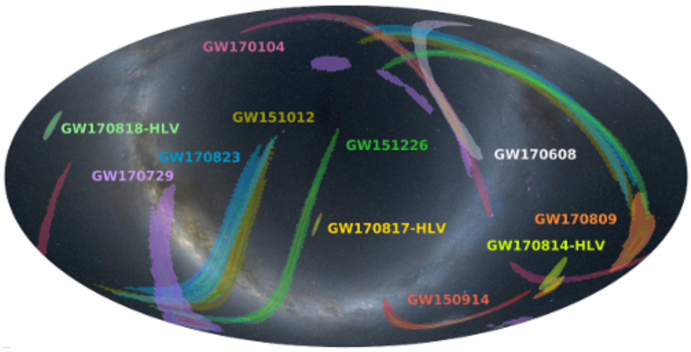
Universiteit Leiden

Nikhef

Gravitational waves (2)



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Bonn, Febr. 2023

Weak gravity: linear approximation of GR

space-time geometry: small deviations from Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$$

Minkowski metric $\eta = \mathrm{diag}(-1, +1, +1, +1)$

metric fluctuation (e.g., grav. wave)

using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to raise and lower indices

⟷ treat $h_{\mu\nu}$ as a symmetric tensor field in Minkowski space

in this approximation:

$$\Gamma_{\mu\nu}^{\;\;\lambda} = \kappa \left(\partial_\mu h_\nu^{\;\lambda} + \partial_\nu h_\mu^{\;\lambda} - \partial^\lambda h_{\mu\nu} \right) + \mathcal{O}(\kappa^2)$$

$$R_{\mu\nu\kappa}^{\;\;\;\;\lambda} = \kappa \left(\partial_\mu \partial_\kappa h_\nu^{\;\lambda} - \partial_\nu \partial_\kappa h_\mu^{\;\lambda} + \partial^\lambda \partial_\nu h_{\mu\kappa} - \partial^\lambda \partial_\mu h_{\nu\kappa} \right) + \mathcal{O}(\kappa^2)$$

Einstein equations:

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \right) = \Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h_\lambda^{\lambda} - \eta_{\mu\nu} \left(\Box h_\lambda^{\lambda} - \partial^\kappa \partial^\lambda h_{\kappa\lambda} \right) = -\kappa T_{\mu\nu}$$

In linear approximation:

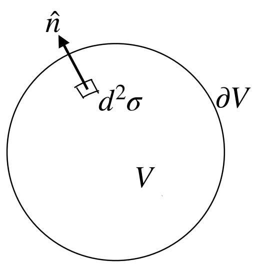
Ricci tensor: $R_{\mu\nu} = \kappa \left(\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial^\nu h_\lambda^{\ \lambda}\right)$

Ricci scalar: $R = R_\mu^{\ \mu} = 2\kappa \left(\Box h_\mu^{\ \mu} - \partial^\mu \partial^\nu h_{\mu\nu}\right)$

Bianchi identity: $\longrightarrow \quad \partial^\mu \left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R\right) = 0 \quad \longleftrightarrow \quad \partial^\mu T_{\mu\nu} = 0$

Energy-momentum conservation: $P_\mu = \int_V d^3x\, T_{0\mu}$

$$\longrightarrow \quad \frac{dP_\mu}{dt} = \int_V d^3x\, \partial_0 T_{0\mu} = \int_V d^3x\, \partial_k T_{k\mu} = \oint_{\partial V} d^2\sigma\, T_{n\mu}$$



forced by Einstein equations

$T_{n\mu} = \hat{n}_k T_{k\mu}$

Energy-momentum conservation of matter
related to gauge symmetry of Einstein equations

local gauge transformations $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

for arbitrary vector field $\xi_\mu(x)$ leave Riemann tensor invariant: $R'_{\mu\kappa\nu}{}^{\lambda} = R_{\mu\kappa\nu}{}^{\lambda}$

Gauge fixing $\longrightarrow$ *De Donder gauge:* $\partial_\nu h^\nu{}_\mu = \frac{1}{2}\partial_\mu h^\nu{}_\nu$

reduces Einstein equations to inhomogeneous wave equation:

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda{}_\lambda\right) = -\kappa T_{\mu\nu} \quad \longleftrightarrow \quad \Box h_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda\right)$$

Field redefinition:

$$\underline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda{}_\lambda \quad \longleftrightarrow \quad h_{\mu\nu} = \underline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \underline{h}^\lambda{}_\lambda .$$

simplification: $\boxed{\Box\, \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu}}$

Action

Einstein: $$S[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \simeq \frac{1}{\kappa^2}\int d^4x\, g^{\mu\nu}\left(\Gamma_{\mu\lambda}{}^{\kappa}\Gamma_{\nu\kappa}{}^{\lambda} - \Gamma_{\mu\nu}{}^{\lambda}\Gamma_{\lambda\kappa}{}^{\kappa}\right)$$

[Proof: use $$\frac{1}{2}\int\sqrt{-g}\,g^{\mu\nu}\partial_\lambda\Gamma^\lambda_{\mu\nu} = \int\sqrt{-g}\,g^{\mu\nu}\left(\Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\kappa\nu} - \frac{1}{2}\Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}\right),$$
$$\int\sqrt{-g}\,g^{\mu\nu}\partial_\mu\Gamma^\lambda_{\nu\lambda} = \int\sqrt{-g}\,g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}.$$]

weak-gravity limit:

$$S[h] = \int d^4x\left[-\frac{1}{2}\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu} + \partial^\mu h^{\nu\lambda}\partial_\nu h_{\mu\lambda} - \partial^\mu h_{\mu\nu}\partial^\nu h_\lambda{}^\lambda + \frac{1}{2}\partial^\lambda h_\mu{}^\mu\partial_\lambda h_\nu{}^\nu\right]$$

This action is invariant (up to boundary terms) under the gauge transformations

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

interaction term: $S_{int} = \kappa\int d^4x\, h_{\mu\nu}T^{\mu\nu}$ invariant if $\partial_\mu T^{\mu\nu} = 0$

Rewriting action and field equation in terms of $\underline{h}_{\mu\nu}$

$$S[\underline{h}] = \int d^4x\left[-\frac{1}{2}\partial^\lambda \underline{h}^{\mu\nu}\partial_\lambda \underline{h}_{\mu\nu} + \partial^\mu \underline{h}^{\nu\lambda}\partial_\nu \underline{h}_{\mu\lambda} + \frac{1}{4}\partial^\lambda \underline{h}_\mu{}^\mu\partial_\lambda \underline{h}_\nu{}^\nu + \kappa\underline{h}_{\mu\nu}\left(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T_\lambda{}^\lambda\right)\right]$$

after some rewriting $\longrightarrow$ $\Box\underline{h}_{\mu\nu} - \partial_\mu\partial^\lambda\underline{h}_{\lambda\nu} - \partial_\nu\partial^\lambda\underline{h}_{\lambda\mu} + \eta_{\mu\nu}\partial^\kappa\partial^\lambda\underline{h}_{\kappa\lambda} = -\kappa T_{\mu\nu}$

invariant under $\underline{h}_{\mu\nu} \to \underline{h}'_{\mu\nu} = \underline{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial^\lambda\xi_\lambda$

Free fields: plane-wave decomposition

$$\underline{h}_{\mu\nu} = \int\frac{d^4k}{(2\pi)^2}\,\varepsilon_{\mu\nu}(k)e^{-ik\cdot x} \qquad \varepsilon^*_{\mu\nu}(k) = \varepsilon_{\mu\nu}(-k)$$

$\longrightarrow$ $$k^2\varepsilon_{\mu\nu} - k_\mu k^\lambda\varepsilon_{\lambda\nu} - k_\nu k^\lambda\varepsilon_{\lambda\mu} + \eta_{\mu\nu}k^\kappa k^\lambda\varepsilon_{\kappa\lambda} = 0$$

invariant under $$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu\alpha_\nu + k_\nu\alpha_\mu - \eta_{\mu\nu}k^\lambda\alpha_\lambda.$$

with $$\xi_\mu = i\int\frac{d^4k}{(2\pi)^2}\,\alpha_\mu(k)e^{-ik\cdot x}$$

gauge fixing

take $k^\mu \varepsilon_{\mu\nu} = 0$ $\longrightarrow$ $k^2 \varepsilon_{\mu\nu} = 0$

$$\varepsilon_{\mu\nu}(k) = e_{\mu\nu}(\mathbf{k}, \omega)\,\delta(k^2)$$

with $\omega = \sqrt{\mathbf{k}^2}$

$$k^\mu e_{\mu 0} = \omega e_{00} + k_i e_{i0} = 0 \qquad k^\mu e_{\mu j} = \omega e_{0j} + k_i e_{ij} = 0$$

residual transformations: use $\alpha_\mu(k) = a_\mu(\mathbf{k}, \omega_\mathbf{k})\,\delta(k^2)$

$$e'_{00} = e_{00} - \omega a_0 + \mathbf{k}\cdot\mathbf{a}, \qquad e'_{0i} = e_{0i} - \omega a_i + k_i a_0$$

$$e'_{ij} = e_{ij} + k_i a_j + k_j a_i - \delta_{ij}\,(\omega a_0 + \mathbf{k}\cdot\mathbf{a})$$

can arrange $\longrightarrow$

$$e'_{00} = e'_{0i} = e'_{ii} = 0$$
$$k_j e'_{ji} = 0$$

(*TT* - gauge)

reality: $e^*_{\mu\nu}(\mathbf{k}, \omega) = e_{\mu\nu}(-\mathbf{k}, -\omega)$

$$\longrightarrow \quad \underline{h}_{\mu\nu} = \int \frac{d^3k}{8\pi^2\omega}\left(e_{\mu\nu}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + e^*_{\mu\nu}(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$$

in canonical quantum gravity $\longrightarrow$ annihilation/creation operators

in *TT* - gauge:

$$\underline{h}_{00} = \underline{h}_{0i} = \underline{h}_{ii} = 0$$
$$\partial_j \underline{h}_{ji} = 0$$

in this gauge $\underline{h}^\lambda{}_\lambda = 0 \longrightarrow \underline{h}_{\mu\nu} = h_{\mu\nu}$

(recall: this holds for *free* fields = external lines in QG)

Results for monochromatic waves

- the free physical plane-wave amplitudes can be restricted to the sets $e_{\mu\nu}(\mathbf{k}, \omega)$ subject to the following conditions

$$e_{00} = e_{0i} = e_{ii} = 0, \qquad k_j e_{ji} = 0$$

- without loss of generality we can take the z-direction in the direction of $\mathbf{k}$:

$$\mathbf{k} = (0, 0, \omega) \longrightarrow e_{i3} = 0$$

- the resulting physical amplitude takes the form

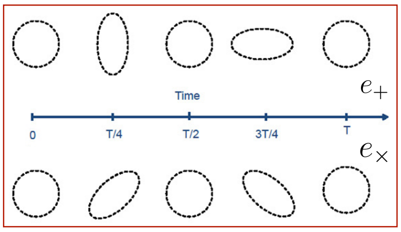
$$e'_{\mu\nu}(\mathbf{k}, \omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+(\omega) & e_\times(\omega) & 0 \\ 0 & e_\times(\omega) & -e_+(\omega) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- there are only 2 independent physical free-wave modes, transverse to the wave-propagation:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \underline{h}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + 2\kappa e_+ \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 2\kappa e_\times \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 0 \\ 0 & 2\kappa e_\times \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 1 - 2\kappa e_+ \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Detection principle

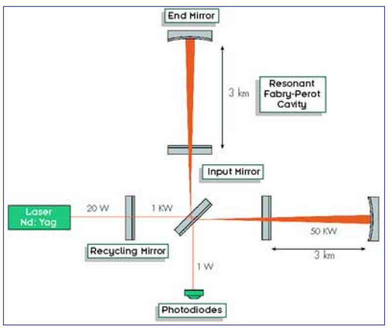
Distances between points on a ring in the x-y-plane change by passage of a monochromatic gravitational wave in the z-direction as follows:



$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dx^2 + dy^2 + dz^2 + 2\kappa e_+ \cos\omega(z-t)\,(dx^2 - dy^2) + 4\kappa e_\times \cos\omega(z-t)\,dx dy \end{aligned}$$

(fix $z = 0$)

This provides a method to detect gravitational waves by comparing distances in 2 perpendicular directions using interferometry:



+-mode: one arm gets longer, the other shorter
⟶ difference in travelling time of the laser beams: phase difference creates change in the output of the interferometer

(disadvantage: does not see diagonal x-mode
↓
need more than one detector with different orientations)

Virgo detector (Pisa, It.)
arm-length: 3 km

curvature dynamics
in the linear approximation

$$R_{\mu\kappa\nu\lambda} = \kappa\left(\partial_\mu\partial_\kappa h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\kappa} - \partial_\nu\partial_\kappa h_{\mu\lambda} + \partial_\nu\partial_\lambda h_{\mu\kappa}\right)$$

is invariant under $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$

Bianchi identity: $\partial_\sigma R_{\mu\kappa\nu\lambda} + \partial_\mu R_{\kappa\sigma\nu\lambda} + \partial_\kappa R_{\sigma\mu\nu\lambda} = 0$

$$\longrightarrow \left[\begin{array}{l} \partial^\mu R_{\mu\kappa\nu\lambda} = \partial_\nu R_{\lambda\kappa} - \partial_\lambda R_{\nu\kappa} \\ \Box R_{\mu\kappa\nu\lambda} = \partial_\mu\partial_\kappa R_{\nu\lambda} - \partial_\mu\partial_\lambda R_{\nu\kappa} - \partial_\nu\partial_\kappa R_{\mu\lambda} + \partial_\nu\partial_\lambda R_{\mu\kappa} \end{array}\right.$$

Einstein equations: $\Box R_{\mu\kappa\nu\lambda} = -8\pi G\left(\partial_\mu\partial_\kappa \underline{T}_{\nu\lambda} - \partial_\mu\partial_\lambda \underline{T}_{\nu\kappa} - \partial_\nu\partial_\kappa \underline{T}_{\mu\lambda} + \partial_\nu\partial_\lambda \underline{T}_{\mu\nu}\right)$

$$\underline{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_\lambda{}^\lambda$$

(Follows directly from definition using De Donder gauge)

space-time split

spatial components: $R_{klmn} = \varepsilon_{kli}\varepsilon_{mnj}P_{ij}$

$\longleftrightarrow$ $P_{ij} = P_{ji} = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn} = \kappa\,\varepsilon_{ikl}\varepsilon_{jmn}\,\partial_k\partial_m h_{ln}$

Bianchi identity: $\partial_i P_{ij} = \varepsilon_{ikl}\left(\partial_i R_{klmn}\right)\varepsilon_{jmn} = 0$

free fields: $P_{jj} = \frac{1}{2}R = 0$

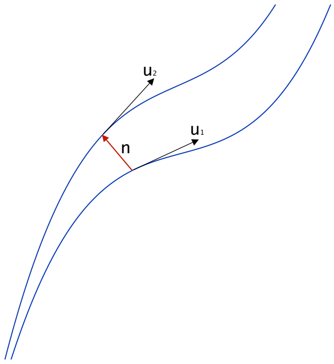
↓

time-components: $R_{0jkl} = \left(\frac{\partial_0\partial_i}{\Delta}P_{mn}\right)\varepsilon_{mij}\varepsilon_{nkl}$

$$R_{0i0j} = -P_{ij} = (\partial_0^2 h_{ij})^{TT}$$

- In empty space all components of the Riemann (= Weyl) tensor are expressed in terms of 2 independent d.o.f. contained in transverse and traceless gauge invariant 3-tensor P_{ij}
- These d.o.f. are expressed by the TT-part of h_{ij}

detection principle: geodesic deviation



continuous set of geodesics $x^\mu(\tau;\lambda)$

proper velocity $u^\mu = \dfrac{dx^\mu}{d\tau}$

geodesic deviation $n^\mu = \dfrac{dx^\mu}{d\lambda}$

$$\longrightarrow \quad \frac{Dn^\mu}{D\tau} = u^\nu \nabla_\nu n^\mu = n^\lambda \nabla_\lambda u^\mu = \frac{Du^\mu}{D\lambda}$$

$$\frac{D^2 n^\mu}{D\tau^2} = u^\lambda \nabla_\lambda (u^\nu \nabla_\nu n^\mu) = R_{\lambda\nu\kappa}{}^{\mu} u^\lambda u^\kappa n^\nu$$

in rest frame of free falling test mass: $u^\mu = (1,0,0,0)$ and $\Gamma_{\lambda\nu}{}^{\mu} = 0$

$$\longrightarrow \quad \boxed{\frac{d^2 n^0}{d\tau^2} = 0, \quad \frac{d^2 n^i}{d\tau^2} = P_{ij} n^j = -(\ddot{h}_{ij})^{TT} n^j}$$