# ***Bootstrapping Gauge Theory Observables with Lattice Path Combinatorics***

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# Motivation

- ✔ Computing observables in gauge theories at finite coupling remains a challenging task
- ✔ Powerful techniques (localization, integrability) have been developed for supersymmetric theories but a complete solution is still out of reach
- ✔ However, there exists a broad class of observables in four-dimensional $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal Yang–Mills theories for which the problem becomes tractable
- ✔ These observables admit a unified representation as Fredholm determinants of semi-infinite (Bessel) matrices

$$D(g,s_{ij}) \sim \det_{1\le n,m<\infty}\Big(\delta_{nm} - K_{nm}(g,s_{ij})\Big)$$

The goal of this talk is to explain connections between (seemingly unrelated) areas:

- ✗ Fredholm determinants
- ✗ Iterated Chen integrals/symbols
- ✗ Enumerative combinatorics of Dyck paths

# Four-point correlation functions in $\mathcal{N} = 4$ SYM

- Half-BPS operators

$$O_1 = \mathrm{tr}(Z^{K/2}\bar{X}^{K/2}) + \text{permutations}\,, \qquad O_2 = \mathrm{tr}(X^{K/2}Z^{K/2}) + \dots$$

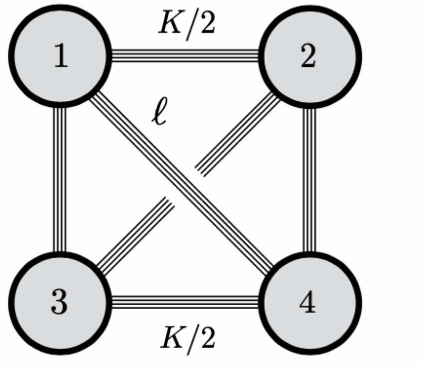
$$O_3 = \mathrm{tr}(\bar{Z}^{K/2}\bar{Y}^{K/2}) + \dots \qquad O_4 = \mathrm{tr}(Y^{K/2}\bar{Z}^{K/2}) + \dots$$

Exact scaling dimension (R-charge) $\Delta = K$

Two– and three–point functions are protected

- "Simplest" four-point function

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \frac{\mathcal{G}_K(u,v)}{(x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2)^{K/2}} = \sum_{\ell \geq 0}$$




Depends on 't Hooft coupling $g^2 = g_{\mathrm{YM}}^2 N_c/(4\pi)^2$ and cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

# Weak coupling expansion

✔ Conventional Feynman diagram approach is inefficient:

The contribution of each individual diagram is gauge dependent

A lot of contributing diagrams... but their sum is remarkably simple

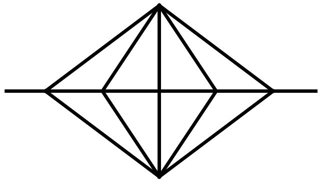
✔ More efficient (bootstrap) approach [Eden,Heslop,GK,Sokatchev]

- ✗ Construct the basis of conformal integrals
- ✗ Expand $\mathcal{G}_K$ over this basis and use the additional conditions to fix the coefficients

$$\mathcal{G}_K = \sum_\ell (g^2)^\ell \times [\text{conformal integrals at } \ell \text{ loops}]$$

Basis of conformal integrals:

- ✗ 'simple' ladder integrals

$f_L =$ 

$$= \frac{1}{z-\bar z}\sum_{m=0}^{L}\frac{(-1)^m(2L-m)!}{L!(L-m)!m!}\ln^m(z\bar z)\Big[\underbrace{\mathrm{Li}_{2L-m}(z)}_{\text{polylog}}-\mathrm{Li}_{2L-m}(\bar z)\Big]$$

- ✗ complicated integrals (a few pages of Goncharov's polylogs)

# Infinitely heavy operators

For finite $K$ the four-point correlation function $\mathcal{G}_K(u,v)$ is known at three loops

Examine the limit $K \to \infty$ (infinitely heavy operators) with $g^2$ kept fixed

Key observations:

- ✔ complicated integrals only appear at $(K+1)-$loops
- ✔ Infinitely heavy operator limit:

$$\lim_{K\to\infty} \mathcal{G}_K = \sum_{\ell\geq 0} [\mathbb{O}_\ell(z,\bar{z})]^2$$

- ✔ $\mathbb{O}_\ell(z,\bar{z})$ is a multilinear combination of *ladder* integrals [Coronado]

$$\mathbb{O}_{\ell=0}(z,\bar{z}) = 1 + g^2 f_1 - 2g^4 f_2 + 6g^6 f_3 + g^8(-20 f_4 - \tfrac{1}{2} f_2^2 + f_1 f_3) + \dots$$

$$= 1 + \sum_{n\geq 1} (g^2)^n \times \sum_{i_1+\dots+i_n=n} d_{i_1\dots i_n} f_{i_1}\dots f_{i_n} \qquad \text{(for general } \ell\text{)}$$

The expansion coefficients $d_{i_1\dots i_n}$ can be found to all loops from different OPE limits of $\mathcal{G}_K$

- ✔ The weak coupling expansion can be resummed to all orders in the coupling

# Octagon in planar $\mathcal{N} = 4$ SYM

Limit of infinitely heavy operators

$$\lim_{K\to\infty} \mathcal{G}_K = \sum_{\ell\geq 0} [\mathbb{O}_\ell(z,\bar z)]^2$$

$\mathbb{O}(z,\bar z) =$ 'octagon' is a multilinear combination of ladder integrals [Coronado]

$$\mathbb{O}_\ell(z,\bar z) = 1 + \sum_{n\geq 1} (g^2)^n \times \sum_{i_1+\dots+i_m=n} d_{i_1\dots i_m} f_{i_1}\dots f_{i_m}$$

For finite 't Hooft coupling, the octagon admits an *exact* representation

[Kostov,Petkova,Serban],[Belitsky,GK]

$$\mathbb{O}_\ell(z,\bar z) = \det_{1\leq n,m<\infty} (\delta_{nm} - K_{nm}(g))$$

$$= 1 - \operatorname{tr} K(g) + \frac{1}{2}\left[\operatorname{tr}(K^2(g)) - (\operatorname{tr} K(g))^2\right] + \dots$$

Fredholm determinant of a semi-infinite (Bessel) matrix $K(g)$ depending on cross ratios $z$ and $\bar z$

At weak coupling, $\operatorname{tr}(K^n(g)) = O(g^{2n})$ is a multilinear combination of ladder integrals

Similar det–representation has been found for other observables in SYM theories

# Truncated Bessel matrix

Across different observables, the primary object is

$$D_\ell(g) = \det\left(\delta_{nm} - K_{nm}(g)\right)\Big|_{1\le n,m<\infty}$$

The matrix elements admit an integral representation in terms of Bessel functions $J_n(x)$

$$K_{nm}(g) = \int_0^\infty dx\, \psi_n(x)\chi\Big(\frac{\sqrt{x}}{2g}\Big)\psi_m(x)$$

$$\psi_n(x) = \sqrt{2n+\ell-1}\,\frac{J_{2n+\ell-1}(\sqrt{x})}{\sqrt{x}}$$

The determinant depends on the coupling $g$, the bridge length $\ell$ and the *symbol* of the matrix $\chi(x)$

For generic symbol $\chi(x)$, the function $D_\ell(g)$ is known as generalized Tracy-Widom distribution

Symbol for the octagon

$$\chi\,(x) = \frac{\cosh y + \cosh \xi}{\cosh y + \cosh(\sqrt{x^2+\xi^2})}$$

Kinematical variables : $\quad z = -\,\mathrm{e}^{-y-\xi}\,, \qquad \bar{z} = -\,\mathrm{e}^{+y-\xi}$

# Why Fredholm determinants?

- What is an advantage of having a determinant representation of the correlation function?

$$\mathbb{O}_\ell(z,\bar{z}) = D_\ell(g) = \det\left(\delta_{nm} - K_{nm}(g)\right)\Big|_{1\le n,m<\infty}$$

- The Fredholm determinants of the Bessel matrix have previously appeared in the study of integrable models
- Their appearance in planar $\mathcal{N}=4$ SYM is not surprising since the theory is believed to be integrable
- Weak coupling expansion of the determinant reproduces ladder integral expansion of the correlator
- Strong coupling expansion is in agreement with AdS/CFT expectations
- Exploit integrable properties of the Fredholm determinants to compute the correlation functions for finite 't Hooft coupling $g=\sqrt{\lambda}$

# Master equation

*For arbitrary symbol function $\chi(x)$, the determinant satisfies a differential-difference equation*

$$g\partial_g \log\left(\frac{D_{\ell+1}}{D_{\ell-1}}\right) = 2\ell\left(\frac{D_\ell^2}{D_{\ell-1}D_{\ell+1}} - 1\right)$$

For the special choice $\chi(x) = -1/\sinh^2(\frac{x}{2})$, emerged in the study of correlation functions in $\mathcal{N}=2$ SYM using integrability [Ferrando,Komatsu,Lefundes,Serban] and localization [GK,Testa]

Supplemented with the boundary condition at weak coupling

$$D_\ell(g) = 1 + O(g^{2(\ell+1)})$$

it allows for a recursive determination of the function $D_{\ell+n}(g)$ for arbitrary $n \geq 1$, e.g.

$$D_{\ell+1}(g) = 2\ell D_{\ell-1}(g)\int_0^1 dx\, x^{2\ell-1}\left(\frac{D_\ell(xg)}{D_{\ell-1}(xg)}\right)^2$$

The function $D_{\ell+n}(g)$ can be expressed in terms of $D_{\ell-1}(g)$ and $D_\ell(g)$

Due to nonlinearity of the integrand, there is little hope of finding a closed-form expression, unless...

# Some simplifications

Examine separately even and odd $n \geq 0$ and introduce the ratios (up to normalizaton factors)

$$\frac{D_{\ell+2n-1}}{D_{\ell-1}} = \frac{d_{2n}(g)}{g^{2n(n+\ell-1)}}\,, \qquad\qquad \frac{D_{\ell+2n}}{D_{\ell}} = \frac{d_{2n+1}(g)}{g^{2n(n+\ell)}}\,,$$

The functions $d_n(g)$ and $d'_n = \partial_g d_n$ satisfy the equations $(d_0 = d_1 = 1)$

$$d_{2n} d'_{2n+2} - d_{2n+2} d'_{2n} = d^2_{2n+1} \, (\log f_+)'$$

$$d_{2n-1} d'_{2n+1} - d_{2n+1} d'_{2n-1} = d^2_{2n} \, (\log f_-)'$$

The functions $f_{\pm}(g)$ depend on the initial conditions $D_{\ell-1}(g)$ and $D_{\ell}(g)$

$$d \log f_+(g) = g^{2\ell-1} \frac{2D^2_{\ell}(g)}{D^2_{\ell-1}(g)} dg\,,$$

$$d \log f_-(g) = g^{1-2\ell} \frac{2D^2_{\ell-1}(g)}{D^2_{\ell}(g)} dg$$

The solutions for $d_{2n}$ and $d_{2n+1}$ can be expanded over the basis of iterated (Chen) integrals built from $d\, f_{\pm}(g)$

# First attempt

Start with $d_2(g)$

$$d_2(g) = \int_0^g dg_1 (\log f_+(g_1))' = \int_0^g d\log f_+(g_1)$$

Continue with $d_3(g)$

$$\begin{aligned} d_3(g) &= \int_0^g d\log f_-(g_1)\, d_2^2(g_1) \\ &= 2\int_0^g d\log f_-(g_1) \int_0^{g_1} d\log f_+(g_2) \int_0^{g_2} d\log f_+(g_3) \end{aligned}$$

Examine $d_4(g)$

$$d_4(g) = d_2(g)\int_0^g dg_1 \frac{d_3^2(g_1)}{d_2^2(g_1)} (\log f_+(g_1))' = d_2(g)\int_0^g dg_1 \frac{d_3^2(g_1)}{d_2^2(g_1)} d_2'(g_1)$$

Integrate by parts

$$d_4(g) = -d_3^2(g) + 2d_2(g)\int_0^g dg_1\, d_2(g_1) d_3(g_1)(\log f_-(g_1))' \,.$$

Nonlinearity disappeared and $d_4(g)$ can be expanded into a linear combination of iterated integrals!

## Iterated (Chen) integrals

The integrals $I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$ depend on a sequence of signs $\sigma_i = \pm$

They are constructed by integrating the product of the derivatives $(\log f_\pm(g))'$

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) = \int_0^g d\log f_{\sigma_1}(g_1) \int_0^{g_1} d\log f_{\sigma_2}(g_2) \cdots \int_0^{g_{k-1}} d\log f_{\sigma_k}(g_k)$$

Satisfy a recurrence relation

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) = \int_0^g d\log f_{\sigma_1}(g_1)\, I_{\sigma_2\cdots\sigma_k}(g_1)$$

Their product can be expanded into a linear combination of integrals using the shuffle product

$$I_{\sigma_1\sigma_2\cdots\sigma_k}(g) I_{\sigma_{k+1}\cdots\sigma_{k+m}}(g) = \sum_{\sigma' \in (k,m)\ \text{shuffles}} I_{\sigma'_1\sigma'_2\cdots\sigma'_{k+m}}(g)$$

Example

$$I_+(g) I_{-++}(g) = 3I_{-+++}(g) + I_{+-++}(g)$$

The total number of '$+$' and '$-$' entries on both sides is preserved

# The use of the iterated integrals

So far we obtained

$$d_2(g) = I_+(g)$$

$$d_3(g) = 2I_{-++}(g)$$

$$d_4(g) = 12I_{+--+++}(g) + 4I_{+-+-++}(g)$$

*A general solution for $d_n$ is given by linear combinations of $I_{\sigma_1\sigma_2\ldots}$ with integer positive coefficients*

$$d_n(g) = \sum c_{\sigma_1\sigma_2\ldots\sigma_k} I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$$

*The sum runs over sequences $(\sigma_1\sigma_2\ldots\sigma_k)$ of length $k = n(n-1)/2$*

The number of '$+$' and '$-$' entries depends on the parity of $n$

$$n = 2p: \qquad k_+ = p^2\,, \qquad k_- = p(p-1)\,,$$

$$n = 2p+1: \qquad k_+ = p(p+1)\,, \qquad k_- = p^2\,.$$

The equations for $d_n(g)$ lead to an overdetermined linear system for the coefficients $c_{\sigma_1\sigma_2\ldots\sigma_k}$

The excess of equations over unknowns grows rapidly with $n$

The resulting expression for $d_5(g)$ is

$$\begin{aligned} d_5(g) = 16\Big( & I_{-+-+-++-++} + I_{-++-+-+-++} + I_{-+-++-+-++} \\ & + 3I_{-+-+-+-+++} + 3I_{-+-++--+++} + 3I_{-++-+--+++} + 3I_{-++--++-++} \\ & + 6I_{-+-+--++++} + 9I_{-++--+-+++} + 18I_{-++---++++}\Big) \end{aligned}$$

The total number of terms equals $(k_+ + k_-)!/(k_+!k_-!) = 210$, but numerous $c-$coefficients vanish

For different $n$, the total number of coefficients and the number of non-zero coefficients are

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| Total | 1 | 3 | 15 | 210 | 5005 | 293930 |
| Non-zero | 1 | 1 | 2 | 10 | 120 | 3276 |

These numbers admit a simple interpretation in terms of path counting on a 2d square lattice lattice

# Solution for $n = 6$

$d_6(g) = 64$

$$\times \big(30I_{+--+-+-+--+++++} + 18I_{+--+-+-+-+-++++} + 9I_{+--+-+-+-++-+++} + 3I_{+--+-+-+-+++-++}$$
$$+ 24I_{+--+-+-++--++++} + 12I_{+--+-+-++-+-+++} + 4I_{+--+-+-++-++-++} + 9I_{+--+-+-+++--+++}$$
$$+ 3I_{+--+-+-+++-+-++} + 90I_{+--+-++---+++++} + 54I_{+--+-++--+-++++} + 27I_{+--+-++--++-+++}$$
$$+ 9I_{+--+-++--+++-++} + 42I_{+--+-++-+--++++} + 21I_{+--+-++-+-+-+++} + 7I_{+--+-++-+-++-++}$$
$$+ 12I_{+--+-++-++--+++} + 4I_{+--+-++-++-+-++} + 54I_{+--+-+++---++++} + 27I_{+--+-+++--+-+++}$$
$$+ 9I_{+--+-+++--++-++} + 9I_{+--+-+++-+--+++} + 3I_{+--+-+++-+-+-++} + 90I_{+--++--+--+++++}$$
$$+ 54I_{+--++--+-+-++++} + 27I_{+--++--+-++-+++} + 9I_{+--++--+-+++-++} + 72I_{+--++--++--++++}$$
$$+ 36I_{+--++--++-+-+++} + 12I_{+--++--++-++-++} + 27I_{+--++--+++--+++} + 9I_{+--++--+++-+-++}$$
$$+ 270I_{+--++-+---+++++} + 162I_{+--++-+--+-++++} + 81I_{+--++-+--++-+++} + 27I_{+--++-+--+++-++}$$
$$+ 96I_{+--++-+-+--++++} + 48I_{+--++-+-+-+-+++} + 16I_{+--++-+-+-++-++} + 21I_{+--++-+-++--+++}$$
$$+ 7I_{+--++-+-++-+-++} + 72I_{+--++-++---++++} + 36I_{+--++-++--+-+++} + 12I_{+--++-++--++-++}$$
$$+ 12I_{+--++-++-+--+++} + 4I_{+--++-++-+-+-++} + 540I_{+--+++----+++++} + 324I_{+--+++---+-++++}$$
$$+ 162I_{+--+++---++-+++} + 54I_{+--+++---+++-++} + 162I_{+--+++--+--++++} + 81I_{+--+++--+-+-+++}$$
$$+ 27I_{+--+++--+-++-++} + 27I_{+--+++--++--+++} + 9I_{+--+++--++-+-++} + 54I_{+--+++-+---++++}$$
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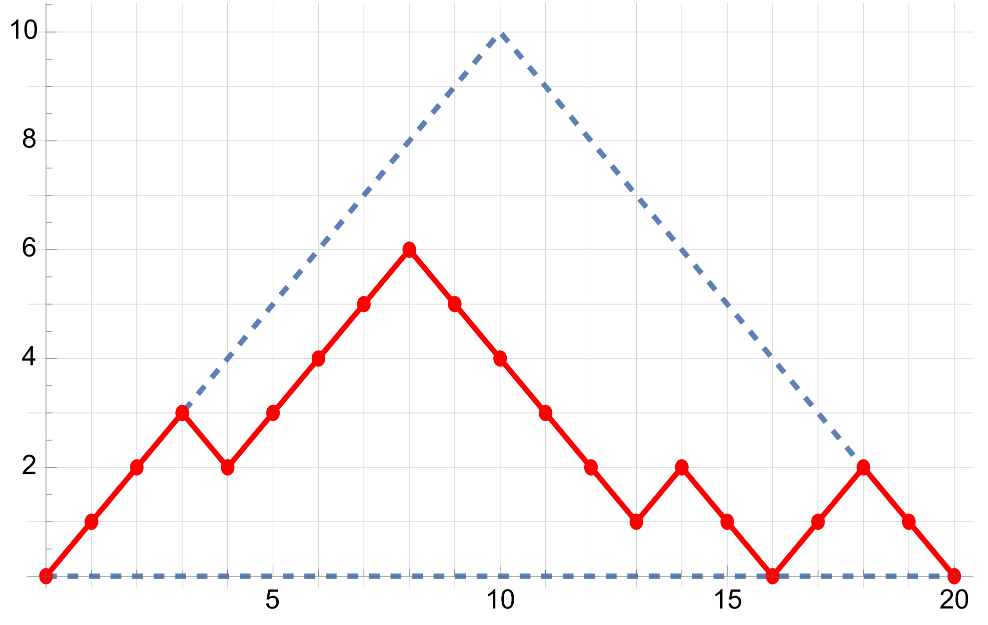
$$+ 3I_{+-+--+-+++-+-++} + 90I_{+-+--++---+++++} + 54I_{+-+--++--+-++++} + 27I_{+-+--++--++-+++}$$
$$+ 9I_{+-+--++--+++-++} + 42I_{+-+--++-+--++++} + 21I_{+-+--++-+-+-+++} + 7I_{+-+--++-+-++-++}$$
$$+ 12I_{+-+--++-++--+++} + 4I_{+-+--++-++-+-++} + 54I_{+-+--+++---++++} + 27I_{+-+--+++--+-+++}$$
$$+ 9I_{+-+--+++--++-++} + 9I_{+-+--+++-+--+++} + 3I_{+-+--+++-+-+-++} + 30I_{+-+-+--+--+++++}$$
$$+ 18I_{+-+-+--+-+-++++} + 9I_{+-+-+--+-++-+++} + 3I_{+-+-+--+-+++-++} + 24I_{+-+-+--++--++++}$$
$$+ 12I_{+-+-+--++-+-+++} + 4I_{+-+-+--++-++-++} + 9I_{+-+-+--+++--+++} + 3I_{+-+-+--+++-+-++}$$
$$+ 90I + 54I + 27I + 9I$$

# Dyck path

A particular example of a lattice path




Starts at the origin $(0, 0)$, ends on the $x-$axis, and never dips below it

Assign signs '$+$' and '$-$' to the up- and down-steps, respectively

A path can be represented as a sequence $(\sigma_1\sigma_2 \ldots \sigma_k)$, where $\sigma_i = \pm$ corresponds to the $i$th step:

$$(+++-++++-----+--++--)$$

# Relation to lattice paths

General solution

$$d_n(g) = \sum c_{\sigma_1\sigma_2\ldots\sigma_k} I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$$

The iterated integrals $I_{\sigma_1\sigma_2\cdots\sigma_k}(g)$ depend on the initial conditions $D_{\ell-1}(g)$ and $D_\ell(g)$

The coefficients $c_{\sigma_1\sigma_2\ldots\sigma_k}$ are *positive integers* and *universal*: they depend only on nonnegative integer $n$

Q: *Is it possible to construct $d_n(g)$ without performing any explicit calculations?*

Main idea: interpret each term in the sum as corresponding to a path on the square lattice, uniquely determined by the sequence $(\sigma_1\sigma_2\cdots\sigma_k)$

*The function $d_n(g)$ is a partition function (or generating function in enumerating combinatorics) of an ensemble of lattice paths confined to a nontrivial domain*

# Warm up example

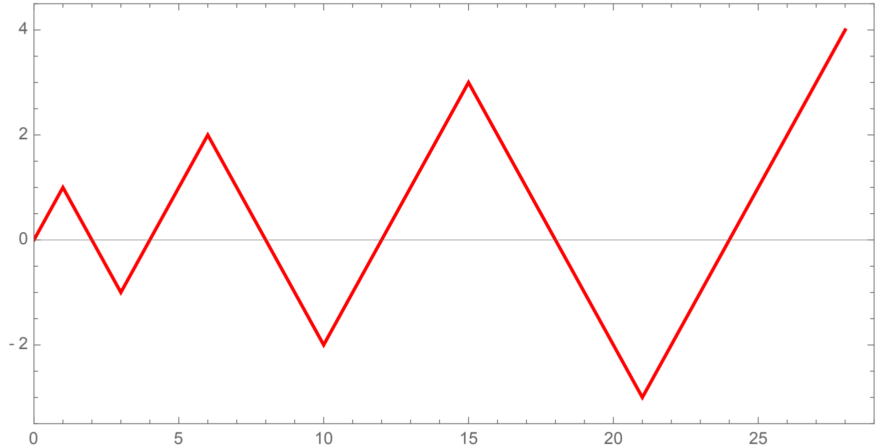
Examples of paths for $d_n$ at $n = 8$:

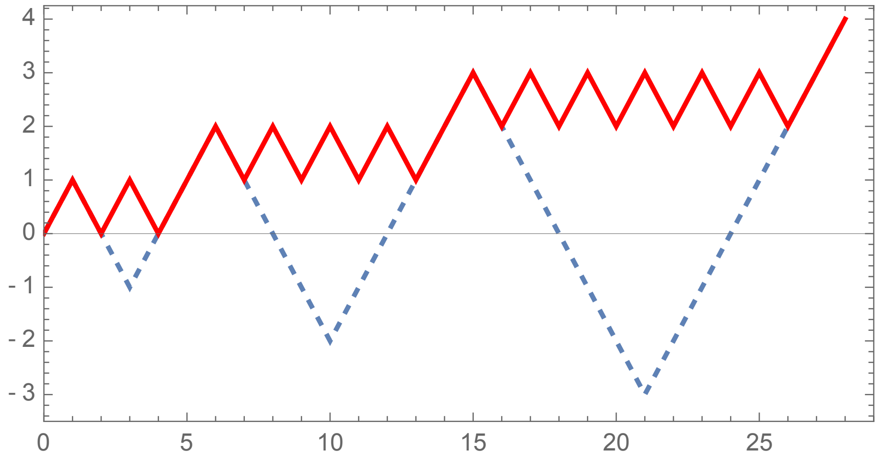
$$c_{+--+++----+++++------+++++++}$$

$$c_{+-+-++-+-+-+-++-+-+-+-+-+-++}$$

Total length $k = n(n-1)/2 = 28$, number of '$+$' and '$-$' is $k_+ = 16$ and $k_- = 12$

The corresponding paths are




For all coefficients in $d_8$, the paths begin at the origin $(0, 0)$ and terminate at the same point

$$p_n = (n(n-1)/2, (-1)^n \lfloor n/2 \rfloor)$$

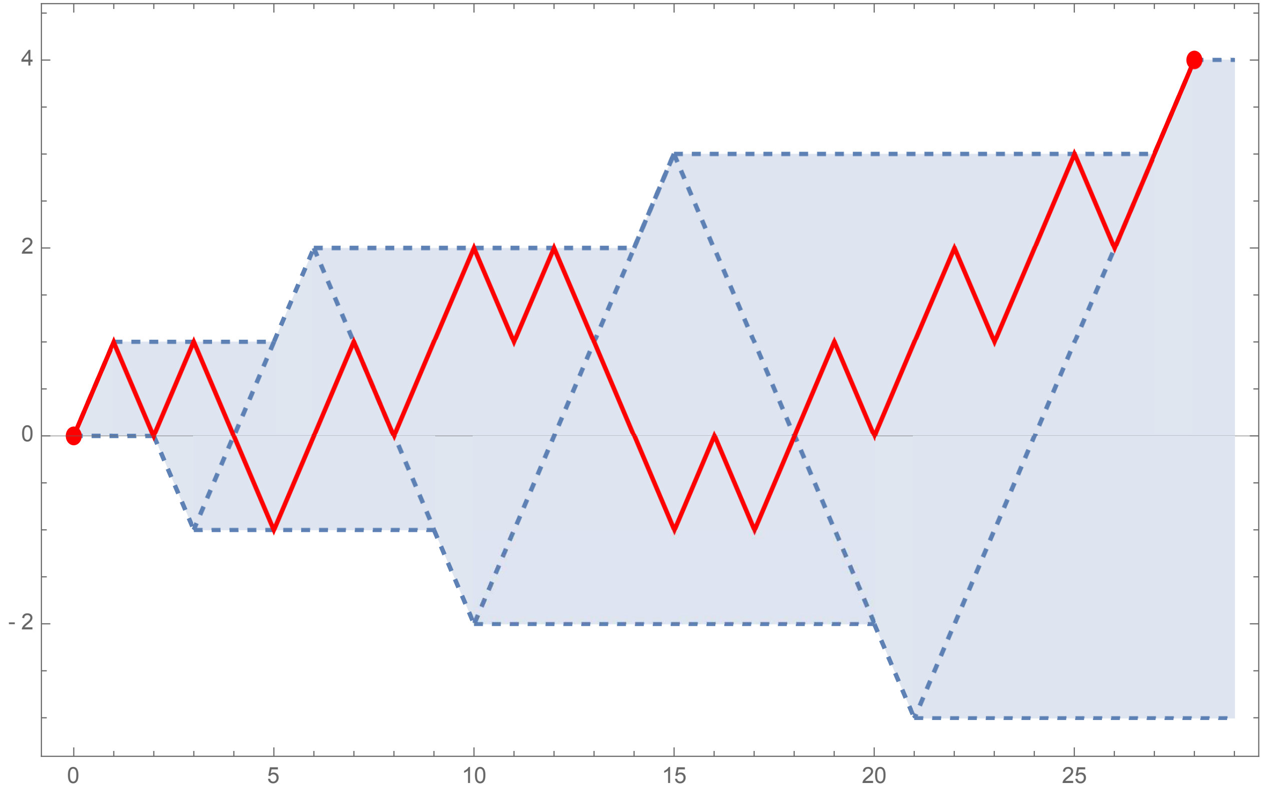
regardless of the order in which the '$+$' and '$-$' signs appear in the sequence.

# Admissible paths

The total number of possible coefficients is $(k_+ + k_-)!/(k_+!k_-!)$

The number of nonzero coefficients is significantly smaller:

*All admissible paths must terminate at the point $p_n$ and remain entirely within the envelope defined by the shaded region. Paths that violate this rule do not contribute.*



## Symbols of symbols

Various quantities in weakly coupled gauge theories are given by iterated (polylog) integrals

$$D(g, s_{ij}) = 1 + g^2 D^{(1)}(s_{ij}) + g^4 D^{(2)}(s_{ij}) + \dots$$

$$D^{(n)}(s_{ij}) = \sum c_{a_1 a_2 \dots a_n} \int d\log w_{a_1} \int d\log w_{a_2} \cdots \int d\log w_{a_n}$$

Symbol map

$$\mathcal{S}[D^{(n)}] = \sum c_{a_1 a_2 \dots a_L} w_{a_1} \otimes w_{a_2} \otimes \cdots \otimes w_{a_L}$$

The letters of the alphabet $w_{a_i}$ are functions of the kinematical invariants $s_{ij}$ and $c_{a_1 a_2 \dots a_L}$ are rational functions

*Take-home message:*

- ✔ For finite coupling, the correlator can be written as a (finite) sum of iterated integrals of the form

$$D(g, s_{ij}) = \sum c'_{\sigma_1 \sigma_2 \sigma_3 \dots} \int_0^g d\log f_{\sigma_1}(g_1) \int_0^{g_1} d\log f_{\sigma_2}(g_2) \int_0^{g_2} d\log f_{\sigma_3}(g_3) \cdots$$

- ✔ The letters of the new alphabet are themselves iterated integrals depending on $s_{ij}$

## Conclusions

- ✔ A broad class of observables in four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal Yang-Mills theories are computable in the planar limit at finite 't Hooft coupling
- ✔ These observables admit a representation as Fredholm determinants of semi-infinite Bessel matrices and satisfy a universal differential-difference equation
- ✔ This powerful equation allows for the recursive determination of its solutions in terms of iterated Chen integrals
- ✔ The symbol representation has a surprisingly simple interpretation in terms of enumerative combinatorics: it defines the partition function (or generating function) of an ensemble of lattice paths

$$\text{[Observables]} \quad \Longrightarrow \quad \text{[Symbology]} \quad \Longrightarrow \quad \text{[Enumerative combinatorics]}$$