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Symbol bootstrap: from N=4 sYM to QCD

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2510.20565, 2602.02783

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State of the art Five-particle scattering at NNLO

Dramatic progress for massless 5-particle scattering

Analytic results for all Feynman integrals

[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

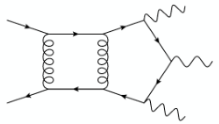
Special function basis

[Gehrmann, Henn, Lo Presti '18; Chicherin, Sotnikov '20]

[see M. Marcoli's talk]

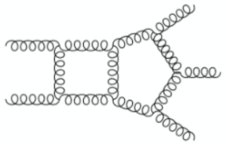
Analytic results for scattering amplitudes

3γ planar



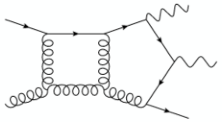
[Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$ planar

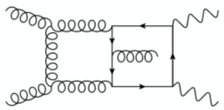


[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; Badger, Brönnum-Hansen, Bayu Hartanto, Peraro, Moodie, **SZ**, *to appear*]

$2\gamma + j$ full colour



[Agarwal, Buccioni, von Manteuffel, Tancredi 2021 x2; Chawdhry, Czakon, Mitov, Poncelet 2021]



[Badger, Brönnum-Hansen, Chicherin, Gehrmann, B. Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

$d\sigma$ **@NNLO QCD:**

$pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]

$pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]

$pp \to 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

[slide from S. Zoia, LoopFest 2022]

Perturbative Structure for QFT

$$\mathcal{A}_n^{(L)} = \boxed{\sum_{w=0}^{2L}} \sum_{ij} \boxed{c_{ij}^{(w)}} \boxed{R_i^{(w)}} \boxed{\mathcal{I}_j^{(w)}}.$$

Graded Transcendental function

rational numbers

Kinematic prefactors

Transcendental functions from Canonical Differential Equations (CDE)

- Challenges:
- Complicated **multivariate transcendental functions**
- Proliferation of **prefactors**
- For higher loops/multiplicity amplitudes, both conceptual and practical advances needed

Challenge: Proliferation of prefactors (1/2)

Challenge

- Rationality of integral coefficients in momenta

$$\mathcal{N}_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots\right), \quad \mathcal{D}_i(\vec{p}) = \sum_{\vec{\alpha}} d_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots\right)$$

$$r_i(\vec{p}) = \frac{\mathcal{N}_i(\vec{p})}{\mathcal{D}_i(\vec{p})}, \quad s_{ij} = (p_i + p_j)^2$$

↪ linear in numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $r_i(\vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}$$

↪ linear system for required coefficients $n_{i,\vec{\alpha}}$ and $d_{i,\vec{\alpha}}$

- Bottleneck:
  - Run times
  - Complexity of coefficients = number of unknowns $n_{i,\vec{\alpha}}$

9

Five-gluon amplitudes

[De Laurentis, Ita, Klinkert, Sotnikov '23]

Helicity remainder	dim(basis)	ansatz size
$R^{(2),(2,0)}_{+++--}$	31	21,910
$R^{(2),(2,0)}_{++-+-}$	54	54,148
$R^{(2),(1,0)}_{+++--}$	274	163,635
$R^{(2),(1,0)}_{+-++-}$	270	241,156
$R^{(2),(1,0)}_{--+++}$	203	82,180
$R^{(2),(1,1)}_{+++--}$	31	21,910
$R^{(2),(1,1)}_{++-+-}$	54	54,148
$R^{(2),(0,1)}_{+++--}$	226	118,880
$R^{(2),(0,1)}_{+-++-}$	240	209,018
$R^{(2),(0,1)}_{--+++}$	157	76,845
$R^{(2),(-1,1)}_{+++--}$	25	5,320
$R^{(2),(-1,1)}_{++-+-}$	35	9,384

[slide from H. Ita, LoopFest 2025]

Challenge: Proliferation of prefactors (2/2)

$$
\begin{aligned}
\{ & R[1] \to -\frac{5}{2}, \\
& R[2] \to -\frac{1}{4\, s12^{3}\, (s12 - s34 - s45)^{3}} \big(6 s12^{5} s15 + 15 s12^{4} s15^{2} + 10 s12^{3} s15^{3} + 6 s12^{5} s23 + 15 s12^{4} s23^{2} + 10 s12^{3} s23^{3} - 12 s12^{4} s15 s34 - 15 s12^{3} s15^{2} s34 - 18 s12^{4} s23 s34 - 45 s12^{3} s23^{2} s34 - 30 s12^{2} s23^{3} s34 + 6 s12^{3} s15 s34^{2} + 18 s12^{3} s23 s34^{2} + \\
& 45 s12^{2} s23^{2} s34^{2} + 30 s12 s23^{3} s34^{2} - 6 s12^{2} s23 s34^{3} - 15 s12 s23^{2} s34^{3} - 10 s23^{3} s34^{3} - 18 s12^{4} s15 s45 - 45 s12^{3} s15^{2} s45 - 30 s12^{2} s15^{3} s45 - 12 s12^{4} s23 s45 - 15 s12^{3} s23^{2} s45 + 6 s12^{4} s34 s45 + 54 s12^{3} s15 s34 s45 + \\
& 60 s12^{2} s15^{2} s34 s45 + 54 s12^{3} s23 s34 s45 + 60 s12^{2} s15 s23 s34 s45 + 30 s12 s15^{2} s23 s34 s45 + 60 s12^{2} s23^{2} s34 s45 + 30 s12 s15 s23^{2} s34 s45 - 12 s12^{3} s34^{2} s45 - 36 s12^{2} s15 s34^{2} s45 - 72 s12^{2} s23 s34^{2} s45 - \\
& 60 s12 s15 s23 s34^{2} s45 - 75 s12 s23^{2} s34^{2} s45 - 30 s15 s23^{2} s34^{2} s45 + 6 s12^{2} s34^{3} s45 + 30 s12 s23 s34^{3} s45 + 30 s23^{2} s34^{3} s45 + 18 s12^{3} s15 s45^{2} + 45 s12^{2} s15^{2} s45^{2} + 30 s12 s15^{3} s45^{2} + 6 s12^{3} s23 s45^{2} - 12 s12^{3} s34 s45^{2} - \\
& 72 s12^{2} s15 s34 s45^{2} - 75 s12 s15^{2} s34 s45^{2} - 36 s12^{2} s23 s34 s45^{2} - 60 s12 s15 s23 s34 s45^{2} - 30 s15^{2} s23 s34 s45^{2} + 27 s12^{2} s34^{2} s45^{2} + 60 s12 s15 s34^{2} s45^{2} + 60 s12 s23 s34^{2} s45^{2} + 60 s15 s23 s34^{2} s45^{2} - \\
& 15 s12 s34^{3} s45^{2} - 30 s23 s34^{3} s45^{2} - 6 s12^{2} s15 s45^{3} - 15 s12 s15^{2} s45^{3} - 10 s15^{3} s45^{3} + 6 s12^{2} s34 s45^{3} + 30 s12 s15 s34 s45^{3} + 30 s15^{2} s34 s45^{3} - 15 s12 s34^{2} s45^{3} - 30 s15 s34^{2} s45^{3} + 10 s34^{3} s45^{3} - 6 s12^{4} tr5 - \\
& 15 s12^{3} s15 tr5 - 10 s12^{2} s15^{2} tr5 - 15 s12^{3} s23 tr5 - 10 s12^{2} s15 s23 tr5 - 10 s12^{2} s23^{2} tr5 + 12 s12^{3} s34 tr5 + 15 s12^{2} s15 s34 tr5 + 30 s12^{2} s23 s34 tr5 + 10 s12 s15 s23 s34 tr5 + 20 s12 s23^{2} s34 tr5 - \\
& 6 s12^{2} s34^{2} tr5 - 15 s12 s23 s34^{2} tr5 - 10 s23^{2} s34^{2} tr5 + 12 s12^{3} s45 tr5 + 30 s12^{2} s15 s45 tr5 + 20 s12 s15^{2} s45 tr5 + 15 s12^{2} s23 s45 tr5 + 10 s12 s15 s23 s45 tr5 - 27 s12^{2} s34 s45 tr5 - 35 s12 s15 s34 s45 tr5 - \\
& 35 s12 s23 s34 s45 tr5 - 20 s15 s23 s34 s45 tr5 + 15 s12 s34^{2} s45 tr5 + 20 s23 s34^{2} s45 tr5 - 6 s12^{2} s45^{2} tr5 - 15 s12 s15 s45^{2} tr5 - 10 s15^{2} s45^{2} tr5 + 15 s12 s34 s45^{2} tr5 + 20 s15 s34 s45^{2} tr5 - 10 s34^{2} s45^{2} tr5\big), \\
& R[4] \to \frac{1}{6\, s12^{3}\, s34^{3}} \big(10 s12^{3} s15^{3} - 30 s12^{3} s15^{2} s23 + 30 s12^{3} s15 s23^{2} - 10 s12^{3} s23^{3} - 15 s12^{3} s15^{2} s34 + 30 s12^{3} s15 s23 s34 - 15 s12^{3} s23^{2} s34 + 6 s12^{3} s15 s34^{2} - 6 s12^{3} s23 s34^{2} + 3 s12^{3} s34^{3} - 6 s12^{2} s23 s34^{3} - \\
& 15 s12 s23^{2} s34^{3} - 10 s23^{3} s34^{3} - 30 s12^{2} s15^{3} s45 + 60 s12^{2} s15^{2} s23 s45 - 30 s12^{2} s15 s23^{2} s45 + 60 s12^{2} s15^{2} s34 s45 - 60 s12^{2} s15 s23 s34 s45 + 30 s12 s15^{2} s23 s34 s45 - 30 s12 s15 s23^{2} s34 s45 - \\
& 36 s12^{2} s15 s34^{2} s45 - 60 s12 s15 s23 s34^{2} s45 - 30 s15 s23^{2} s34^{2} s45 + 6 s12^{2} s34^{3} s45 + 30 s12 s23 s34^{3} s45 + 30 s23^{2} s34^{3} s45 + 30 s12 s15^{3} s45^{2} - 30 s12 s15^{2} s23 s45^{2} - 75 s12 s15^{2} s34 s45^{2} + \\
& 30 s12 s15 s23 s34 s45^{2} - 30 s15^{2} s23 s34 s45^{2} + 60 s12 s15 s34^{2} s45^{2} + 60 s15 s23 s34^{2} s45^{2} - 15 s12 s34^{3} s45^{2} - 30 s23 s34^{3} s45^{2} - 10 s15^{3} s45^{3} + 30 s15^{2} s34 s45^{3} - 30 s15 s34^{2} s45^{3} + 10 s34^{3} s45^{3} - 10 s12^{2} s15^{2} tr5 + \\
& 20 s12^{2} s15 s23 tr5 - 10 s12^{2} s23^{2} tr5 + 15 s12^{2} s15 s34 tr5 - 15 s12^{2} s23 s34 tr5 + 10 s12 s15 s23 s34 tr5 - 10 s12 s23^{2} s34 tr5 - 6 s12^{2} s34^{2} tr5 - 15 s12 s23 s34^{2} tr5 - 10 s23^{2} s34^{2} tr5 + 20 s12 s15^{2} s45 tr5 - \\
& 20 s12 s15 s23 s45 tr5 - 35 s12 s15 s34 s45 tr5 + 10 s12 s23 s34 s45 tr5 - 20 s15 s23 s34 s45 tr5 + 15 s12 s34^{2} s45 tr5 + 20 s23 s34^{2} s45 tr5 - 10 s15^{2} s45^{2} tr5 + 20 s15 s34 s45^{2} tr5 - 10 s34^{2} s45^{2} tr5\big), \\
& R[5] \to -\frac{1}{s12^{3}\, s45^{3}} \big(10 s12^{3} s15^{3} - 30 s12^{3} s15^{2} s23 + 30 s12^{3} s15 s23^{2} - 10 s12^{3} s23^{3} + 30 s12^{2} s15^{2} s23 s34 - 60 s12^{2} s15 s23^{2} s34 + 30 s12^{2} s23^{3} s34 + 30 s12 s15 s23^{2} s34^{2} - 30 s12 s23^{3} s34^{2} + 10 s23^{3} s34^{3} + \\
& 15 s12^{3} s15^{2} s45 - 30 s12^{3} s15 s23 s45 + 15 s12^{3} s23^{2} s45 + 60 s12^{2} s15 s23 s34 s45 + 30 s12 s15^{2} s23 s34 s45 - 60 s12^{2} s23^{2} s34 s45 - 30 s12 s15 s23^{2} s34 s45 - 30 s12 s15 s23 s34^{2} s45 + 75 s12 s23^{2} s34^{2} s45 + \\
& 30 s15 s23^{2} s34^{2} s45 - 30 s23^{2} s34^{3} s45 + 6 s12^{3} s15 s45^{2} - 6 s12^{3} s23 s45^{2} + 36 s12^{2} s23 s34 s45^{2} + 60 s12 s15 s23 s34 s45^{2} + 30 s15^{2} s23 s34 s45^{2} - 60 s12 s23 s34^{2} s45^{2} - 60 s15 s23 s34^{2} s45^{2} + 30 s23 s34^{3} s45^{2} + \\
& 6 s12^{2} s15 s45^{3} + 15 s12 s15^{2} s45^{3} + 10 s15^{3} s45^{3} - 6 s12^{2} s34 s45^{3} - 30 s12 s15 s34 s45^{3} - 30 s15^{2} s34 s45^{3} + 15 s12 s34^{2} s45^{3} + 30 s15 s34^{2} s45^{3} - 10 s34^{3} s45^{3} + 10 s12^{2} s15^{2} tr5 - 20 s12^{2} s15 s23 tr5 + \\
& 10 s12^{2} s23^{2} tr5 + 20 s12 s15 s23 s34 tr5 - 20 s12 s23^{2} s34 tr5 + 10 s23^{2} s34^{2} tr5 + 15 s12^{2} s15 s45 tr5 + 10 s12 s15^{2} s45 tr5 - 15 s12^{2} s23 s45 tr5 - 10 s12 s15 s23 s45 tr5 - 10 s12 s15 s34 s45 tr5 + \\
& 35 s12 s23 s34 s45 tr5 + 20 s15 s23 s34 s45 tr5 - 20 s23 s34^{2} s45 tr5 + 6 s12^{2} s45^{2} tr5 + 15 s12 s15 s45^{2} tr5 + 10 s15^{2} s45^{2} tr5 - 15 s12 s34 s45^{2} tr5 - 20 s15 s34 s45^{2} tr5 + 10 s34^{2} s45^{2} tr5\big)\}
\end{aligned}
$$

From Five-Parton NNLO amplitudes
[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov, 19']

Any understanding of analytic structures for QCD amplitudes?

Planar N=4 sYM scattering amplitudes

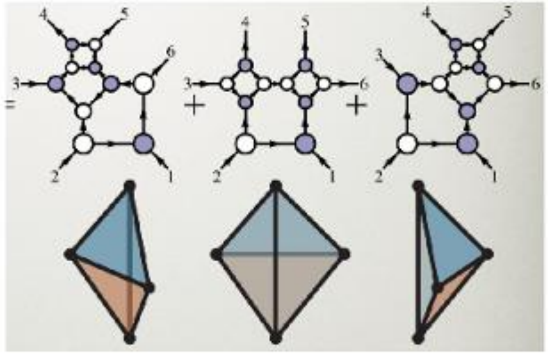
In the past decades, scattering amplitudes in planar N=4 sYM have been well-explored

$$\mathcal{A}_n^{(L)} = \sum_{ij} \boxed{c_{ij}}\, \boxed{R_i}\, \boxed{\mathcal{I}_j^{(2L)}}\,.$$

rational numbers

Yangian-invariants
[0902.2987,Drummond, Henn, Plefka]

- Leading singularity classification for all loops and all multiplicity from Grassmannian geometry
- Carry supersymmetric dependence



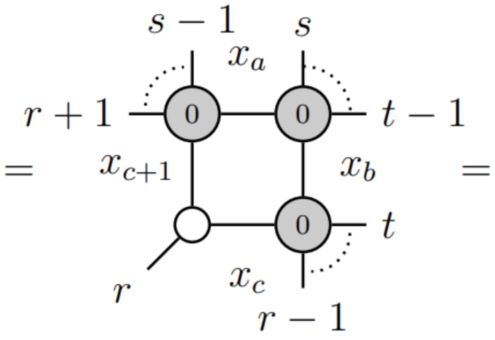
Transcendental functions

- MPL functions in simplest cases; maximal weight
- Steinman/Cluster bootstrap for six- and seven-point amplitudes
- Symbol alphabet and symbology: cluster algebras / Geometric Landau analysis for general multiplicity
- Q-bar anomaly equations: two-loop MHV, eight- and nine- point NMHV amplitudes⋯
- Beyond-MPL: Elliptic symbology

Leading singularity & Grassmannian geometry

- 1.on-shell functions from gluing

$R_{rst} =$



$$= \frac{\langle s-1\, s\rangle \langle t-1\, t\rangle\, \delta^{(4)}(\langle r|x_{ca}x_{ab}|\theta_{bc}\rangle + \langle r|x_{cb}x_{ba}|\theta_{ac}\rangle)}{x_{ab}^2 \langle r|x_{cb}x_{ba}|s-1\rangle \langle r|x_{cb}x_{ba}|s\rangle \langle r|x_{ca}x_{ab}|t-1\rangle \langle r|x_{ca}x_{ab}|t\rangle} \qquad (2.2)$$

- 2.Stratification for cells of Grassmanian CW-complex/Solvable Yangian invariants **[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka,12']**

$$f_\sigma^{(k)} = \oint_{C\subset\Gamma_\sigma} \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{\delta^{k\times 4}(C\cdot\widetilde{\eta})}{(1\cdots k)\cdots(n\cdots k{-}1)} \delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp}). \quad (8.2)$$

Transcendental functions: Bootstrap

- **(Steinmann/cluster) bootstrap for scattering amplitudes/form factors** : **[Golden, Goncharov, Spradlin, Vergu, Volovich; Basso, Caron-Huot, Dixon, Drummond, Duhr, Dulat, Foster, Gürdoğan, Harrington, He, Henn, von Hippel, Jiang, J.Liu, Y.Liu, McLeod, Papathanasiou, Wilhelm⋯]**
- Extremely powerful in calculating six-point (up to **eight loops**) and seven-point (up **five loops**) amplitudes in N=4 SYM theory

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Table 1: The dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

weight n	0	1	2	3	4	5	6	7
First entry	1	7	42	237	1288	6763	?	?
Steinmann	1	7	28	97	322	1030	3192	9570
Ext. Stein.	1	7	28	97	308	911	2555	6826

Table 2: The dimensions of the heptagon, Steinmann heptagon, and extended Steinmann heptagon spaces at symbol level.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathcal{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. $\mathrm{N^3LL}$ MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

- Ansatz from prediction (**Grassmannian geometry** & **symbol letters**) of the amplitudes → Physical/mathematical consistency condition the result should satisfy → unique allowed answer!
- **NO IBP needed anymore!**

Outline

- Bootstrapping four-point NMHV Form factors in N=4 sYM

- Bootstrapping (the maximal weight parts of) six-parton amplitudes in QCD

  Example: <u>--++++ helicity sector at two loops, pure YM</u>

- Discussions about the result

  <u>**Generalized principle of maximal transcendentality**</u>

Bootstrapping Four-Point NMHV Form factor in N=4 sYM

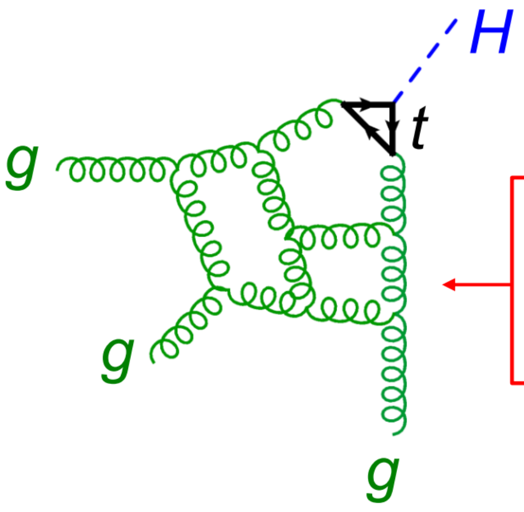
Form factors in N=4 sYM vs Higgs production

"Goldilocks Process" ~ $gg \rightarrow Hg$

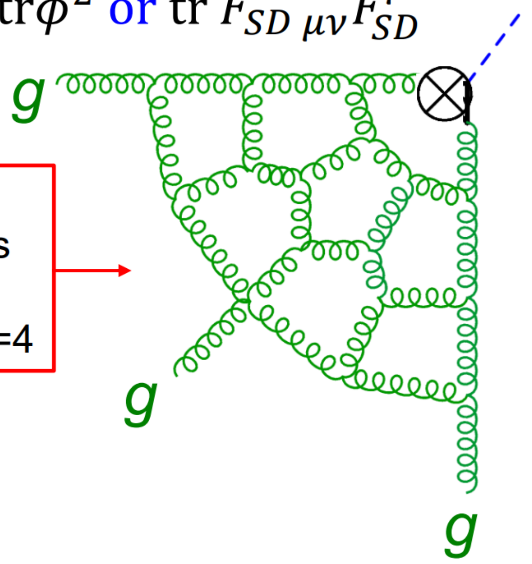
QCD state of art is now three loops @ large N_c (not counting top quark loop)
Chen, Guan, Mistlberger, 2504.06490

Can get to **eight** loops – in analog process: 3-point form factor of $\mathrm{tr}\phi^2$ operator in planar N=4 SYM
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm 2012.12286, 2112.06243, 2204.11901

$\mathcal{O}_2 = \mathrm{tr}\phi^2$ or $\mathrm{tr}\, F_{SD\,\mu\nu} F_{SD}^{\mu\nu}$



Leading weight $2L$ transcendental parts match between QCD ←→ planar N=4



L. Dixon 8 loop form factors... Bayreuth 15 April 2026 4

- **Maximal weight principle** between form factor and effective Higgs-gluon amplitudes **[Brandhuber, Travaglini, Yang, 12']**

Form Factor bootstrap and C2 alphabet

$$u = \frac{s_{12}}{s_{123}}\,, \quad v = \frac{s_{23}}{s_{123}}\,, \quad w = \frac{s_{31}}{s_{123}}\,.$$

When n=3, its alphabet contains 6 elements (C2 alphabet)

$$u + v + w = 1$$

$$\mathcal{S}_3 = \{u,\, v,\, w,\, 1-u,\, 1-v,\, 1-w\}\,.$$

$$\hat{u}(u,v,w) = \frac{vw}{(1-v)(1-w)}\,,$$
$$\hat{v}(u,v,w) = \frac{uw}{(1-u)(1-w)}\,,$$
$$\hat{w}(u,v,w) = \frac{uv}{(1-u)(1-v)}\,.$$

First entries conditions, Extended Steinmann-like conditions etc.

Has been determined up to 8 loops [2012.12286,2204.11901]

Antipodal symmetry:

On the parity even surface, 6pt MHV amplitudes and 3pt form factor satisfy

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)\Big|_{\hat{u}_i \to \hat{u}_i(u,v,w)} \qquad S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m\, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

8-loop Form Factor result→ 8-loop 6-point MHV amplitude.

Four-point NMHV form factor

- We calculate form factor at four points two loops via bootstrap strategy

$$\mathcal{F}_n(p_i, q, \eta_i) := \int \mathrm{d}^4x \mathrm{d}^4\theta^+ \; \mathrm{e}^{-\mathrm{i}(qx+\theta^+\gamma)} \langle \Omega_n | \mathcal{T}(x, \theta^+) | 0 \rangle$$

$$\mathcal{T}(x, \theta^+) = \mathrm{Tr}(\phi(x)^2) + \cdots + (\theta^+)^4 \mathcal{L}(x)$$

- We expand the supersymmetric observable by its Grassmann degree

$$\mathcal{F}_4 = \mathcal{F}_{4,0} + \mathcal{F}_{4,1} + \mathcal{F}_{4,2},$$

$\mathcal{F}_{4,k}$ has Grassmann degree $4k + 8$.

$\mathcal{F}_{4,0}$ is known up to four loops

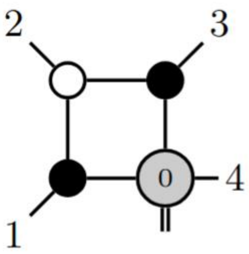
- Subtraction: $\mathcal{F}_{4,1} = \mathcal{F}_{4,0} \, \mathcal{R}_4 \, .$

$$\mathcal{R}_4^{(2)} = \sum_{ij} c_{ij} \; R_i \; \mathcal{I}_j^{(4)} \, .$$

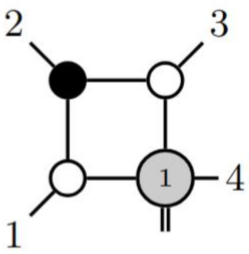
Ratio function: IR and UV finite function

Dihedral Ansatz: (susy) prefactors

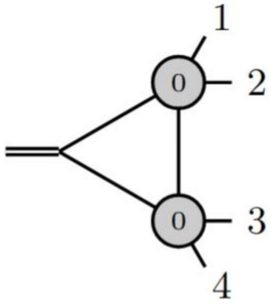
- At one loop, the prefactors are calculated from 4D generalized unitarity **[Bork 13'; Bianchi, Brandhuber, Panerai, Travaglini, 18']**



$R_{241} \equiv R_1$



$R_{144} \equiv \bar{R}_1$



$R_{12,34} \equiv S_1$

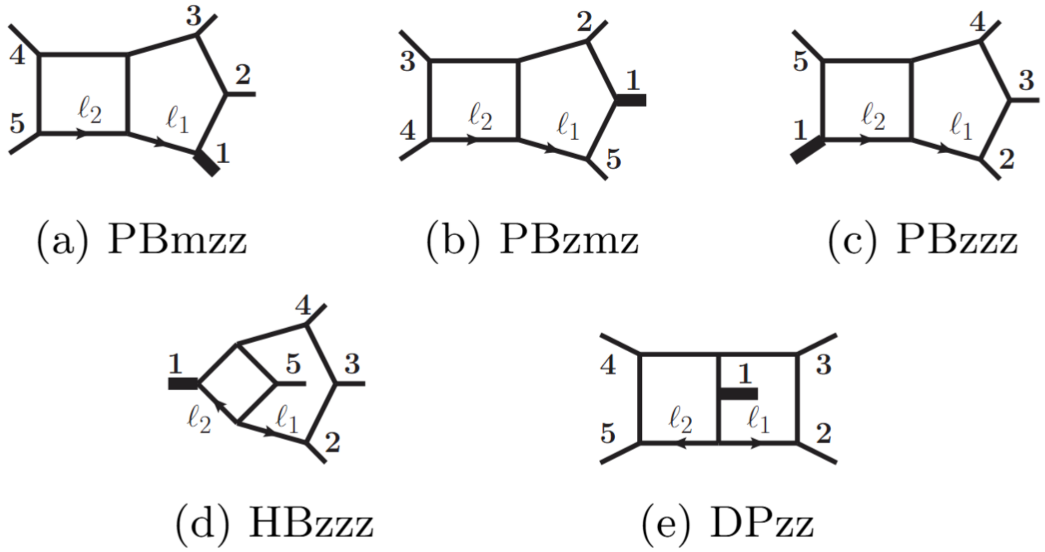
- Our conjecture: these prefactors are enough for all-loop NMHV four-point form factors!

$$\mathcal{R}_4^{(L)} = \sum_{i=1}^{4} \left(R_i\, f_i^{(L)} + \bar{R}_i\, \bar{f}_i^{(L)} \right) + \sum_{i=1}^{2} S_i\, f_{s,i}^{(L)} \,.$$

Dihedral Ansatz: transcendental function basis

- We use the one-mass two-loop pentagon functions as the basis for transcendental functions **[S.Abreu et al, 2306.15431]**
- Calculated from Canonical Differential equations

- In total, there are 989 independent master integrals



(a) PBmzz (b) PBzmz (c) PBzzz

(d) HBzzz (e) DPzz

- We only need finite combinations

- 638 unknowns left for each functions (1914 unknowns in the whole ansatz.)

Fixing the ansatz

$$\mathcal{R}_4^{(2)} = \sum_{ij} \boxed{c_{ij}}\,\boxed{R_i}\,\boxed{\mathcal{I}_j^{(4)}}\,.$$

1914 Coefficients (to be fixed)

Independent prefactors (from one-loop input)

Master integral basis (from one-mass pentagon CDE input)

TABLE I: Numbers of unknowns left under successive bootstrap constraints.

Condition	Unknowns left
Galois and parity	605
Dihedral symmetry	308
Spurious-pole cancellation	88
Collinear limit $p_4 \| p_3$	7
Triple-collinear $p_4 \| p_3 \| p_2$	0
Soft limit	0
Double-soft	0

- We impose the parity, symmetry

spurious-pole cancellation and

multi-collinear limit, fix the ansatz uniquely

- We check our result by soft, double-soft, directional dual conformal symmetry, etc.

Result at three loops via symbol bootstrap

Important observation: two-loop four-point NMHV form factor has 78 letters, all falls in the 88-letter alphabet that previously found for MHV form factors up to four loops

- Prefactors: from one-loop input
- Transcendental functions: from symbol letter input
- In total, there are **163809** symbols at three loops
- We need almost the same set of physical conditions to fix the three-loop result!

Condition	unknowns left
ES-satisfied integrable symbols	54603×3
Galois, parity, and dihedral symmetry	22241
Spurious-pole cancellation and well-defined multi-collinear limits	318
Eliminate redundancy of the ansatz	60
Explicit collinear limit	21
Explicit triple-collinear limit	0
Explicit soft and double-soft limits (check)	0
Directional dual conformal invariance (check)	0

Bootstrapping Six-gluon QCD amplitudes

Planar Six-Parton Scattering Amplitudes

- We calculate QCD helicity scattering amplitudes in leading color approximation under dim-reg from bootstrap strategy

$$\mathcal{A}(1_g, 2_g, 3_g, 4_g, 5_g, 6_g)|_{\text{leading order}} = \sum_{\sigma} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(6)}}) \times A(\sigma(1)_g, \cdots, \sigma(6)_g)$$

- We keep the leading term in the large limit of number of color and number of flavors, while keeping their ratio fixed

$$A_6^{(2)} = A_6^{(2),0} + \frac{N_f}{N_c} A_6^{(2),1} + \left(\frac{N_f}{N_c}\right)^2 A_6^{(2),2}$$

- Subtraction:

$$I_n^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^{n} \left(\frac{-s_{i\,i+1}}{\mu^2}\right)^{-\epsilon}$$

$$H^{(1)} = A^{(1)} - I^{(1)} A^{(0)} \,,$$
$$H^{(2)} = A^{(2)} - I^{(1)} A^{(1)} - I^{(2)} A^{(0)} \,,$$

Leading singularities and prefactors at maximally-transcendental weight

Maximally-transcendental weight part

$$\mathcal{A}_n^{(L)} = \boxed{\sum_{w=0}^{2L}} \sum_{ij} c_{ij}^{(w)} \, R_i^{(w)} \, \mathcal{I}_j^{(w)} \,.$$

Only focus on w=2L in this talk

- **At maximally-transcendental weight part**, **prefactors** (**Leading singularity**) are related to **maximal residues of the integrand**! **[Henn, Torres Bobadilla,21']**

$$\mathcal{A}^{(L)} = \int \omega^{(L)} \,.$$

integrand

$$\mathcal{I}_i^{(L)} = \sum_j b_{ij} \prod_{k=1}^{4L} d\log \alpha_{ijk} \,.$$

dlog-form integrand basis

$$\omega^{(L)} = \sum_{i=1}^{m} c_i \, \mathcal{I}_i^{(L)} + \ldots \,.$$

Maximal weight part + double-pole part in ⋯

$$\mathcal{P}\left(\omega^{(L)}\right) = \sum_{i=1}^{m} c_i \, \mathcal{I}_i^{(L)} \,.$$

Maximal weight projection

“Principle of Maximal Transcendentality”

- Kotikov, Lipatov et *al* noticed that the *‚most complicated terms‘* in terms in QCD twist-two anomalous dimensions were the same as in maximally supersymmetric Yang-Mills. The argument is that gluons give the relevant dominant contribution to the BFKL equation.

[Kotikov, Lipatov, Nucl. Phys. B 661 19 (2003)]
[Kotikov, Lipatov, Onishchenko, Velizhanin, Phys. Lett. B 595, 521 (2004)]

- Similar observations hold for certain form factors and anomalous dimensions; but already one-loop, +-+- helicity amplitudes have a richer structure.

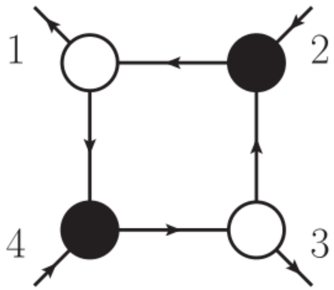
[Gehrmann, JMH, Huber, 1112.4524, JHEP 03 (2012)] [Dixon, 1712.07274, JHEP 01 (2018)]

- This corresponds to keeping the leading transcendental terms in QCD amplitudes only. In this talk, I show how to fix those ‚most complicated terms‘ for two-loop QCD amplitudes, working in the planar limit.

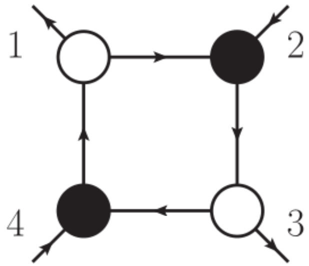
[slide from J.Henn, New Frontiers in Quantum Field and Gravity, 26]

On-shell diagrams and leading singularity

- All non-trivial leading singularities are calculated by **on-shell diagrams** from gluing tree-level amplitudes
- Non-trivial prefactors come from box diagrams with **cycle in its loop**



$$= \frac{\langle 14\rangle^4 \langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle\langle 13\rangle^4}$$

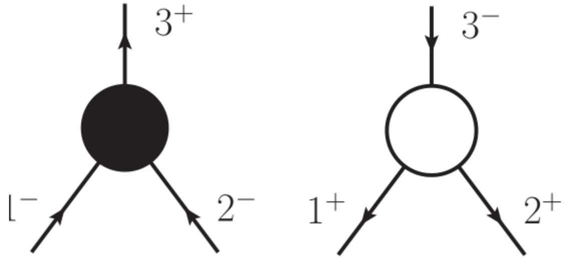


$$= \frac{\langle 12\rangle^4 \langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle\langle 13\rangle^4}$$

All one-loop prefactors at maximal weight part!

$$\boxed{\mathrm{PT}_{n,ij} \times \left(\left(\frac{\langle ik\rangle\langle jl\rangle}{\langle kl\rangle\langle ij\rangle} \right)^4 + \left(\frac{\langle il\rangle\langle jk\rangle}{\langle kl\rangle\langle ij\rangle} \right)^4 \right), \quad 1 \le i < k < j < l \le n}$$

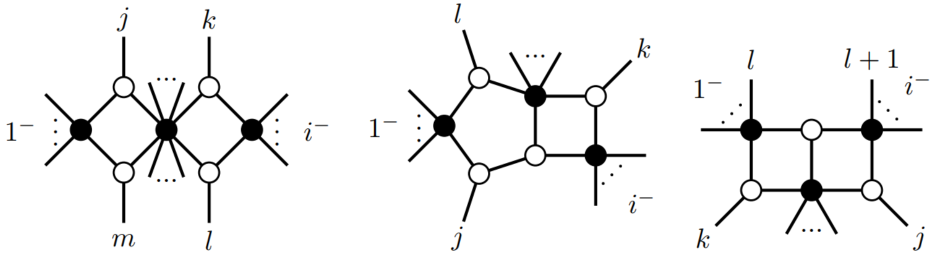
- **Conformal invariants** under action of $\displaystyle\sum_{i=1}^{n} \frac{\partial^2}{\partial \lambda_\alpha^i \partial \tilde{\lambda}_{\dot\alpha}^i}$



[Witten 03';Henn, Power, Zoia, 19']

Two-loop MHV on-shell diagrams and prefactors

- At two loops, we summarize all non-trivial two-loop pure YM MHV on-shell diagrams as three types (Also applicable to general QCD MHV amplitudes!):



(a) Kissing-box (b) Penta-box (c) Double-box

- Naturally simplified by spinor-helicity variables in this calculation!

$$\Big\{R[1]\to-\frac{5}{2},$$

$$R[2]\to-\frac{1}{4\,s_{12}^3\,(s_{12}-s_{34}-s_{45})^3}\big(6 s_{12}^5 s_{15}+15 s_{12}^4 s_{15}^2+10 s_{12}^3 s_{15}^3+6 s_{12}^5 s_{23}+15 s_{12}^4 s_{23}^2+10 s_{12}^3 s_{23}^3-12 s_{12}^4 s_{15} s_{34}-15 s_{12}^3 s_{15}^2 s_{34}-18 s_{12}^4 s_{23} s_{34}-45 s_{12}^3 s_{23}^2 s_{34}-30 s_{12}^2 s_{23}^3 s_{34}+6 s_{12}^3 s_{15} s_{34}^2+18 s_{12}^3 s_{23} s_{34}^2+45 s_{12}^2 s_{23}^2 s_{34}^2+30 s_{12} s_{23}^3 s_{34}^2-6 s_{12}^2 s_{23} s_{34}^3-15 s_{12} s_{23}^2 s_{34}^3-10 s_{23}^3 s_{34}^3-18 s_{12}^4 s_{15} s_{45}-45 s_{12}^3 s_{15}^2 s_{45}-30 s_{12}^2 s_{15}^3 s_{45}-12 s_{12}^4 s_{23} s_{45}-15 s_{12}^3 s_{23}^2 s_{45}+6 s_{12}^4 s_{34} s_{45}+54 s_{12}^3 s_{15} s_{34} s_{45}+60 s_{12}^2 s_{15}^2 s_{34} s_{45}+54 s_{12}^3 s_{23} s_{34} s_{45}+60 s_{12}^2 s_{15} s_{23} s_{34} s_{45}+30 s_{12} s_{15}^2 s_{23} s_{34} s_{45}+60 s_{12}^2 s_{23}^2 s_{34} s_{45}+30 s_{12} s_{15} s_{23}^2 s_{34} s_{45}-12 s_{12}^3 s_{34}^2 s_{45}-36 s_{12}^2 s_{15} s_{34}^2 s_{45}-72 s_{12}^2 s_{23} s_{34}^2 s_{45}-60 s_{12} s_{15} s_{23} s_{34}^2 s_{45}-75 s_{12} s_{23}^2 s_{34}^2 s_{45}-30 s_{15} s_{23}^2 s_{34}^2 s_{45}+6 s_{12}^2 s_{34}^3 s_{45}+30 s_{12} s_{23} s_{34}^3 s_{45}+30 s_{23}^2 s_{34}^3 s_{45}+18 s_{12}^3 s_{15} s_{45}^2+45 s_{12}^2 s_{15}^2 s_{45}^2+30 s_{12} s_{15}^3 s_{45}^2+6 s_{12}^3 s_{23} s_{45}^2-12 s_{12}^3 s_{34} s_{45}^2-72 s_{12}^2 s_{15} s_{34} s_{45}^2-75 s_{12} s_{15}^2 s_{34} s_{45}^2-36 s_{12}^2 s_{23} s_{34} s_{45}^2-60 s_{12} s_{15} s_{23} s_{34} s_{45}^2-30 s_{15}^2 s_{23} s_{34} s_{45}^2+27 s_{12}^2 s_{34}^2 s_{45}^2+60 s_{12} s_{15} s_{34}^2 s_{45}^2+60 s_{12} s_{23} s_{34}^2 s_{45}^2+60 s_{15} s_{23} s_{34}^2 s_{45}^2-15 s_{12} s_{34}^3 s_{45}^2-30 s_{23} s_{34}^3 s_{45}^2-6 s_{12}^2 s_{15} s_{45}^3-15 s_{12} s_{15}^2 s_{45}^3-10 s_{15}^3 s_{45}^3+6 s_{12}^2 s_{34} s_{45}^3+30 s_{12} s_{15} s_{34} s_{45}^3+30 s_{15}^2 s_{34} s_{45}^3-15 s_{12} s_{34}^2 s_{45}^3-30 s_{15} s_{34}^2 s_{45}^3+10 s_{34}^3 s_{45}^3-6 s_{12}^4\,\mathrm{tr5}-15 s_{12}^3 s_{15}\,\mathrm{tr5}-10 s_{12}^2 s_{15}^2\,\mathrm{tr5}-15 s_{12}^3 s_{23}\,\mathrm{tr5}-10 s_{12}^2 s_{15} s_{23}\,\mathrm{tr5}-10 s_{12}^2 s_{23}^2\,\mathrm{tr5}+12 s_{12}^3 s_{34}\,\mathrm{tr5}+15 s_{12}^2 s_{15} s_{34}\,\mathrm{tr5}+30 s_{12}^2 s_{23} s_{34}\,\mathrm{tr5}+10 s_{12} s_{15} s_{23} s_{34}\,\mathrm{tr5}+20 s_{12} s_{23}^2 s_{34}\,\mathrm{tr5}-6 s_{12}^2 s_{34}^2\,\mathrm{tr5}-15 s_{12} s_{23} s_{34}^2\,\mathrm{tr5}-10 s_{23}^2 s_{34}^2\,\mathrm{tr5}+12 s_{12}^3 s_{45}\,\mathrm{tr5}+30 s_{12}^2 s_{15} s_{45}\,\mathrm{tr5}+20 s_{12} s_{15}^2 s_{45}\,\mathrm{tr5}+15 s_{12}^2 s_{23} s_{45}\,\mathrm{tr5}+10 s_{12} s_{15} s_{23} s_{45}\,\mathrm{tr5}-27 s_{12}^2 s_{34} s_{45}\,\mathrm{tr5}-35 s_{12} s_{15} s_{34} s_{45}\,\mathrm{tr5}-35 s_{12} s_{23} s_{34} s_{45}\,\mathrm{tr5}-20 s_{15} s_{23} s_{34} s_{45}\,\mathrm{tr5}+15 s_{12} s_{34}^2 s_{45}\,\mathrm{tr5}+20 s_{23} s_{34}^2 s_{45}\,\mathrm{tr5}-6 s_{12}^2 s_{45}^2\,\mathrm{tr5}-15 s_{12} s_{15} s_{45}^2\,\mathrm{tr5}-10 s_{15}^2 s_{45}^2\,\mathrm{tr5}+15 s_{12} s_{34} s_{45}^2\,\mathrm{tr5}+20 s_{15} s_{34} s_{45}^2\,\mathrm{tr5}-10 s_{34}^2 s_{45}^2\,\mathrm{tr5}\big),$$

$$R[4]\to\frac{1}{6\,s_{12}^3\,s_{34}^3}\big(10 s_{12}^3 s_{15}^3-30 s_{12}^3 s_{15}^2 s_{23}+30 s_{12}^3 s_{15} s_{23}^2-10 s_{12}^3 s_{23}^3-15 s_{12}^3 s_{15}^2 s_{34}+30 s_{12}^3 s_{15} s_{23} s_{34}-15 s_{12}^3 s_{23}^2 s_{34}+6 s_{12}^3 s_{15} s_{34}^2-6 s_{12}^3 s_{23} s_{34}^2+3 s_{12}^3 s_{34}^3-6 s_{12}^2 s_{23} s_{34}^3-15 s_{12} s_{23}^2 s_{34}^3-10 s_{23}^3 s_{34}^3-30 s_{12}^2 s_{15}^3 s_{45}+60 s_{12}^2 s_{15}^2 s_{23} s_{45}-30 s_{12}^2 s_{15} s_{23}^2 s_{45}+60 s_{12}^2 s_{15}^2 s_{34} s_{45}-60 s_{12}^2 s_{15} s_{23} s_{34} s_{45}+30 s_{12} s_{15}^2 s_{23} s_{34} s_{45}-30 s_{12} s_{15} s_{23}^2 s_{34} s_{45}-36 s_{12}^2 s_{15} s_{34}^2 s_{45}-60 s_{12} s_{15} s_{23} s_{34}^2 s_{45}-30 s_{15} s_{23}^2 s_{34}^2 s_{45}+6 s_{12}^2 s_{34}^3 s_{45}+30 s_{12} s_{23} s_{34}^3 s_{45}+30 s_{23}^2 s_{34}^3 s_{45}+30 s_{12} s_{15}^3 s_{45}^2-30 s_{12} s_{15}^2 s_{23} s_{45}^2-75 s_{12} s_{15}^2 s_{34} s_{45}^2+30 s_{12} s_{15} s_{23} s_{34} s_{45}^2-30 s_{15}^2 s_{23} s_{34} s_{45}^2+60 s_{12} s_{15} s_{34}^2 s_{45}^2+60 s_{15} s_{23} s_{34}^2 s_{45}^2-15 s_{12} s_{34}^3 s_{45}^2-30 s_{23} s_{34}^3 s_{45}^2-10 s_{15}^3 s_{45}^3+30 s_{15}^2 s_{34} s_{45}^3-30 s_{15} s_{34}^2 s_{45}^3+10 s_{34}^3 s_{45}^3-10 s_{12}^2 s_{15}^2\,\mathrm{tr5}+20 s_{12}^2 s_{15} s_{23}\,\mathrm{tr5}-10 s_{12}^2 s_{23}^2\,\mathrm{tr5}+15 s_{12}^2 s_{15} s_{34}\,\mathrm{tr5}-15 s_{12}^2 s_{23} s_{34}\,\mathrm{tr5}+10 s_{12} s_{15} s_{23} s_{34}\,\mathrm{tr5}-10 s_{12} s_{23}^2 s_{34}\,\mathrm{tr5}-6 s_{12}^2 s_{34}^2\,\mathrm{tr5}-15 s_{12} s_{23} s_{34}^2\,\mathrm{tr5}-10 s_{23}^2 s_{34}^2\,\mathrm{tr5}+20 s_{12} s_{15}^2 s_{45}\,\mathrm{tr5}-20 s_{12} s_{15} s_{23} s_{45}\,\mathrm{tr5}-35 s_{12} s_{15} s_{34} s_{45}\,\mathrm{tr5}+10 s_{12} s_{23} s_{34} s_{45}\,\mathrm{tr5}-20 s_{15} s_{23} s_{34} s_{45}\,\mathrm{tr5}+15 s_{12} s_{34}^2 s_{45}\,\mathrm{tr5}+20 s_{23} s_{34}^2 s_{45}\,\mathrm{tr5}-10 s_{15}^2 s_{45}^2\,\mathrm{tr5}+20 s_{15} s_{34} s_{45}^2\,\mathrm{tr5}-10 s_{34}^2 s_{45}^2\,\mathrm{tr5}\big),$$

$$R[5]\to-\frac{1}{s_{12}^3\,s_{45}^3}\big(10 s_{12}^3 s_{15}^3-30 s_{12}^3 s_{15}^2 s_{23}+30 s_{12}^3 s_{15} s_{23}^2-10 s_{12}^3 s_{23}^3+30 s_{12}^2 s_{15}^2 s_{23} s_{34}-60 s_{12}^2 s_{15} s_{23}^2 s_{34}+30 s_{12}^2 s_{23}^3 s_{34}+30 s_{12} s_{15} s_{23}^2 s_{34}^2-30 s_{12} s_{23}^3 s_{34}^2+10 s_{23}^3 s_{34}^3+15 s_{12}^3 s_{15}^2 s_{45}-30 s_{12}^3 s_{15} s_{23} s_{45}+15 s_{12}^3 s_{23}^2 s_{45}+60 s_{12}^2 s_{15} s_{23} s_{34} s_{45}+30 s_{12} s_{15}^2 s_{23} s_{34} s_{45}-60 s_{12}^2 s_{23}^2 s_{34} s_{45}-30 s_{12} s_{15} s_{23}^2 s_{34} s_{45}-30 s_{12} s_{15} s_{23} s_{34}^2 s_{45}+75 s_{12} s_{23}^2 s_{34}^2 s_{45}+30 s_{15} s_{23}^2 s_{34}^2 s_{45}-30 s_{23}^2 s_{34}^3 s_{45}+6 s_{12}^3 s_{15} s_{45}^2-6 s_{12}^3 s_{23} s_{45}^2+36 s_{12}^2 s_{23} s_{34} s_{45}^2+60 s_{12} s_{15} s_{23} s_{34} s_{45}^2+30 s_{15}^2 s_{23} s_{34} s_{45}^2-60 s_{12} s_{23} s_{34}^2 s_{45}^2-60 s_{15} s_{23} s_{34}^2 s_{45}^2+30 s_{23} s_{34}^3 s_{45}^2+6 s_{12}^2 s_{15} s_{45}^3+15 s_{12} s_{15}^2 s_{45}^3+10 s_{15}^3 s_{45}^3-6 s_{12}^2 s_{34} s_{45}^3-30 s_{12} s_{15} s_{34} s_{45}^3-30 s_{15}^2 s_{34} s_{45}^3+15 s_{12} s_{34}^2 s_{45}^3+30 s_{15} s_{34}^2 s_{45}^3-10 s_{34}^3 s_{45}^3+10 s_{12}^2 s_{15}^2\,\mathrm{tr5}-20 s_{12}^2 s_{15} s_{23}\,\mathrm{tr5}+10 s_{12}^2 s_{23}^2\,\mathrm{tr5}+20 s_{12} s_{15} s_{23} s_{34}\,\mathrm{tr5}-20 s_{12} s_{23}^2 s_{34}\,\mathrm{tr5}+10 s_{23}^2 s_{34}^2\,\mathrm{tr5}+15 s_{12}^2 s_{15} s_{45}\,\mathrm{tr5}+10 s_{12} s_{15}^2 s_{45}\,\mathrm{tr5}-15 s_{12}^2 s_{23} s_{45}\,\mathrm{tr5}-10 s_{12} s_{15} s_{23} s_{45}\,\mathrm{tr5}-10 s_{12} s_{15} s_{34} s_{45}\,\mathrm{tr5}+35 s_{12} s_{23} s_{34} s_{45}\,\mathrm{tr5}+20 s_{15} s_{23} s_{34} s_{45}\,\mathrm{tr5}-20 s_{23} s_{34}^2 s_{45}\,\mathrm{tr5}+6 s_{12}^2 s_{45}^2\,\mathrm{tr5}+15 s_{12} s_{15} s_{45}^2\,\mathrm{tr5}+10 s_{15}^2 s_{45}^2\,\mathrm{tr5}-15 s_{12} s_{34} s_{45}^2\,\mathrm{tr5}-20 s_{15} s_{34} s_{45}^2\,\mathrm{tr5}+10 s_{34}^2 s_{45}^2\,\mathrm{tr5}\big)\Big\}$$

Conformal invariance!

$$\Big\{1,\ -\frac{(-2\,\mathrm{ab}[1,3]\times\mathrm{ab}[2,4]+\mathrm{ab}[1,2]\times\mathrm{ab}[3,4])\,(10\,\mathrm{ab}[1,3]^2\,\mathrm{ab}[2,4]^2-10\,\mathrm{ab}[1,2]\times\mathrm{ab}[1,3]\times\mathrm{ab}[2,4]\times\mathrm{ab}[3,4]+\mathrm{ab}[1,2]^2\,\mathrm{ab}[3,4]^2)}{\mathrm{ab}[1,2]^3\,\mathrm{ab}[3,4]^3},$$

$$-\frac{(-2\,\mathrm{ab}[1,3]\times\mathrm{ab}[2,5]+\mathrm{ab}[1,2]\times\mathrm{ab}[3,5])\,(10\,\mathrm{ab}[1,3]^2\,\mathrm{ab}[2,5]^2-10\,\mathrm{ab}[1,2]\times\mathrm{ab}[1,3]\times\mathrm{ab}[2,5]\times\mathrm{ab}[3,5]+\mathrm{ab}[1,2]^2\,\mathrm{ab}[3,5]^2)}{\mathrm{ab}[1,2]^3\,\mathrm{ab}[3,5]^3},$$

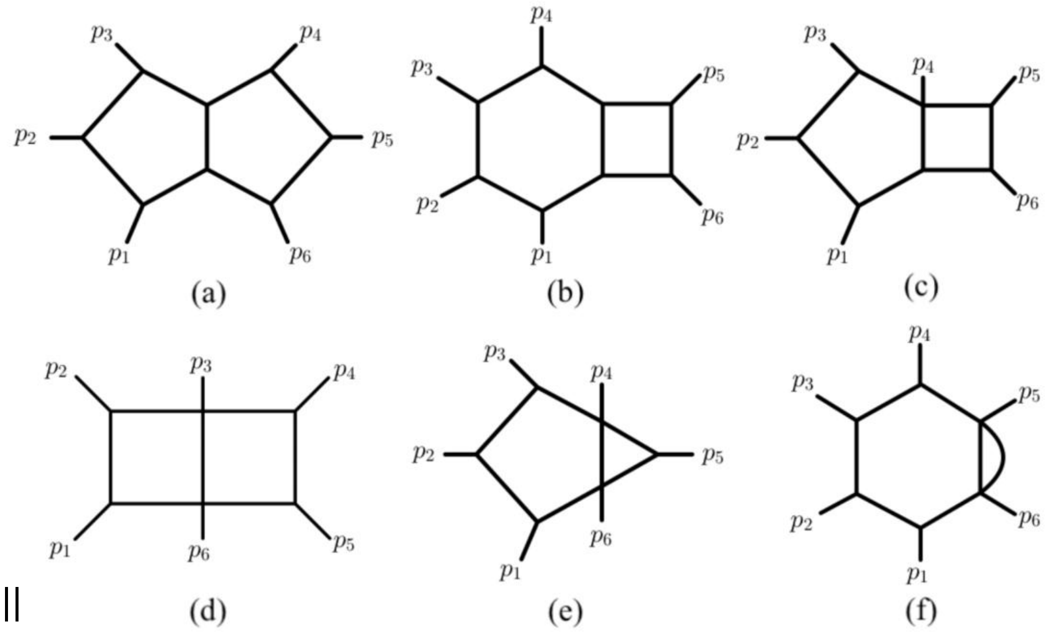
$$-\frac{(-2\,\mathrm{ab}[1,4]\times\mathrm{ab}[2,5]+\mathrm{ab}[1,2]\times\mathrm{ab}[4,5])\,(10\,\mathrm{ab}[1,4]^2\,\mathrm{ab}[2,5]^2-10\,\mathrm{ab}[1,2]\times\mathrm{ab}[1,4]\times\mathrm{ab}[2,5]\times\mathrm{ab}[4,5]+\mathrm{ab}[1,2]^2\,\mathrm{ab}[4,5]^2)}{\mathrm{ab}[1,2]^3\,\mathrm{ab}[4,5]^3}\Big\}$$

Hexagon functions at two loops [Abreu, Monni, Page, Usovitsch, 24'; Henn, Matijašić, Miczajka, Peraro, Xu, Zhang, 25']

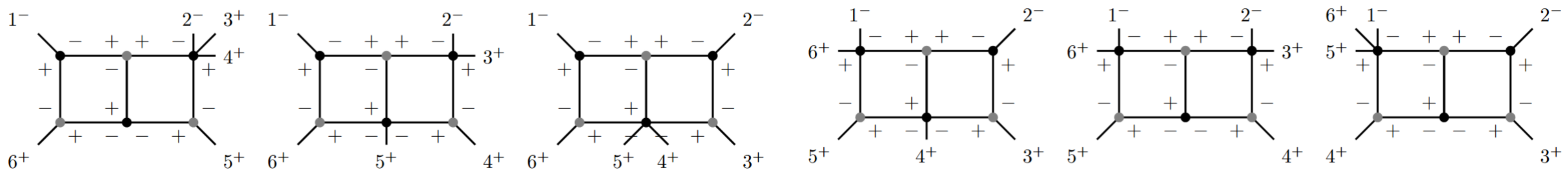
- **Two-loop massless scattering processes with six particles,** State-of-art for Feynman integral calculation
- Up to weight four, **hexagon function space** can be constructed
- **dlog-iterative integrals (4D external legs)**

Transcendental weight	1	2	3	4
# All symbols	9	62	319	$\boxed{945}$
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Basis for all planar six-point two-loop observables, at maximal weight, symbol level



A bootstrap example: pure YM amplitudes with --++++(1/2)



- In this sector, there are **7 linear-independent leading singularities** from on-shell diagrams and **945 hexagon functions (at symbol level)** from CDE

$$H_6^{(2)} = \sum_{i=1}^{7}\sum_{j=1}^{945} \boxed{c_{i,j}}\,\boxed{R_i}\,\boxed{\mathcal{I}_j^{(2)}}$$

Coefficients (to be fixed)

Independent prefactors (from unitarity input)

Master integral basis (from CDE input)

$$R_1 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

$$R_{i,j}/R_1 = -1 + 12u_{i,j} - 30u_{i,j}^2 + 20u_{i,j}^3$$

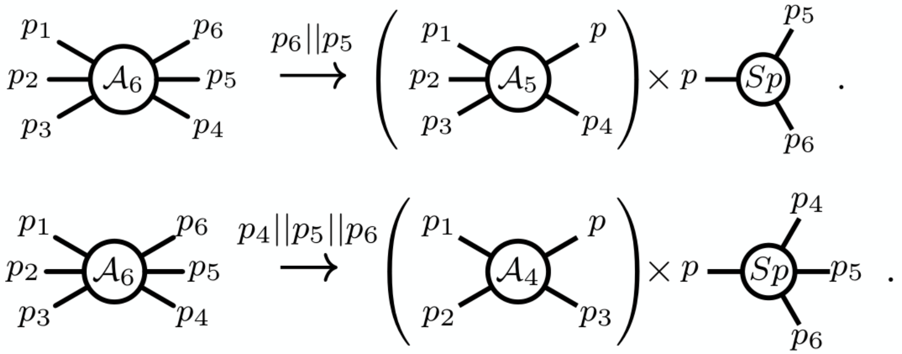
$$u_{i,j} := \langle 1i\rangle\langle 2j\rangle/(\langle 12\rangle\langle ij\rangle)$$

- **2412** genuine unknown coefficients by reflection symmetry and ruling out unnecessary products of one-loop functions

A bootstrap example: pure YM amplitudes with --++++(2/2)

$$\mathcal{H}^{(2)}_{\mathrm{YM}} = R_1 G_1 + \sum_{2<i<j\leq 6} R_{i,j} G_{i,j}\ .$$

We impose cancellations of spurious poles, as well as consistency with soft and collinear limits.



The answer is fixed uniquely!

Table I. Two-loop bootstrap constraints for $\mathcal{H}^{(2)}_{\mathrm{YM}}$

Condition on $\mathcal{H}^{(2)}_{\mathrm{YM}}$	No. of constraints
dimensionless symbols G	996
no spurious/high-order poles	
$\langle 36\rangle = 0$	333
$\langle 35\rangle = 0$	614
$\langle 34\rangle = 0$	629
$\langle 45\rangle = 0$	343
collinear limit $p_5\|\|p_6$	
$--++++$	1785
$+--+++$	1307
$++--++$	1785
$+++--+$	1646
$-++++-$	1646
triple collinear limit $p_4\|\|p_5\|\|p_6$	
$--++++$	1836
$++--++$	724
Total	2412

Similar calculations are performed for other two MHV helicity sectors, as well as Nf terms of gluonic amplitudes in QCD

Maximal transcendentality principle & prescriptive unitarity (1/3)

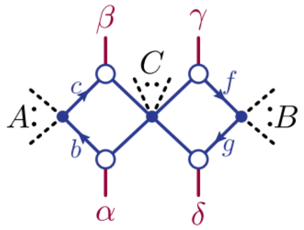
- In the bootstrap result, we find a same set of integrals appear in different helicity sectors
- **Prescriptive unitarity** approach! [Bourjaily,Herrmann,Trnka]

$$H^{(2)}_{--++++} = \mathrm{PT}_{6,12}\, f_0 + \sum_{2<i<j\le 6} R^{(2),\mathrm{db}}_{12,ij1} I^{\mathrm{db}}_{6,ij1}, \tag{6.4}$$

$$\begin{aligned} H^{(2)}_{-+-+++} = {}& \mathrm{PT}_{6,13} g_0 + R^{(1)}_{13,24} g_{2,4} + R^{(1)}_{13,25} g_{2,5} + R^{(1)}_{13,26} g_{2,6} \\ & + \sum_{\substack{l=1,2\\ 3<j<k\le 6}} R^{(2),\mathrm{db}}_{13,jkl} I^{\mathrm{db}}_{6,jkl} + \sum_{\{j,k,l\}\in\sigma_1} R^{(2),\mathrm{pb}}_{13,jkl} (I^{\mathrm{pb}}_{6,jkl} + I^{\mathrm{pb}}_{6,jlk}) , \end{aligned} \tag{6.5}$$

$$\begin{aligned} H^{(2)}_{-++-++} = {}& \mathrm{PT}_{6,14} h_0 + R^{(1)}_{14,25} h_{2,5} + R^{(1)}_{14,26} h_{2,6} + R^{(1)}_{14,35} h_{3,5} + R^{(1)}_{14,36} h_{3,6} \\ & + R^{(2),\mathrm{kb}}_{14,2356} I^{\mathrm{kb}}_{6,2356} + \sum_{1\le l\le 3} R^{(2),\mathrm{db}}_{14,56l} I^{\mathrm{db}}_{6,56l} + \sum_{4\le l\le 6} R^{(2),\mathrm{db}}_{41,23l} I^{\mathrm{db}}_{6,23l} \\ & + \sum_{\{j,k,l\}\in\sigma_2} R^{(2),\mathrm{pb}}_{14,jkl} (I^{\mathrm{pb}}_{6,jkl} + I^{\mathrm{pb}}_{6,jlk}) , \end{aligned} \tag{6.9}$$

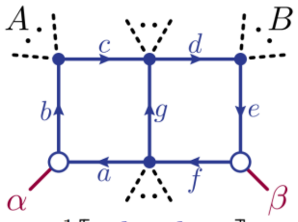
kissing boxes



$\mathcal{I}_{\mathrm{kb}}[\alpha,\beta,\gamma,\delta,A,B,C]$

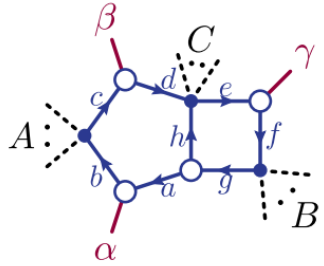
$\mathfrak{n} := [\![\alpha,b,c,\beta]\!][\![\gamma,f,g,\delta]\!]$

double boxes



$\mathfrak{n} := \frac{1}{2}[\![\alpha,b,c,d,e,\beta]\!]$

pentabox



$\mathcal{I}_{\mathrm{pb}}[\alpha,\beta,\gamma,A,B,C]$

$\mathfrak{n} := -[\![\alpha,b,c,\beta]\!][\![\gamma,f,g,a]\!] + \frac{1}{2}[\![\alpha,b,c,\beta,\gamma,f,g,a]\!] + \mathfrak{n}^{\mathrm{c}}_{\mathrm{pb}}$

[Bourjaily, Herrmann, Langer, McLeod, Trnka, 19']

Analytic structure at ALL multiplicity for two-loop MHV amplitudes!

Maximal transcendentality principle & prescriptive unitarity (2/3)

- In the bootstrap result, if we replace all prefactors by 1 or 0 directly, we recover the six-point two-loop hard function for N=4 sYM

$$H^{(2)}_{h_1\cdots h_6} \xrightarrow[R^{(2),\mathrm{db}}\to 0]{\mathrm{PT},\quad R^{(1)},\quad R^{(2),\mathrm{pb}},\quad R^{(2),\mathrm{kb}}\to 1} \left(H^{(2)}_6\right)^{\mathcal{N}=4\ \mathrm{sYM}}. \tag{6.21}$$

- For different helicity sectors, essentially, pure YM and N=4 sYM are formed by the same set of function basis, but with differential coefficients

Generalization of the principle of maximal transcendentality!

Maximal transcendentality principle & prescriptive unitarity (3/3)

- We also see the correspondence between four-point form factor and Effective Higgs-four-gluon amplitudes [De Laurentis, Ita, Kuschke, Ruf, Sotnikov, 26']

$$\mathcal{H}^{(L)}_{++++} = \mathcal{H}^{(0)}_{++++}\, F_4^{(L)}, \qquad L = 1, 2.$$

$$\mathcal{H}^{(2)}_{--++} = \mathcal{H}^{(0)}_{--++}\, F_4^{(2)} + R_{\mathrm{db},H}\, f^{(2)}_{\mathrm{db},1} + \overline{R}_{\mathrm{db},H}\, f^{(2)}_{\mathrm{db},2}.$$

$$\mathcal{H}^{(1)}_{+++-} = \sum_{i=1}^{4} \widetilde{R}_i f^{(1)}_{+,i} + \sum_{i=1}^{2} \widetilde{S}_i f^{(1)}_{s,i} + \widetilde{R}_5\, \widetilde{f}^{(1)}_5, \qquad \mathcal{H}^{(2)}_{+++-}\Big|_{\mathrm{alg}} = \widetilde{S}_1 f^{(2)}_{s,1} + \widetilde{S}_2 f^{(2)}_{s,2},$$

Summary

- Two key simplification in our cases are

1. Symbol bootstrap based on function basis instead of IBP reducing the integrand

2. Leading singularities classified by calculation from on-shell methods

   (Nice properties for prefactors are always expected at maximal transcendental weight part)

- How to determine the prefactors at lower weight parts & rational terms?
- Uplifting to function level, which allows us to explore various properties (positivity, complete monotonicity **[Henn, Raman]** ..)

Thanks!