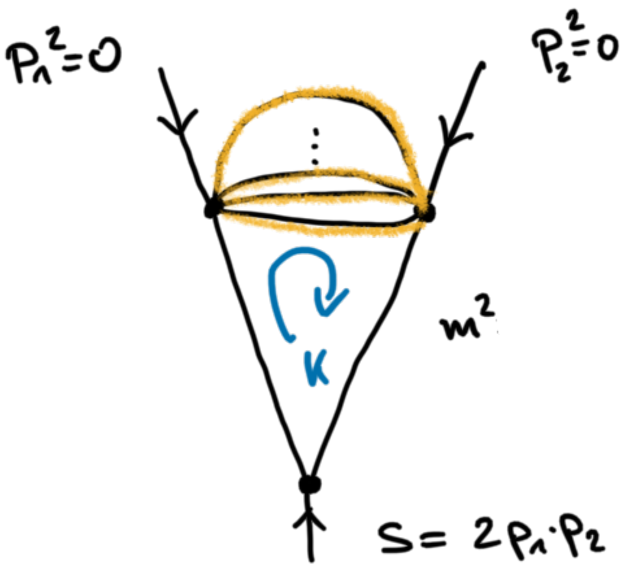
- We will find the ice cone geometry from analyzing **maximal cuts**.

The Ice Cone Family

Q1: Calabi-Yau geometry

- Consider the following representation of the ice cone:

$$I_{\text{ice}}^{(l)} = \int \frac{d^2k}{((k-p_1)^2 - m^2)((k+p_2)^2 - m^2)} I_{\text{ban}}^{(l-1)}(k^2)$$



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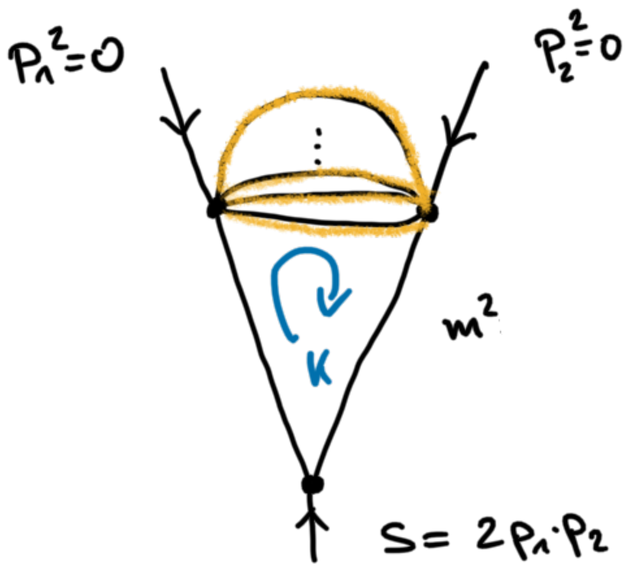


- We analyze the **maximal cuts** in with the Baikov representation:

$$I_{\text{ice, cut}}^{(l)} = \oint \frac{du}{(u - m^2 x)(u - m^2/x)} I_{\text{ban, cut}}^{(l-1)}(u)$$

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

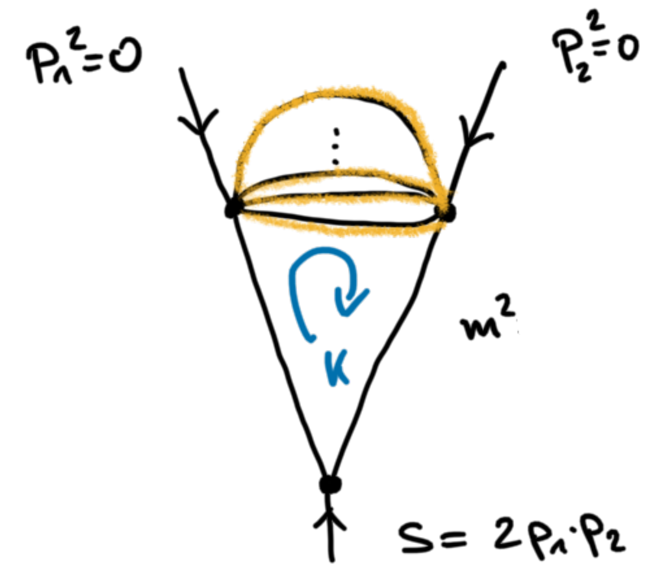
Landau variable

The Ice Cone Family

Q1: Two banana Calabi-Yau geometries

- Consider the following representation of the ice cone:

$$I^{(l)}_{\text{ice}} = \int \frac{d^2k}{((k-p_1)^2 - m^2)((k+p_2)^2 - m^2)} I^{(l-1)}_{\text{ban}}(k^2)$$



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$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

Landau variable

have two choose two different residues

CY periods

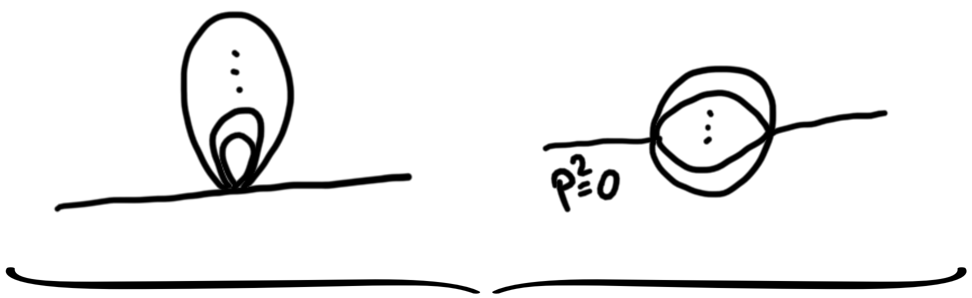
→ We have **two copies** of the cut banana integrals appearing in the cuts of ice cone:

$$\left\{ I^{(l)}_{\text{cut, ice}} \right\} = \left\{ I^{(l-1)}_{\text{ban, cut}}(m^2 x), I^{(l-1)}_{\text{ban, cut}}(m^2/x) \right\}, \quad 2(l-1)$$

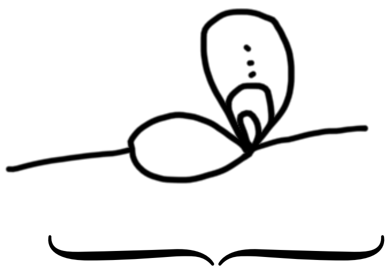
The Ice Cone Family

- We found that **a good basis** of master integrals is given by: [1]

  trivial master integrals:



constant



simple
algebraic & log

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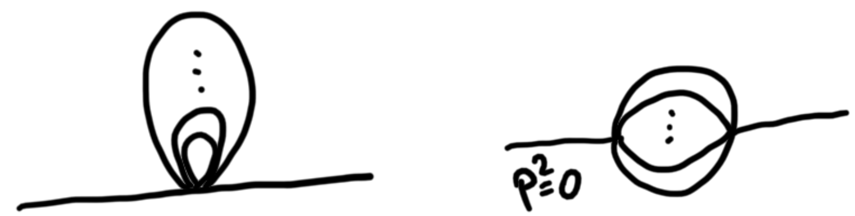

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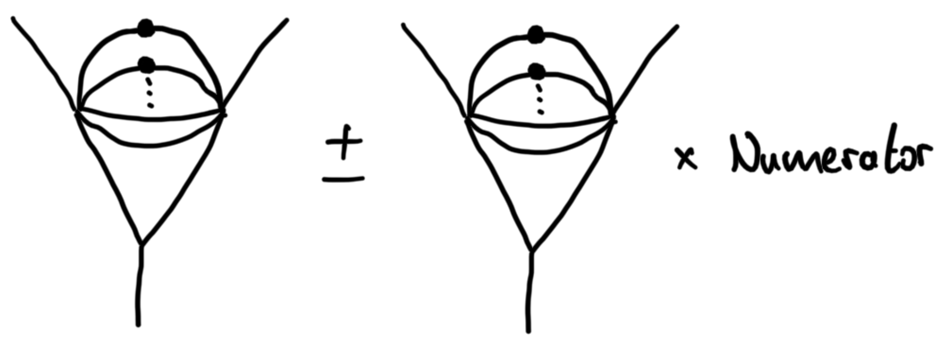
non-trivial master integrals:



correspond to the two
copies of the bananas

"Gram master"

vanishes in two
dimensions

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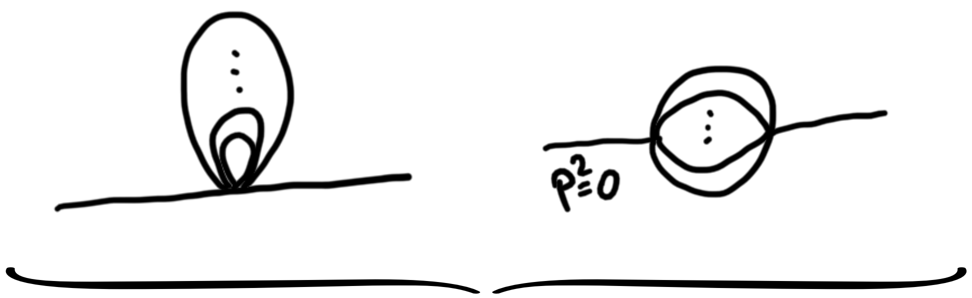

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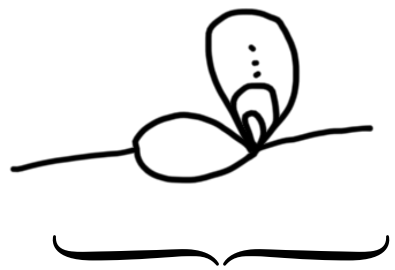

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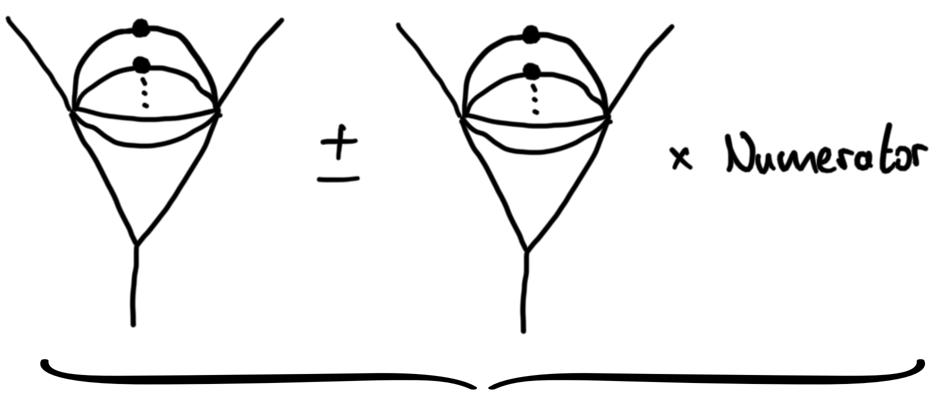

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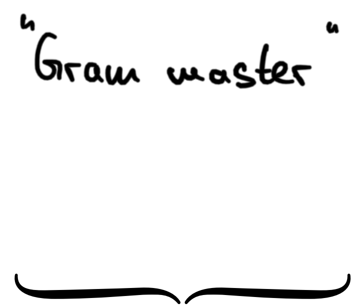

vanishes in two
dimensions

- For this basis we can (conjecturally) write down the full GM system in two dimensions.

The Ice Cone Family

Q2: Iterated Calabi-Yau Period Integrals

- The only non-trivial part of the GM system takes the simple form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^+ = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(x)\underline{\mathcal{I}}_l^+ + \underline{N}_l^+ I_0 + \mathcal{O}(d-2)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^- = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(1/x)\underline{\mathcal{I}}_l^- + \underline{N}_l^- I_0 + \mathcal{O}(d-2)$$

- To fix the boundary condition we notice:

$$\mathcal{L}_{\mathrm{ice},l}\mathcal{I}_{l,1}^+ = (\theta-1)^2\mathcal{L}_{\mathrm{ban},l-1}\mathcal{I}_{l,1}^+ = 0$$

Double extension of CY operator

- As in the banana case the master integrals of the ice cone family are **iterated CY period integrals**:

$$\underline{\mathcal{I}}_l^+ \sim \mathbf{W}_{l-1}^+\mathbf{\Sigma}_{l-1}\int_0^x \frac{\log(x')}{x'^2}\underline{\Pi}_{l-1}(x')\mathrm{d}x' + \mathbf{W}_{l-1}^+\underline{c}_l^+$$

The Ice Cone Family

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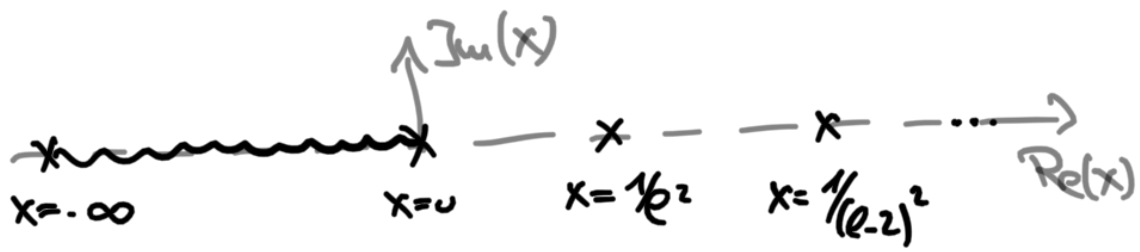
Pure function of weight l

The Ice Cone Family

Q2: Special Calabi-Yau Monodromies

- We used **numerics** to fix the boundary values. $(l \leq 7)$

- **Special monodromies** of the ice cone integrals: $(l \leq 3)$

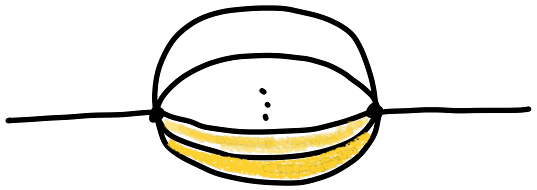


- We found a **generating series**:

$$1 + \sum_{l+2}^{\infty} (-1)^{l+1} c^{+}_{l,1} \frac{t^l}{l!} = \Gamma(1-t)^2 e^{-2\gamma t} \qquad \text{and} \qquad c^{+}_{l+1,k+1} = (l+1) c^{+}_{l,k}$$

- $\hat{\Gamma}$**-class**?

Bananas vs. Ice Cones



maximal cut geometry: $(l-1)$-dimensional CY M_{l-1}	maximal cut geometry: two $(l-2)$-dimensional CYs M_{l-2}
$I_{\mathrm{ban},l} \sim \varpi_0(q)\Big(\sum_{k=1}^{l}\lambda_k I(1,Y_1,\ldots,Y_{l-k-1};q) + I(1,Y_1,\ldots,Y_1,1,g_{\mathrm{ban}};q)\Big)$ pure function of weight l	$\mathcal{I}^{+}_{l,1} \sim \varpi_0(q)\Big(\sum_{k=1}^{l-1}c^{+}_{l-k} I(1,Y_1,\ldots,Y_{l-k-2};q) - l!\,I(1,Y_1,\ldots,Y_1,1,g_{\mathrm{ice}};q)\Big)$ pure function of weight l
$\mathcal{L}_{\mathrm{ban},l} = (\theta-1)\mathcal{L}_{\mathrm{CY},l-1}$ single extension	$\mathcal{L}_{\mathrm{ice},l} = (\theta-1)^2\mathcal{L}_{\mathrm{CY},l-2}$ double extension
generating series: $-\frac{\Gamma(1-t)}{\Gamma(1+t)}e^{-2\gamma t - i\pi t}$ + $\widehat{\Gamma}$-class	generating series: $\Gamma(1-t)^2 e^{-2\gamma t}$

Fishnet Integrals

- **Fishnet graphs** are obtained from a cut of a tiling of the plane: [2]

- These graphs yield **conformal integrals** if at each vertex the propagator powers add up to the dimension:

$$D = 1 : \quad \nu_i = \tfrac{1}{4}\,, \quad D = 2 : \quad \nu_i = \tfrac{1}{2}\,, \quad D = 4 : \quad \nu_i = 1\,, \ldots$$

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- We consider fishnet integrals in **two Euclidean dimensions**. Here we can express everything through complex quantities:
  - External points: $a_i \in \mathbb{C}$
  - Internal points: $X_j \in \mathbb{C}$
  - Propagators: $\frac{1}{|X_i - X_j|}$ or $\frac{1}{|X_i - a_j|}$

$$I_G(a) = \int_{\mathbb{C}^l} \left(\prod_{i=1}^{l} \frac{\mathrm{d}\bar{X}_i \wedge \mathrm{d}X_i}{-2i} \right) \frac{1}{\sqrt{|P_G(X,a)|^2}}$$

- These integrals have a **permutation** and **Yangian** Symmetry.

Fishnet Integrals

Q1: Calabi-Yau geometry

- The **Calabi-Yau geometry** follows immediately from our integral representation:

$$I_G(a) = \int_{(\mathbb{P}^1)^l} \left(\prod_{i=1}^{l} \frac{\mathrm{d}\bar{X}_i \wedge \mathrm{d}X_i}{-2i} \right) \frac{1}{\sqrt{|P_G(X,a)|^2}}$$

$$\{Y^2 = P_G(X,a)\} \subset \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$$

$$\Omega = \frac{\mu}{\sqrt{P_G(X,a)}}$$

This yields a CY with Ω iff P_G has degree four in each $\mathbb{P}^1$.

Four-valence of vertices ✓

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- Unfortunately, the constraint $Y^2 = P_G(X,a)$ is **singular**:
  - Take Newton polytope and Batyrev's mirror construction. → too many parameters
  - Resolve singularities by a small resolution. → complicated

- But fortunately, we can compute the **Picard-Fuchs ideal** from the "torus period":

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> The **Picard-Fuchs ideal** equals the ideal of differential operators derived from the **Yangian symmetry** and the **Permutation symmetries** of the graph.

Fishnet Integrals

Q2: Monodromy invariance

- Fishnet integrals are **monodromy invariant**:

$$I_G(a) = \int_{(\mathbb{P}^1)^l} \left(\prod_{i=1}^{l} \frac{\mathrm{d}\bar{X}_i \wedge \mathrm{d}X_i}{-2i} \right) \frac{1}{\sqrt{|P_G(X,a)|^2}}$$

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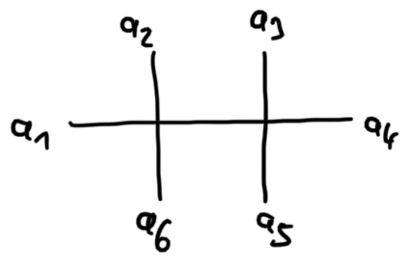
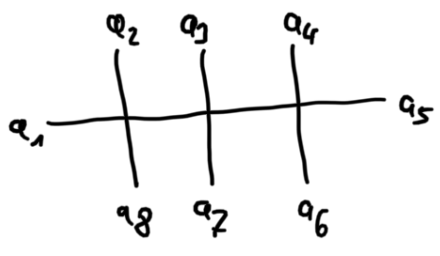
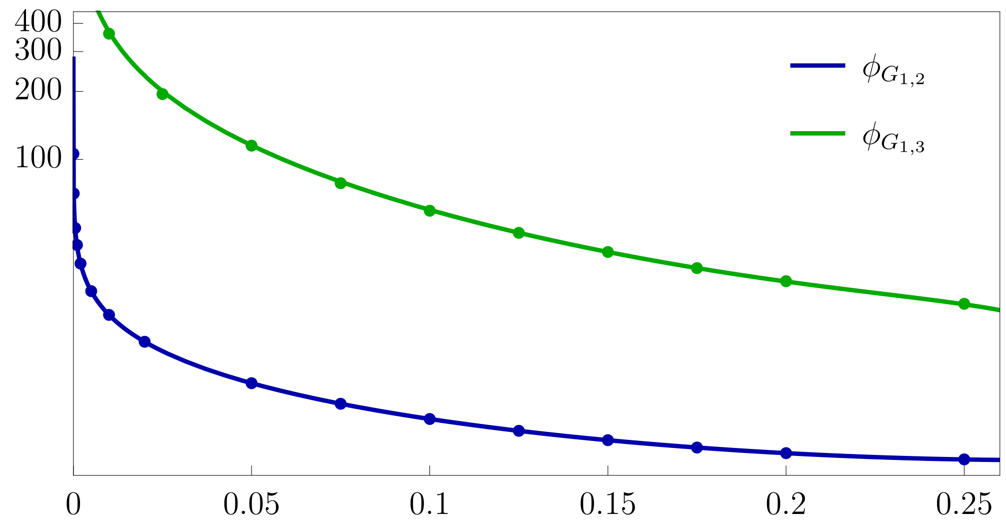
- "We can compute Fishnet integrals from the the Yangian and permutation symmetry and combine them in a monodromy invariant way."

new results for two- and three-loop fishnets







Fishnet Integrals

Q2: Additional observations

- Using mirror symmetry we can interpret a fishnet integral as a **quantum volume** of the mirror CYs:

$$
\begin{aligned}
I_G \sim \Pi^\dagger \Sigma \Pi &\sim |\Pi_0|^2 \mathrm{Vol_q}(W) \\
&\sim |\Pi_0|^2 \mathrm{Vol_{cl}}(W) + \mathcal{O}(e^{-t(z)})
\end{aligned}
$$

quantum corrections in string theory

  - For elliptic curves and K3s these quantum corrections are absent:

$$I_{\mathcal{E}} \sim |\Pi_0|^2 \mathrm{Im}(t) \qquad \text{and} \qquad I_{\mathrm{K3}} \sim |\Pi_0|^2 (\mathrm{Im}(t))^2$$

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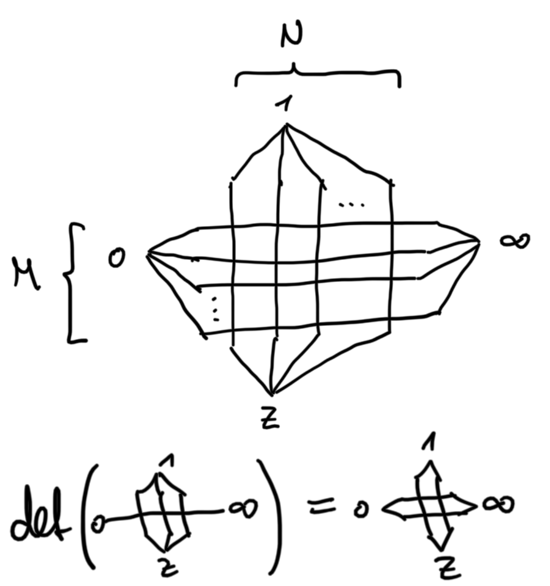
- There are interesting **relations** between different fishnet integrals:

The periods of a **(MxN)-fishnet graphs** can be constructed from **(MxM)-determinants** of the period matrix of the **(M+n-1)-ladder graphs**.

Basso-Dixon formula



  - What do these relations mean on the level of the CY geometries?

Feynman Integrals in Dimensional Regularization

- So far we have only considered Feynman integrals in exactly $\mathbf{d=2}$.
- But many Feynman integrals are **divergent** and **regularization** is required:

Dimensional regularization

$$\mathbf{d=2} \longrightarrow \mathbf{d=2-2}\epsilon$$

$$I(z,\epsilon) = \sum_{k=-m}^{\infty} I_k(z)\epsilon^k$$

$$dI(z,\epsilon) = A(z,\epsilon)I(z,\epsilon)$$

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- The ϵ-expansion can easily be solved if the GM system is ϵ**-factorized**: [Henn]

$\tilde{I}(z,\epsilon) = T(z,\epsilon)I(z,\epsilon)$ such that $d\tilde{I}(z,\epsilon) = \epsilon\tilde{A}(z)\tilde{I}(z,\epsilon)$

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- For **equal-mass bananas** a ϵ-factorized GM was found by Weinzierl et al.. [Weinzierl]

- We focus with our new approach more on **elliptic Feynman integrals** of phenomenological relevance in **one and more parameters**. But our approach works also for at least some non-elliptic examples.

Feynman Integrals in Dimensional Regularization

- Our approach is based on constructing the rotation $T(z,\epsilon)$ in **different steps**: [new]

$$T(z,\epsilon) = T_{\text{new objects}}\, T_{\text{tot. deri.}}\, T_{\epsilon-\text{scalings}}\, T_{\text{semi-simple}}\, T_{\text{lead. sing.}}$$

$$\varpi_0$$

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  - We perform a "**leading singularities**"-type analysis: $T_{\text{lead. sing.}}$
  - Multiplication with the inverse **semi-simple part** of the **Wronskian**: $T_{\text{semi-simple}}$

$$W = SU \quad \text{with} \quad S = \begin{pmatrix} \varpi_0 & 0 \\ \varpi_0' & f(\varpi_0) \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & t = \frac{\varpi_1}{\varpi_0} \\ 0 & 1 \end{pmatrix}$$

  - **Simple rescalings** with ϵ: $T_{\epsilon-\text{scalings}}$
  - **Total derivatives/simple rotations** remove nearly all disturbing ϵ terms: $T_{\text{tot. deri.}}$
  - The remaining wrong ϵ terms can be removed by introducing **new objects.** These new objects are **integrals over** ϖ_0: $T_{\text{new objects}}$

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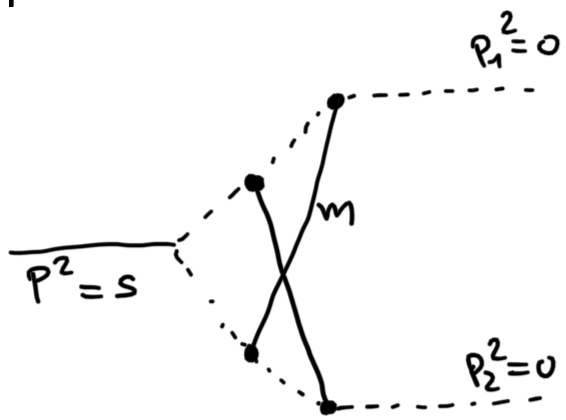
- As for $\epsilon = 0$ we need integrals over ϖ_0.
- Advantage of our approach is that one sees what new objects one needs and that they are independent.
- Also in Weinzierl's approach for non-elliptic geometries one needs these new objects.

Feynman Integrals in Dimensional Regularization

- Example: **Electroweak form factor**



$d = 4 - 2\epsilon$

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18 master integrals

elliptic top sector
with residue

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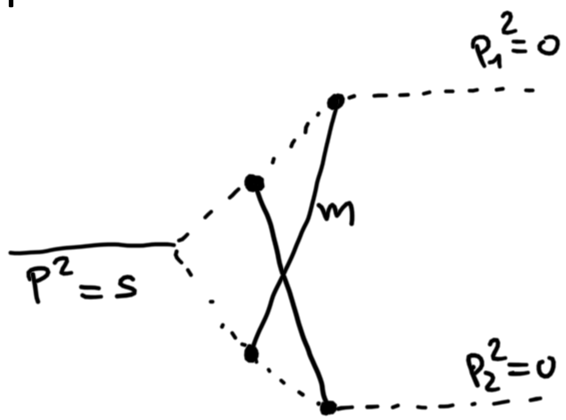

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18 master integrals

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$$T(z,\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{\text{new objects}}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3}(z+1)\varpi_0(z) & 1 & 0 \\ \frac{1}{24}\left(5z^2 - 44z - 76\right)\varpi_0(z)^2 & 0 & 1 \end{pmatrix} \qquad T_{\text{tot. deri.}}$$

$$\cdot \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & 0 & \epsilon^4 \\ 0 & \epsilon^3 & 0 \end{pmatrix} \qquad T_{\epsilon-\text{scalings}}$$

$$\cdot \begin{pmatrix} \frac{1}{\varpi_0(z)} & 0 & 0 \\ \frac{1}{8}(z-8)(z+1)\left(\varpi_0(z) + z\varpi_0'(z)\right) & \frac{1}{4}(z-8)(z+1)\varpi_0(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{\text{semi-simple}}$$

$$\cdot \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} \qquad T_{\text{lead. sing.}}$$

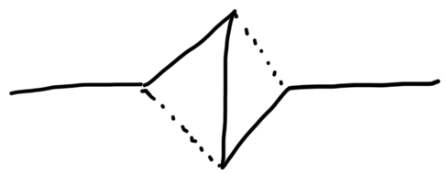
Full ϵ-factorized GM differential equation:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\epsilon}{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\epsilon}{z} & -\frac{\epsilon}{z+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6\epsilon}{z} & -\frac{4\epsilon}{z+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{2(z-1)} & 0 & 0 & \frac{2\epsilon}{z-1} & -\frac{\epsilon}{2(z-1)} & -\frac{\epsilon}{(z-1)z} & 0 & 0 & -\frac{\epsilon}{z-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon}{z} & 0 & 0 & \frac{\epsilon}{\sqrt{z-4}\sqrt{z}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\epsilon}{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\epsilon}{2(z-1)} & 0 & 0 & -\frac{(z+1)\epsilon}{(z-1)z} & \frac{\epsilon}{2(z-1)} & \frac{\epsilon}{(z-1)z} & 0 & 0 & -\frac{(z-2)\epsilon}{(z-1)z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{\sqrt{z-4}\sqrt{z}} & 0 & \frac{\epsilon}{\sqrt{z-4}\sqrt{z}} & 0 & 0 & 0 & -\frac{3\epsilon}{\sqrt{z-4}\sqrt{z}} & 0 & 0 & -\frac{(3z-4)\epsilon}{(z-4)z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\epsilon}{2z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon}{z} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\epsilon}{z} & \frac{2\epsilon}{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\epsilon}{8z} & 0 & -\frac{3\epsilon}{2z} & \frac{\epsilon}{4z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\epsilon}{z} & 0 & -\frac{\epsilon}{z} & 0 & 0 & 0 \\
\frac{\epsilon}{(z-1)(z+1)} & -\frac{\epsilon}{4(z+1)} & 0 & \frac{\left(2z^2+z+1\right)\epsilon}{(z-1)z(z+1)} & -\frac{z\epsilon}{(z-1)(z+1)} & -\frac{2\epsilon}{(z-1)(z+1)} & 0 & 0 & -\frac{2z\epsilon}{(z-1)(z+1)} & 0 & -\frac{2\epsilon}{z(z+1)} & 0 & 0 & \frac{(z+3)\epsilon}{z(z+1)} & 0 & 0 & 0 & 0 \\
0 & \frac{\epsilon}{4(z+1)} & -\frac{\epsilon}{z+1} & -\frac{\epsilon}{z+1} & \frac{\epsilon}{2(z+1)} & 0 & 0 & -\frac{2\epsilon}{z+1} & 0 & 0 & 0 & 0 & \frac{2\epsilon}{z+1} & 0 & -\frac{2\epsilon}{z+1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\left(z^2+2z+28\right)\epsilon}{3(z-8)z(z+1)} & 0 & -\frac{8\epsilon}{(z-8)z(z+1)\varpi_0(z)^2} \\
0 & \frac{\epsilon}{12z} & 0 & -\frac{4\epsilon}{3z} & \frac{\epsilon}{6z} & \frac{4\epsilon}{3z} & -\frac{4\epsilon}{3z} & 0 & 0 & 0 & 0 & 0 & \frac{2\epsilon}{3z} & \frac{2\epsilon}{3z} & -\frac{2\epsilon}{3z} & \frac{4(z+1)\epsilon\varpi_0(z)}{9z} & \frac{2\epsilon}{3z} & 0 \\
\frac{1}{2}\epsilon\varpi_0(z) & -\frac{(13z-8)\epsilon\varpi_0(z)}{48z} & -\epsilon\varpi_0(z) & \frac{(73z-20)\epsilon\varpi_0(z)}{12z} & -\frac{(37z-8)\epsilon\varpi_0(z)}{24z} & \frac{(5z+8)\epsilon\varpi_0(z)}{3z} & -\frac{(59z+32)\epsilon\varpi_0(z)}{12z} & -2\epsilon\varpi_0(z) & -2\epsilon\varpi_0(z) & -\frac{(7z-8)\epsilon\varpi_0(z)}{4\sqrt{z-4}\sqrt{z}} & \frac{2\epsilon\varpi_0(z)}{z} & 0 & -\frac{(7z-8)\epsilon\varpi_0(z)}{6z} & \frac{2(2z-1)\epsilon\varpi_0(z)}{3z} & \frac{(7z-8)\epsilon\varpi_0(z)}{6z} & -\frac{(z-2)\left(11z^3-66z^2-84z-88\right)\epsilon\varpi_0(z)^2}{24(z-8)z(z+1)} & -\frac{2(z+1)\epsilon\varpi_0(z)}{3z} & -\frac{\left(z^2+2z+28\right)\epsilon}{3(z-8)z(z+1)}
\end{pmatrix}$$

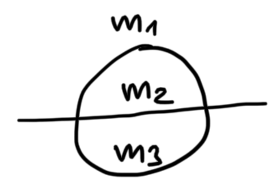
Feynman Integrals in Dimensional Regularization

- Our examples include so far (two-loop Higgs+jet-production in QCD): [new]

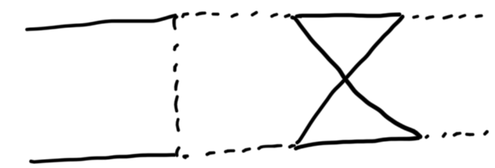
Kite



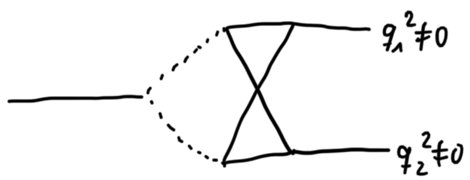
Sunsets



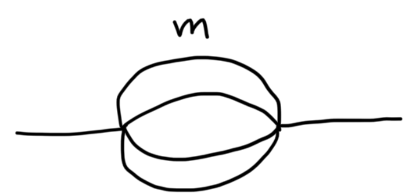
Crossed Box



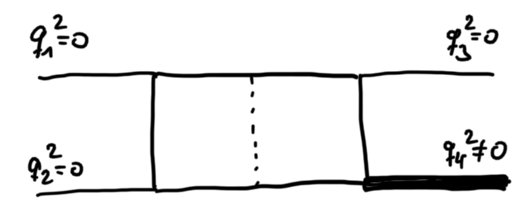
Triangles



Bananas



Double Box



- For multi-parameter elliptic or more complicated geometries we need as new kernels:

$$\int^z R_2(z') \int^{z'} R_2(z'')\varpi_0(z'')^3 dz' dz''$$

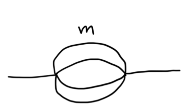
[Weinzierl]

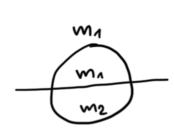
$$\int \varpi_0(z_1, z_2) dz_1$$

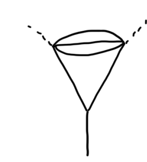
$$\int R(z)\varpi_0(z)\varpi_0(1/z)dz$$

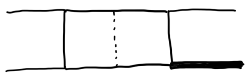
$$\int \varpi_0'(z)\varpi_0(1/z)dz$$

2 new kernels





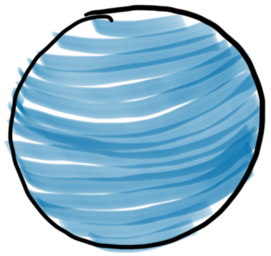




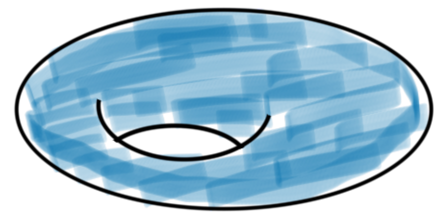
- Number of new kernels depends on number of parameters and the total elliptic sector with residues.

Conclusion

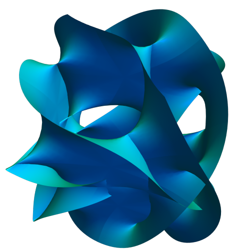
- Unterstanding **CY geometries** is **essential** for understanding **higher loop Feynman integrals**.



Riemann sphere

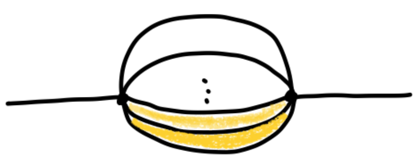


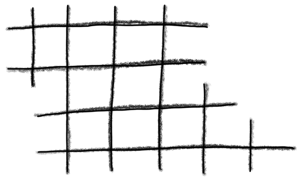
elliptic curve



Calabi-Yau

- The geometry behind a Feynman integral tells us the **function space** and **boundary condition**.
- **Dictionary** between Feynman integrals and non-trivial geometries.
- Using CY techniques we can solve so far **three** different **families** of Feynman graphs:





- Our new approach gives, in particular for elliptic Feynman integrals, ϵ-**factorized differential equations**.

  → For this we have to introduce new objects which are (iterated) integrals of ϖ_0 .

- Open questions:
  - Other families with underlying Calabi-Yau geometry?
  - What are the limits of our method for ϵ-factorized differential equations?
  - Need mathematical definition of iterated Calabi-Yau periods similar to elliptic polylogs.

Thank you for your attention