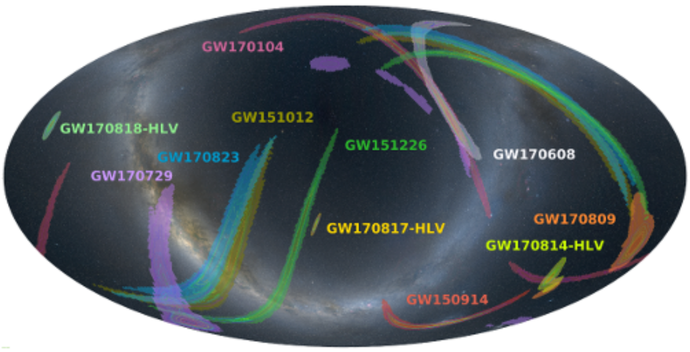
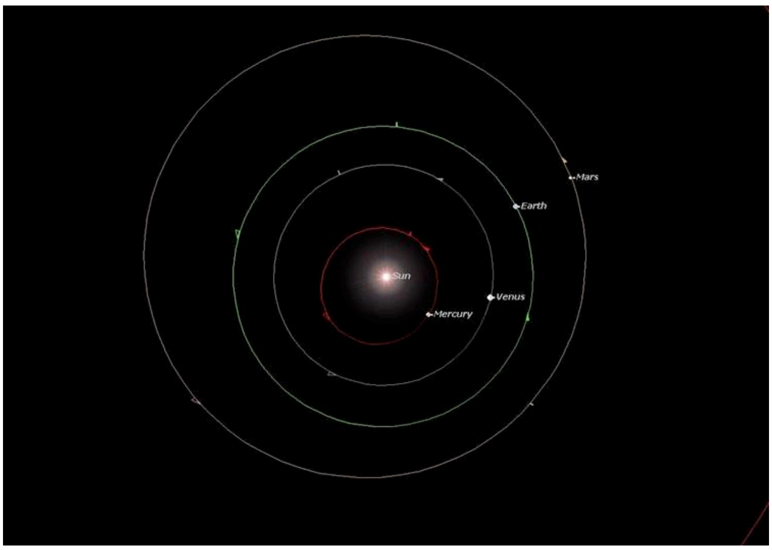
Gravitational waves



Jan W. van Holten

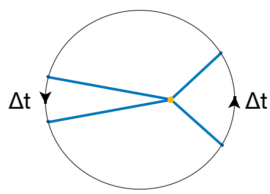
Bonn, Febr. 2023

Kepler's laws of planetary orbits



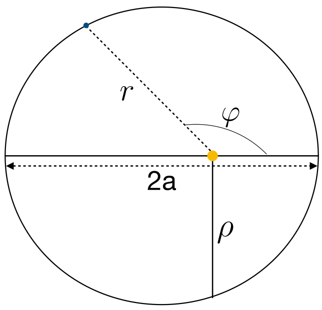
inner solar system

- planets move on elliptic orbits with the sun in a focal point
- the radius connecting the planet and the sun sweeps out equal areas in equal times



- the square of the period of the orbit is proportional to the cube of the major axis

Newton's theory of gravity



$$r = \frac{\rho}{1 - e\cos\varphi}$$

- masses interact by a *central force*

  → conservation of angular momentum

- inverse square law

  → period-radius relation

$$\mathbf{F} = \frac{GmM}{r^2}\,\hat{\mathbf{r}} \quad \longrightarrow \quad \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$$

	a^3/T^2
Mercury	3.362
Venus	3.362
Earth	3.362
Mars	3.362
	(10^{18} m³ s⁻²)

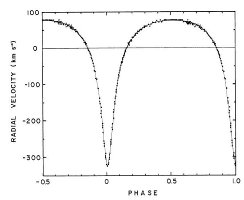
actually motion around common center of mass

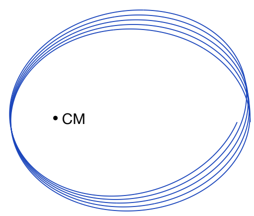
$$\longrightarrow \quad M = m_1 + m_2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

Instantaneous action at a distance

Binary pulsar 1913 +16:
violations of Kepler's laws and evidence for GR

Binary pulsar data:



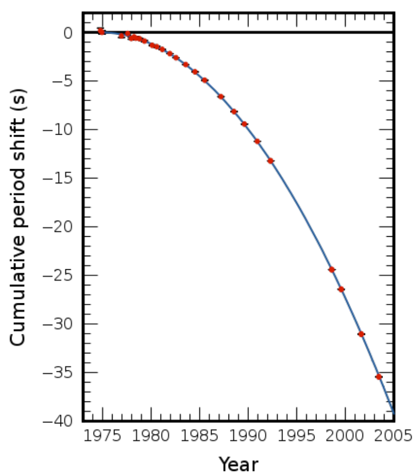


masses	$m_1 = 1.441\,M_\odot, \quad m_2 = 1.387\,M_\odot$
period	$7.75\,\mathrm{hr}$
semi-major axis	$a = 1\,950\,000\,\mathrm{km}$
eccentricity	$e = 0.617$
distance	$6.4\,\mathrm{kpc}$

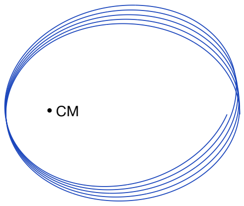
quasi-elliptic orbit

- precesses $4.2°/\mathrm{yr}$ → corrections to Newton's law
- shrinks $\Delta a = 3.5\,\mathrm{m/yr}$ → emission of gravitational waves

$7.35 \times 10^{24}\ \mathrm{W}$



Binary black holes



$$M_{BH}^2 \simeq 50\, m_1 m_2$$

GW energy flux of binaries ~ product of the masses

→ binary black holes with masses $M_{BH} = 10\, M_\odot$
same orbit would emit ~ 50 times more energy:
3.7×10^{26} W ~ solar luminosity in e.m. radiation

- frequencies of these massive compact binaries are very low:

$$f_{binary} \sim 0.7 \times 10^{-4}\ \text{Hz} \qquad f_{light} \sim 0.5 \times 10^{6}\ \text{Hz}$$

$$\lambda_{binary} \sim 4 \times 10^{9}\ \text{km} \qquad \lambda_{light} \sim 600\ \text{nm}$$

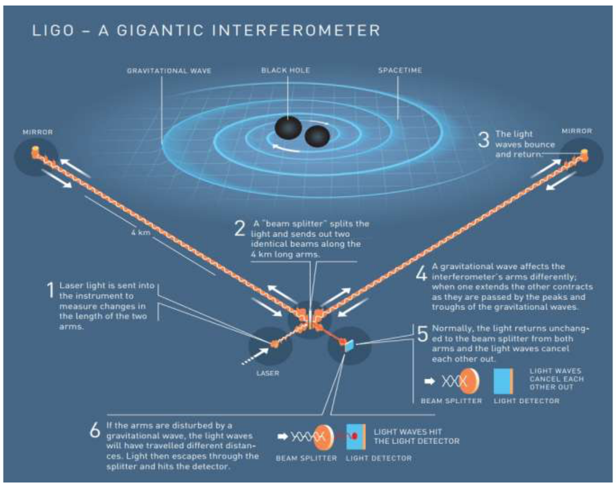
- Gravitational forces are extremely weak:

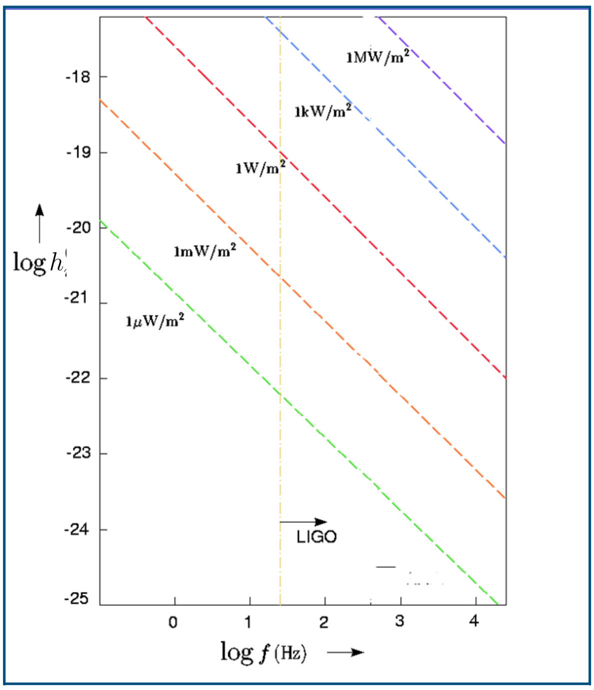
in hydrogen atom $\dfrac{F_{newton}}{F_{coulomb}} \simeq 0.45 \times 10^{-39}$

gravitational radiation very difficult to detect!

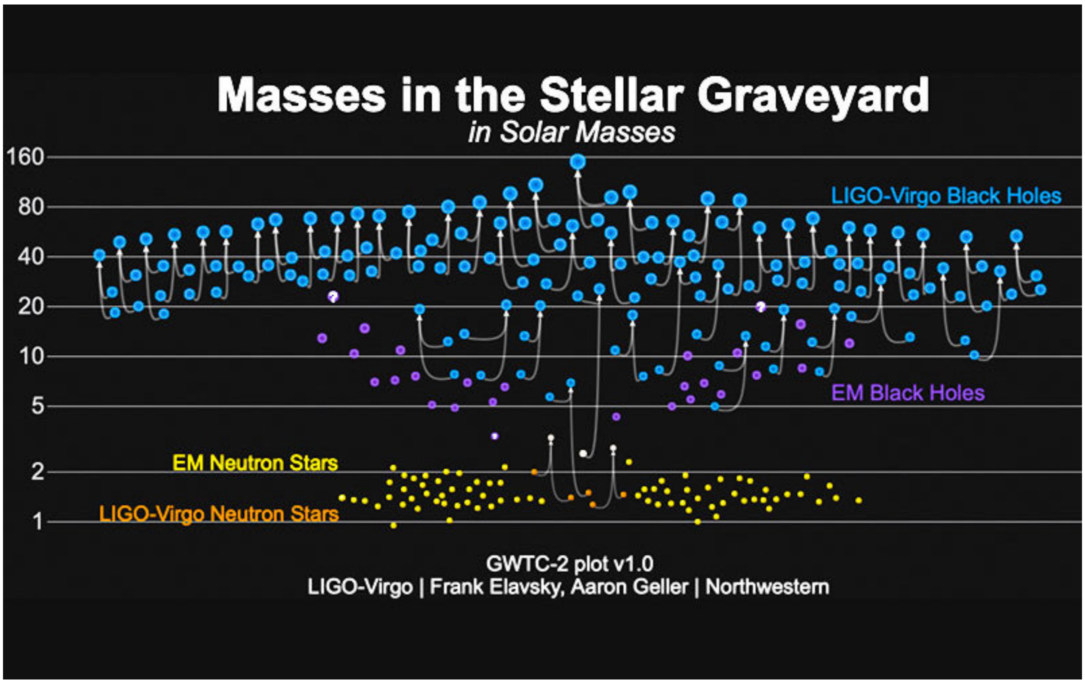
Direct detection of gravitational waves

Amplitudes expressed by metric deformations (strain) $h = \dfrac{\Delta l}{l}$





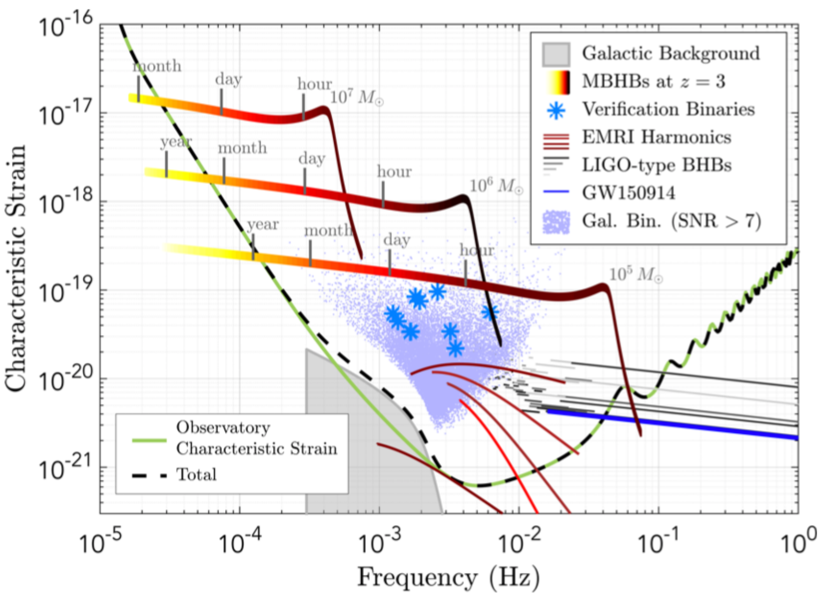
Selection of observed mergers of black-hole and neutron-star binaries
(LIGO-Virgo-KAGRA)

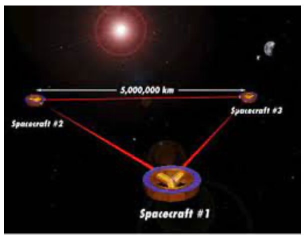


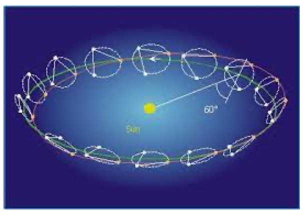
mergers last up to minutes and reach frequencies up to ~ 300 Hz.

Observing the inspiral phase of compact binaries:

space mission (LISA)

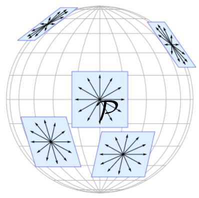




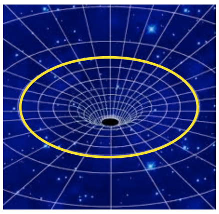


General Relativity

gravitational field ⟷ space-time geometry



local flat geometry + gravitational forces ⟷ global curvature



Static Schwarzschild geometry:

non-euclidean relation between circumference and radius of circular orbits ⟶ corrections to Newton's law

Gravitational fields are dynamical ⟷ geometry can *fluctuate*

fluctuations can propagate as *gravitational waves*

Tools for dynamical space-time: differential geometry

metric: line element $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$ (signature $(-,+,+,+)$)

connection: geodesics $\ddot{x}^\mu + \Gamma_{\lambda\nu}{}^{\mu}\, \dot{x}^\lambda \dot{x}^\nu = 0$

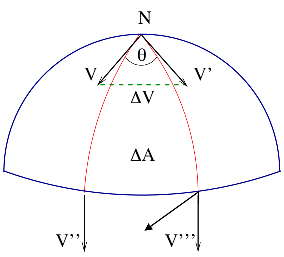
with $\Gamma_{\lambda\nu}{}^{\mu} = \frac{1}{2} g^{\mu\kappa} \left(\partial_\lambda g_{\kappa\nu} + \partial_\nu g_{\kappa\lambda} - \partial_\kappa g_{\lambda\nu} \right)$

⟷ covariant derivative $\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma_{\nu\mu}{}^{\lambda} A_\lambda$ (parallel displacement)

in particular $\nabla_\lambda g_{\mu\nu} = 0$

curvature: Riemann tensor

non-commuting covariant derivatives $[\nabla_\mu, \nabla_\lambda] A_\nu = - R_{\mu\lambda\nu}{}^{\kappa} A_\kappa$



with $R_{\mu\lambda\nu}{}^{\kappa} = \partial_\mu \Gamma_{\lambda\nu}{}^{\kappa} - \partial_\lambda \Gamma_{\mu\nu}{}^{\kappa} - \Gamma_{\mu\nu}{}^{\sigma} \Gamma_{\lambda\sigma}{}^{\kappa} + \Gamma_{\lambda\nu}{}^{\sigma} \Gamma_{\mu\sigma}{}^{\kappa}$

$= (\partial_\mu \Gamma_\lambda - \partial_\lambda \Gamma_\mu - [\Gamma_\mu, \Gamma_\lambda])_\nu{}^{\kappa}$

Einstein equations

Ricci tensor and scalar $\quad R_{\mu\nu} = R_{\nu\mu} = R_{\mu\lambda\nu}{}^{\lambda}, \qquad R = R_{\mu}{}^{\mu}$

Bianchi identity $\quad \nabla_\sigma R_{\mu\lambda\nu}{}^{\kappa} + \nabla_\mu R_{\lambda\sigma\nu}{}^{\kappa} + \nabla_\lambda R_{\sigma\mu\nu}{}^{\kappa} = 0$

$$\longrightarrow \quad \nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$$

Einstein tensor $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad \longrightarrow \quad \nabla^\mu G_{\mu\nu} = 0$

$$\boxed{G_{\mu\nu} = -\,\kappa^2\, T_{\mu\nu}}$$

energy-momentum tensor

$$\kappa^2 = \frac{\hbar}{m_{planck}^2 c^3} = \frac{8\pi G}{c^4} \simeq 2.1 \times 10^{-43}\ \ \mathrm{kg}^{-1}\ \mathrm{m}^{-1}\ \mathrm{s}^2$$

local energy-momentum conservation: $\quad \nabla^\mu T_{\mu\nu} = 0$