Last session of Bethe Lectures 2023: Double Copy

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Outline

Part I Brief Introduction to the Double Copy

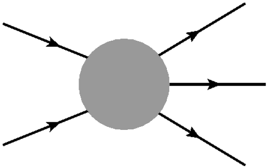
Part II Application to Exact Classical Solutions

Part III From Scattering Amplitudes DC to Classical DC

Part IV Application to String Amplitudes

Part I

Brief Introduction to the Double Copy



Perturbative gravity is hard!

Feynman rules: expand Einstein-Hilbert Lagrangian $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ [DeWitt '66]

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],
\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\
&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad\qquad-P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\Big].
\end{aligned}$$

$+$ infinite number of higher-point vertices...

Gravity $\sim$ (Yang-Mills)^2 in Scattering Amplitudes

Asymptotic states

- Yang-Mills theory: gluon $A_\mu = e^{ik\cdot x}\,\epsilon_\mu\,T^a$
  colour index a, polarisation ϵ_μ has $D-2$ dof.

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  Contains graviton $h_{\mu\nu}$ + dilaton Φ + B-field $B_{\mu\nu}$, $(D-2)^2$ dof.

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Scattering amplitudes

- "Factorisation" of ϵ^μ, $\tilde{\epsilon}^\nu$ preserved by interactions!
- **Double copy** $\boxed{\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\, \mathcal{A}_{\text{YM}}(\epsilon_i^\mu) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^\mu)\, \Big|_{\text{colour stripped}}}$
- Famous application: supergravity UV behaviour. [Bern,Carrasco,Johansson,Roiban,. . .]

String theory origin

QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.

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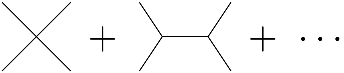
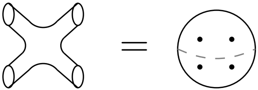


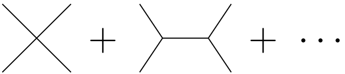
particle scattering
(many Feynman diagrams)



string scattering
(one “world-sheet”, 2D CFT)

String theory origin

QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.



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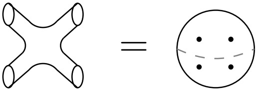

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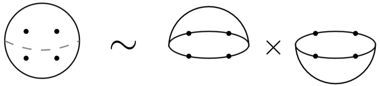
Gravity (closed strings) vs. gauge theory (open strings):

Asymptotic states (vertex operators): $V_{\text{closed}}(\varepsilon^{\mu\nu} = \epsilon^{\mu}\tilde{\epsilon}^{\nu}) \sim V_{\text{open}}(\epsilon^{\mu})\,\bar{V}_{\text{open}}(\tilde{\epsilon}^{\nu})$

Scattering amplitudes:

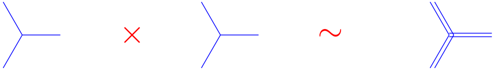
KLT relations
[Kawai, Lewellen, Tye 86]



Field theory limit: **Gravity $\sim$ (Yang-Mills)^2** (KLT, BCJ, CHY, . . .)

Why simpler?

Basic example: 3-pt interactions.



Gauge theory field A^a_μ

3-pt vertex: $\boxed{f^{abc}\, V^{\mu\nu\lambda}\, A^a_\mu(p_1)\, A^b_\nu(p_2)\, A^c_\lambda(p_3)}$

$$V^{\mu\nu\lambda} = (p_1 - p_2)^\lambda \eta^{\mu\nu} + (p_2 - p_3)^\mu \eta^{\nu\lambda} + (p_3 - p_1)^\nu \eta^{\lambda\mu}$$

Gravity field $H_{\mu\mu'} \sim$ graviton + dilaton + B-field 'fat graviton'

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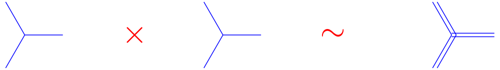

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3-pt vertex: $V^{\mu\nu\lambda}\, V^{\mu'\nu'\lambda'}\, H_{\mu\mu'}(p_1)\, H_{\nu\nu'}(p_2)\, H_{\lambda\lambda'}(p_3)$

Great simplification: **index factorisation**, c.f. $\sim$100 terms in GR 3-pt vertex!

Powerful implementation: colour-kinematics duality. [Bern, Carrasco, Johansson ‘08] [...]

New directions in (classical) perturbative gravity

Generically, double copy applies in perturbation theory.

- Double-copy-like field theory for gravity.

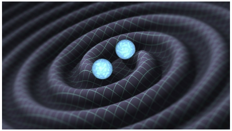
  [Bern et al] [Goldberger et al] [Luna et al] [Cheung et al] [Plefka et al] [Borsten et al] [. . .]

- Gauge-invariant approach: classical physics from scattering amplitudes.

  [Bjerrum-Bohr et al] [Kosower et al] [Di Vecchia et al] [Guevara et al] [Huang et al] [Arkani-Hamed et al] [Brandhuber et al] [. . .]

- Beyond Minkowski: “amplitudes” on plane waves / (A)dS.

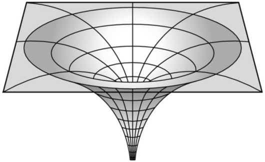
  [Farrow et al] [Adamo et al] [Armstrong et al] [Alday et al] [Gomez et al] [Cristofoli et al] [Roiban et al] [. . .]



- Highlight: G^3, G^4 (3PM, 4PM) corrections to 2-body potential. [Bern et al] [. . .]

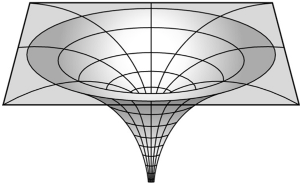
Part II

Application to Exact Classical Solutions



Beyond perturbation theory

Question: is a black hole a double copy of something?



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- What is “graviton” in exact solution?
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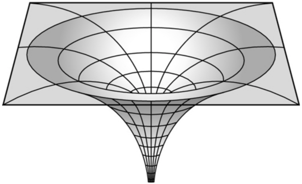

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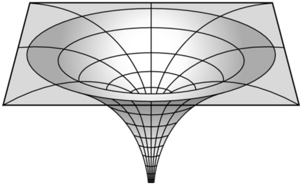

Still...

- Can relate to perturbation theory, examples:

  Schwarzchild [Duff 73; Neill, Rothstein 13; Mougiakakos, Vanhove 20; . . .] ,

  shockwave [Saotome, Akhoury 12; Cristofoli 20; . . .] .

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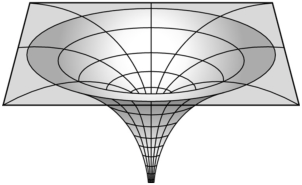

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- Direct map of exact solutions? Need miracle!   Symmetry

Stationary Kerr-Schild spacetimes

[RM, O'Connell, White 14]

"Exact perturbation" $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu$$

where k_μ is null and geodesic wrt $\eta_{\mu\nu}$ and $g_{\mu\nu}$. $(k^\mu = g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\nu)$

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Einstein equations linearise:

- $g^{\mu\nu} = \eta^{\mu\nu} - \phi\, k^\mu k^\nu$
- $R^\mu{}_\nu = \frac{1}{2}\partial_\alpha \left[\partial^\mu \left(\phi k^\alpha k_\nu\right) + \partial_\nu \left(\phi k^\alpha k^\mu\right) - \partial^\alpha \left(\phi k^\mu k_\nu\right)\right]$ $\quad \partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$

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Stationary vacuum case (take $k_0 = 1$): $\qquad R^0{}_0 = \frac{1}{2}\,\nabla^2 \phi = 0$

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"Single copy" $\quad$ linear $\leftrightarrow$ Abelian, $\quad$ $$A^a_\mu = \phi\, k_\mu\, c^a$$ $\quad c^a$ const

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$$0 = D_\mu F^{a\mu\nu} = c^a \begin{cases} -\nabla^2\phi & \nu = 0 \\ -\partial_\ell \left[\partial^i \left(\phi k^\ell\right) - \partial^\ell \left(\phi k^i\right)\right] & \nu = i \end{cases} \qquad \checkmark$$

Simplest example: point charge

Check spherically symmetric solutions sourced by point charge.

Einstein theory: Schwarzschild solution

- $$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu, \qquad \phi(r) = \frac{2M}{r}, \qquad k = dt + dr$$

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YM theory: Coulomb solution

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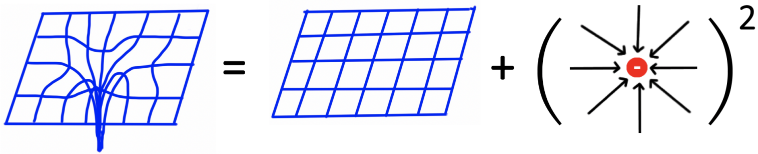
Schwarzschild $\sim$ (Coulomb)^2



Many more examples

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- Kerr black hole: (M, J).
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  Other black holes families? Black rings, etc...

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Cosmological constant $\leftrightarrow$ constant charge density. [Luna, RM, O'Connell, White '15]
[Bahjat-Abbas, Luna, White '17; Carrillo-Gonzalez, Penco, Trodden 17]

NUT charge $\leftrightarrow$ magnetic monopole: multi-Kerr-Schild [Luna, RM, O'Connell, White '15]

$$g_{\mu\nu}^{\text{(Taub-NUT)}} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu + \psi\, \ell_\mu \ell_\nu\,, \quad \phi \propto M\,, \quad \psi \propto N \quad \Rightarrow \quad A_\mu^{\text{(dyon)}} = \phi\, k_\mu + \psi\, \ell_\mu$$

Radiation from accelerated particle: correct Bremsstrahlung.
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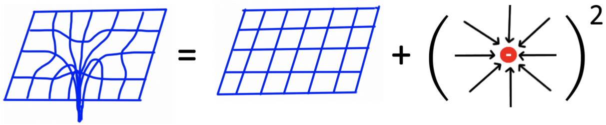
Much related work [Adamo et al, Alawadhi et al, Alfonsi et al, Anastasiou et al, Andrzejewski et al, Bah et al, Bahjat-Abbas et al, Berman et al, Borsten et al, Cardoso et al, Casali et al, Chacon et al, Cho et al, Easson et al, Elor et al, Emond et al, Goldberger et al, Gonzalez et al, Gurses et al, Keeler et al, Kim et al, Lescano, Luna et al, Cristofoli et al, Godazgar et al, Ilderton et al, Lee et al, Mafra et al, Mizera et al, Pasarin et al, Pasterski et al, Prabhu, P.V. et al, Sabharwal et al, White, . . .]

Beyond vacuum solutions

Simplest example: $\quad (\text{Coulomb})^2 \;\sim\;$ Schwarzschild .

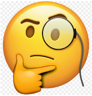


But $\quad (\text{YM})^2 \;\sim\;$ Einstein $h_{\mu\nu}$ + dilaton Φ + B-field $B_{\mu\nu}$.

Other fields conveniently packaged in Double Field Theory.

double copy

Gravity $=$ YM $\times$ YM

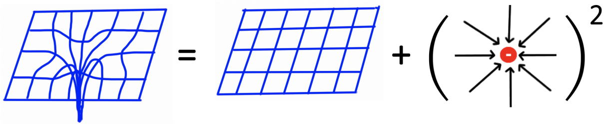


double field theory

doubled geometry $(x^{\mu}, \tilde{x}_{\mu})$

Beyond vacuum solutions

Simplest example: (Coulomb)^2 $\sim$ Schwarzschild .



But (YM)^2 $\sim$ Einstein $h_{\mu\nu}$ $+$ dilaton Φ $+$ B-field $B_{\mu\nu}$.

Other fields conveniently packaged in Double Field Theory.

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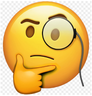

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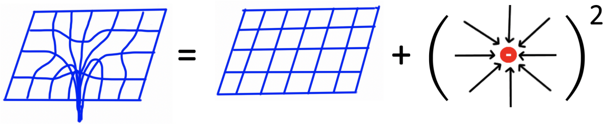

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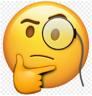

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$\Rightarrow$ Extended classical double copy with **KLT interpretation**!

Non-unique: $(\text{Coulomb})^2 \sim$ JNW (graviton + dilaton) [Janis, Newman, Winicour '68] .

Back to vacuum: alternative approach [Luna, RM, Nicholson, O'Connell '18]

Try double copy of curvatures:

$$A_\mu = \epsilon_\mu \, e^{ik\cdot x}, \quad F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)\, e^{ik\cdot x}$$

$$h_{\mu\nu} = \epsilon_\mu \epsilon_\nu \, e^{ik\cdot x}, \quad R_{\mu\nu\rho\lambda} = \frac{1}{2}(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)(k_\rho \epsilon_\lambda - k_\lambda \epsilon_\rho)\, e^{ik\cdot x}$$

Obvious relation: $e^{ik\cdot x} R_{\mu\nu\rho\lambda} \sim F_{\mu\nu} F_{\rho\lambda}$

More general? Not so simple: symmetries of $R_{\mu\nu\rho\lambda}$, non-linear gauge, ...

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Spinorial approach to GR ($D=4$) [Penrose '60]

Basic object is $\sigma^\mu_{A\dot{A}}$ such that $\left(\sigma^\mu_{A\dot{A}}\sigma^\nu_{B\dot{B}} + \sigma^\nu_{A\dot{A}}\sigma^\mu_{B\dot{B}}\right) \varepsilon^{\dot{A}\dot{B}} = g^{\mu\nu} \varepsilon_{AB}$

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Want formula: curvature $R \sim \dfrac{1}{\text{scalar}}$ (curvature F)^2

Weyl spinor and algebraic classification

Weyl curvature $W_{\mu\nu\rho\lambda}$:

$$W_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \text{terms}(R_{\mu\nu}, g_{\mu\nu}) = R_{\mu\nu\rho\lambda} \ \text{ in vacuum as } \ R_{\mu\nu} = 0$$

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Can decompose into four rank 1 spinors: $C_{ABCD} = \mathbf{a}_{(A}\mathbf{b}_B\mathbf{c}_C\mathbf{d}_{D)}$

$\rightarrow$ Four *principal null directions*: $a_{A\dot{A}} = \mathbf{a}_A\,\bar{\mathbf{a}}_{\dot{A}}$ and same for $b_{A\dot{A}}, c_{A\dot{A}}, d_{A\dot{A}}$.

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Algebraic classification of spacetimes [Petrov '54]

How many principal null directions are aligned? Types I, II, D, III, N, O.

Type D: $\mathbf{a}_A \propto \mathbf{c}_A$, $\mathbf{b}_A \propto \mathbf{d}_A$, then $C_{ABCD} = y_{(AB}\,y_{CD)}$, where $y_{AB} = \mathbf{a}_{(A}\mathbf{b}_{B)}$.

Weyl double copy: vacuum Type D spacetimes

Take Minkowski space: $\sigma^a = \frac{1}{\sqrt{2}}(\mathbb{1}, \sigma^i)$.

Maxwell spinor f_{AB}: $F_{ab} \rightarrow F_{A\dot{A}B\dot{B}} = f_{AB}\,\varepsilon_{\dot{A}\dot{B}} + \bar{f}_{\dot{A}\dot{B}}\,\varepsilon_{CD}$

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$$C_{ABCD} \;=\; \frac{1}{S}\, f_{(AB}\, f_{CD)}$$

Type D solutions: 2 principal null directions of multiplicity 2,

Coulombic, no functional freedom. Eg. Kerr-Taub-NUT.

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- Origin: $\exists$ Killing rank-2 spinor, $\nabla_{(A}{}^{\dot{A}}\,\chi_{BC)} = 0$. [Walker, Penrose 70]
  Then $C_{ABCD} = \chi^{-5}\,\chi_{(AB}\,\chi_{CD)}\,,\quad f_{AB} = \chi^{-3}\,\chi_{AB}\,,\quad S = \chi^{-1}\,.$
  Admit complex double-Kerr-Schild form [Plebanski, Demianski 75].

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Weyl tensor is type N $\Leftrightarrow$ $\exists$ degenerate Maxwell field

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Non-uniqueness due to functional freedom in S.

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- $s = 2$, $T^{(2)} = \Psi_4$: Bianchi identity for Weyl tensor
- $s = 1$, $T^{(1)} = \Phi_2$: Maxwell equation
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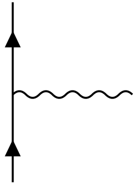
Weyl double copy has twistorial formulation. [White 20; Chacon, Nagy, White 21]

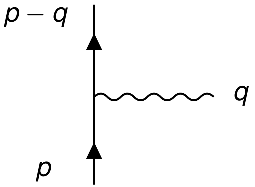
Part III

From Scattering Amplitudes to Classical Double Copy



3-point scattering amplitudes

On-shell 3-pt interaction: massive particle emits gauge boson.



$$p^2 = (p-q)^2 = m^2 \qquad q^2 = 0$$

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Eg. BCFW recursion.

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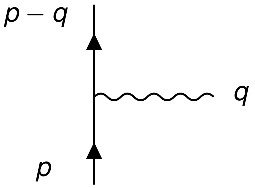

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Lorentzian signature: 3-pt amplitudes supported on complex kinematics.

Split signature (t^1, t^2, x^1, x^2): 3-pt amplitudes supported on real kinematics,
eg, $p = m\,(0,1,0,0)\,,\; q \propto (1,0,0,1)\,.$

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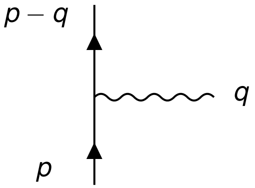

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eg, $p = m\,(0,1,0,0)$, $q \propto (1,0,0,1)$.

Classical limit: $q = \hbar\, k$, $\hbar \to 0$. KMOC formalism [Kosower, Maybe, O'Connell 18]

Classical fields from 3-pt amplitudes

[RM, O'Connell, Peinador, Sergola 20]

What classical objects do 3-pt amplitudes compute?

$\Rightarrow$ Linearised curvature (gravity) and field strength (EM) in split signature.

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KMOC formalism: $\langle \mathcal{O} \rangle \equiv {}_{\text{in}}\langle S^\dagger \mathcal{O} S \rangle_{\text{in}}$ $\quad S = 1 + i\,T \quad$ [eg. $\mathcal{O} = F_{\mu\nu}(x)$]

$$\xrightarrow{\text{calculation}} \quad \langle\, \mathcal{O}(x)\, \rangle_{\text{classical}} = \text{Re} \int d^4k\, \delta(k^2)\, \theta(k_1)\, \underbrace{\tilde{\mathcal{O}}(k)}_{\text{includes 3-pt amp}}\, e^{-ik\cdot x}$$

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and $k^2 = 0: \; k^\mu \mapsto |k\rangle_A\, [k|_{\dot{A}}$, we find for 'static' particle

$$\tilde{C}_{ABCD}(k) = |k\rangle_A |k\rangle_B |k\rangle_C |k\rangle_D\, \mathcal{A}^{(+)}_{3,grav}(k) \qquad \text{Schwarzschild}^*$$

$$\tilde{f}_{AB}(k) = |k\rangle_A |k\rangle_B\, \mathcal{A}^{(+)}_{3,EM}(k) \qquad \text{Coulomb}^*$$

$$\tilde{S}(k) = 1 \qquad 1/r^{\,*}$$

eg, $\mathcal{A}^{(+)}_{3,EM}(k) \sim p \cdot \epsilon^{(+)}(k)$

* analytic continuation to split signature

Double copy: from amplitudes to classical solutions

Amplitudes

double copy

$$\mathcal{A}^{(\pm)}_{3,grav} = \left(\mathcal{A}^{(\pm)}_{3,EM}\right)^2$$

Classical solutions

Weyl double copy
in on-shell momentum space

$$\tilde{C}_{ABCD} = \frac{1}{\tilde{S}}\, \tilde{f}_{(AB}\, \tilde{f}_{CD)}$$

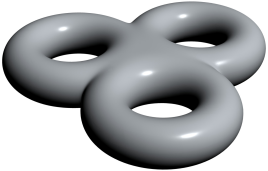
Back to coordinate space $\longrightarrow$ Schwarzschild $\sim$ (Coulomb)^2

Why simplicity in coordinate space examples? Symmetry!

Expect generic double copy to be non-local in coordinate space. [Anastasiou et al 14]

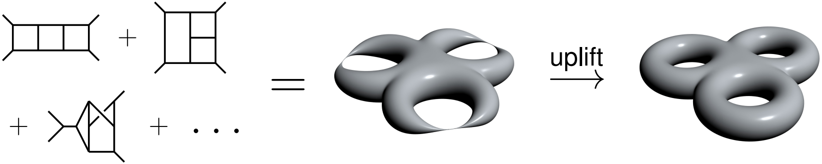
Part IV

Application to String Amplitudes



Importing field theory into string theory [Geyer, RM, Stark-Muchão 21]

Supergravity loop integrand $\longrightarrow$ Superstring moduli-space integrand.



(i) Take sugra g-loop integrand obtained from BCJ double copy, (superYM)^2.

(ii) Translate it into Riemann sphere with g nodes cf. ambitwistor string.

(iii) Uplift from nodal sphere to genus g guided by modular invariance.

New conjecture for 3-loop 4-pt type II superstring amplitude.

16 years after 2 loops [D'Hoker, Phong; Berkovits 05] , 39 years after 1 loop [Green, Schwarz, Brink 82] .

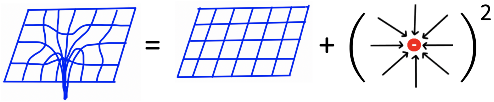
Conclusion

Conclusion of Double Copy Lecture

- Double copy (DC) **useful** and **insightful**, from LIGO to superstrings.

  Pervasive in perturbative gauge theory, gravity, string theory.

- DC of exact classical solutions possible for some classes.

  Approaches exploit algebraic structure, 'stringy' aspects, . . .

- DC of classical solutions $=$ DC of amplitudes.

- **Much more to explore**