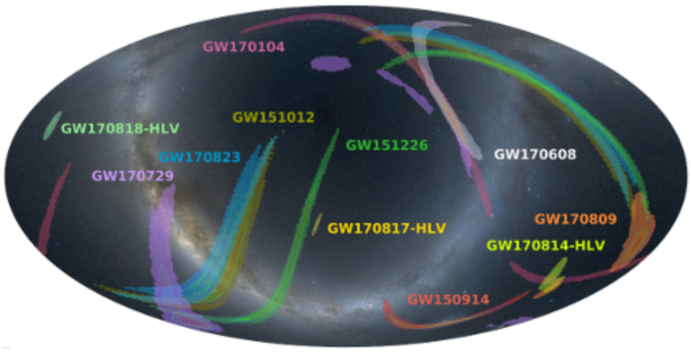
Gravitational waves (4)



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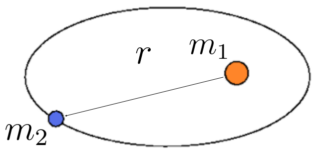
Bonn, Febr. 2023

Compact binaries



binary systems

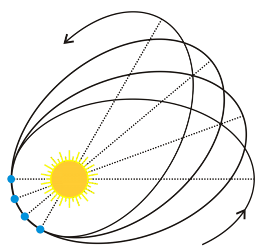
$$M = m_1 + m_2 \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$r = |\mathbf{r}_2 - \mathbf{r}_1|$$

3 stages:

formation	(n-body interactions)
inspiral	(quasi-newtonian)
merger	(relativistic)

quasi-newtonian: Kepler + precession



$$r(\varphi) = \frac{\rho}{1 - e \cos n\varphi}$$

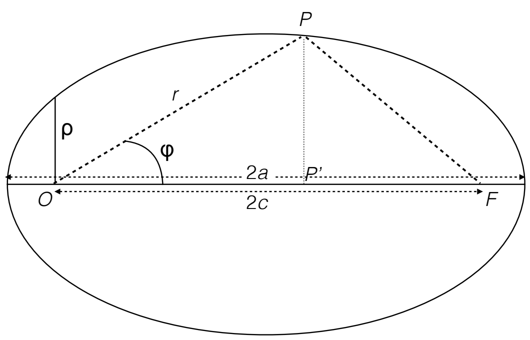
eccentricity precession rate

find central force with such orbits

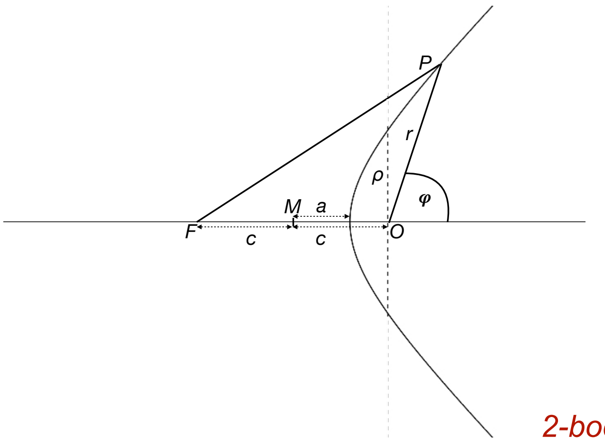
$$\mathbf{F}(r) = F(r)\hat{\mathbf{r}} \longrightarrow r^2 \dot{\varphi} = \ell$$

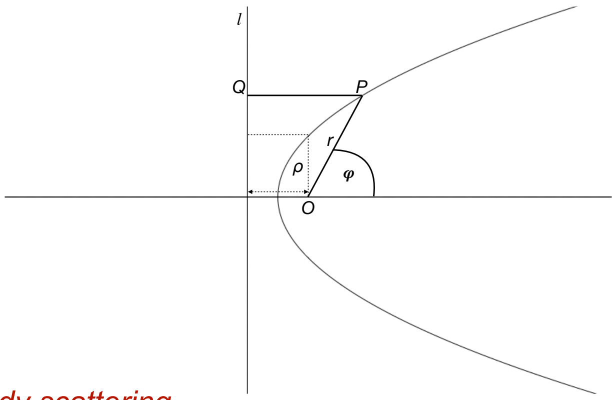
quasi-newtonian plane orbits

$$\boxed{r = \frac{\rho}{1 - e\cos n\varphi}}$$



bound system





2-body scattering

$$\boxed{F(r) = -\frac{GM\mu}{r^2} - \frac{\beta\mu}{r^3}}$$

$$\dot{\varphi} = \frac{\ell}{r^2}$$ angular momentum/unit of mass

$$\dot{r} = r'\dot{\varphi} = -\frac{en\ell}{\rho}\sin n\varphi$$ radial velocity

$$\frac{1}{n^2} = 1 + \frac{\beta}{GM\rho} = \frac{\ell^2}{GM\rho}$$ precession

$$\longrightarrow n\ell = \sqrt{GM\rho}$$

Relativistic precession (Schwarzschild) $$\frac{1}{n^2} \simeq 1 + \frac{6GM}{c^2\rho}$$

but precession may also arise because of many-body forces.

Quadrupole approximation

$$Q_{ij}(t) = m_1\left(r_{1i}r_{1j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_1^2\right) + m_2\left(r_{2i}r_{2j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_2^2\right)$$

$$\xrightarrow[CM]{} \quad \mu\left(r_i r_j - \frac{1}{3}\,\delta_{ij} r^2\right)$$

T T - gauge:

$$2\kappa h_{ij}(\vec{\mathsf{r}},t) = \frac{2G}{c^4 \mathsf{r}}\left[\ddot{Q}_{ij} - \hat{\mathsf{r}}_i(\ddot{Q}\cdot\hat{\mathsf{r}})_j - \hat{\mathsf{r}}_j(\ddot{Q}\cdot\hat{\mathsf{r}})_i + \frac{1}{2}\,(\delta_{ij} + \hat{\mathsf{r}}_i\hat{\mathsf{r}}_j)\hat{\mathsf{r}}\cdot Q\cdot\hat{\mathsf{r}}\right]_{t_{ret}}$$

Note: $r = |\mathbf{r}_2 - \mathbf{r}_1| = \dfrac{\rho}{1 - e\cos n\varphi}$ = separation between masses

$\vec{\mathsf{r}} = \mathsf{r}\,\hat{\mathsf{r}}$ with $\mathsf{r} = |\mathbf{x}|$ = distance to observer

$\hat{\mathsf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ = direction of observer

kinematical relations

$$r(\varphi) = \frac{\rho}{1 - e\cos n\varphi}$$

$$\dot{\varphi} = \frac{\ell}{r^2} \qquad \dot{r} = r'(\varphi)\dot{\varphi} = -\frac{n\ell}{\rho}\sqrt{e^2 - \left(1 - \frac{\rho}{r}\right)^2}$$

$$\ddot{\varphi} = -\frac{2\ell}{r^3}\,\dot{r} \qquad \ddot{r} = -\frac{n\ell^2}{\rho r^2}\left(1 - \frac{\rho}{r}\right)$$

$$\longrightarrow \quad \ddot{\mathbf{Q}} = \frac{\mu n\ell^2}{\rho^2}\left[n\left(e^2 - 1 + \frac{\rho}{r}\right)\mathbf{E} + n\left((e^2 - 1 + \frac{\rho}{r} - \frac{2\rho^2}{r^2}\right)\mathbf{M} - \frac{2\rho}{r}\sqrt{e^2 - 1 + \frac{2\rho}{r} - \frac{\rho^2}{r^2}}\,\mathbf{N}\right]$$

where

$$\mathbf{E} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} -\sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example I: circular orbits

$$\mathbf{r} = R\,(\cos\omega t, \sin\omega t, 0) \quad \text{with} \quad \omega^2 = \frac{GM}{R^3}$$

$$2\kappa h_{ij}^{(x)} = \frac{2G^2M\mu}{c^4 \mathsf{r} R} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\omega t & 0 \\ 0 & 0 & -\cos 2\omega t \end{pmatrix}_{t_{ret}}$$

+-mode only

$$2\kappa h_{ij}^{(y)} = \frac{2G^2M\mu}{c^4 \mathsf{r} R} \begin{pmatrix} -\cos 2\omega t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos 2\omega t \end{pmatrix}_{t_{ret}}$$

+-mode only

180° out of phase

$$2\kappa h_{ij}^{(z)} = -\frac{4G^2M\mu}{c^4 \mathsf{r} R} \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{t_{ret}}$$

+- and x-mode
90° out of phase

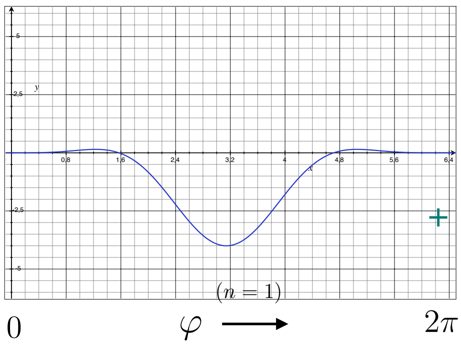
Binary neutron stars of Hulse-Taylor type : $\dfrac{G^2M\mu}{c^4 R\,\mathsf{r}} = \dfrac{2\times 10^{-19}}{\mathsf{r}[\mathrm{lyr}]}$

The frequency of gravitational waves = 2 x orbital frequency

Example II: parabolic orbits

$$2\kappa h_{ij}^{(x)} = -\frac{2G^2M\mu}{c^4 \mathsf{r}\rho}(1-\cos\varphi)^2\cos\varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

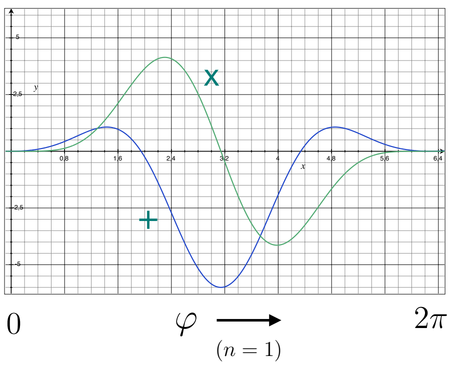
+-mode only



$$2\kappa h_{ij}^{(z)} = \frac{GM\mu}{c^4 \mathsf{r}\rho}(1-\cos\varphi)$$

$$\times \begin{pmatrix} 2\cos\varphi - \cos 2\varphi & 2\sin\varphi - \sin 2\varphi & 0 \\ 2\sin\varphi - \sin 2\varphi & -2\cos\varphi + \cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

+- and x-mode
90° out of phase



Parabolic orbit representing the motion of the effective 1-body reduced mass in the CM frame of two masses scattering in the x-y-plane: Graphs: gravitational-wave amplitudes seen along the axis of the parabola (x-axis) and from above (along the z-axis) as a function of the evolution parameter φ, measuring the progression of the effective mass in its orbit.

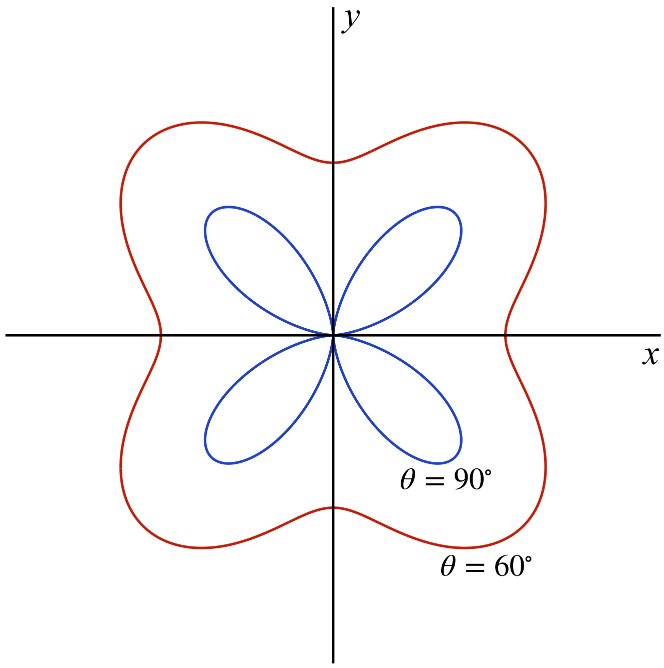
Energy loss

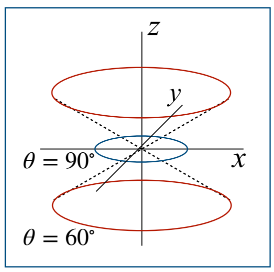
Differential flux $$\frac{dE}{d\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathrm{r}}\cdot\dddot{Q}^2\cdot\hat{\mathrm{r}} + \frac{1}{2}(\hat{\mathrm{r}}\cdot\dddot{Q}\cdot\hat{\mathrm{r}})^2\right]$$

$$\begin{aligned}\frac{dE}{d\Omega dt} \;=\; -\frac{G\mu^2\ell^6}{8\pi c^5 r^8}\Big[& 2A^2 + 2B^2 \\ & - 2\sin^2\theta\left(A^2 + B^2 + A^2\cos 2(\phi-\varphi) + AB\sin 2(\phi-\varphi)\right) \\ & + \frac{1}{2}\sin^4\theta\,\big(A^2 + B^2 + 2\,A^2\cos 2(\phi-\varphi) + 2\,AB\sin 2(\phi-\varphi) \\ & + \left(A^2 - B^2\right)\cos^2 2(\phi-\varphi) + 2\,AB\sin 2(\phi-\varphi)\cos 2(\phi-\varphi)\big)\;\Big]\end{aligned}$$

$$A = \frac{n^3 r^2}{\rho^2}\sqrt{(e^2-1) + \frac{2\rho}{r} - \frac{\rho^2}{r^2}} \qquad\qquad B = -\frac{4n^2 r}{\rho} + 4(n^2-1)$$

normalised angular distribution of intensity of radiation
in periastron and apastron for PSR 1913+16





view perpendicular to orbital plane
absolute intensity periastron:apastron: 220:1

Total flux $\boxed{\dfrac{dE}{dt} = -\dfrac{G}{5c^5}\,\mathrm{Tr}\,\dddot{Q}^2}$

$$\longrightarrow \quad \frac{dE}{dt} = -\frac{1}{30n^6}\left(\frac{2GM}{c^2\rho}\right)^4\frac{\mu^2c^3}{M\rho}\left[n^6\left(e^2-1\right)\frac{\rho^4}{r^4} + 2n^6\frac{\rho^5}{r^5} - n^4\left(n^2-12\right)\frac{\rho^6}{r^6} - 24n^2\left(n^2-1\right)\frac{\rho^7}{r^7} + 12(n^2-1)^2\frac{\rho^8}{r^8}\right]$$

$$\xrightarrow{n=1} \quad \frac{dE}{dt} = -\frac{1}{30}\left(\frac{2GM}{c^2\rho}\right)^4\frac{\mu^2c^3}{M\rho}\left[\left(e^2-1\right)\frac{\rho^4}{r^4} + 2\frac{\rho^5}{r^5} + 11\frac{\rho^6}{r^6}\right]$$

(Peters-Mathews, 1963)

closed orbits (*e* < *1*): energy loss per period

$$\Delta E = -\frac{4\pi\sqrt{2}}{5n^6}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\left[1 + \frac{e^2}{24}\left(n^6+12n^4-120n^2+180\right) + \frac{e^4}{96}\left(n^6+216n^4-720n^2+540\right) + \frac{5e^6}{16}\left(n^2-1\right)^2\right]$$

[PSR 1913 + 16 (Hulse-Taylor) 3.16×10^{46} **J**]

average power: ~ 10^{25} W

circular orbits:

$$\frac{dE}{dt} = -\frac{32G^4M^3\mu^2}{5c^5R^5} \quad \longrightarrow \quad \Delta E = -\frac{4\pi\sqrt{2}}{5}\left(\frac{2GM}{c^2R}\right)^{7/2}\frac{\mu^2c^2}{M}$$

orbital energy: $E = -\dfrac{GM\mu}{2R}$

$$\longrightarrow \quad \frac{\Delta E}{E} = \frac{\pi\sqrt{2}}{5}\,\frac{\mu}{M}\left(\frac{2GM}{c^2R}\right)^{5/2}$$

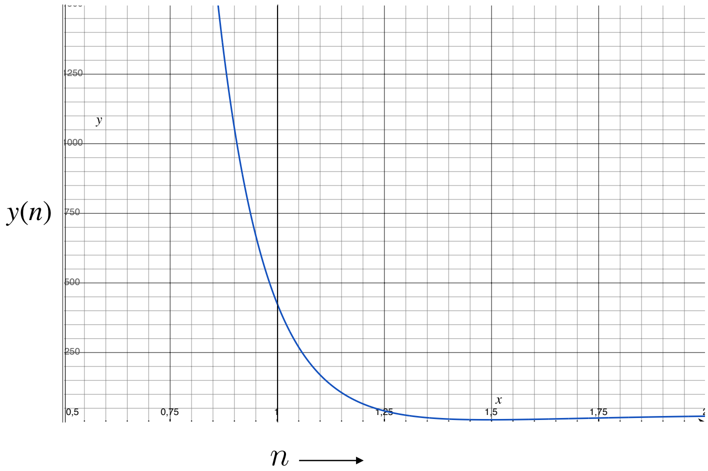
rate of inspiral: $\dfrac{dR}{dt} = \dfrac{dE/dt}{dE/dR} = -\dfrac{64}{5}\dfrac{G^3M^2\mu}{c^5R^3}$

$$\longrightarrow \quad \boxed{R(t) = 4\left[\frac{G^3M^2\mu}{5c^5}\left(t_0-t\right)\right]^{1/4}}$$

parabolic orbits:

$$\Delta E = -\frac{\pi\sqrt{2}}{120}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2 c^2}{M}\underbrace{\left(5+\frac{294}{n^2}-\frac{1260}{n^4}+\frac{1386}{n^6}\right)}_{y(n)}$$

for $n = 1$: ~ 4.5 times the energy/turn of equivalent circular orbit



but: any radiative energy loss results in capture!

Loss of angular momentum

Differential flux
$$\frac{dL_i}{d\Omega dt} = -\frac{G}{4\pi c^5}\varepsilon_{kij}\left[\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij} - \left(\ddot{Q}\cdot\hat{\mathrm{r}}\right)_i\left(\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j + \hat{\mathrm{r}}_i\left(\ddot{Q}\cdot\dddot{Q}\cdot\hat{\mathrm{r}} - \frac{1}{2}\ddot{Q}\cdot\hat{\mathrm{r}}\,\hat{r}\cdot\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j\right]$$

Total flux

$$\longrightarrow \quad \frac{dL_z}{dt} = -\frac{4G}{5c^5}\left(\ddot{Q}\cdot\dddot{Q}\right)_{xy}$$

$$= -\frac{8G\mu^2\ell^5}{5c^5 r^6}\left[n^4(1-e^2)\frac{r^3}{\rho^3} - 2n^2(n^2-1)(1-e^2)\frac{r^2}{\rho^2} + n^2(n^2+2)\frac{r}{\rho} - 4(n^2-1)\right]$$

$$\frac{dL_x}{dt} = \frac{dL_y}{dt} = 0$$

closed orbits ($e < 1$): angular momentum loss per period

$$\Delta L_z = -\frac{8\pi}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\left[1+\frac{e^2}{8}\left(3n^4-20n^2+24\right)+\frac{e^4}{8}\left(2n^2-3\right)\left(n^2-1\right)\right]$$

circular orbits:

$$\frac{dL_z}{dt} = -\frac{32\,G^{7/2}M^{5/2}\mu^2}{5c^5R^{7/2}} \longrightarrow \Delta L_z = -\frac{8\pi}{5}\left(\frac{2GM}{c^2R}\right)^3\frac{\mu^2Rc}{M}$$

orbital angular momentum: $L_z = \mu\sqrt{GMR}$

$$\longrightarrow \frac{\Delta L_z}{L_z} = -\frac{4\pi\sqrt{2}}{5}\frac{\mu}{M}\left(\frac{2GM}{c^2R}\right)^{5/2}$$

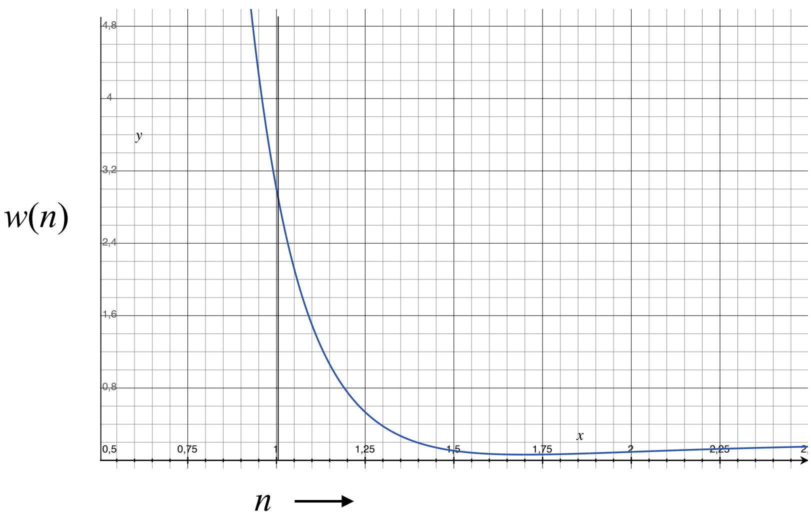
rate of inspiral: $$\frac{dR}{dt} = \frac{dL_z/dt}{dL_z/dR} = -\frac{64}{5}\frac{G^3M^2\mu}{c^5R^3}$$

$$\longrightarrow \boxed{R(t) = 4\left[\frac{G^3M^2\mu}{5c^5}(t_0 - t)\right]^{1/4}}$$

parabolic orbits:

$$\Delta L_z = -\pi\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\underbrace{\left[\frac{7}{n^5} - \frac{5}{n^3} + \frac{1}{n}\right]}_{w(n)}$$

for *n* = *1*: ~ 2 times the angular momentum/turn of equivalent circular orbit



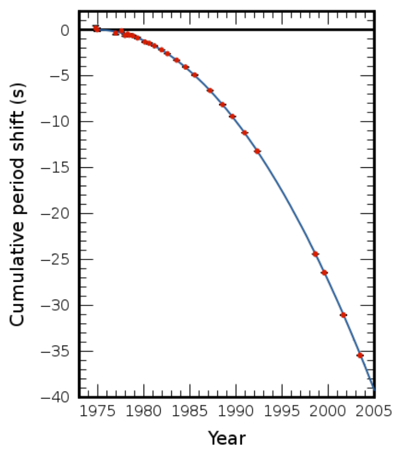
bound orbits: $a = \frac{\rho}{1-e^2}$

$\longrightarrow \quad \frac{da}{dt} = \frac{1}{1-e^2}\frac{d\rho}{dt} + \frac{2\rho e}{(1-e^2)^2}\frac{de}{dt}$

Kepler: $\omega^2 = \frac{4\pi^2}{T^2} = \frac{GM}{a^3}$

$\longrightarrow \quad \frac{dT}{dt} = 3\pi\sqrt{\frac{a}{GM}}\,\frac{da}{dt}$

$$\frac{\Delta T}{T} = -\frac{192\pi}{5}\frac{\mu}{M}\left(\frac{2\pi GM}{c^3T}\right)^{5/3}(1-e^2)^{-7/2}\left[\frac{1}{n^6}+\frac{e^2}{24}\left(1+\frac{12}{n^2}-\frac{120}{n^4}+\frac{180}{n^6}\right)\right.$$
$$\left.+\frac{e^4}{96}\left(1+\frac{216}{n^2}-\frac{720}{n^4}+\frac{540}{n^6}\right)+\frac{5e^6}{16n^6}\left(n^2-1\right)^2\right]$$



for PSR1913+16:

$\frac{\Delta T}{T} = 2.40\times10^{-12} \longrightarrow \Delta T = 0.67\times10^{-7}\ \mathrm{s/orbit}$

$\frac{\Delta a}{a} = 1.60\times10^{-12} \longrightarrow \Delta a = 3.1\ \mathrm{mm/orbit}$

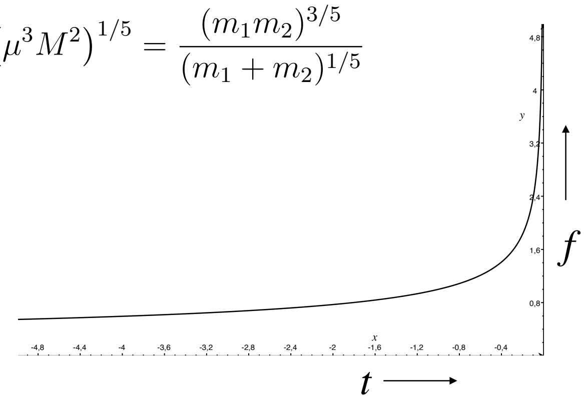
Circular orbits

$$\rho = R,\ \ e=0,\ \ n=1 \longrightarrow R(t) = R_S\left[\frac{32\mu}{5M}\,\frac{c(t_0-t)}{R_S}\right]^{1/4}$$

$\longleftrightarrow$ time till coalescence: $c(t_0-t) = \frac{5}{32}\frac{\mu}{M}\frac{R^4}{R_S^3} \qquad (R \gg R_S)$

orbital frequency: $f(t) = \frac{c}{16\pi}\left(\frac{G\mathcal{M}}{c^2}\right)^{-5/8}\left(\frac{5}{c(t_0-t)}\right)^{3/8}$

chirp mass $\mathcal{M} \equiv \left(\mu^3M^2\right)^{1/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$

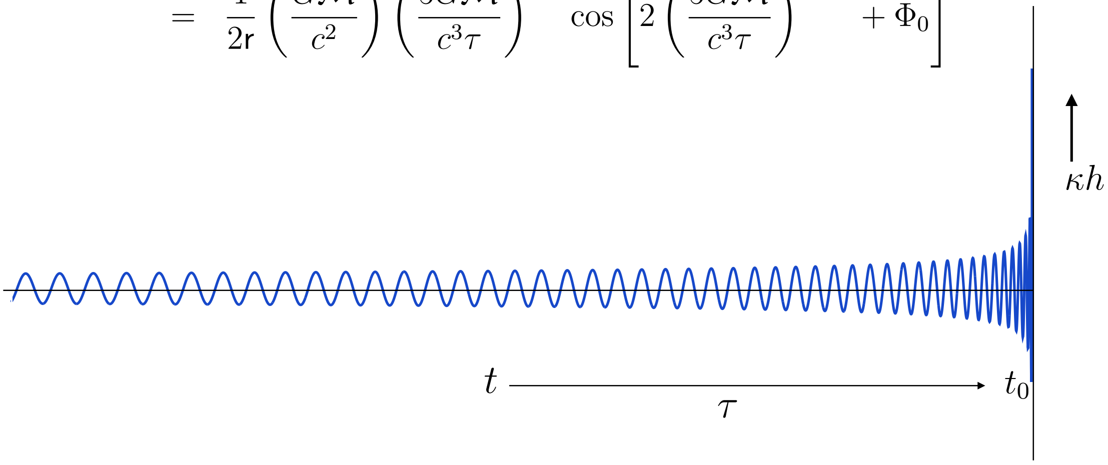


wave form

$$\tau = (t_0 - t)_{ret} = t_0 - t \longrightarrow$$

$$2\kappa h_+ = \frac{2G^2 M\mu}{c^4 \mathrm{r} R(\tau)} \cos\left(4\pi \int_0^\tau d\tau' f(\tau')\right)$$

$$= \frac{1}{2\mathrm{r}} \left(\frac{G\mathcal{M}}{c^2}\right) \left(\frac{5G\mathcal{M}}{c^3\tau}\right)^{1/4} \cos\left[2\left(\frac{5G\mathcal{M}}{c^3\tau}\right)^{-5/8} + \Phi_0\right]$$



A sample of binary merger signals