The Four-Loop Non-Singlet Splitting Functions in QCD

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Based on

Thomas Gehrmann, Andreas von Manteuffel, VS, Tong-Zhi Yang [arXiv:2604.09534]

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Outline

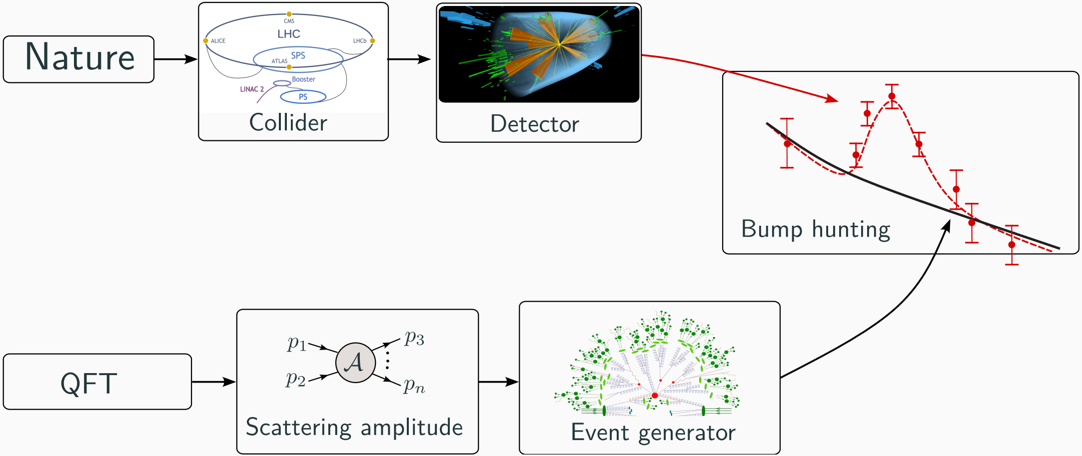
1. Introduction
2. Off-shell OME method
3. The four-loop computation
4. Results
5. Towards complete four-loop splitting functions
6. Conclusions and outlook

Introduction

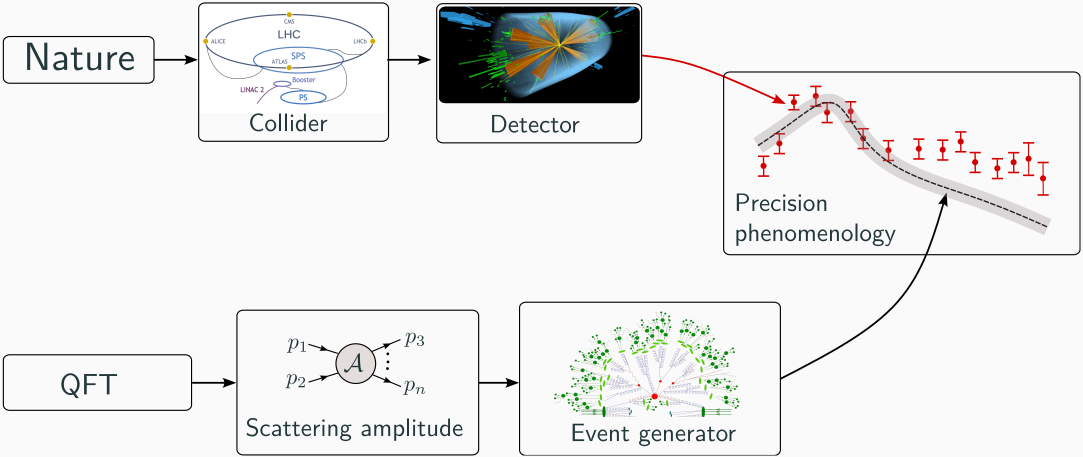
Introduction

General motivation

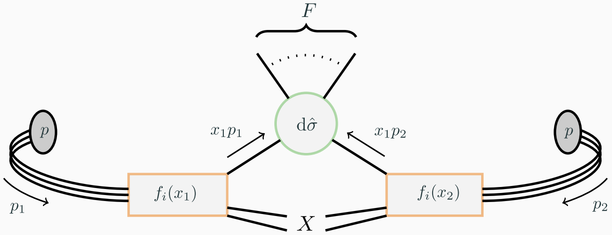
Precision phenomenology at particle colliders



Precision phenomenology at particle colliders



QCD factorization at the LHC



$$\mathrm{d}\sigma_{pp\to F+X} = \sum_{i,j}\int \mathrm{d}x_1\,\mathrm{d}x_2\; f_i(x_1,\mu) f_j(x_2,\mu)\; \mathrm{d}\hat{\sigma}_{ij\to F+X}(x_1p_1, x_2p_2,\mu) \;+\; \mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)$$

Parton distribution functions (PDFs)
(non-perturbative, universal)

Hard scattering
(pertubation theory)

Non-perturbative effects
(power suppressed)

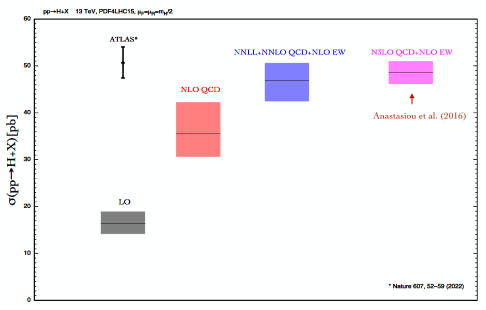
$$\mathrm{d}\hat{\sigma}_0\Big(1 + \alpha_s\,\sigma^{(1,0)} + \alpha_s^2\,\sigma^{(2,0)} + \alpha_s^3\,\sigma^{(3,0)} + \alpha\,\sigma^{(0,1)} + \alpha\,\alpha_s\,\sigma^{(1,1)} + \dots\Big)$$

$\alpha_s(M_Z) \sim 0.1$

$\alpha(M_Z) \sim 0.01$

Higher order corrections and PDF uncertainties

Higgs production in gluon fusion $pp \to H + X$



[Grazzini's talk at Higgs10]

Higher order corrections and PDF uncertainties

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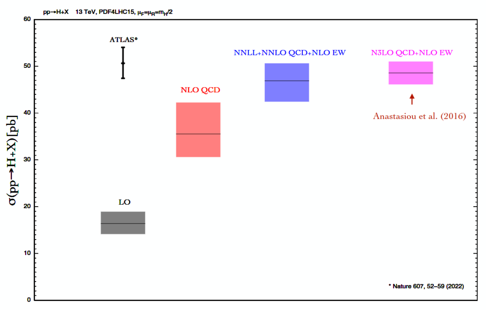

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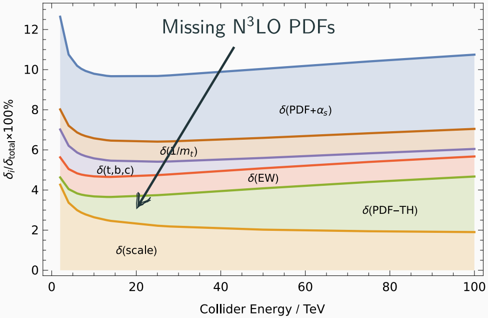

[Dulat, Lazopoulos, Mistlberger '18]

Higher order corrections and PDF uncertainties

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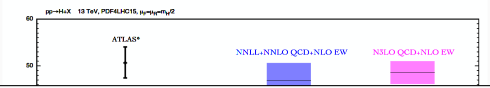
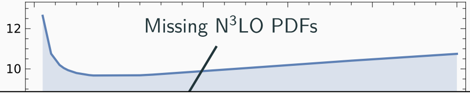
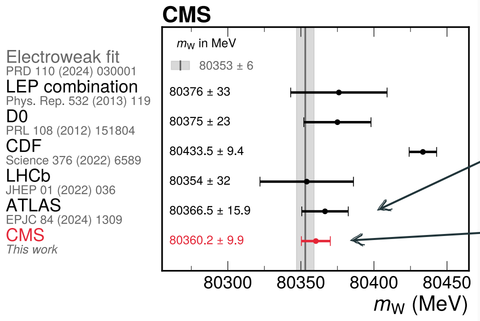




W mass measurement



Unc. [MeV]	Total	Stat.	Syst.	PDF
p_T^ℓ	16.2	11.1	11.8	4.9
m_T	24.4	11.4	21.6	11.7
Combined	15.9	9.8	12.5	5.7

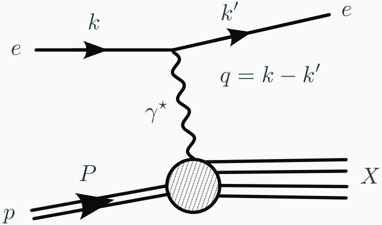
MCFM [51]. Uncertainties due to the PDFs, including their impact on the W boson helicity states, are evaluated by propagating the Hessian eigenvectors of the CT18Z PDF set [52]. Their contribution to the uncertainty in m_W is 4.4 MeV. We have repeated the m_W measurement using seven alternative PDF sets. Additional details on these studies, corrections, and uncertainties

[CMS Nature 652 (2026) 321]

Introduction

Parton distributions and their evolution

DIS and parton model

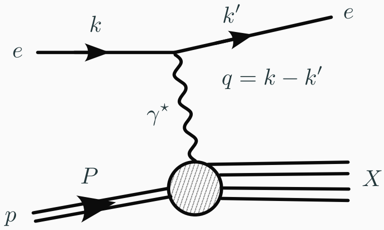


$$Q^2 = -q^2, \quad x = \frac{Q^2}{2P\cdot q}, \quad y = \frac{P\cdot q}{P\cdot k}$$

$$\frac{\mathrm{d}^2\sigma_{\mathrm{DIS}}}{\mathrm{d}x\,\mathrm{d}y} \simeq \frac{\alpha^2}{Q^4}\left[xy^2\,F_1(x,Q^2) + (1-y)\,F_2(x,Q^2)\right].$$

$$F_i(x,Q^2) \sim \Pi_i^{\mu\nu}(P,q)\int d^4z\, e^{\mathrm{i}\,q\cdot z}\,\langle h(P)|[J^\mu(z),J^\nu(0)]|h(P)\rangle.$$

DIS and parton model



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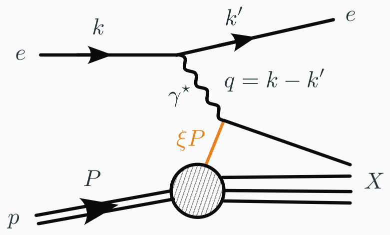

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Collinear factorization with universal parton distributions

$$F_i(x,Q^2) = \sum_{a=q,\bar{q},g}\int_x^1 \frac{\mathrm{d}\xi}{\xi}\,f_a(\xi,\mu)\,\widehat{F}_{i,a}\left(\frac{x}{\xi},\frac{Q^2}{\mu^2}\right) = \sum_a\left[\widehat{F}_{i,a}\otimes f_a\right](x)$$

Renormalized PDFs

Calculate bare partonic structure function ($z = \frac{x}{\xi}$)

$$\widehat{F}_{2,q}^{\text{bare}}(z, \frac{Q^2}{\mu^2}) = \delta(1 - z)$$

- Naive parton model at LO.

Renormalized PDFs

Calculate bare partonic structure function ($z = \frac{x}{\xi}$)

$$\widehat{F}_{2,q}^{\text{bare}}(z, \frac{Q^2}{\mu^2}) = \delta(1-z) + a_s \left[-\frac{1}{\epsilon} P_{qq}^{(0)}(z) + \ln\frac{Q^2}{\mu^2} P_{qq}^{(0)}(z) + \text{finite} \right] + \mathcal{O}(a_s^2),$$

- Naive parton model at LO.
- Initial-state collinear singularities do not cancel at NLO.

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Divergences absorbed into renormalized PDF:

$$f^{\text{bare}}_q = Z_{qq}(\mu) \otimes f_q(\mu), \qquad Z_{qq}(\mu) = \delta(1-z) + a_s(\mu) \frac{1}{\epsilon} P^{(0)}_{qq}(z) + \mathcal{O}(a_s^2).$$

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Renormalized quantities depend on factorization scale $\mu_F = \mu_R = \mu$.

DGLAP equation and splitting functions

Scale evolution of PDFs is given by DGLAP equation

$$\mu \frac{\mathrm{d} f_i(x,\mu)}{\mathrm{d}\mu} = 2 \sum_j \left[P_{ij} \otimes f_j(\mu) \right](x), \qquad i,j \quad \in \{q_1, \ldots, q_{n_f}, \bar{q}_1, \ldots, \bar{q}_{n_f}, g\}\,.$$

The coefficients are the splitting functions

$$P_{ij} = a_s P_{ij}^{(0)} + a_s^2 P_{ij}^{(1)} + a_s^3 P_{ij}^{(2)} + a_s^4 P_{ij}^{(3)} \ldots\,.$$

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Mellin transform turns convolutions into products

$$f_i(n) = - \int_0^1 \mathrm{d}x\, x^{n-1} f_i(x),$$
$$\gamma_{ij}(n) = - \int_0^1 \mathrm{d}x\, x^{n-1} P_{ij}(x),$$

$$\mu \frac{\mathrm{d} f_i(n,\mu)}{\mathrm{d}\mu} = -2 \sum_j \gamma_{ij}(n) f_j(n,\mu)\,.$$

Determination of PDFs

1. Parametrize $f_i(x, Q_0)$ at one scale $Q = Q_0$, e.g.

$$f_i(x, Q_0) = A_i \, x^{\alpha_i} (1-x)^{\beta_i} \left(1 + \gamma_i \sqrt{x} + \delta_i x + \eta_i x^2\right) .$$

2. Evolve with DGLAP to other (hard) scales Q $\leftarrow$ splitting functions through N$^{\mathrm{k}}$LO.
3. Compare to data, fit parameters $\leftarrow$ partonic cross sections through N$^{\mathrm{k}}$LO.

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State of the art

DIS data from HERA; W/Z,jets from LHC.

- NNLO well established: NNPDF, CTEQ, MSHT, ABMP, ...
- Approximate $\mathrm{N^3LO}$ actively developed: MSHT [McGowan et al. '22], NNPDF [Ball et al. '24]

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$\mathrm{N^3LO}$ splitting functions are known only approximately.

Evolution sectors

- Evolution of PDFs is coupled $\frac{\mathrm{d} f_i}{\mathrm{d}\ln\mu} = 2\sum_j P_{ij} \otimes f_j, \quad i,j \in \{q_1,\dots,q_{n_f},\bar{q}_1,\dots,\bar{q}_{n_f} g\}$.
- Convenient to change the basis to decouple non-singlet sector.

$$
\begin{aligned}
q^{\pm}_{\mathrm{ns},ij} &= q_i \pm \bar{q}_i - (q_j \pm \bar{q}_j),\\
q^V_{\mathrm{ns}} &= \sum_{i=1}^{n_f} (q_i - \bar{q}_i),\\
\Sigma &= \sum_{i=1}^{n_f} (q_i + \bar{q}_i)
\end{aligned}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\begin{pmatrix} \Sigma \\ g \\ \hline q^+_{\mathrm{ns}} \\ q^-_{\mathrm{ns}} \\ q^V_{\mathrm{ns}} \end{pmatrix} = \left(\begin{array}{cc|ccc} P_{qq} & P_{qg} & 0 & 0 & 0 \\ P_{gq} & P_{gg} & 0 & 0 & 0 \\ \hline 0 & 0 & \boxed{P^+_{\mathrm{ns}}} & 0 & 0 \\ 0 & 0 & 0 & \boxed{P^-_{\mathrm{ns}}} & 0 \\ 0 & 0 & 0 & 0 & \boxed{P^V_{\mathrm{ns}}} \end{array}\right) \otimes \begin{pmatrix} \Sigma \\ g \\ \hline q^+_{\mathrm{ns}} \\ q^-_{\mathrm{ns}} \\ q^V_{\mathrm{ns}} \end{pmatrix}
$$

- $P_{qq} = P^+_{\mathrm{ns}} + P_{\mathrm{ps}}$

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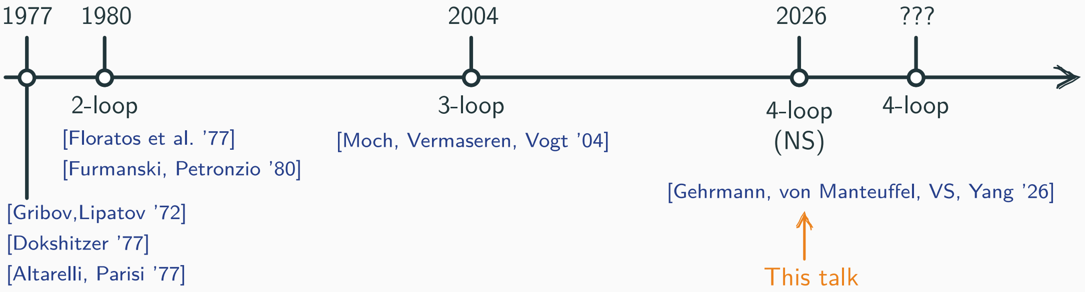
- $P_{qq} = P^+_{\mathrm{ns}} + P_{\mathrm{ps}}$
- At LO (one loop): $P^{(0)}_{\mathrm{ps}} = 0, \quad P^{(0)}_{qq} = P^{(0)\pm,V}_{\mathrm{ns}} = C_{\mathrm{F}}\left((1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\,\delta(1-x)\right)$

Timeline: splitting functions



Partial 4-loop results and approximations since 2017:

[Moch, Ruijl, Ueda, Vermaseren, Vogt '17] [Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt 18']
[Basdew-Sharma, Pelloni, Herzog, Vogt '23] [Falcioni, Herzog, Moch, Vermaseren, Vogt '23]
[Falcioni, Herzog, Moch, Vogt '23] [Gehrmann, Manteuffel, VS, Yang '23] [Falcioni, Herzog, Moch, Pelloni, Vogt '24]
[Gehrmann, Manteuffel, VS, Yang '24] [Falcioni, Herzog, Moch, Pelloni, Vogt '24] [Kniehl, Moch, Velizhanin, Vogt '25]
[Falcioni, Herzog, Moch, Pelloni, Vogt '26] [Kniehl, Moch, Velizhanin, Vogt '26] , . . .

Four-loop non-singlet splitting functions: partial results

- n_f^2 [Davies, Vogt, Ruijl, Ueda, Vermaseren '16] [Gehrmann, von Manteuffel, VS, Yang '23]
- Leading color [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]
- Fixed Mellin moments $n \leq 16$ [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]
- $n_f C_F^3$ [Gehrmann, von Manteuffel, VS, Yang '24]
- n_f in $P_{\text{ns}}^{(3)\pm}$ [Kniehl, Moch, Velizhanin, Vogt '25]

- Small x limit: $\ln^k x,\ k = \{6, 5, 4\}$ [Davies, Kom, Moch, Vogt '22]
- Large x limit: [Dokshitzer, Marchesini, Salam '05] [Grozin '18] [Henn, Korchemsky, Mistlberger '19] [von Manteuffel, Panzer, Schabinger '20] [Das, Moch, Vogt '19]

- aN^{3}LO PDFs use approximations based on these partial results (see e.g. [Cooper-Sarkar, Cridge, Giuli, Harland-Lang, Hekhorn, Huston, Magni, Moch, Thorne '24]).

Off-shell OME method

Operator definition of PDFs

Parton densities can be defined as matrix elements of a non-local operator

$$f_i^{\mathrm{bare}}(x) = \frac{1}{2} \int \frac{dt}{2\pi} \, e^{-ixt\, P\cdot\Delta} \, \langle h(P)|\mathcal{O}_i^{\mathrm{bare}}(t)|h(P)\rangle, \qquad \Delta^2 = 0$$

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Quark PDF:

$$\mathcal{O}_q^{\mathrm{bare}}(t) = \bar{\psi}(t\Delta)\, \not{\Delta}\, W(t)\, \lambda\, \psi(0), \qquad W(t) = \mathcal{P}\exp\left[ig\int_0^t dt'\, \Delta\cdot A(t'\Delta)\right],$$

- Probes quark content of hadron at light-like separation.
- $\not{\Delta}$ projects leading unpolarized collinear component.
- Wilson line needed for non-local gauge invariance.
- Matrix in flavor space λ projects onto (non-singlet) combinations.

Operator renormalization

Renormalization of PDFs corresponds to renormalization of the operator $\mathcal{O}_i$:

$$\mathcal{O}_i^{\text{bare}}(x) = \sum_j \left[Z_{ij} \otimes \mathcal{O}_j\right](x, \mu)$$

$\mu \frac{d}{d\mu} \mathcal{O}_i^{\text{bare}}(x) = 0 \implies$ they satisfy RGE

$$\mu \frac{d}{d\mu} \mathcal{O}_i = \sum_j \gamma_{ij} \otimes \mathcal{O}_j, \qquad \gamma = -Z^{-1} \otimes \left(\mu \frac{dZ}{d\mu}\right).$$

Upon taking matrix elements $\langle h|\mathcal{O}_i|h\rangle$ we see that splitting functions are anomalous dimensions the composite non-local operators $\mathcal{O}_i$.

Mellin moments of PDFs

Recall $f(x) = \int \mathrm{d}t\, e^{-ixt\, P\cdot\Delta} \langle h(P)|\mathcal{O}_i(t)|h(P)\rangle$.

Calculating Mellin moment n picks up series coefficients of $\mathcal{O}(t)$ around $t = 0$:

$$\int \mathrm{d}x\, x^{n-1} f(x) \sim \left(\frac{1}{P\cdot\Delta}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-1} \langle P|\mathcal{O}(t)|P\rangle\Big|_{t=0} .$$

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The coefficients are given by local spin n operators:

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\mathcal{O}(t)\bigg|_{t=0} = \Delta_{\mu_1}\cdots\Delta_{\mu_n}\mathcal{O}^{\mu_1\cdots\mu_n}.$$

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(Non-singlet) quark operator:

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\bar{\psi}(t\Delta)\,\not{\Delta}\,\mathcal{P}\exp\left[ig\int_0^t dt'\,\Delta\cdot A(t'\Delta)\right]\,\lambda\,\psi(0)\bigg|_{t=0} = \bar{\psi}(0)\,\not{\Delta}\,(\Delta\cdot D)^{n-1}\,\lambda\,\psi(0)$$

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(Non-singlet) quark operator:

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\bar{\psi}(t\Delta)\,\not{\Delta}\,\mathcal{P}\exp\left[ig\int_0^t dt'\,\Delta\cdot A(t'\Delta)\right]\lambda\,\psi(0)\Big|_{t=0} = \bar{\psi}(0)\,\not{\Delta}\,(\Delta\cdot D)^{n-1}\,\lambda\,\psi(0)$$

Mellin moments of PDFs are matrix elements of local twist two operators.

Renormalization in Mellin space

For each Mellin moment n we have local operator $\implies$ "normal" multiplicative RGE

$$\mathcal{O}_i(\mu, n) = \sum_j Z_{ij}(\mu, n)\, \mathcal{O}_j(n), \qquad \gamma_{ij}(n) = -\frac{1}{2} \sum_k Z_{ik}^{-1}(\mu, n) \frac{\mathrm{d}Z_{kj}(\mu, n)}{\mathrm{d}\ln\mu}$$

The anomalous dimensions are Mellin transforms of splitting functions

$$\gamma_{ij}(n) = -\int_0^1 \mathrm{d}x\, x^{n-1} P_{ij}(x)$$

Renormalization in Mellin space: non-singlet

Non-singlet operator does not mix:

$$O_{\rm ns}^{\rm bare}(n) \sim \bar{\psi}\, \not{\Delta}\, (\Delta \cdot D)^{n-1}\, \lambda\, \psi, \qquad O_{\rm ns}(\mu, n) = Z_{\rm ns}(\mu, n) O_{\rm ns}^{\rm bare}(n)\,,$$

Four-loop anomalous dimension $\gamma_{\mathsf{ns}}^{(3)}$ enters through $\frac{1}{\epsilon}$ pole of $Z_{\mathsf{ns}}^{(4)}$:

$$\begin{aligned} Z_{\mathsf{ns}} =& 1 + a_s \frac{\gamma_{\mathsf{ns}}^{(0)}}{\epsilon} + a_s^2 \left(\frac{1}{2\epsilon^2} \left[-\beta_0 \gamma_{\mathsf{ns}}^{(0)} + \left(\gamma_{\mathsf{ns}}^{(0)}\right)^2 \right] + \frac{\gamma_{\mathsf{ns}}^{(1)}}{2\epsilon} \right) \\ & + a_s^3 \left(\frac{1}{6\epsilon^3} \left[\dots \right] + \frac{1}{6\epsilon^2} \left[-2\beta_1 \gamma_{\mathsf{ns}}^{(0)} - 2\beta_0 \gamma_{\mathsf{ns}}^{(1)} + 3\gamma_{\mathsf{ns}}^{(0)} \gamma_{\mathsf{ns}}^{(1)} \right] + \frac{1}{3\epsilon} \gamma_{\mathsf{ns}}^{(2)} \right) \\ & + a_s^4 \left(\frac{1}{\epsilon^4} \left[\dots \right] + \frac{1}{\epsilon^3} \left[\dots \right] + \frac{1}{\epsilon^2} \left[\dots \right] + \frac{\gamma_{\mathsf{ns}}^{(3)}}{4\epsilon} \right) + \mathcal{O}(a_s^5) \end{aligned}$$

Extraction from off-shell OMEs

Main idea [Gross, Wilczek '74] [Potitzer '74]

Extract $\gamma_{\rm ns}^{(l-1)}$ from off-shell OME

$$A_{\rm ns}(n) = \langle q(p)|\, O_{\rm ns}(n)\, |q(p)\rangle \text{ with } p^2 < 0$$

- RGE independent of state in matrix elements $\implies$ free to replace by quark.
- Off-shell to avoid IR singularities.

⚠ Gauge dependent: complicated renormalization in singlet sector.

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Off-shell OME contains non-redundant information to extract splitting functions.

Tracing parameter

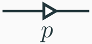
$$A_{\text{ns}}(n) = \langle q(p)|\, \bar{\psi}\, \not{\Delta}\, (\Delta \cdot D)^{n-1}\, \frac{\lambda}{2}\psi\, |q(p)\rangle$$

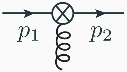
- Infinite number of OME computations at fixed n.
- Introduce tracing parameter t to compute all simultaneously
  [Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]

Rank $n-1$ numerator tensors	$\Longleftrightarrow$	Additional denominators
Fixed n moments		Coefficients of Taylor series in t

Tracing parameter

$$\sim (\Delta \cdot p)^{n-1} \to \sum_{n=0}^{\infty} t^n (\Delta \cdot p)^n = \frac{1}{1 - t\,\Delta \cdot p} \Rightarrow$$



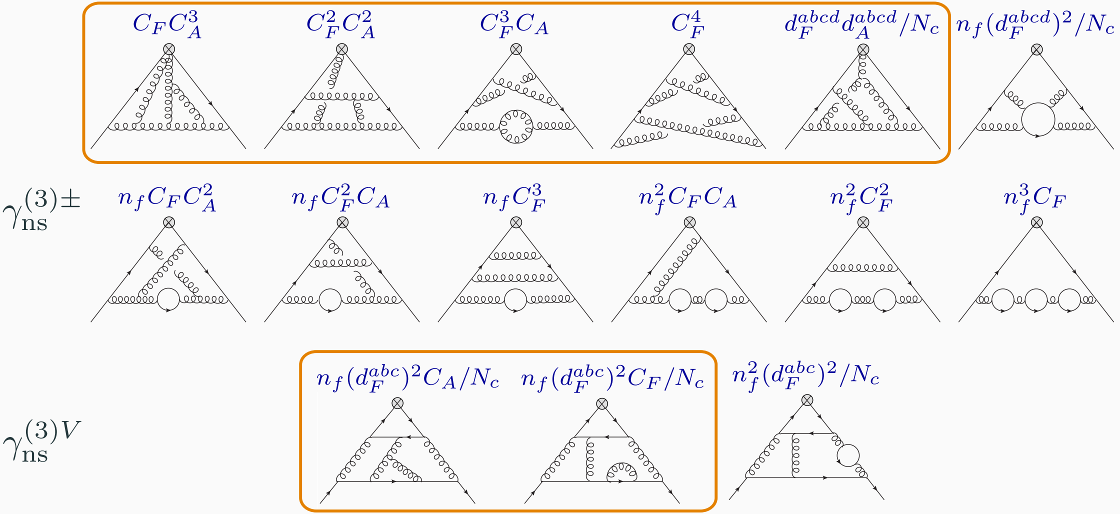


$$\to \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (\Delta \cdot p_1)^n (\Delta \cdot p_2)^{n-k} = \frac{1}{1 - t\,\Delta \cdot p_1} \frac{1}{1 - t\,\Delta \cdot p_2} \Rightarrow$$

- Diagrams with up to 4 additional linear denominators, standard IBP relations.
- Effectively one-scale problem ($p^2 = -1$).
- Goal: expand in t to sufficiently high powers to recover all-n form of $\gamma_{\mathrm{ns}}^{(3)}(n)$.

The four-loop computation

Feynman diagrams



Integral reduction

⚠ Main technical challenge: integration-by-parts (IBP) reduction of around $3 \cdot 10^6$ integrals:

$$\int \mathrm{d}^d\ell_1 \cdots \mathrm{d}^d\ell_4 \, \frac{\partial}{\partial \ell_i^\mu} \left(v^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0, \qquad D_j = \{q_j^2, \; 1 - t(\Delta \cdot q_j)\} \, .$$

- Custom private reduction code.
- Finite-field sampling and analytic reconstruction [von Manteuffel, Schabinger '14] [Peraro '15].
- Aggressively optimized identity generation and filtering
  [Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch '24][Guan, Liu, Ma, Wu '24]
  [Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng '24] [von Hippel, Wilhelm '25] [Song, Yang, Cat, Leo, Zhu '25] [Lange, Usovitsch, Wu '25]
- Spanning cuts and sectors [Larsen, Zhang '15] [Guan, Liu, Ma, Wu '24].
- Avoiding mixed ϵ and t dependence in denominators through improved basis choice [Smirnov, Smirnov '20] [Usovitsch '20].

Differential equations

- Reduce OME to basis of around 6k master integrals $$A = \sum_i c_i(\epsilon, t)\, I_i(\epsilon, t)\,.$$
- Derive differential equations in tracing parameter $$\partial_t I \;=\; M(\epsilon, t)\, I\,.$$
- Use DE to derive recursion relations [Blümlein, Kauers, Klein, Schneider '09] [Lee, Smirnov, Smirnov '17] for Laurent series in ϵ and Taylor series in t coefficients of $I_i(\epsilon, t)$.
- Boundary condition: at $t \to 0$ four-loop self-energy integrals [Baikov, Chetyrkin '10] [Lee, Smirnov, Smirnov '11].

Stats for experts

Integrand

- Complete sets of standard and linear propagators (“integral families”) : 52.
- Top-level sectors with 13 different denominators: 549.
- Unreduced integrals: $\sim 3 \cdot 10^6$, up to rank 5 numerators, up to 4 “dots”.

Reduction

- Standard IBP relations with $\sim 9k$ spanning sectors: reduce to $\sim 120k$ integrals.
- Symmetries and “anomalous” (aka “magic”) relations leave $\sim 6k$ master integrals.

Differential equations

- Derived ~ 300 differential equations.
- Up to $2k$ master integrals in each.
- Homogeneous blocks of size 12 common.

Surprise: elliptic geometry

- Homogeneous DE with $\epsilon \to 0$ related to class of special functions.
- OME @ three loops: rational and (rationalizeable) square roots
  $\implies$ solutions through multiple polylogarithms in t.
- OME @ four loops: more square roots, elliptic curves.

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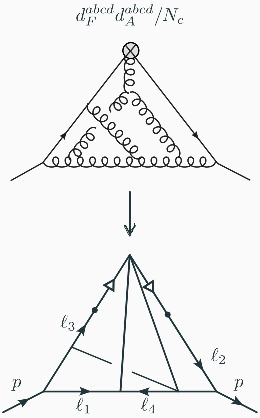
Example

Nonplanar graph with 9 standard (massless) propagators and 2 denominators $1 - t(\Delta \cdot \ell_i)$.

Found coupled 2×2 DE block, solutions are periods of elliptic curves.



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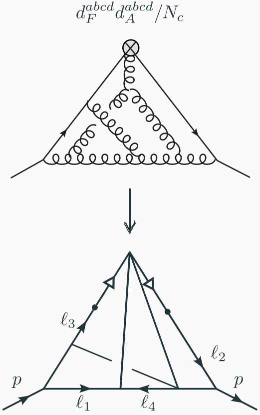
- No special treatment in our approach, derive series in t regardless.
- Elliptic contributions must drop out in OME, at least in $1/\epsilon$ poles.



Harmonic sums

Splitting functions are expected to be harmonic polylogarithms (HPLs) [Remiddi, Vermaseren '00]

$$H_{a_1,\ldots,a_n}(x) = \int_0^x \mathrm{d}x' \frac{1}{1-a_1}\, H_{a_2,\ldots,a_n}(x'), \quad H(x) = 1\,.$$

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Nested harmonic sums [Vermaseren '99] are their Mellin space counterparts

$$S_{\vec{i}}(n) = S_{i_1,\dots,i_d}(n) = \sum_{j=1}^{n} \frac{(\operatorname{sgn} i_1)^j}{j^{|i_1|}}\, S_{i_2,\dots,i_d}(j), \quad \left|\vec{i}\right| = \sum_j |i_j|\,.$$

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They describe all-n information of HPL's

- Mellin transform: $$\int_0^1 \mathrm{d}x\; x^n H_{1,1}(x) = \frac{S_{1,1}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{1}{(n+1)^3}$$
- Series coefficients: $$H_{1,1}(t) = \sum_{n=1}^{\infty} t^n \left(\frac{S_1(n)}{n} - \frac{1}{n^2} \right)$$

Reconstruction of all-n form

Series in $t \implies$ samples of fixed moments: $\gamma_{\text{ns}}^{(3)}(n_i) = s_i^{(7)} + \sum_{j=3}^{5} s_i^{(7-j)} \zeta_j$

Match $s_i^{(w)}$ samples to generic ansatz of monomials of harmonic sums and denominators

$$\mathcal{A}^{(w)}(n) = \sum_{j=0}^{w} \sum_{k+|\vec{i}|=j} \sum_{a} c^{j}_{k,\vec{i},a} \frac{1}{(n+a)^k} S_{\vec{i}}(n),$$

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- Poles in n: $\{0,-1\}$ for $\gamma_{\text{ns}}^{(3)\pm}$, $\{0,\pm1,\pm2\}$ for $\gamma_{\text{ns}}^{(3)s} = \gamma_{\text{ns}}^{(3)V} - \gamma_{\text{ns}}^{(3)-}$.
- Not using constraints from limits or reciprocity relations [Moch, Ruijl, Ueda, Vermaseren, Vogt '17] [Kniehl, Moch, Velizhanin, Vogt '25].
- No "elliptic" contributions.

Results

Exact four-loop non-singlet splitting functions

n space

Around 4k fixed moments to successfully fit the ansatz to obtain all-n form, confirming expected analytic structure.

Inverse Mellin transform to x space [Ablinger '09]

$P_{\rm ns}^{(3)\pm}$

- HPLs weight 6, ζ values weight 7.
- Denominators $1 \pm x$.
- Distribution contributions $\delta(1-x), [1/(1-x)]_+$.

$P_{\rm ns}^{(3)s}$

- HPLs weight 6, ζ values weight 6.
- Denominators $\{1 \pm x, x\}$.

✓ All earlier partial results independently confirmed*

Large x limit

Universal all-order structure in $x \to 1$ limit [Dokshitzer, Marchesini, Salam '05]:

$$P_{\mathsf{ns}}^{\pm,V} = A \left[\frac{1}{1-x} \right]_+ + B\,\delta(1-x) + C \log(1-x) + D - A + \mathcal{O}(1-x)$$

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- Cusp anomalous dimension A_4 reproduced [Grozin '18] [Henn, Korchemsky, Mistlberger '19] [von Manteuffel, Panzer, Schabinger '20].
- Analytic result for virtual anomalous dimension B_4 new, consistent with numerical [Das, Moch, Vogt '19].

Virtual and rapidity anomalous dimensions

$$
\begin{aligned}
B_4 = {} & C_A^3 C_F \left(295.7 \pm 0.5 \right) \\
& + C_A^2 C_F^2 \left(1219.5 \pm 2.0 \right) \\
& + C_A C_F^3 \left(-687.5 \pm 1.5 \right) \\
& + C_F^4 \left(196.5 \pm 1.0 \right) \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_c} \left(-998.0 \pm 0.2 \right) \\
& + C_A^2 C_F n_f \left(-274.466 \pm 0.01 \right) \\
& + C_A C_F^2 n_f \left(-455.247 \pm 0.005 \right) \\
& + C_F^3 n_f \left(80.780 \pm 0.005 \right) \\
& + \frac{(d_F^{abcd})^2 \, n_f}{N_c} \left(-143.6 \pm 0.2 \right) \\
& + C_A C_F n_f^2 \left(-\frac{88}{9}\zeta_5 + \frac{80}{3}\zeta_3\zeta_2 - \frac{80}{9}\zeta_4 - \frac{320}{9}\zeta_3 + \frac{3170}{81}\zeta_2 - \frac{193}{54} \right) \\
& + C_F^2 n_f^2 \left(\frac{368}{9}\zeta_5 - \frac{160}{9}\zeta_3\zeta_2 - \frac{2104}{27}\zeta_4 + \frac{56}{27}\zeta_3 + \frac{1244}{27}\zeta_2 - \frac{188}{27} \right) \\
& + C_F n_f^3 \left(-\frac{32}{27}\zeta_4 + \frac{304}{81}\zeta_3 + \frac{32}{81}\zeta_2 - \frac{131}{81} \right)
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[Das, Moch, Vogt '19]

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B_4 = C_A^3 C_F \Big(& -\frac{8960}{3}\zeta_7 + \frac{1472}{3}\zeta_5\zeta_2 + \frac{32}{3}\zeta_4\zeta_3 + \frac{73333}{108}\zeta_6 + \frac{1672}{3}\zeta_3^2 + \frac{11522}{9}\zeta_5 + \frac{584}{3}\zeta_3\zeta_2 - \frac{11206}{27}\zeta_4 - \frac{152284}{81}\zeta_3 + \frac{13864}{9}\zeta_2 - \frac{373793}{648}\Big) \\
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+ C_A C_F^3 \Big(& -10920\zeta_7 + 2064\zeta_5\zeta_2 + 128\zeta_4\zeta_3 + \frac{79297}{18}\zeta_6 + 3220\zeta_3^2 - 976\zeta_5 - \frac{1988}{3}\zeta_3\zeta_2 + 2167\zeta_4 - 3260\zeta_3 + 1167\zeta_2 - \frac{2085}{4}\Big) \\
+ C_F^4 \Big(& 5880\zeta_7 - 384\zeta_5\zeta_2 + 64\zeta_4\zeta_3 - 2111\zeta_6 - 1152\zeta_3^2 - 2520\zeta_5 - 120\zeta_3\zeta_2 - 342\zeta_4 + 2004\zeta_3 - 450\zeta_2 + \frac{4873}{24}\Big) \\
+ \frac{d_F^{abcd} d_A^{abcd}}{N_c} \Big(& 2800\zeta_7 + 320\zeta_5\zeta_2 - 64\zeta_3\zeta_4 - \frac{1562}{9}\zeta_6 - 704\zeta_3^2 - 400\zeta_5 - 896\zeta_3\zeta_2 + \frac{32}{3}\zeta_4 - \frac{1232}{3}\zeta_3 - \frac{944}{3}\zeta_2 + 96\Big) \\
+ C_A^2 C_F n_f \Big(& -\frac{3913}{27}\zeta_6 + \frac{416}{3}\zeta_3^2 + \frac{1130}{9}\zeta_5 - \frac{580}{3}\zeta_3\zeta_2 + \frac{1234}{9}\zeta_4 + \frac{9554}{27}\zeta_3 - \frac{41092}{81}\zeta_2 + \frac{20027}{108}\Big) \\
+ C_A C_F^2 n_f \Big(& 351\zeta_6 - \frac{1232}{3}\zeta_3^2 - \frac{7432}{9}\zeta_5 + \frac{2672}{9}\zeta_3\zeta_2 + \frac{27854}{27}\zeta_4 - \frac{15400}{27}\zeta_3 - \frac{3892}{27}\zeta_2 - \frac{7751}{54}\Big) \\
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[Gehrmann, von Manteuffel, VS, Yang '26]

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& + C_A C_F^3 \left(-10920\zeta_7 + 2064\zeta_5\zeta_2 + 128\zeta_4\zeta_3 + \frac{79297}{18}\zeta_6 + 3220\zeta_3^2 - 976\zeta_5 - \frac{1988}{3}\zeta_3\zeta_2 + 2167\zeta_4 - 3260\zeta_3 + 1167\zeta_2 - \frac{2085}{4} \right) \\
& + C_F^4 \left(5880\zeta_7 - 384\zeta_5\zeta_2 + 64\zeta_4\zeta_3 - 2111\zeta_6 - 1152\zeta_3^2 - 2520\zeta_5 - 120\zeta_3\zeta_2 - 342\zeta_4 + 2004\zeta_3 - 450\zeta_2 + \frac{4873}{24} \right) \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_c} \left(2800\zeta_7 + 320\zeta_5\zeta_2 - 64\zeta_3\zeta_4 - \frac{1562}{9}\zeta_6 - 704\zeta_3^2 - 400\zeta_5 - 896\zeta_3\zeta_2 + \frac{32}{3}\zeta_4 - \frac{1232}{3}\zeta_3 - \frac{944}{3}\zeta_2 + 96 \right)
\end{aligned}
$$

Last missing ingredient to get analytic result for four-loop rapidity anomalous dimension

$$\gamma_r^{(3)q}\Big|_{n_f=4} = 0.61362 \pm 5\cdot 10^{-5} \qquad \gamma_r^{(3)g}\Big|_{n_f=4} = 1.58840 \pm 5\cdot 10^{-5}$$

[Moult, Zhu, Zhu '21]
[Duhr, Mistlberger, Vita '21]

(and other anomalous dimensions [Moch, Vogt '26]).

$$+ C_F n_f^3 \left(-\frac{32}{27}\zeta_4 + \frac{304}{81}\zeta_3 + \frac{32}{81}\zeta_2 - \frac{131}{81} \right)$$

[Gehrmann, von Manteuffel, VS, Yang '26]

Small x limit

Expand exact results to obtain all logarithmic contributions, for $N_c = 3$:

$$P_{\rm ns}^{(3)+} = \frac{256}{729}\ln^6 x + \left(\frac{3200}{243} - \frac{256}{243}n_f\right)\ln^5 x + \left(-\frac{14240}{243}\zeta_2 + \frac{89008}{243} - \frac{7616}{243}n_f + \frac{64}{81}n_f^2\right)\ln^4 x$$
$$+ \left\{\frac{1024}{243}\zeta_3 - \frac{85504}{81}\zeta_2 + \frac{1062692}{243} + \left(\frac{2368}{27}\zeta_2 - \frac{43208}{81}\right)n_f + \frac{4016}{243}n_f^2 - \frac{32}{243}n_f^3\right\}\ln^3 x$$
$$+ \quad \cdots$$

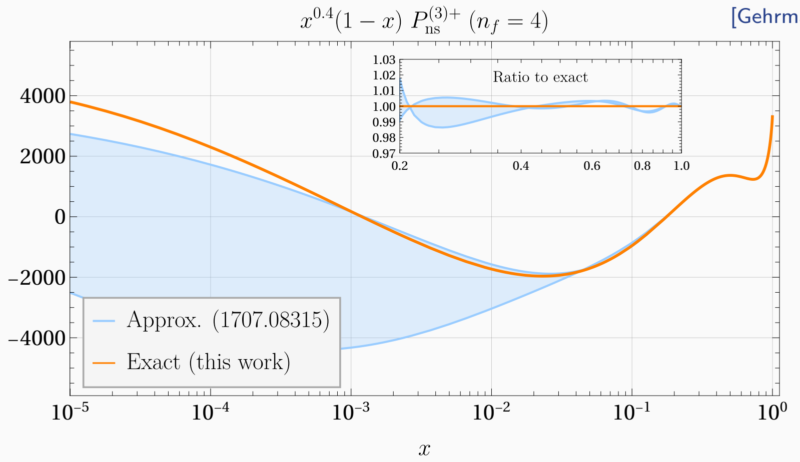
- Subleading power in x compared to singlet (no $1/x$ pole).
- $\ln^3 x$ and beyond not previously known.

✓ LL and NLL agree with [Davies, Kom, Moch, Vogt '22].

⚠ NNLL: deviations proportional to ζ_2 for all color structures, conspire to agree at leading color.

Comparison to earlier approximations I

[Gehrmann, von Manteuffel, VS, Yang '26]

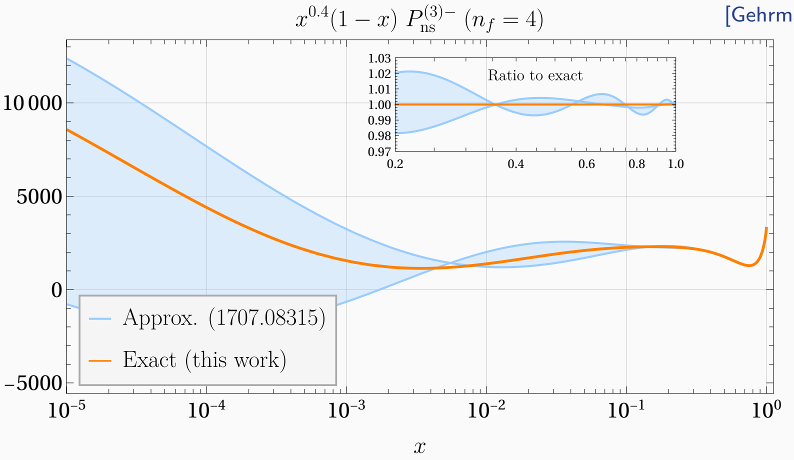


- Good agreement within uncertainty in the bulk.
- Deviation at small x, suppressed w.r.t. singlet.

Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

Comparison to earlier approximations II

[Gehrmann, von Manteuffel, VS, Yang '26]



- Excellent agreement within uncertainty.

Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

Towards complete four-loop splitting functions

Renormalization in singlet sector

Recall evolution in singlet sector

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\begin{pmatrix}\Sigma\\ g\end{pmatrix} = \begin{pmatrix}P_{qq} & P_{qg}\\ P_{gq} & P_{gg}\end{pmatrix}\otimes\begin{pmatrix}\Sigma\\ g\end{pmatrix} \quad \xleftrightarrow{\text{naive}} \quad \begin{pmatrix}\mathcal{O}_q\\ \mathcal{O}_g\end{pmatrix} = \begin{pmatrix}Z_{qq} & Z_{qg}\\ Z_{gq} & Z_{gg}\end{pmatrix}\begin{pmatrix}\mathcal{O}_q^{\text{bare}}\\ \mathcal{O}_g^{\text{bare}}\end{pmatrix}.$$

Renormalization in singlet sector

Recall evolution in singlet sector

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\begin{pmatrix}\Sigma\\ g\end{pmatrix} = \begin{pmatrix}P_{qq} & P_{qg}\\ P_{gq} & P_{gg}\end{pmatrix}\otimes\begin{pmatrix}\Sigma\\ g\end{pmatrix} \quad\overset{\text{naive}}{\longleftrightarrow}\quad \begin{pmatrix}\mathcal{O}_q\\ \mathcal{O}_g\end{pmatrix} = \begin{pmatrix}Z_{qq} & Z_{qg}\\ Z_{gq} & Z_{gg}\end{pmatrix}\begin{pmatrix}\mathcal{O}_q^{\text{bare}}\\ \mathcal{O}_g^{\text{bare}}\end{pmatrix}.$$

Gauge dependence of OMEs $\Longrightarrow$ mixing with "gauge variant" (a.k.a "alien") operators:

$$\begin{pmatrix}\mathcal{O}_q\\ \mathcal{O}_g\\ \mathcal{O}_{\mathrm{A}}\\ \vdots\end{pmatrix} = \begin{pmatrix}Z_{qq} & Z_{qg} & Z_{q\mathrm{A}} & \cdots\\ Z_{gq} & Z_{gg} & Z_{g\mathrm{A}} & \cdots\\ 0 & 0 & Z_{\mathrm{AA}} & \cdots\\ 0 & 0 & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\mathcal{O}_q^{\text{bare}}\\ \mathcal{O}_g^{\text{bare}}\\ \mathcal{O}_{\mathrm{A}}^{\text{bare}}\\ \vdots\end{pmatrix}.$$

⚠ Operator basis in Mellin space grows with loop order and n.
Worked out for fixed n [Falcioni, Herzog '22].

Renormalization in singlet sector

How to do all-n renormalization?

New idea [Gehrmann, von Manteuffel, Yang '23]:
no explicit basis needed, derive counter term Feynman rules from other (lower loop) OMEs.

$$\begin{pmatrix} \mathcal{O}_q \\ \mathcal{O}_g \\ \mathcal{O}_{ABC} \end{pmatrix} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{pmatrix} \begin{pmatrix} \mathcal{O}_q \\ \mathcal{O}_g \\ \mathcal{O}_{ABC} \end{pmatrix}^{\text{bare}} + \begin{pmatrix} [Z\mathcal{O}]_q^{\text{GV}} \\ [Z\mathcal{O}]_g^{\text{GV}} \\ [Z\mathcal{O}]_A^{\text{GV}} \end{pmatrix}^{\text{bare}}$$

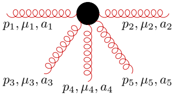
$$Z_{gA} = \mathcal{O}(a_s), \qquad Z_{qA} = \mathcal{O}(a_s^2), \qquad [Z\mathcal{O}]_{g,q}^{\text{GV}} = \sum_{l=2,3}^{\infty} a_s^l \, [Z\mathcal{O}]_{g,q}^{\text{GV},(l)}$$

- $\mathcal{O}_{ABC} = \mathcal{O}_A + \mathcal{O}_B + \mathcal{O}_C$,
  $\mathcal{O}_A$(gluon fields only), $\mathcal{O}_B$(quark + gluon fields), $\mathcal{O}_C$ (ghost + gluon fields)
- $[Z\mathcal{O}]_{g,q}^{\text{GV},(l)}$: collection of counterterms

Example: counter term Feynman rule

Feynman rules for O_{ABC} with five legs

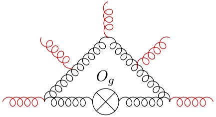
[Falcioni, Herzog, Moch, Thurenhout, 24'; Gehrmann, Manteuffel, TZY, 24']



17074 lines in mma

$$\rightarrow \frac{1+(-1)^n}{2} i g_s^3 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5}\Bigg[\frac{1}{C_A} f^{a a_1 a_2} d_4^{a a_3 a_4 a_5}\Bigg\{ \frac{3}{32}\sum_{j_1=0}^{n-4}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}(-\Delta\cdot p_1)^{n-4-j_1}(-\Delta\cdot(p_1+p_2))^{j_1-j_2} \times (\Delta\cdot(p_4+p_5))^{j_2-j_3}(\Delta\cdot p_5)^{j_3}+\cdots\Bigg\} + 11 \text{ color structures}\Bigg] + 30 \text{ Lorentz Structures}$$

extracted from



$$\sum_n^{\infty} t^n(\text{all-}n \text{ Feynman rules}) \rightarrow \text{linear propagator in } t\text{-space}$$

[Tong-Zhi's seminar talk]

⚠ Non-trivial computation: lower loops, but higher multiplicity.

Work in progress

Conclusions and outlook

Conclusions and outlook

Conclusions

- Non-singlet four-loop splitting functions complete, also timelike (fragmentation functions).
- Analytical result for virtual and rapidity anomalous dimensions.
- Precise fit for N^{3}LO PDF evolution provided.

Outlook

- Towards complete four-loop splitting functions.
- Renormalization in singlet sector.
- Gaps in understating of small-x resummation (hint at “super-leading” logarithms)?

Recognition by leading figures in the field

Properties and implications of the four-loop non-singlet splitting functions in QCD

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37 pages

e-Print: 2605.03889 [hep-ph]

Abstract

We have studied the recently completed analytic all-N expressions for the four-loop anomalous dimensions corresponding to the next-to-next-to-next-to-leading order splitting functions for the non-singlet quark distribution in perturbative QCD. The results agree with fixed-N values beyond those published so far. Their structural consistency with theoretical requirements is established. They are used to cast the four-loop gluon virtual anomalous dimension and the next-to-next-to-

1 Introduction

Recently, the penultimate milestone has been reached of research projects spanning more than a decade: the exact expressions have been completed of the fourth-order contributions to the splitting functions for the scale dependence (evolution) of the non-singlet quark distributions of hadrons [1].

Acknowledgements

We compliment the authors of ref. [1] on their achievement.

[1] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T. Yang, *The four-loop non-singlet splitting functions in QCD*, arXiv:2604.09534

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European Research Council

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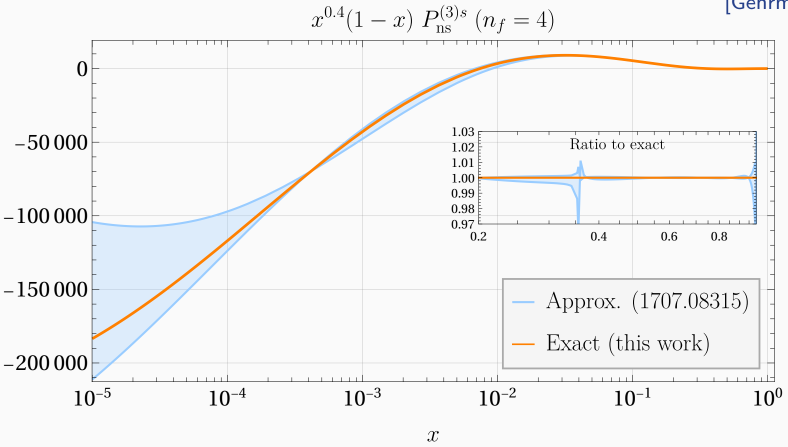
Backup

Comparison to earlier approximations: $P_{\rm ns}^{(3)s}$

[Gehrmann, von Manteuffel, VS, Yang '26]



Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]