Period Geometry of Calabi-Yau n-folds for Feynman integrals

Bethe Forum: Geometries and Special functions for Physics and Mathematics

Albrecht Klemm, BCTP/HCM Bonn University

March 20 2023

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Florian Loebbert, Christoph Nega, Franzika Porkert, Reza Safari, Lorenzo Tancredi

[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, published JHEP
[4]=arXiv:2209.05291 in PRL and [5]=arXiv: 2212.09550 in JHEP, in progress

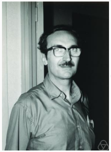
Aspects of Calabi-Yau manifolds (historically)

Differential geometry question

On which Kähler n-folds
$\exists g$ with $R_{i\bar{\jmath}}(g) = 0$?

Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question



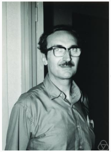
On which Kähler n-folds

$\exists g$ with $R_{i\bar{\jmath}}(g) = 0$?

$\exists!$ g in given Kähler class, if $c_1(TM) = 0$?

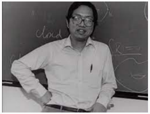
Aspects of Calabi-Yau manifolds (historically)

1954 — Differential geometry question — 1976



On which Kähler n-folds

$\exists g$ with $R_{i\bar{j}}(g) = 0$?

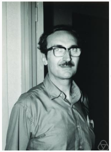


$\exists!$ g in given Kähler class, if $c_1(TM) = 0$?

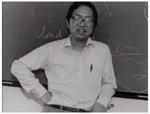
$\exists!$ $g \leftrightarrow c_1(TM) = 0!$
Existence proof.

Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question 1976



On which Kähler n-folds $\exists g$ with $R_{i\bar{\jmath}}(g) = 0$?



$\exists!$ g in given Kähler class, if $c_1(TM) = 0$?

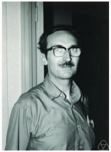
$\exists!$ $g \leftrightarrow c_1(TM) = 0!$ Existence proof.

String Compactification: Candelas, Horowitz, Strominger, Witten 1985

String Theory

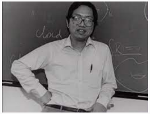
Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question 1976



On which Kähler n-folds

$\exists g$ with $R_{i\bar{j}}(g) = 0$?



$\exists !\ g$ in given Kähler class, if $c_1(TM) = 0$?

$\exists !\ g \leftrightarrow c_1(TM) = 0!$
Existence proof.

Localisation: Kontsevich, Givental, Lian, Yau,.. 1993

String Compactification: Candelas, Horowitz, Strominger, Witten 1985

Enumerative Geometry

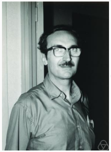
Black hole Entropy

String Theory

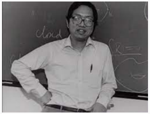
Mirror Symmetry 1990

Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question 1976



On which Kähler n-folds $\exists g$ with $R_{i\bar{j}}(g) = 0$?



$\exists!\ g$ in given Kähler class, if $c_1(TM) = 0$?

$\exists!\ g \leftrightarrow c_1(TM) = 0!$ Existence proof.

Localisation: Kontsevich, Givental, Lian, Yau,.. 1993

String Compactification: Candelas, Horowitz, Strominger, Witten 1985

Physicists,..

Variation of mix. Hodge structure: Griffiths,Tian/Todorov, Bloch, Kerr ..

Enumerative Geometry

Black hole Entropy

String Theory

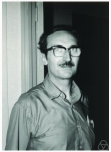
WS→WL Dixon,..

Feynman integrals

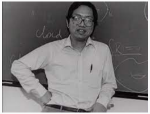
Mirror Symmetry 1990

Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question 1976



On which Kähler n-folds
$\exists g$ with $R_{i\bar{j}}(g) = 0$?



$\exists!$ g in given Kähler class, if $c_1(TM) = 0$?

$\exists!$ $g \leftrightarrow c_1(TM) = 0$! Existence proof.

p-adic periods def, Dwork, Deligne,..1965

Localisation: Kontsevich, Giventhal, Lian, Yau,.. 1993

String Compactification: Candelas, Horowitz, Strominger, Witten 1985

Physicists,..

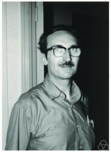
Variation of mix. Hodge structure: Griffiths,Tian/Todorov, Bloch, Kerr ..

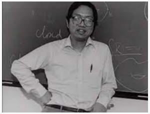
Arithmetic Geometry — modularity / p-adic Analysis — Enumerative Geometry — Black hole Entropy — String Theory — WS→WL Dixon,.. — Feynman integrals

Mirror Symmetry 1990

Aspects of Calabi-Yau manifolds (historically)

1954 Differential geometry question 1976





On which Kähler n-folds
$\exists g$ with $R_{i\bar{\jmath}}(g) = 0$?

$\exists!\ g$ in given Kähler class, if $c_1(TM) = 0$?

$\exists!\ g \leftrightarrow c_1(TM) = 0!$
Existence proof.

p-adic periods def, Dwork, Deligne,..1965

Localisation: Kontsevich, Giventhal, Lian, Yau,.. 1993

String Compactification: Candelas, Horowitz, Strominger, Witten 1985

Physicists,..

Variation of mix. Hodge structure: Griffiths,Tian/Todorov, Bloch, Kerr ..

Arithmetic Geometry

modularity

p-adic Analysis

Enumerative Geometry

Black hole

Entropy

String Theory

WS→WL

Dixon,..

Feynman integrals

Mirror Symmetry 1990

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau *n*-fold M is a compact *complex manifold* of complex dimension *n* that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.
The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau *n*-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.
- We use $SU(n)$ rather then $\subset SU(n)$ to avoid trivial products of lower CY n-folds in the generalisation.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2 H^2} + \ldots$$

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2 H^2} + \ldots$$

$\Rightarrow$ 1) cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2H^2} + \ldots$$

⇒ 1) cubic $d=3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

⇒ 2) quartic in $\mathbb{P}^3$ is a CY 2-fold with 19 complex moduli.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2H^2} + \ldots$$

⇒ 1) cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

⇒ 2) quartic in $\mathbb{P}^3$ is a CY 2-fold with 19 complex moduli.

⇒ 3) quintic in $\mathbb{P}^4$ is a CY 3-fold with 101 complex moduli.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

Euler number (Gauss Bonnet): $\chi = \int_{M_n} c_n(TM) = c_n d$,

1) $\chi = 0$, $\chi = 2g - 2 \Rightarrow g = 1$ one topological type E.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

Euler number (Gauss Bonnet): $\chi = \int_{M_n} c_n(TM) = c_n d$,

1) $\chi = 0$, $\chi = 2g - 2 \Rightarrow g = 1$ one topological type E.
2) By $c_2(TM) = 6H^2 \Rightarrow \chi = 24$. HRR for arithmetic genus of surface $\chi_0 = \sum_{i=0}^{2}(-1)^i h^{0,i} = \frac{1}{12}\int_{M_2}(c_1^2 + c_2)$. Now by definition $h^{00} = h^{02} = 1$, $h^{01} = 0$ because of $SU(2)$ hol, i.e. $\chi_0(M_2) = 2$ and since $c_1 = 0 \Rightarrow \chi(M_2) = 24$ and we have only one topological type the $K3$ surface
3) By $c_3(TM) = -40H^3 \Rightarrow \chi = -200$. Hirzebruch Riemann Roch $\chi_0 = \frac{1}{24}\int_{M_3} c_1 c_2 = 1 - 0 + 0 - 1\checkmark$, $\chi_1 = -h^{11} + h^{21} = \frac{1}{24}\int_{M_3} c_1 c_2 - 12 c_3 \Rightarrow \chi = 2(h^{11} - h^{21})\checkmark$

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$.

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

CICYs: Complete intersections: The vanishing locus of r polynomials $P_k = 0$, $k = 1, \ldots, r$ in $\mathbb{P} = \otimes_{l=1}^{m} \mathbb{P}_l^{n_l}$ define a CY $(\sum_{l=1}^{m} n_l - r)$ -fold if $\sum_{k=1}^{r} d_{kl} = n_l + 1$, $\forall l = 1, \ldots, m$, with d_{kl} are degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

- CICYs: Complete intersections: The vanishing locus of r polynomials $P_k = 0$, $k = 1, \ldots, r$ in $\mathbb{P} = \otimes_{l=1}^m \mathbb{P}_l^{n_l}$ define a CY $(\sum_{l=1}^m n_l - r)$ -fold if $\sum_{k=1}^r d_{kl} = n_l + 1$, $\forall l = 1, \ldots, m$, with d_{kl} are degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.
- BCs: Branched covers: Let $\mathbb{P}$ be a n-dimensional Fano variety with positive canonical class $K(\mathbb{P}) = c_1(\mathbb{P}) > 0$ then a b-fold cover that is branched at $bK(\mathbb{P})$ is a non necessarily smooth CY n-fold: 2d n loop fishnets .

Hypersurface in toric ambient spaces

Hypersurface in toric ambient spaces

Huge classified construction: CY 3folds as hypersurfaces in toric ambient spaces $\mathbb{P}_\Delta$, $\mathbb{P}_{\hat{\Delta}}$ defined by reflexive pairs $(\Delta, \hat{\Delta})$:

Hypersurface in toric ambient spaces

Huge classified construction: CY 3folds as hypersurfaces in toric ambient spaces $\mathbb{P}_\Delta$, $\mathbb{P}_{\hat{\Delta}}$ defined by reflexive pairs $(\Delta, \hat{\Delta})$:

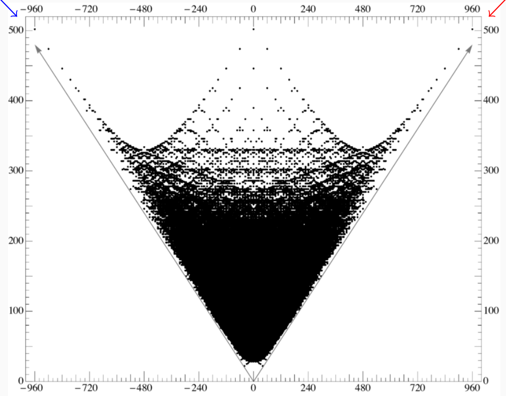
$(h^{11}, h^{2,1}) = (11, 491)$

$(h^{11}, h^{2,1}) = (491, 11)$

$M = \{[p_{\hat{\Delta}} = 0] = [\sum_i H_i] \subset \mathbb{P}_\Delta\}$

Batyrev: (M, W) mirror pairs

$W = \{[p_\Delta = 0] = [\sum_i \hat{H}_i] \subset \mathbb{P}_{\hat{\Delta}}\}$



Hypersurface in toric ambient spaces

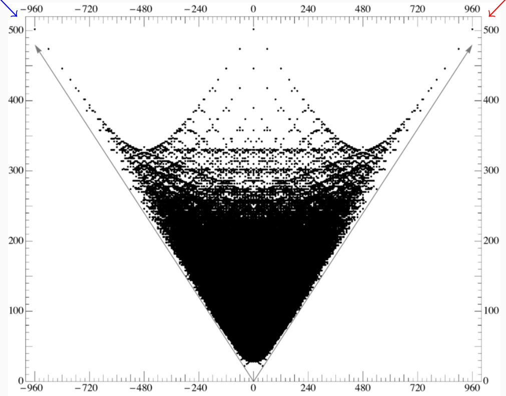
Huge classified construction: CY 3folds as hypersurfaces in toric ambient spaces $\mathbb{P}_\Delta$, $\mathbb{P}_{\hat{\Delta}}$ defined by reflexive pairs $(\Delta, \hat{\Delta})$:

Geom.	$\#(\Delta_k, \hat{\Delta}_k)$
2 *pts*	1
ell crv	16
K3	4319[1]
CY 3-flds	473800776[1]
CY 4-flds	$\mathcal{O}(10^{22})$?
⋮	⋮

[1] Kreuzer & Skarke '02, $k = 3, 4$

$(h^{11}, h^{2,1}) = (11, 491)$

$(h^{11}, h^{2,1}) = (491, 11)$



General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

Example: We counted $101 = h^{2,1}$ complex deformation parameters for the quintic in $\mathbb{P}^4$ and by the Lefshetz hyperplane theorem $h^{1,1} = 1$ (inherited from $\mathbb{P}^4$), hence $\chi = 2(h^{1,1} - h^{21}) = -200$ in accordance with Gauss-Bonnet.

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

Example: We counted $101 = h^{2,1}$ complex deformation parameters for the quintic in $\mathbb{P}^4$ and by the Lefshetz hyperplane theorem $h^{1,1} = 1$ (inherited from $\mathbb{P}^4$), hence $\chi = 2(h^{1,1} - h^{21}) = -200$ in accordance with Gauss-Bonnet.

Application: The complex moduli dependent period integrals on CY n-fold families generalize elliptic functions. They are identified for important examples with the maximal cut Feynman $n-1$-loop integrals, where the complex moduli z are identified with the scale invariant physical parameters $z_i = p^2/m_i^2, \ldots$.

Periods integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing between between (middle) homology and (middle) cohomology well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{K}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

It is possible and natural to have $\mathbb{K}$ to be $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{K}) \times H_n(M_n, \mathbb{K}) \to \mathbb{K},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is a symmetric on the even self dual lattice $H_n(M_n, \mathbb{K})$. E.g. for $K3$ $b_2 = 22$ and $\sigma = b_2^+ - b_2^- = \frac{1}{3}\int_{M_2} c_1^2 - 2c_2 = -16$ hence b_2 has signature $(3, 19)$ and is $E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

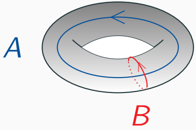
It is clearly defined up only to an $\mathsf{Sp}(b_n(M), \mathbb{Z})$ choice.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$

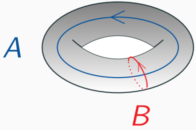


If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x - w)(x - wz) = 0 \subset \mathbb{P}^2$



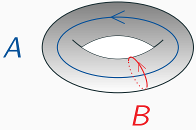
$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z \Omega(z) \sim \frac{xdx}{y}$

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z \Omega(z) \sim \frac{xdx}{y}$

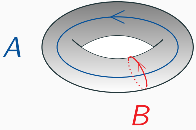
$E_1(z) = \oint_A \Omega$, $E_2(z) = \oint_B \Omega$ Elliptic integrals.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y},\ \partial_z \Omega(z) \sim \frac{xdx}{y}$$

$$E_1(z) = \oint_A \Omega,\ E_2(z) = \oint_B \Omega \quad \text{Elliptic integrals.}$$

Well studied in part because they solve Keplers problem

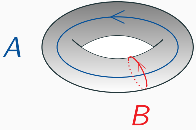
Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z \Omega(z) \sim \frac{xdx}{y}$

$E_1(z) = \oint_A \Omega$, $E_2(z) = \oint_B \Omega$ Elliptic integrals.

Well studied in part because they solve Keplers problem

Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

$$L^{(2)} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4}\right] \int_\Gamma \Omega = 0 \; .$$

The Picard-Fuchs differential ideal

We can always expand $\Omega = \sum_{i=1}^{b_3(W)} \Pi_i(z)\gamma_i$ in terms of periods $\Pi_i(z) = \int_{\Gamma_i} \Omega(z)$.

The $b_n(M_n)$ periods span a vector space that is identified with the solutions space of linear Picard-Fuchs differential ideal $\mathcal{L}\Pi_i(z) = 0$.

For one parameter families $\mathcal{L}$ is generated by a $b_n(M_n) + 1$ order Picard-Fuchs operator $L^{(b_n(M_n)+1)}$, while for multiparameter families $\mathcal{L} = \{L_i^{(k)}, i = 1, \ldots, |\mathcal{L}|, k = 2, \ldots, b_n(M_n) + 1\}$ with several $L_i^{(k)}$.

The latter can derived using the Griffiths reduction method and for CY embedded in toric ambient space also as a reduction of a Gelfand Kapranov Zelevinskii system.

Finding and integral basis

The Feynman integrals correspond i.a. to periods over integral cycles, e.g. $\underline{\Gamma} = \{A_I, B^I\}$. Such are not specified by $\mathcal{L}$ alone.

Finding and integral basis

The Feynman integrals correspond i.a. to periods over integral cycles, e.g. $\underline{\Gamma} = \{A_I, B^I\}$. Such are not specified by $\mathcal{L}$ alone.

On CY moduli space the monodromy of the periods in $\overline{\mathcal{M}_{cs}(M_n)}$ is irreducible and monodromies $\vec{\Pi} \to M_i \vec{\Pi}$ along loops λ_i encircling singular (normal crossing) divisor $D_i \in \overline{\mathcal{M}_{cs}(M_n)}$ respect the intersection form Σ, i.e. $M_i^t \Sigma M_i = \Sigma$, $\forall\ M_i$. In an integral bases the $\{M_i\}$ generate a subgroup of $O(\Sigma, \mathbb{Z})$. Since the latter acts irreducible it is possible to find an integral basis up to $O(\Sigma, \mathbb{Z})$ by caculating all $\{M_i\}$ in an arbitrary basis and conjugating them simultaneously into $O(\Sigma, \mathbb{Z})$.

Finding and integral basis

The Feynman integrals correspond i.a. to periods over integral cycles, e.g. $\underline{\Gamma} = \{A_I, B^I\}$. Such are not specified by $\mathcal{L}$ alone.

On CY moduli space the monodromy of the periods in $\overline{\mathcal{M}_{cs}(M_n)}$ is irreducible and monodromies $\vec{\Pi} \to M_i \vec{\Pi}$ along loops λ_i encircling singular (normal crossing) divisor $D_i \in \overline{\mathcal{M}_{cs}(M_n)}$ respect the intersection form Σ, i.e. $M_i^t \Sigma M_i = \Sigma$, $\forall\ M_i$. In an integral bases the $\{M_i\}$ generate a subgroup of $O(\Sigma, \mathbb{Z})$. Since the latter acts irreducible it is possible to find an integral basis up to $O(\Sigma, \mathbb{Z})$ by caculating all $\{M_i\}$ in an arbitrary basis and conjugating them simultaneously into $O(\Sigma, \mathbb{Z})$.

If $\overline{\mathcal{M}_{cs}(M_n)}$ has a point of maximal unipotent mondromy (MUM) with a known mirror W_n one can calculate an integral period vector using the $\hat{\Gamma}(TW_n)$-class.

One Parameter CY 3 fold operators

Examples: There are 14 hyper geometric ${}_3F_4$ CY 3-fold operators given by

$$L^{(4)} \;=\; \theta^4 - \mu^{-1} z \prod_{k\,=\,1}^{4} (\theta + a_k)\;,$$

where $\theta = z\frac{d}{dz}$ and z parametrizes $\mathcal{M}_{cs}(M_3) = \mathbb{P}^1 \setminus \{0, \mu, \infty\}$

One Parameter CY 3 fold operators

Examples: There are 14 hyper geometric ${}_3F_4$ CY 3-fold operators given by

$$L^{(4)} = \theta^4 - \mu^{-1} z \prod_{k=1}^{4} (\theta + a_k) ,$$

where $\theta = z\frac{d}{dz}$ and z parametrizes $\mathcal{M}_{cs}(M_3) = \mathbb{P}^1 \setminus \{0, \mu, \infty\}$ and

$\#$	W	κ	$c_2 \cdot D$	$\chi(W)$	a_1, a_2, a_3, a_4	μ^{-1}	dT_∞
1	$X_5(1^5)$	5	50	-200	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	5^5	O_5^{DG}
2	$X_{4,2}(1^6)$	8	56	-176	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	2^{10}	C_4
3	$X_{3,3}(1^6)$	9	54	-144	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	3^6	K_3
4	$X_{2,2,2,2}(1^8)$	16	64	-128	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	2^8	M_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

One Parameter CY 3 fold operators

Their Riemann symbols are

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \mu & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_4 \end{array}\right\} \quad .$$

One Parameter CY 3 fold operators

Their Riemann symbols are

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \mu & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_4 \end{array}\right\} \quad .$$

At $z = 0$ the local exponents are completely degenerate and we have a MUM point. A Frobenius $\mathbb{C}$-basis of solutions is

$$\vec{\Pi}_0(z) = \begin{pmatrix} f_0(z) \\ f_0(z)\log(z) + f_1(z) \\ \frac{1}{2}f_0(z)\log^2(z) + f_1(z)\log(z) + f_2(z) \\ \frac{1}{6}f_0(z)\log^3(z) + \frac{1}{2}f_1(z)\log^2(z) + f_2(z)\log(z) + f_3(z) \end{pmatrix}$$

for power series normalized by $f_0(0) = 1$ and $f_1(0) = f_2(0) = f_3(0) = 0$.

One Parameter CY 3 fold operators

The $\hat{\Gamma}(TW)$ class determines an integral basis at $z = 0$

$$\vec{\Pi} = \begin{pmatrix} \int_{B^0} \Omega \\ \int_{B^1} \Omega \\ \int_{A_0} \Omega \\ \int_{A_1} \Omega \end{pmatrix} = (2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)\chi(M)}{(2\pi i)^3} & \frac{c_2 \cdot D}{24 \cdot 2\pi i} & 0 & \frac{\kappa}{(2\pi i)^3} \\ \frac{c_2 \cdot D}{24} & \frac{\sigma}{2\pi i} & -\frac{\kappa}{(2\pi i)^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \end{pmatrix} \Pi_0 \,. \qquad (1)$$

in terms of the C.T.C Wall data.

One Parameter CY 3 fold operators

The $\hat{\Gamma}(TW)$ class determines an integral basis at $z = 0$

$$\vec{\Pi} = \begin{pmatrix} \int_{B^0} \Omega \\ \int_{B^1} \Omega \\ \int_{A_0} \Omega \\ \int_{A_1} \Omega \end{pmatrix} = (2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)\chi(M)}{(2\pi i)^3} & \frac{c_2\cdot D}{24\cdot 2\pi i} & 0 & \frac{\kappa}{(2\pi i)^3} \\ \frac{c_2\cdot D}{24} & \frac{\sigma}{2\pi i} & -\frac{\kappa}{(2\pi i)^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \end{pmatrix} \Pi_0 \,. \quad (1)$$

in terms of the C.T.C Wall data. The monodromies in $\mathrm{SP}(4,\mathbb{Z}) = O(\Sigma,\mathbb{Z})$ are generated by

$$M_0 = \begin{pmatrix} 1 & -1 & \frac{\kappa}{6} + \frac{c_2\cdot D}{12} & \frac{\kappa}{2} + \sigma \\ 0 & 1 & \sigma - \frac{\kappa}{2} & -\kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; M_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $M_\infty = (M_0 M_\mu)^{-1}$. Note that by HRR $\frac{\kappa}{6} + \frac{c_2\cdot D}{12} = \chi(\mathcal{O}_D) + 1 \in \mathbb{Z}$

One Parameter CY 3 fold operators

One parameter CY 3-fold differential operators $L^{(4)} = \sum_{i=0}^{4} c_i(z)\partial_z^i$ have been classified by Almkvist, Enckevort, van Straten and Zudilin (AESZ list) at least to finite order in $c_i(z)$ in z. E.g. the AESZ34 operator

$$\begin{aligned} L^{(4)} = \quad & 1 - 5z - (4-28z)\theta + \left(6 - 63z + 26z^2 - 225z^3\right)\theta^2 - \left(4 - 70z + 450z^3\right)\theta^3 \\ & + (1-z)(1-9z)(1-25z)\theta^4 \end{aligned}$$

with Riemann symbol

$$\mathcal{P}_4 \left\{ \begin{array}{ccccc} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{array} \right\}$$

One Parameter CY 3 fold operators

One parameter CY 3-fold differential operators $L^{(4)} = \sum_{i=0}^{4} c_i(z)\partial_z^i$ have been classified by Almkvist, Enckevort, van Straten and Zudilin (AESZ list) at least to finite order in $c_i(z)$ in z. E.g. the AESZ34 operator

$$\begin{aligned} L^{(4)} = \quad & 1 - 5z - (4 - 28z)\theta + \left(6 - 63z + 26z^2 - 225z^3\right)\theta^2 - \left(4 - 70z + 450z^3\right)\theta^3 \\ & + (1-z)(1-9z)(1-25z)\theta^4 \end{aligned}$$

with Riemann symbol

$$\mathcal{P}_4 \left\{ \begin{array}{ccccc} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{array} \right\}$$

corresponds to the 4 loop equal mass Banana maximal cut integral with $z = m^2/p^2$. Which itself is the diagonal specialisation of the five parameter GKZ system of the complete inter section of two degree $d_{1,k} = (1,1,1,1,1)$, $d_{2,k} = (1,1,1,1,1)$ constraints in $(\mathbb{P}^1)^5$ describing the general mass case $z_i = m_i^2/p^2$.

Period geometry on CY n-fold

The main constrains which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \qquad (2)$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

Period geometry on CY n-fold

The main constrains which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \qquad (2)$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$. As well as from relations on holomorphic bilinears and their derivatives that follow from Griffiths transversality

$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(z) & \text{if } k = n\,. \end{cases} \qquad (3)$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^k H^{n-p,p}(W)$ are arbitrary combinations of derivatives w.r.t. to the z_i, $i = 1, \ldots, r$.

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Exercise: Show that

$$C_{111} = \frac{\kappa}{z^3(1-\mu^{-1}z)} \; .$$

for the hypergeometric cases.

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Exercise: Show that

$$C_{111} = \frac{\kappa}{z^3(1-\mu^{-1}z)} \ .$$

for the hypergeometric cases.

Remark 1: W.r.t the Hodge decomposition the pairings (2) and (3) have the property that if $\alpha_{m,n} \in H^{m,n}(M_n)$ and $\beta_{p,q} \in H^{r,s}(M_n)$ then $\int_W \alpha_{m,n} \wedge \beta_{p,q} = 0$ unless $m+p = n+q = 3$.

Remark 2: In terms of a basis of periods compatible with Σ they can be written as

$$\int_{M_n} \Omega \wedge \bar{\Omega} = \vec{\Pi}^\dagger \Sigma \vec{\Pi}, \qquad \int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = -\vec{\Pi}^T \Sigma \underline{\partial}^k_{I_k} \vec{\Pi} \ ,$$

Period geometry on CY n-fold

The pairings (2) and (3) together with Remark 1 give rise to what is know for CY 3-folds as Special Geometry.

Period geometry on CY n-fold

The pairings (2) and (3) together with Remark 1 give rise to what is know for CY 3-folds as Special Geometry.

We will focus here of one aspect of the latter that is relevant for Feynman integral for the following reason: While the maximal integrals are periods and as such solutions of the the homogeneous differential equations $\mathcal{L}\Pi = 0$ the actual Feynman integral is a solution of an inhomogeneous extension $\mathcal{L}\Pi = g(z)$.

Period geometry on CY n-fold

The pairings (2) and (3) together with Remark 1 give rise to what is know for CY 3-folds as Special Geometry.

We will focus here of one aspect of the latter that is relevant for Feynman integral for the following reason: While the maximal integrals are periods and as such solutions of the the homogeneous differential equations $\mathcal{L}\Pi = 0$ the actual Feynman integral is a solution of an inhomogeneous extension $\mathcal{L}\Pi = g(z)$.

When determining the inhomogeous solutions by the variation of constants methods one considers the Wronskian $[W(z)]_{i,j} = \partial_z^i \Pi_j$, $i, j = 0, \ldots, r$ and in particular its inverse.

A simple consequence of Griffiths transversality

Let us define the skew symmetric matrix

$$Z = W\Sigma W^T, \qquad \text{i.e. } [Z(z)]_{ij} = \partial_z^i \Pi^T \Sigma \partial_z^j \Pi, \quad \text{for } i,j = 0, \ldots, r$$

Then (3) implies that Z is rational and its entries are calculated recursively from derivatives of (3) using $\Pi^T \Sigma \underline{\partial}_{l_k}^k \mathcal{L} \Pi = 0$.

A simple consequence of Griffiths transversality

Let us define the skew symmetric matrix

$$Z = W\Sigma W^T, \qquad \text{i.e. } [Z(z)]_{ij} = \partial_z^i \Pi^T \Sigma \partial_z^j \Pi, \quad \text{for } i,j = 0,\ldots,r$$

Then (3) implies that Z is rational and its entries are calculated recursively from derivatives of (3) using $\Pi^T \Sigma \underline{\partial}^k_{I_k} \mathcal{L} \Pi = 0$.

E.g. for the one parameter case with $r = 3$, with the abbreviations $C = C_{111}$, $C' = \partial_z C$ one finds

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{c_2}{c_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{c_2}{c_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

A simple consequence of Griffiths transversality

Let us define the skew symmetric matrix

$$Z = W\Sigma W^T, \qquad \text{i.e. } [Z(z)]_{ij} = \partial_z^i \Pi^T \Sigma \partial_z^i \Pi, \quad \text{for } i,j = 0,\ldots,r$$

Then (3) implies that Z is rational and its entries are calculated recursively from derivatives of (3) using $\Pi^T \Sigma \underline{\partial}^k_{I_k} \mathcal{L} \Pi = 0$.

E.g. for the one parameter case with $r = 3$, with the abbreviations $C = C_{111}$, $C' = \partial_z C$ one finds

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{c_2}{c_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{c_2}{c_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad W^{-1} = \Sigma W^T Z^{-1} \, .$$

depends up to rational functions linear on the periods and its derivatives and the inhomogeous solution becomes an iterated integral.

Special coordinates in special geometry

The local Torelli theorem states that a sufficiently small domain $U_* \subset \mathcal{M}_{cs}(W)$ can be identified with a chart in $\mathbb{P}^r$ using the period map $z \mapsto (X_*^0(z) : \ldots : X_*^r(z)) \in \mathbb{P}^r$ and parametrized by inhomogeneous coordinates $t_*^i(z) = X_*^i / X_*^0$. Clearly the $P_I^* = \int_{B_I} \Omega$ are then homogeneous functions of the $X_*^I = \int_{A^I} \Omega$.

Special coordinates in special geometry

The local Torelli theorem states that a sufficiently small domain $U_* \subset \mathcal{M}_{cs}(W)$ can be identified with a chart in $\mathbb{P}^r$ using the period map $z \mapsto (X^0_*(z) : \ldots : X^r_*(z)) \in \mathbb{P}^r$ and parametrized by inhomogeneous coordinates $t^i_*(z) = X^i_*/X^0_*$. Clearly the $P^*_I = \int_{B_I} \Omega$ are then homogeneous functions of the $X^I_* = \int_{A^I} \Omega$. E.g. for CY 3-folds (3) for $k = 1, 2$ implies the existence of a prepotential $F^*(X_*)$ so that

$$P^*_I(X_*) = \frac{\partial}{\partial X^I_*} F^*(X_*), \qquad 2F^*(X_*) = X^I_* P^*_I(X_*),$$

where the Newton equation with Einstein sum conventions implies that $F(X_*)$ is of degree two in the X^I_*.

Special coordinates in special geometry

Writing $F_*(X_*) = \mathcal{F}_*(t_*)(X_*^0)^2$ $\vec{\Pi} = (P_I^*, X_*^I)^t$ becomes $\vec{\Pi} = X^0(2\mathcal{F}_* - t^i\partial_{t_*^i}\mathcal{F}_*, \partial_{t_*^i}\mathcal{F}_*, 1, t_*^i)^t$, and inserting this into (**??**), changing variables from z_k to t^k and using the transformations properties of $C_{z_i z_j z_k}$ one establishes $C_{t_*^i t_*^j t_*^k} = \partial_{t_*^i}\partial_{t_*^j}\partial_{t_*^k}\mathcal{F}_*(t_*)$

Special coordinates in special geometry

Writing $F_*(X_*) = \mathcal{F}_*(t_*)(X_*^0)^2$ $\vec{\Pi} = (P_I^*, X_*^I)^t$ becomes $\vec{\Pi} = X^0(2\mathcal{F}_* - t^i\partial_{t_*^i}\mathcal{F}_*, \partial_{t_*^i}\mathcal{F}_*, 1, t_*^i)^t$, and inserting this into (**??**), changing variables from z_k to t^k and using the transformations properties of $C_{z_i z_j z_k}$ one establishes $C_{t_*^i t_*^j t_*^k} = \partial_{t_*^i}\partial_{t_*^j}\partial_{t_*^k}\mathcal{F}_*(t_*)$

By change of the dependent variable one defines a vector $\underline{\mathcal{V}} = (2\mathcal{F}_* - t_*^c\partial_c\mathcal{F}_*, \partial_j(2\mathcal{F}_* - t_*^c\partial_c\mathcal{F}_*), t_*^j, 1)^T$, and with $\mathcal{V}^j := \mathcal{V}_{b_3(M_3)/2+j}$, $\mathcal{V}^0 := \mathcal{V}_{b_3(M_3)}$ one gets trivially

$$\partial_{t_*^i}\begin{pmatrix}\mathcal{V}_0\\ \mathcal{V}_j\\ \mathcal{V}^j\\ \mathcal{V}^0\end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0\\ 0 & 0 & C_{ijk} & 0\\ 0 & 0 & 0 & \delta_i^j\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathcal{V}_0\\ \mathcal{V}_k\\ \mathcal{V}^k\\ \mathcal{V}^0\end{pmatrix}.$$

Special coordinates in special geometry

Writing $F_*(X_*) = \mathcal{F}_*(t_*)(X_*^0)^2$ $\vec{\Pi} = (P_I^*, X_*^I)^t$ becomes $\vec{\Pi} = X^0(2\mathcal{F}_* - t^i \partial_{t_*^i} \mathcal{F}_*, \partial_{t_*^i} \mathcal{F}_*, 1, t_*^i)^t$, and inserting this into (**??**), changing variables from z_k to t^k and using the transformations properties of $C_{z_i z_j z_k}$ one establishes $C_{t_*^i t_*^j t_*^k} = \partial_{t_*^i} \partial_{t_*^j} \partial_{t_*^k} \mathcal{F}_*(t_*)$

By change of the dependent variable one defines a vector $\underline{\mathcal{V}} = (2\mathcal{F}_* - t_*^c \partial_c \mathcal{F}_*, \partial_j(2\mathcal{F}_* - t_*^c \partial_c \mathcal{F}_*), t_*^j, 1)^T$, and with $\mathcal{V}^j := \mathcal{V}_{b_3(M_3)/2+j}$, $\mathcal{V}^0 := \mathcal{V}_{b_3(M_3)}$ one gets trivially

$$\partial_{t_*^i} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

This is the Gauss-Manin connection in projective flat coordinates and in special Kähler gauge. These formulas allow simpler iterated integrals and generalise to all n provided one knows Σ.

Omisions:

The theory of mixed Hodge structure gives very detailed information about the possible degeneration of the periods. Here we brushed only over the maximal unipotent degeneration.

Omisions:

The theory of mixed Hodge structure gives very detailed information about the possible degeneration of the periods. Here we brushed only over the maximal unipotent degeneration.

CY n-folds exhibit modularity in a similar sense than elliptic curves. As as consequence at certain rational z values the periods can be given by Hecke L function values or better by periods and quasi periods of modular objects.

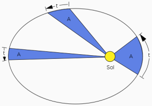
Omisions:

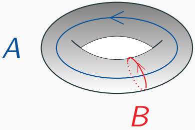
The theory of mixed Hodge structure gives very detailed information about the possible degeneration of the periods. Here we brushed only over the maximal unipotent degeneration.

CY n-folds exhibit modularity in a similar sense than elliptic curves. As as consequence at certain rational z values the periods can be given by Hecke L function values or better by periods and quasi periods of modular objects.

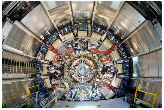
So far we explored mainly the holomorphic story following from (3) and not the combinations of (3) with (2) and *Remark 1*. This leads to an integrable structure, tt^*-equations, Kodaira Spencer gravity, topological string theory and related topics ....

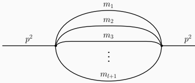
Conclusion and Outlook





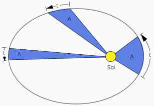
Conclusion and Outlook

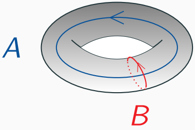




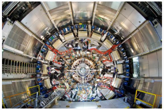
Physics

Mathematics



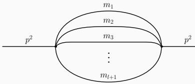


Conclusion and Outlook

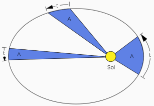


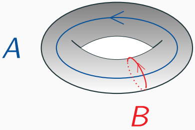
String Theory

Physics



Mathematics



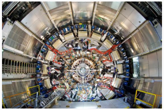


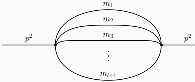
Enumerative Geometry

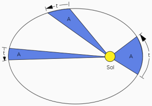
Conclusion and Outlook

String Theory

Black Hole Physics

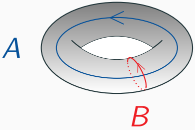






Physics

Mathematics



Enumerative Geometry

Arithmetic Geometry