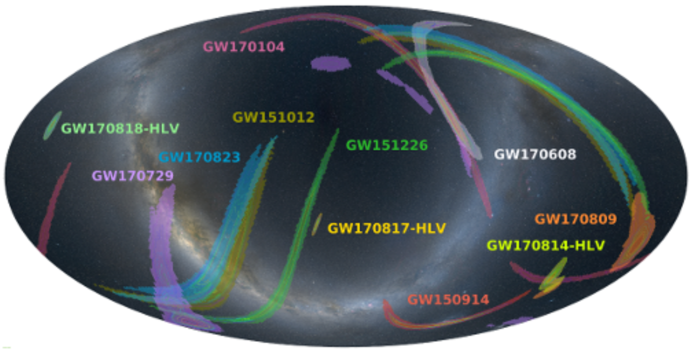
Universiteit Leiden

Nikhef

Gravitational waves (3)



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Energy-momentum and angular momentum of gravitational radiation

For *free* gravitational waves in the TT-gauge, in a volume *V*, the following quantities are conserved modulo boundary terms:

$$E_V = \int_V d^3x\, \mathcal{E}(x,t) \qquad \mathcal{E} = \frac{1}{2}\left(\partial_t h_{ij}\right)^2 + \frac{1}{2}\left(\partial_k h_{ij}\right)^2$$

energy density

$$P_{V\,i} = \int_V d^3x\, \Pi_i(x,t) \qquad \Pi_i = -\,\partial_i h_{mn}\, \partial_t h_{mn}$$

momentum density

$$L_{V\,i} = \int_V d^3x\, \Lambda_i(x,t) \qquad \Lambda_i = \varepsilon_{ijk}\left[2h_{jm}\partial_t h_{km} - x_j \partial_k h_{mn} \partial_t h_{mn}\right]$$

angular-momentum density

Each integrand satisfies an equation of continuity:

$$\partial_t \mathcal{E} = -\partial_i \Pi_i$$

$$\partial_t \Pi_i = -\partial_k \mathcal{S}_{ki}$$

$$\partial_t \Lambda_i = -\partial_k \mathcal{J}_{ki}$$

where Π_i as above, and

$$\mathcal{S}_{ki} = \partial_k h_{mn} \partial_i h_{mn} + \frac{1}{2}\left[\left(\partial_t h_{mn}\right)^2 - \left(\partial_j h_{mn}\right)^2\right]$$

$$\mathcal{J}_{ki} = \varepsilon_{ijl}\left[h_{ln} \overset{\leftrightarrow}{\partial_k} h_{jn} + x_j \partial_l h_{mn} \partial_k h_{mn} + \frac{1}{2}\,\delta_{kl} x_j \left(\left(\partial_t h_{mn}\right)^2 - \left(\partial_p h_{mn}\right)^2\right)\right]$$

$$\longrightarrow \left(\frac{dE_V}{dt}, \frac{dP_{V\,i}}{dt}, \frac{dL_{V\,i}}{dt}\right) = -\oint_{\partial V} d^2\sigma\, \left(\Pi_n, \mathcal{S}_{ni}, \mathcal{J}_{ni}\right) = 0$$

modulo flow of gravitational-wave energy/momentum/angular momentum across the boundary of *V*

Energy flux

Taking the volume to be a large sphere of radius *r* : $V = S_r$

the surface element becomes a spherical surface element: $d^2\sigma = r^2 \sin\theta d\theta d\varphi = r^2 d\Omega$

then we can write for the outward radial energy flux:

$$\frac{dE}{r^2 d\Omega dt} = \Pi_n = \partial_r h_{ij} \partial_t h_{ij}$$

-Taking monochromatic plane waves in the *z*-direction through an area element $dA = dxdy$ in the plane $z = 0$:

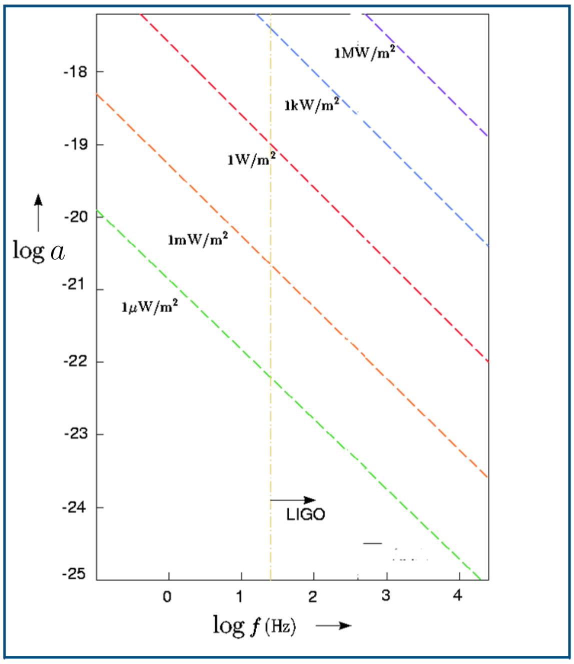
$$\frac{dE}{dAdt} = \Pi_z = \partial_z h_{ij} \partial_t h_{ij} \underset{(z=0)}{=} -\frac{2\omega^2}{c}\left(e_+^2 + e_\times^2\right)\sin^2 \omega t$$

(temporarily restoring c)

recall:
$$\boxed{\begin{aligned} \underline{h}_{11} &= -\underline{h}_{22} = e_+ \cos\omega(z-ct) \\ \underline{h}_{12} &= \underline{h}_{21} = e_\times \cos\omega(z-ct) \end{aligned}}$$

Energy densities in monochromatic plane waves

Averaging the flux of plane waves over an integral number of cycles: $\omega T = 2\pi n$



$$\overline{\frac{dE}{dAdt}} = \frac{1}{T}\int_0^T dt \left.\frac{dE}{dAdt}\right|_{z=0} = -\frac{\omega^2}{c}\left(e_+^2 + e_\times^2\right)$$

amplitude of metric variations: $a_{+,\times} = 2\kappa e_{+,\times}$

frequency: $\omega = 2\pi f$

→ combine in expression for flux

$$\boxed{\overline{\frac{dE}{dAdt}} = \frac{\pi c^3 f^2}{8G}\left(a_+^2 + a_\times^2\right)}$$

full amplitude: $a = \sqrt{a_+^2 + a_\times^2}$

Even small amplitudes correspond to large fluxes: extreme energy densities create tiny deformations of space:

space is `stiffest substance' known

flux $\Phi = \frac{\pi c^3 f^2}{8G} a^2$

$$\frac{\pi c^3}{8G} = 1.6 \times 10^{35}\,\mathrm{W/m^2}$$

Other fluxes

Outward momentum flux: $$\frac{dP_i}{r^2 d\Omega dt} = \mathcal{S}_{ni}$$

Outward angular momentum flux: $$\frac{dL_i}{r^2 d\Omega dt} = \mathcal{J}_{ni}$$

1. For radial flow out of spherical volume S_r involving fields $h_{ij}(t-r)$:

integrated momentum flux vanishes: $$\frac{dP_i}{dt} = -\oint_{\partial S_r} d^2\sigma\, \mathcal{S}_{ni} = 0$$

as momentum density $\mathcal{S}_{ni}$ on the boundary surface in direction of propagation, i.e. radially outward:

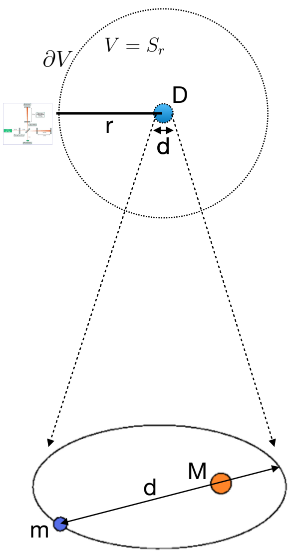
$\mathcal{S}_{ni} \propto \hat{r}_i \longrightarrow$ integrating over a full sphere contributions from opposite points cancel

2. This argument does *not* hold for angular momentum:

$$\frac{dL_i}{dt} = -\oint_{\partial S_r} d^2\sigma\, \mathcal{J}_{ni} \neq 0$$

as $\mathcal{J}_{ni}$ directed *orthogonal* to direction of propagation: *tangent* to surface

Sources of gravitational waves



binary star system

here: consider an isolated source of maximal size d observed from a distance r with $r \gg d$

e.g., a binary system of compact objects like white dwarfs, neutron stars or black holes

note: for PSR 1913+16 $\frac{d}{r} \sim 10^{-8}$

Assume the observer is at rest w.r.t. to the CM of the source; if not: signals are Doppler shifted.

We have to solve an inhomogeneous wave equation if type

$$\Box\, \phi(\mathbf{x}, t) = \rho(\mathbf{x}, t)$$

Retarded (causal) solution:

position of source element

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi} \int_{V = S_r} d^3x' \, \frac{\rho(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|}$$

position of observer

distance between source element and observer

solving the gravitational-wave equation

$$\boxed{\Box \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu} \longrightarrow \underline{h}_{\mu\nu}(\mathbf{x},t) = \frac{\kappa}{4\pi}\int_{S_r} d^3x' \, \frac{T_{\mu\nu}(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}}$$

to evaluate, note:

- $\underline{h}_{\mu\nu}(\mathbf{x},t)$ observed in far region where $T_{\mu\nu}(\mathbf{x},t) = 0$

- in that region $\Box \underline{h}_{\mu\nu} = 0$

and the *TT* - gauge applies: $\underline{h}_{\mu\nu}(\mathbf{x},t) = h_{\mu\nu}(\mathbf{x},t)$

- only spherical waves falling off as $1/r$ survive:

$$h_{ij}(\mathbf{x},t) \sim \int dk \, e_{ij}(k) \frac{e^{ik(r-t)}}{r} \quad \text{and} \quad h_{00} = h_{0i} = 0$$

with $h_{jj} = 0$ and $\hat{r}_i h_{ij} = 0$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$e_{jj} = 0$ $\qquad\qquad$ $k_i e_{ij} = 0$

General form of amplitude

$$h_{\mu\nu} = \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, T_{\mu\nu}(\mathbf{x}', t-r)$$

energy-momentum conservation:

$$\partial_0 h_{0\mu} = \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, \partial_0 T_{0\mu}(\mathbf{x}', t-r)$$

$$= \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, \partial_i' T_{i\mu}(\mathbf{x}', t-r) = 0$$

TT - gauge:

$$h_{ij} = \frac{\kappa}{4\pi r}(\delta_{ik} - \hat{r}_i\hat{r}_k)(\delta_{jl} - \hat{r}_j\hat{r}_l)\left(I_{kl} + \frac{1}{2}\delta_{kl}\hat{r}\cdot I\cdot\hat{r}\right)$$

with $h_{kk} = 0 \longleftrightarrow I_{kk} = 0$

and $I_{ij}(t-r) = \int_{S_r} d^3x' \left(T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}\right)_{t-r}$

Quadrupole approximation

use: $\partial_0^2 T_{00} = \partial_0 \partial_i T_{i0} = \partial_i \partial_j T_{ij}$

$$\longrightarrow \quad \frac{1}{2}\,\partial_0^2 \int d^3x\, x_i x_j T_{00} = \frac{1}{2} \int d^3x\, x_i x_j\, \partial_k \partial_l T_{kl} = \int d^3x\, T_{ij}$$

for non-relativistic sources $T_{00}(\mathbf{x},t) = \rho(\mathbf{x},t)$ (mass density)

$$\longrightarrow \quad I_{ij} = \frac{1}{2}\frac{\partial^2 Q_{ij}}{\partial t^2}$$

$$Q_{ij}(t-r) = \int_{S_r} d^3x' \left(x'_i x'_j - \frac{1}{3}\,\delta_{ij} \mathbf{x}'^2 \right) \rho(\mathbf{x}', t-r)$$

(mass quadrupole)

final result:

$$\boxed{h_{ij} = \frac{\kappa}{8\pi r}\,(\delta_{ik} - \hat{r}_i \hat{r}_k)\,(\delta_{jl} - \hat{r}_j \hat{r}_l) \left(\ddot{Q}_{kl} + \frac{1}{2}\,\delta_{kl} \hat{r} \cdot \ddot{Q} \cdot \hat{r} \right)}$$

Differential fluxes of energy, momentum and angular momentum

$$\frac{dE}{d\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\, \dddot{Q}^2 - 2\hat{\mathrm{r}} \cdot \dddot{Q}^2 \cdot \hat{\mathrm{r}} + \frac{1}{2}(\hat{\mathrm{r}} \cdot \dddot{Q} \cdot \hat{\mathrm{r}})^2 \right]$$

$$\frac{dP_i}{d\Omega dt} = \frac{G}{8\pi G c^6}\,\hat{r}_i \left[\mathrm{Tr}\, \dddot{Q}^2 - 2\hat{r} \cdot \dddot{Q}^2 \cdot \hat{r} + \frac{1}{2}\,(\hat{r} \cdot \dddot{Q} \cdot \hat{r})^2 \right] = -\frac{1}{c}\frac{dE}{d\Omega dt}\,\hat{r}_i$$

$$\frac{dL_k}{d\Omega dt} = -\frac{G}{4\pi c^5}\,\varepsilon_{kij} \left[(\ddot{Q} \cdot \dddot{Q})_{ij} - (\ddot{Q} \cdot \hat{r})_i (\dddot{Q} \cdot \hat{r})_j + \hat{r}_i\,(\ddot{Q} \cdot \dddot{Q} \cdot \hat{r} - \frac{1}{2}\,\ddot{Q} \cdot \hat{r}\,\hat{r} \cdot \dddot{Q} \cdot \hat{r})_j \right]$$

Integrated fluxes

$$\frac{dE}{dt} = -\frac{G}{5c^5}\,\mathrm{Tr}\, \dddot{Q}^2 \qquad\qquad \frac{dP_k}{dt} = 0$$

$$\frac{dL_i}{dt} = -\frac{2G}{5c^5}\,\varepsilon_{kij} \left(\ddot{Q} \cdot \dddot{Q} \right)_{ij}$$