Solving strong QCD dynamics with Gauge Theory Bootstrap

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Bethe Forum "Bootstrap Approaches to Scattering Amplitude", Bonn 2026.9

initial idea & test
- [YH, Kruczenski, Phys.Rev.Lett.133,191601, Phys.Rev.D.110,096001]
- [YH, Kruczenski, arXiv: 2403.10772]
- [YH, Kruczenski, PTEP 2026 6, 063B05]

m_q: [*to appear* with Nick Evans, Wanxiang Fan, Martin Kruczenski]

N_c: [*to appear* with Rafael Cordoba, Martin Kruczenski, Derek Ping, Jignesh Mohanty, Emil Alegria]

Low energy physics of asymptotically free gauge theory

asymptotically free $SU(N_c)$ gauge theory with

N_f massive quarks $m_q \ll \Lambda_{\mathrm{QCD}}$ in the fundamental representation of gauge group

confinement & chiral symmetry breaking

$$\mathcal{L} = i \sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \text{gauge fixing} + \text{ghost}$$

microscopic parameters: $N_c \quad N_f \quad m_q \quad \Lambda_{\mathrm{QCD}}$

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microscopic parameters: $N_c \quad N_f \quad m_q \quad \Lambda_{\mathrm{QCD}}$

What is the low energy physics?

How does the low energy dynamics depend on the UV parameters?

Physics of Goldstone bosons

chiral symmetry breaking

explicitly — *dynamically*

quark mass $m_q \bar{q}_j q_j$ $\qquad SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \qquad$ *condensate* $\langle \bar{q}_j q_j \rangle$

pseudo-Goldstone bosons dominate the low energy physics

e.g. $N_f = 2$ pions $\quad \pi_0 = \pi^3 \quad \pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$

NLSM

$$\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\} \qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$$

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{1}{2} m_\pi^2 \vec{\pi}^2 \qquad \mathcal{L}_2^{4\pi} = \frac{1}{6 f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24 f_\pi^2}(\vec{\pi}^2)^2 \quad \ldots$$

The EFT approach

non-renormalizable, add new terms with unknown coefficients:

e.g.

$$
\begin{aligned}
\mathcal{L}_4 \;=\; & \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger] \\
& + \frac{l_3}{16}[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu U(D^\mu \chi)^\dagger + D_\mu \chi(D^\mu U)^\dagger] \\
& + l_5\left[\mathrm{Tr}(f^R_{\mu\nu} U f^{\mu\nu}_L U^\dagger) - \frac{1}{2}\mathrm{Tr}(f^L_{\mu\nu} f^{\mu\nu}_L + f^R_{\mu\nu} f^{\mu\nu}_R)\right] \\
& + i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu} D^\mu U(D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U] \\
& - \frac{l_7}{16}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2
\end{aligned}
$$

χPT: unknown coefficients determined from fitting amplitude with experimental data

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χ**PT**: unknown coefficients determined from fitting amplitude with experimental data

in principle should be computed from UV gauge theory

The strong coupling problem

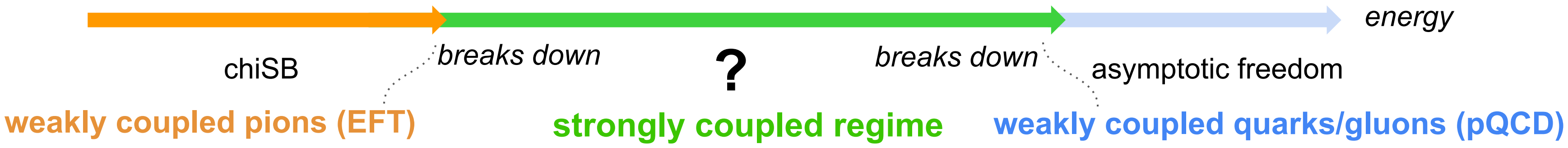
chiSB

asymptotic freedom

weakly coupled pions (EFT)

weakly coupled quarks/gluons (pQCD)

The strong coupling problem



The strong coupling problem → Gauge Theory Bootstrap

energy

chiSB *breaks down* **?** *breaks down* asymptotic freedom

weakly coupled pions (EFT) **strongly coupled regime** **weakly coupled quarks/gluons (pQCD)**

compute the strongly coupled hadron dynamics:
amplitudes, form factors, correlation functions, spectrum/couplings

Gauge Theory Bootstrap

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Gauge Theory Bootstrap

rules:

assume — *chiral symmetry breaking & confinement*

input — $\underbrace{N_c \quad m_q \quad N_f \quad \alpha_s}_{\textit{defining gauge theory}}$

m_π set the energy unit

f_π size of pion

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N_f=2, N_c=3, physical m_q: QCD

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N_f=2, N_c=3, physical m_q: QCD

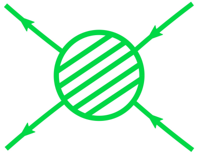
theoretical/numerical computation:

- not using experimental scattering data as input
- no assumption on spectrum

2-to-2 amplitude

$|\psi_1\rangle = |p_1, a; p_2, b\rangle_{\text{in}}$ $\quad$ $|\psi_2\rangle = |p_1, a; p_2, b\rangle_{\text{out}}$ $\quad$ $|\psi_3\rangle = \mathcal{O}^a_\ell(p)|0\rangle$

[Karateev, Kuhn, Penedones, 2019]



${}_{\text{out}}\langle p_3, c|$ $\quad$ $|p_1, a\rangle_{\text{in}}$

${}_{\text{out}}\langle p_4, d|$ $\quad$ $|p_2, b\rangle_{\text{in}}$

$$ {}_{out}\langle p_3, c; p_4, d|p_1, a; p_2, b\rangle_{in} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc} $$

Lorentz invariants: $s = (p_1 + p_2)^2$ $\quad$ $t = (p_1 - p_3)^2$ $\quad$ $u = (p_1 - p_4)^2$

CoM energy: $\sqrt{s}$ $\quad$ $p_i^2 = m_\pi^2$ $\quad$ $s + t + u = 4m_\pi^2$

2-to-2 amplitude

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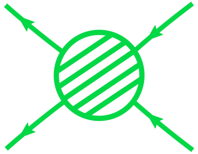

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Lorentz invariants: $\quad s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2 \quad u = (p_1 - p_4)^2$

CoM energy: $\sqrt{s}$ $\qquad p_i^2 = m_\pi^2 \qquad s + t + u = 4m_\pi^2$

alternatively, consider 2-particle irreps states

$$|p_1, a; p_2, b\rangle \to |p, I, \ell\rangle$$

isospin

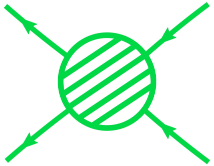
angular momentum

$$_{\text{out}}\langle p', I', \ell'|p, I, \ell\rangle_{\text{in}} = S^I_\ell(s)(2\pi)^4\delta^{(4)}(p - p')\delta^{II'}\delta_{\ell\ell'}$$

phase shift

2-pion partial amplitude $\quad S^I_\ell(s) = \eta^I_\ell(s)\, e^{2i\delta^I_\ell(s)}, \quad \forall \ell, I, s > 4m_\pi^2$

${}_{\text{out}}\langle p, I, \ell|$



$|p, I, \ell\rangle_{\text{in}}$

2-pion form factor and 2-point correlation functions

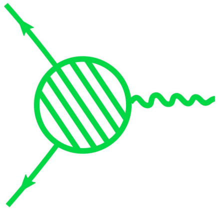
$$|\psi_1\rangle = |p_1, a; p_2, b\rangle_{\text{in}} \qquad |\psi_2\rangle = |p_1, a; p_2, b\rangle_{\text{out}} \qquad |\psi_3\rangle = \mathcal{O}^a_\ell(p)|0\rangle$$

[Karateev, Kuhn, Penedones, 2019]

2-pion form factor

$${}_{\text{out}}\langle p, I, \ell|\mathcal{O}^I_\ell(0)|0\rangle = F^I_\ell(s) \qquad s = p^2$$

${}_{\text{out}}\langle p, I, \ell|$



$\mathcal{O}^I_\ell|0\rangle$

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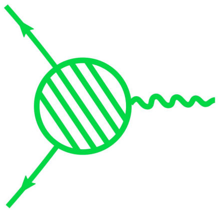


$\mathcal{O}^I_\ell|0\rangle$

2-point correlation function

$$\langle 0|\mathcal{O}^{I\dagger}_\ell(p)\mathcal{O}^I_\ell(p)|0\rangle = \rho^I_\ell(s) \qquad s = p^2$$

$\langle 0|\mathcal{O}^{I\,\dagger}_\ell$

$\mathcal{O}^I_\ell|0\rangle$

time-ordered:

$$\Pi^I_\ell(s) = i\int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x}\langle 0|\hat{T}\left\{\mathcal{O}^{I\dagger}_\ell(x)\mathcal{O}^I_\ell(0)\right\}|0\rangle$$

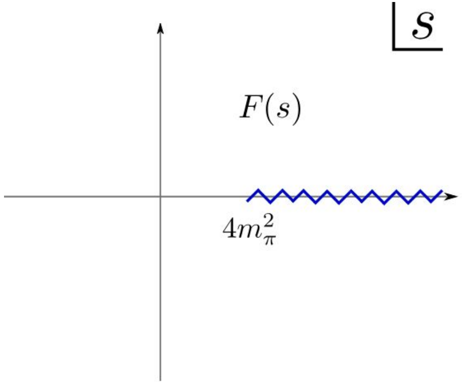
Generic constraints: analyticity

Analyticity $A(s,t,u)$ $s+t+u=4m_\pi^2$ analytic function of two variables

(causality) cuts $s,t,u>4m_\pi^2$

cuts: multi-particle states

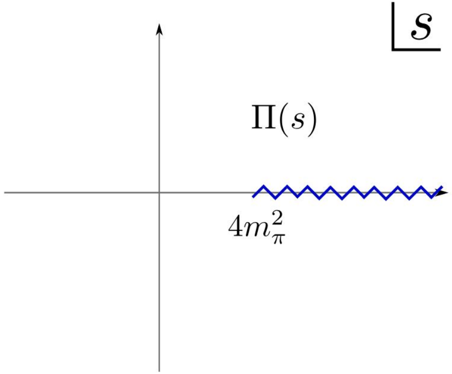
$F(s)$



analytic function

cuts $s>4m_\pi^2$

$\Pi(s)$



$\rho(s)=2\mathrm{Im}\Pi(s+i\epsilon)$

$s>4m_\pi^2$

Generic constraints: crossing and unitarity

Crossing (exchange symmetry) $$A(s,t,u) = A(s,u,t)$$

Unitarity (probability conservation)

$$\langle\psi_a|\psi_b\rangle = \begin{array}{cc} & \begin{array}{ccc} |p,I,\ell\rangle^{\pi\pi}_{\text{out}} & |p,I,\ell\rangle^{\pi\pi}_{\text{in}} & \mathcal{O}^I_\ell|0\rangle \end{array} \\ \begin{array}{c} {}^{\pi\pi}_{\text{out}}\langle p,I,\ell| \\ {}^{\pi\pi}_{\text{in}}\langle p,I,\ell| \\ \langle 0|\mathcal{O}^{I\,\dagger}_\ell \end{array} & \begin{bmatrix} 1 & S^I_\ell(s) & \mathcal{F}^I_\ell(s) \\ S^{I*}_\ell(s) & 1 & \mathcal{F}^{I*}_\ell(s) \\ \mathcal{F}^{I*}_\ell(s) & \mathcal{F}^I_\ell(s) & \rho^I_\ell(s) \end{bmatrix} \end{array} \succeq 0 \qquad \forall \ell, I, s > 4m_\pi^2$$

positive semidefinite matrix

[Karateev, Kuhn, Penedones, 2019]

Non-perturbative parametrizations

$$\rho_2(x,y) = \rho_2(y,x)$$

$$A(s,t,u) = \frac{1}{\pi^2}\int_{4m_\pi^2}^{\infty} dx \int_{4m_\pi^2}^{\infty} dy \left[\frac{\rho_1(x,y)}{(x-s)(y-t)} + \frac{\rho_1(x,y)}{(x-s)(y-u)} + \frac{\rho_2(x,y)}{(x-t)(y-u)}\right]$$ +subtractions

Analyticity & Crossing

$$F_\ell^I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dx \frac{\mathrm{Im}F_\ell^I(x)}{x-s}$$ +subtractions

$\rho_\ell^I(s)$ *supported at* $s > 4m_\pi^2$

$$\begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$$

Unitarity

subject to

parameters: $\{\rho_1(x,y), \rho_2(x,y) = \rho_2(y,x), \mathrm{Im}F_\ell^I(x), \rho_\ell^I(x), \dots\}, \ \forall I, \ell, x, y \in (4m_\pi^2, \infty)$

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numerics: discretize $\longrightarrow$ bootstrap variables

If we can solve these variables, we can use them to reconstruct the full analytic functions, we know these observables at all energies, i.e. along the whole RG flow

S-matrix bootstrap: old

Bootstrap (pre-QCD): solve the theory of **strong interaction** *from these constraints* [Chew… , 1960s]

Symmetry+Analyticity+Crossing+Unitarity

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convex space of generic amplitudes

Symmetry+Analyticity+Crossing+Unitarity

i.e. space of constrained bootstrap variables $\{\rho_{1,2}(x,y),\ldots\}$

generic, no dynamical info about the interaction

$\Downarrow$

infinitely many solutions

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maximize linear functional

extremal amplitudes

convex optimization

modern S-matrix bootstrap: bound physical quantities

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]

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recent pheno application:

fit experimental scattering data [Guerrieri, Haring, Su, 2025]

pion physics application: bound low energy parameters

[Guerrieri, Penedones, Vieira, 2018 & 2020]

large N context (meromorphic amplitude):

[Albert, Rastelli, 2022] [Albert, Henriksson, Rastelli, Vichi, 2023] [Fernandez, Pomarol, Riva, Sciotti, 2023] [Ma, Pomarol, Sciotti, 2023]

Gauge Theory Bootstrap: the philosophy

instead of putting generic bounds

back to the original motivation of bootstrap:

<u>*solve*</u> the theory of strong interaction

<u>compute strong QCD dynamics</u>

using the dynamical UV information of gauge theory

Gauge Theory Bootstrap: the philosophy

instead of putting generic bounds

back to the original motivation of bootstrap:

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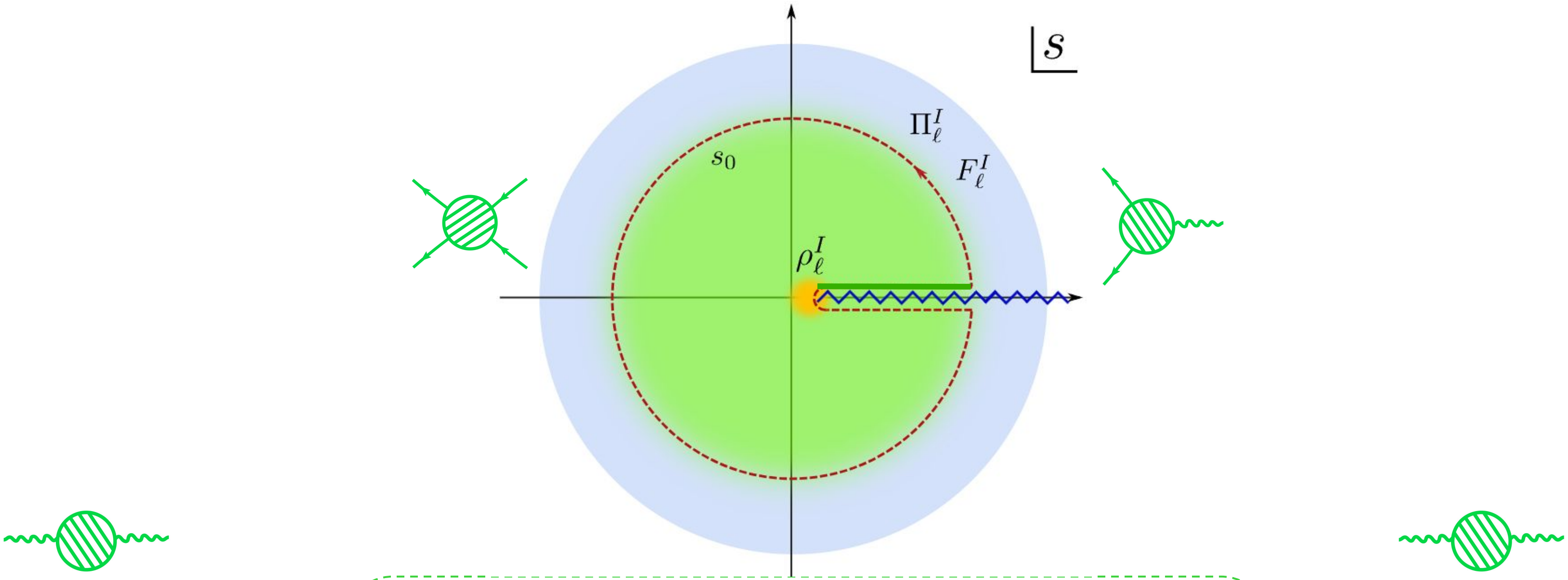
compute strong QCD dynamics

using the dynamical UV information of gauge theory

- naive expectation: given enough theoretical UV input, should find unique solution (within errors)
- by varying UV input, study how the low energy strongly dynamics vary with microscopic parameters

closer to first-principle computation of strongly coupled dynamics of QCD

Gauge Theory Bootstrap: the recipe



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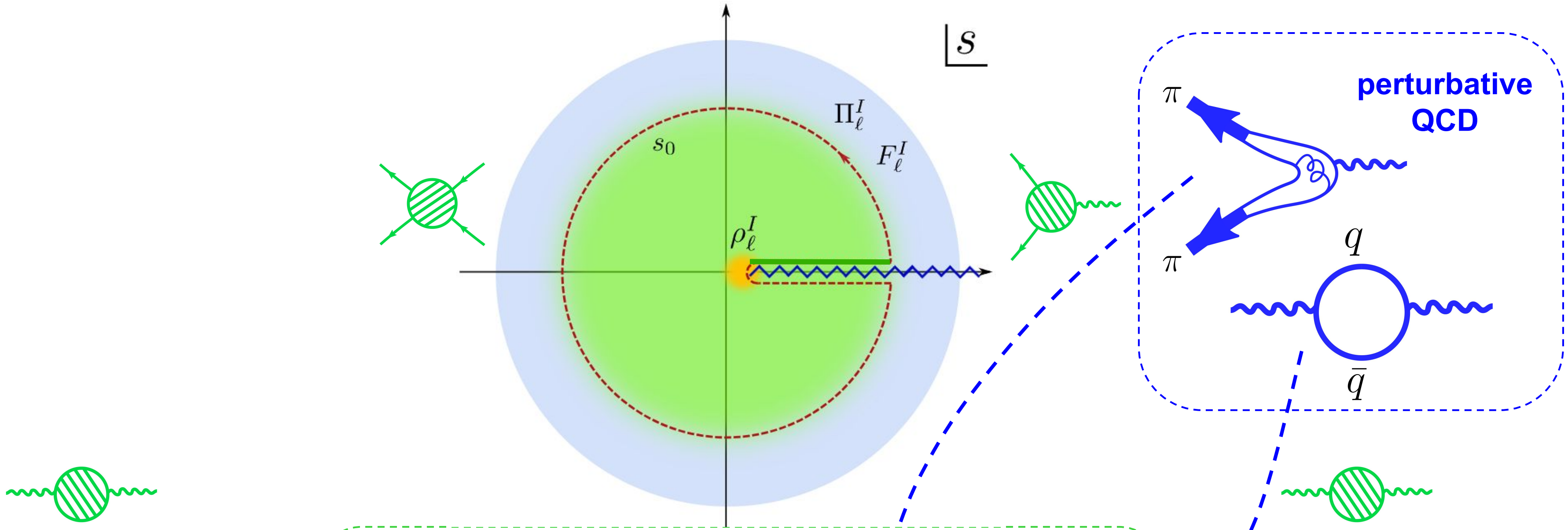
Analyticity, Crossing, Unitarity

Gauge Theory Bootstrap: the recipe



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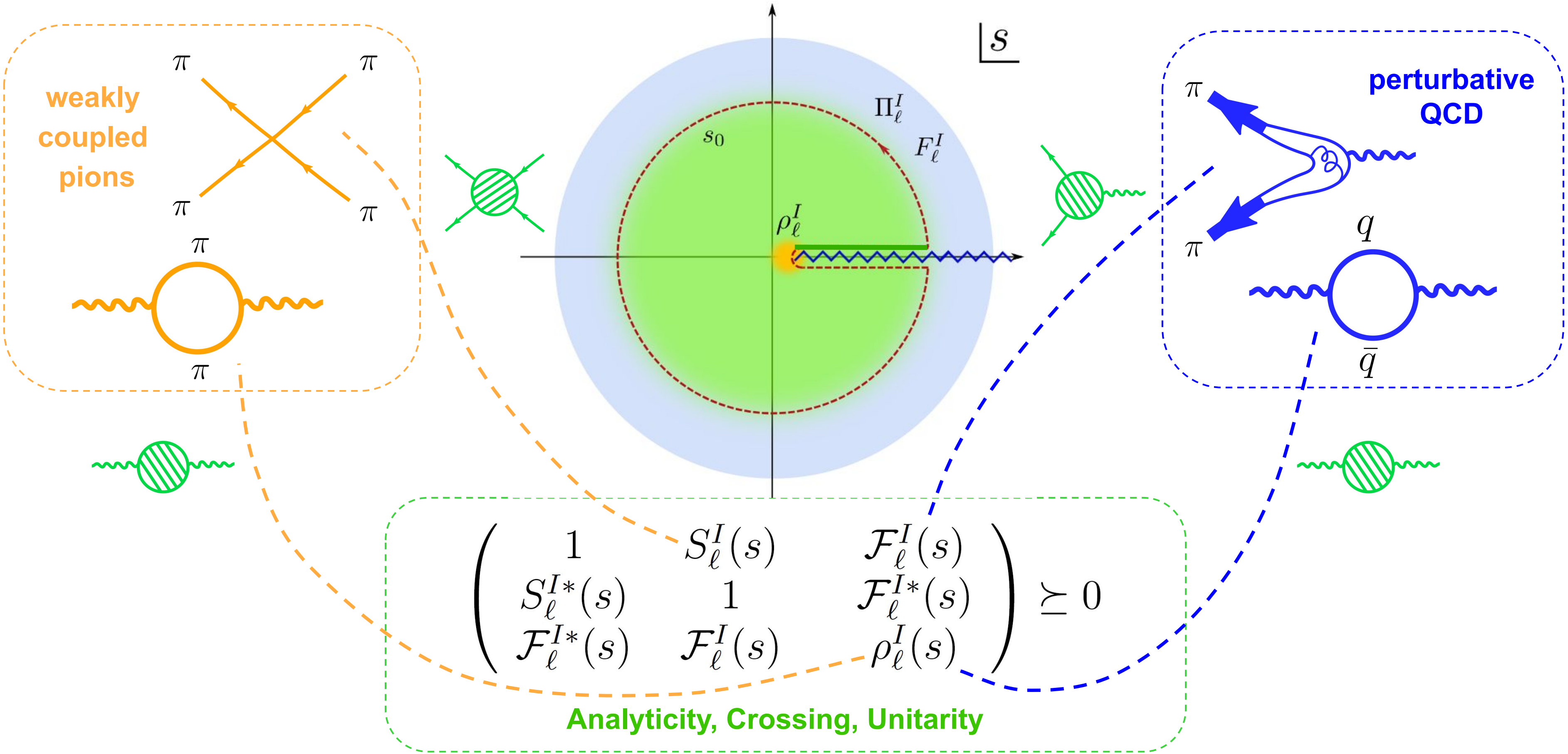
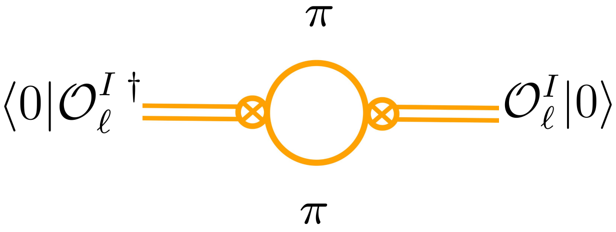


IR limit: free pion current correlator

Free pion Lagrangian $$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{1}{2} m_\pi^2 \vec{\pi}^2$$



e.g.

scalar current $I = 0, \ell = 0$ $$\mathcal{O}_{\ell=0}^{I=0} \simeq \frac{1}{2} m_\pi^2 \pi^a \pi^a + \mathcal{O}(\pi^4)$$

SU(2) vector current $I = 1, \ell = 1$ $$\mathcal{O}_{\ell=1}^{I=1} \simeq \epsilon^{abc} \pi^b \partial_\mu \pi^c + \mathcal{O}(\pi^4)$$

energy-momentum tensor $I = 0, \ell = 2$ $$\mathcal{O}_{\ell=2}^{I=0} \simeq \partial_\mu \pi^a \partial_\nu \pi^a - \frac{1}{2}(\partial_\alpha \pi^a \partial^\alpha \pi^a - m_\pi^2 \pi^a \pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$$

$\vdots$

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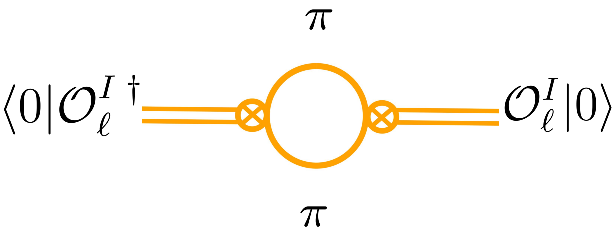

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energy-momentum tensor $I=0, \ell=2$ $\quad \mathcal{O}_{\ell=2}^{I=0} \simeq \partial_\mu \pi^a \partial_\nu \pi^a - \frac{1}{2}(\partial_\alpha \pi^a \partial^\alpha \pi^a - m_\pi^2 \pi^a \pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$

$\vdots$

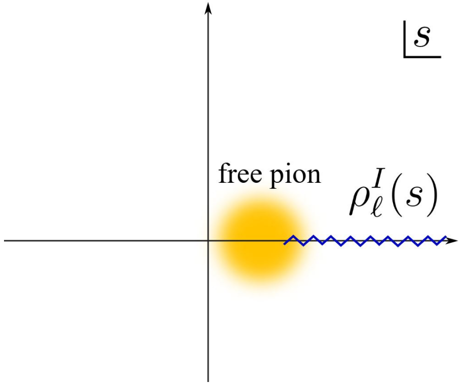
leading low energy behavior of spectral density

[Gasser, Leutwyler, 1983]

$$\rho_0^0(s) \simeq \frac{m_\pi^4}{(2\pi)^4} \frac{3}{16\pi} \left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{1}{2}}$$

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4} \frac{s}{24\pi} \left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

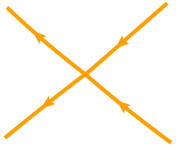
$$\rho_2^0(s) \simeq \frac{1}{(2\pi)^4} \frac{s^2}{160\pi} \left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{5}{2}}$$



Numerics: parameterize the spectral density with this low energy threshold behavior

IR limit: tree-level amplitude

interaction: $$\mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2$$

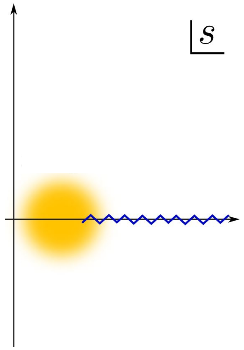
$_{\mathrm{out}}\langle p, I, \ell|$  $|p, I, \ell\rangle_{\mathrm{in}}$

tree-level amplitude: $$A_{\mathrm{tree}}(s,t,u) = \frac{4}{\pi}\frac{s-m_\pi^2}{32\pi f_\pi^2}$$ [Weinberg, 1966]

S0: $f^0_{0,\mathrm{tree}}(s) = \frac{2}{\pi}\frac{2s-m_\pi^2}{32\pi f_\pi^2}$ P1: $f^1_{1,\mathrm{tree}}(s) = \frac{2}{\pi}\frac{s-4m_\pi^2}{96\pi f_\pi^2}$ S2: $f^2_{0,\mathrm{tree}}(s) = \frac{2}{\pi}\frac{2m_\pi^2-s}{32\pi f_\pi^2}$

$$S_\ell^I(s) = 1 + i\pi\sqrt{\frac{s-4m_\pi^2}{s}}\,f_\ell^I(s)$$

good in unphysical region (very low energy) $0 < s < 4m_\pi^2$



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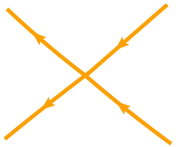
${}_{\text{out}}\langle p, I, \ell|$  $|p, I, \ell\rangle_{\text{in}}$

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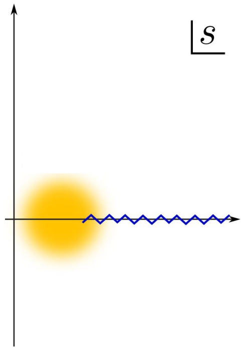

Numerics: require partial waves match this at very low energy

$$\frac{f_0^2(s)}{f_1^1(s)} \simeq \frac{3(2m_\pi^2-s)}{s-4m_\pi^2} \qquad \frac{f_0^0(s)}{f_1^1(s)} \simeq \frac{3(2s-m_\pi^2)}{s-4m_\pi^2}$$

no f_π input here

pion coupling will be computed

UV limit: current correlator at high energy

QCD Lagrangian $$\mathcal{L} = i\sum_{j}^{N_f} \bar{q}_j \not{D} q_j - \sum_{j}^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu}$$

e.g.

scalar current $I = 0, \ell = 0$ $$\mathcal{O}^{I=0}_{\ell=0} = m_q(\bar{u}u + \bar{d}d)$$

SU(2) vector current $I = 1, \ell = 1$ $$\mathcal{O}^{I=1}_{\ell=1} = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$$

energy-momentum tensor $I = 0, \ell = 2$ $$\mathcal{O}^{I=0}_{\ell=2} = \bar{u}\, i\gamma^{(\mu} D^{\nu)} u + \bar{d}\, i\gamma^{(\mu} D^{\nu)} d + G^{a\,\mu\lambda} G^{a\,\nu}{}_{\lambda} + g^{\mu\nu}\mathcal{L}_{\rm QCD}$$

$\vdots$

UV limit: current correlator at high energy

QCD Lagrangian $$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu}$$

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scalar current $I=0, \ell=0 \quad \mathcal{O}^{I=0}_{\ell=0} = m_q(\bar{u}u + \bar{d}d)$

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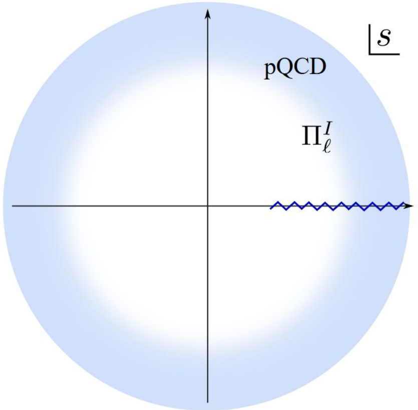
$\vdots$

time-ordered 2-point correlation function $$\Pi^I_\ell(s) = i\int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x} \langle 0|\hat{T}\left\{\mathcal{O}^{I\dagger}_\ell(x)\mathcal{O}^I_\ell(0)\right\}|0\rangle$$



asymptotic freedom: compute in perturbative QCD

good at short distance — large momenta region

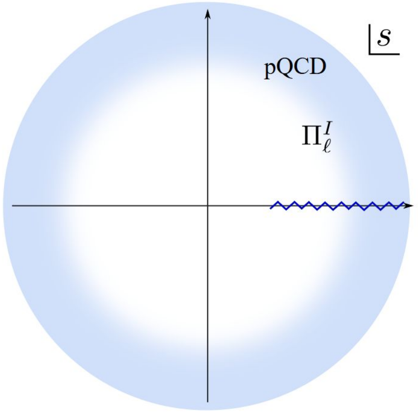
Short distance expansion

[Shifman, Vainshtein, Zakharov, 1979]

at the short distance operator product expansion:

$$T\{\mathcal{O}^\dagger(x)\mathcal{O}(0)\} \overset{x\to 0}{=} C_{\mathbb{1}}(x)\ \mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\ \mathcal{O}(0)$$

$$\langle 0|T\{\mathcal{O}^\dagger(x)\mathcal{O}(0)\}|0\rangle \overset{x\to 0}{=} C_{\mathbb{1}}(x) + \ldots$$



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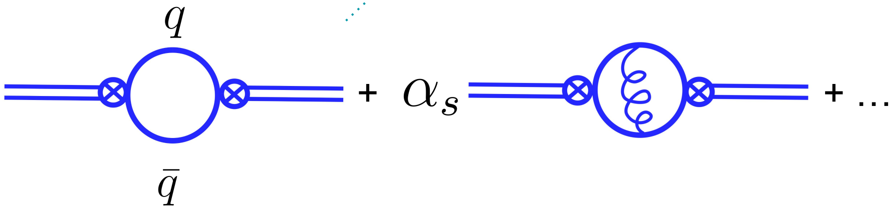
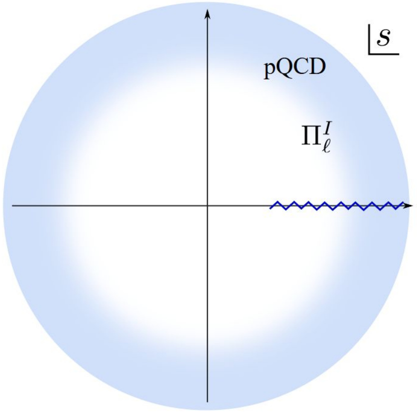
$$\Pi(s) = i\int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x}\langle 0|\hat{T}\left\{\mathcal{O}^\dagger(x)\mathcal{O}(0)\right\}|0\rangle$$

OPE coefficients: perturbative QCD

e.g.:

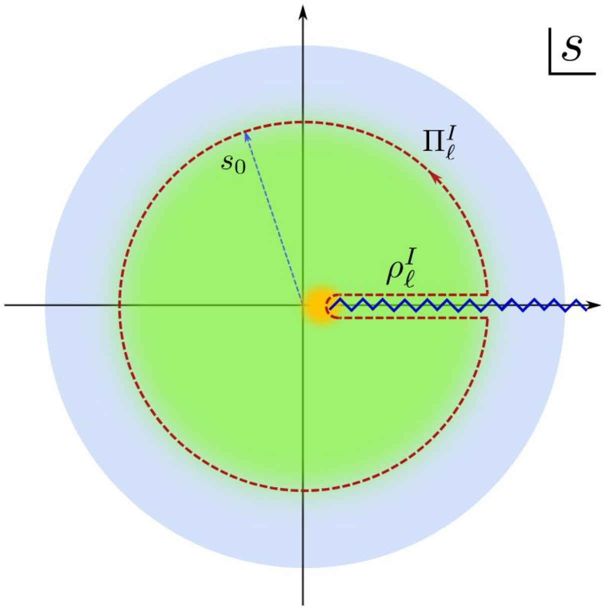




large energy expansion of time-ordered two-point correlator: e.g. spin 1 vector current

$$\Pi^1_1(s) = \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{N_c}{12\pi^2}\left(1+\frac{\alpha_s}{\pi}\right) s\ln\left(-\frac{s}{\mu^2}\right) + \ldots\right\}$$

Finite energy sum rules



connect strongly coupled with short distance region at large s_0

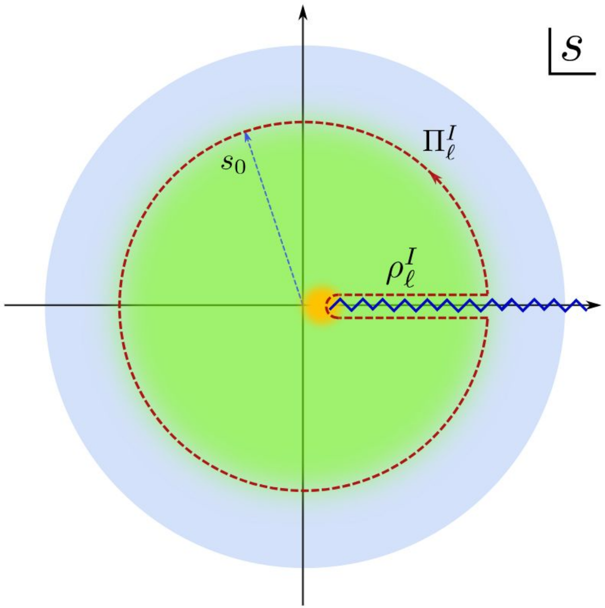
contour integral $s^n\Pi(s)$ vanishes

moments of correlator 

large momenta expansion

$$\int_{4m_\pi^2}^{s_0} \rho(x)x^n dx = -s_0^{n+1}\int_0^{2\pi} e^{i(n+1)\varphi}\Pi(s_0e^{i\varphi})d\varphi$$

Finite energy sum rules



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microscopic theory parameters enter

Numerics: constrain the moments using pQCD results

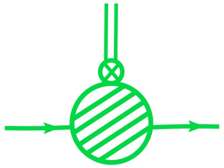
e.g.: spin 1 vector current

$$P1 \quad : \quad \frac{1}{s_0^{n+2}} \int_{4m_\pi^2}^{s_0} \rho_1^1(x) x^n dx = \frac{1}{2(2\pi)^4} \left\{ \frac{N_c}{6\pi(n+2)} \left(1 + \frac{\alpha_s}{\pi}\right) + \dots \right\}, \; n \geq -1$$

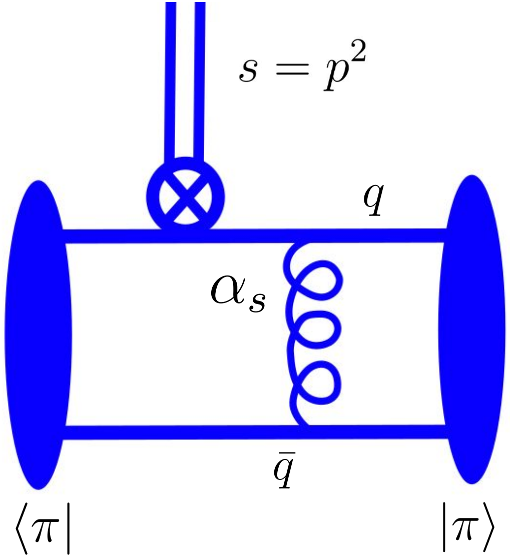
UV limit: form factor at high momentum transfer

[Lepage, Brodsky, 1979]

pQCD predict the the form factors at asymptotically high momentum transfer



Leading contribution:



$$\frac{1}{R_\pi^2} \sim f_\pi^2$$

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[Lepage, Brodsky, 1979]

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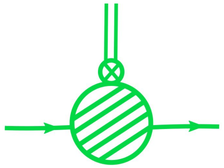

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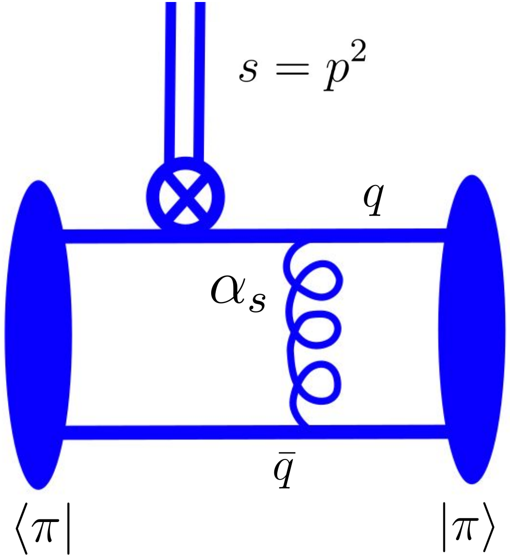


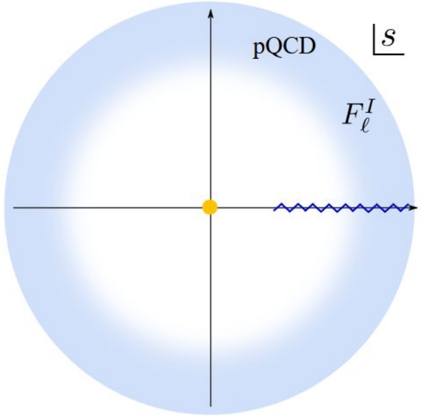
$$\frac{1}{R_\pi^2} \sim f_\pi^2$$

the only f_π input:

the size of pion

e.g. spin 1 vector current

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$



Numerics: impose this behavior above s_0

Summary of the GTB constraints

given a set of quantum numbers

e.g. $I = 1, \ell = 1$ $\quad \epsilon^{abc}\pi^b\partial_\mu\pi^c \simeq \quad j_V^\mu \quad = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$ *identify local operator*

IR **UV**

free pion kinematics

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

two-point function

$\langle 0|\mathcal{O}_\ell^{I\,\dagger}$  $\mathcal{O}_\ell^I|0\rangle$

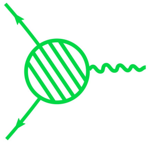
pQCD sum rules

$$\frac{1}{s_0^{n+2}}\int_{4m_\pi^2}^{s_0}\rho_1^1(x)x^n dx \simeq \frac{1}{(2\pi)^4}\frac{1}{4\pi(n+2)}\left(1+\frac{\alpha_s}{\pi}\right)$$

normalization

$$F_1^1(0) = 1$$

form factor

${}_{\mathrm{out}}\langle p, I, \ell|$  $\mathcal{O}_\ell^I|0\rangle$

asymptotics

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$

analyticity
crossing
unitarity

$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \quad s > 4m_\pi^2, \ \forall I, \ \ell$$

Summary of the GTB constraints

given a set of quantum numbers

e.g. $I=1, \ell=1$ $\qquad \epsilon^{abc}\pi^b\partial_\mu\pi^c \simeq \quad j_V^\mu \quad = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$ *identify local operator*

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$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

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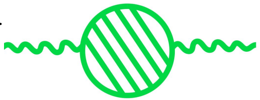
$\langle 0|\mathcal{O}_\ell^{I\,\dagger}$  $\mathcal{O}_\ell^I|0\rangle$

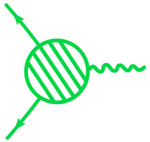
pQCD sum rules

$$\frac{1}{s_0^{n+2}}\int_{4m_\pi^2}^{s_0}\rho_1^1(x)x^n dx \simeq \frac{1}{(2\pi)^4}\frac{1}{4\pi(n+2)}\left(1+\frac{\alpha_s}{\pi}\right)$$

normalization

$$F_1^1(0) = 1$$

form factor

${}_{\text{out}}\langle p, I, \ell|$  $\mathcal{O}_\ell^I|0\rangle$

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$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$

analyticity
crossing
unitarity

$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \quad s > 4m_\pi^2, \ \forall I, \ \ell$$

how to find solution?

Intuition: saturating the matrix of state overlaps

Ignoring other interactions, pion (and nucleon) is the only stable particle in QCD

$$
\begin{array}{c}
\\
\langle \pi | \\ \langle \pi\pi | \\ \langle \pi\pi\pi | \\ \langle \pi\pi\pi\pi | \\ \vdots \\ \langle 0 | \mathcal{O}_1^\dagger \\ \langle 0 | \mathcal{O}_2^\dagger \\ \langle 0 | \mathcal{O}_3^\dagger \\ \vdots
\end{array}
\begin{array}{c}
\overbrace{|\pi\rangle,\ |\pi\pi\rangle,\ |\pi\pi\pi\rangle,\ |\pi\pi\pi\pi\rangle,\ \ldots}^{\text{Hilbert space spanned by n-pion in/out states:}} \quad \overbrace{\mathcal{O}_1|0\rangle,\ \mathcal{O}_2|0\rangle,\ \mathcal{O}_3|0\rangle,\ \ldots}^{\text{States created by local operators encoding QCD information:}} \\
\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right]
\end{array}
\succeq 0
$$

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Hilbert space spanned by n-pion in/out states:

States created by local operators encoding QCD information:

$$
\begin{array}{c}
\\
\langle\pi| \\ \langle\pi\pi| \\ \langle\pi\pi\pi| \\ \langle\pi\pi\pi\pi| \\ \vdots \\ \langle 0|\mathcal{O}_1^\dagger \\ \langle 0|\mathcal{O}_2^\dagger \\ \langle 0|\mathcal{O}_3^\dagger \\ \vdots
\end{array}
\begin{array}{c}
\overbrace{|\pi\rangle,\ |\pi\pi\rangle,\ |\pi\pi\pi\rangle,\ |\pi\pi\pi\pi\rangle,\ \ldots}\quad \overbrace{\mathcal{O}_1|0\rangle,\ \mathcal{O}_2|0\rangle,\ \mathcal{O}_3|0\rangle,\ \ldots} \\
\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right]
\end{array}
\succeq 0
$$

n-pion in/out-states are each complete

- *in/out-states can be expanded in terms of each other*
- *states created by local operators can be expanded in terms of n-pion states*

Intuition: saturating the matrix of state overlaps

Ignoring other interactions, pion (and nucleon) is the only stable particle in QCD

Hilbert space spanned by n-pion in/out states:

States created by local operators encoding QCD information:

$$
\begin{array}{c}
\\
\langle\pi| \\
\langle\pi\pi| \\
\langle\pi\pi\pi| \\
\langle\pi\pi\pi\pi| \\
\vdots \\
\langle 0|\mathcal{O}_1^\dagger \\
\langle 0|\mathcal{O}_2^\dagger \\
\langle 0|\mathcal{O}_3^\dagger \\
\vdots
\end{array}
\begin{array}{c}
\overbrace{|\pi\rangle,\ |\pi\pi\rangle,\ |\pi\pi\pi\rangle,\ |\pi\pi\pi\pi\rangle,\ \ldots}\quad \overbrace{\mathcal{O}_1|0\rangle,\ \mathcal{O}_2|0\rangle,\ \mathcal{O}_3|0\rangle,\ \ldots} \\
\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right]
\end{array}
\succeq 0
$$

n-pion in/out-states are each complete

- *in/out-states can be expanded in terms of each other*
- *states created by local operators can be expanded in terms of n-pion states*

matrix has zero modes

saturate

Physics of the zero modes

Currently, consider only two-pion (in & out) states and one operator

$$\begin{bmatrix} {}_{\rm out}\langle\pi\pi|\pi\pi\rangle_{\rm out} & {}_{\rm out}\langle\pi\pi|\pi\pi\rangle_{\rm in} & {}_{\rm out}\langle\pi\pi|\mathcal{O}|0\rangle \\ {}_{\rm in}\langle\pi\pi|\pi\pi\rangle_{\rm out} & {}_{\rm in}\langle\pi\pi|\pi\pi\rangle_{\rm in} & {}_{\rm in}\langle\pi\pi|\mathcal{O}|0\rangle \\ \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\rm out} & \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\rm in} & \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle \end{bmatrix} \succeq 0 \qquad \forall \ell, I, s > 4m_\pi^2$$

zero modes of the matrix

$$\rho \sim \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle = \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\rm in}\ {}_{\rm in}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq |\mathcal{F}|^2$$

insert a complete set of in-states:

$$\mathbb{1} = |X\rangle\langle X| = |\pi\rangle_{\rm in}\ {}_{\rm in}\langle\pi| + |\pi\pi\rangle_{\rm in}\ {}_{\rm in}\langle\pi\pi| + |\pi\pi\pi\rangle_{\rm in}\ {}_{\rm in}\langle\pi\pi\pi| + \ldots$$

$$\mathcal{F} \sim {}_{\rm out}\langle\pi\pi|\mathcal{O}|0\rangle = {}_{\rm out}\langle\pi\pi|\pi\pi\rangle_{\rm in}\ {}_{\rm in}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq S\mathcal{F}^*$$

Physics of the zero modes

Currently, consider only two-pion (in & out) states and one operator

$$\begin{bmatrix} {}_{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{out}} & {}_{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{in}} & {}_{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle \\ {}_{\text{in}}\langle\pi\pi|\pi\pi\rangle_{\text{out}} & {}_{\text{in}}\langle\pi\pi|\pi\pi\rangle_{\text{in}} & {}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle \\ \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{out}} & \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{in}} & \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle \end{bmatrix} \succeq 0 \qquad \forall \ell, I, s > 4m_\pi^2$$

zero modes of the matrix

$$\rho \sim \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle = \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{in}}\,{}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq |\mathcal{F}|^2$$

insert a complete set of in-states:

$$\mathbb{1} = |X\rangle\langle X| = |\pi\rangle_{\text{in}}\,{}_{\text{in}}\langle\pi| + |\pi\pi\rangle_{\text{in}}\,{}_{\text{in}}\langle\pi\pi| + |\pi\pi\pi\rangle_{\text{in}}\,{}_{\text{in}}\langle\pi\pi\pi| + \ldots$$

$$\mathcal{F} \sim {}_{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle = {}_{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{in}}\,{}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq S\mathcal{F}^*$$

equality exact up to certain energy

approximate above — enlarge the matrix with more states

write QCD operator states in terms of on-shell pion states and vice versa

The zero modes

Zero modes: nonlinear in bootstrap variables, hard to solve. Some observations:

$$G_{\mathrm{SDP}} = \begin{pmatrix} 1 & S_\ell(s) & \mathcal{F}_\ell(s) \\ S_\ell^*(s) & 1 & \mathcal{F}_\ell^*(s) \\ \mathcal{F}_\ell^*(s) & \mathcal{F}_\ell(s) & \rho(s) \end{pmatrix} \longrightarrow G_{\mathrm{z.m.}} = \begin{pmatrix} 1 & \frac{\mathcal{F}_\ell(s)}{\mathcal{F}_\ell^*(s)} & \mathcal{F}_\ell(s) \\ \frac{\mathcal{F}_\ell^*(s)}{\mathcal{F}_\ell(s)} & 1 & \mathcal{F}_\ell^*(s) \\ \mathcal{F}_\ell^*(s) & \mathcal{F}_\ell(s) & \mathcal{F}_\ell(s)\mathcal{F}_\ell^*(s) \end{pmatrix}$$

two zero modes:

$$v_1 = \begin{pmatrix} -\mathcal{F}_\ell(s) \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\frac{\mathcal{F}_\ell(s)}{\mathcal{F}_\ell^*(s)} \\ 1 \\ 0 \end{pmatrix}$$

satisfying

$$\mathrm{Tr}(M_i \cdot G_{\mathrm{z.m.}}) = 0, \quad M_i = v_i v_i^\dagger, \quad i = 1, 2$$

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generically SDP satisfy:

$$\mathrm{Tr}(M \cdot G_{\mathrm{SDP}}) \geq 0, \quad \forall\ M = vv^\dagger$$

minimum

two zero modes:

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Iterative procedure for finding GTB solution

- start by finding an arbitrary initial solution satisfying GTB constraints

e.g. maximizing some linear functional, or an arbitrary feasible point by min 0

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$$\mathcal{F}_\ell^{(\mathrm{old})}(s) \longrightarrow v \longrightarrow M^{(\mathrm{old})} = vv^\dagger$$

linear functional
in new variables

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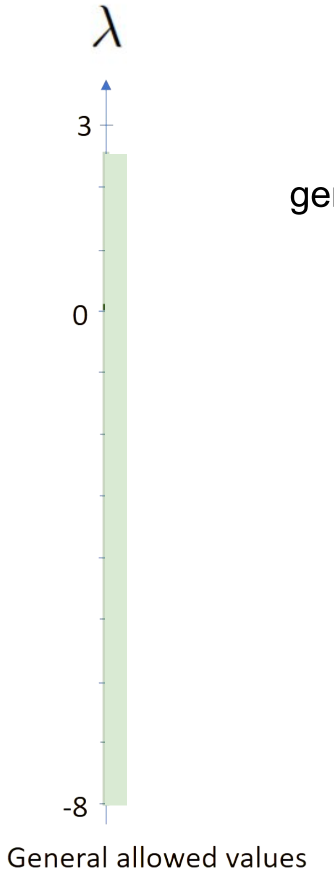
- Iterate until converge

GTB solution is a fix point of the procedure

$N_f = 2 \quad N_c = 3$ Test 1: Computing pion quartic coupling



$$\lambda = \frac{1}{32\pi}\,\mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0)\big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4} A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$

generic bounds from S-matrix bootstrap (ACU): $-8.02 \leq \lambda \leq 2.661$

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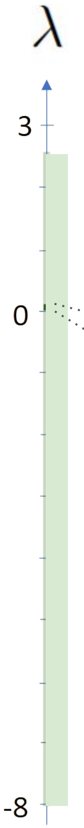
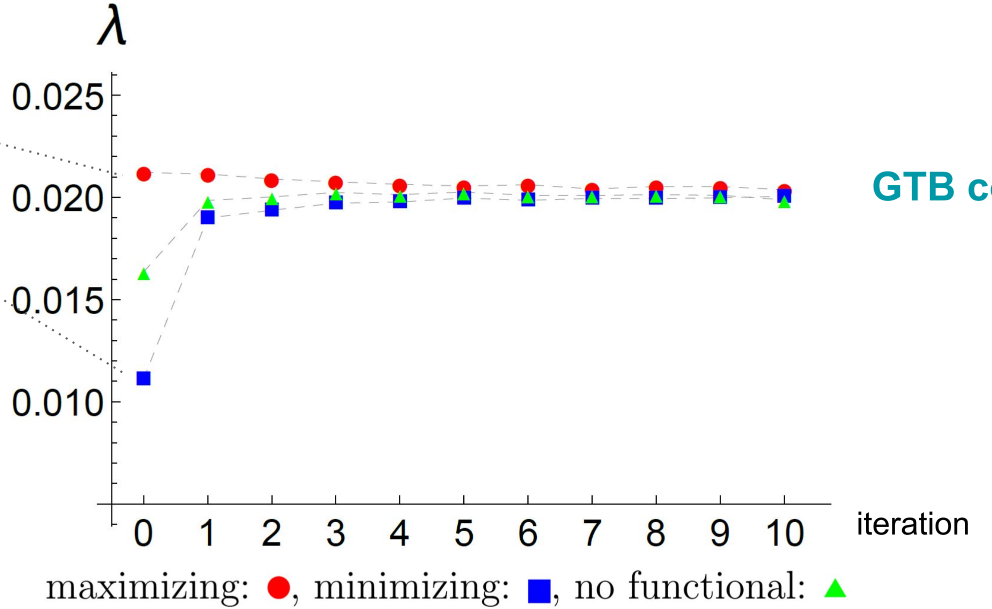


General allowed values



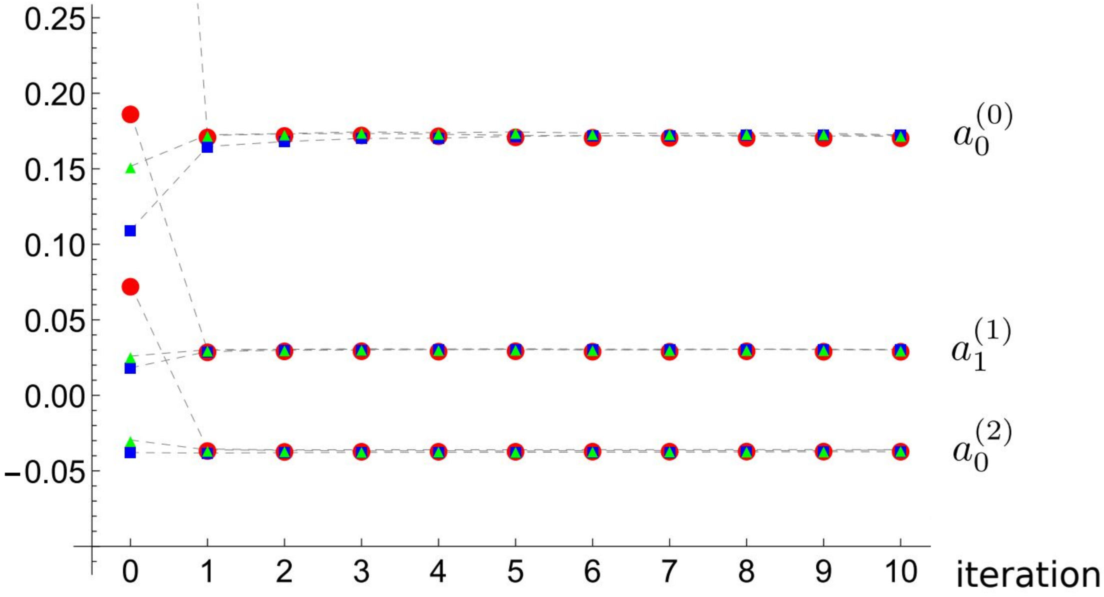
GTB computation

$N_f = 2 \quad N_c = 3$

Test 2: Computing scattering lengths

$$\mathrm{Re} f_\ell^I(s) \overset{k\to 0}{\simeq} k^{2\ell}\left(a_\ell^I + b_\ell^I k^2 + \dots\right), \quad k = \frac{\sqrt{s-4m_\pi^2}}{2}$$

$$S_\ell^I(s) = 1 + i\pi\sqrt{\frac{s-4m_\pi^2}{s}} f_\ell^I(s)$$

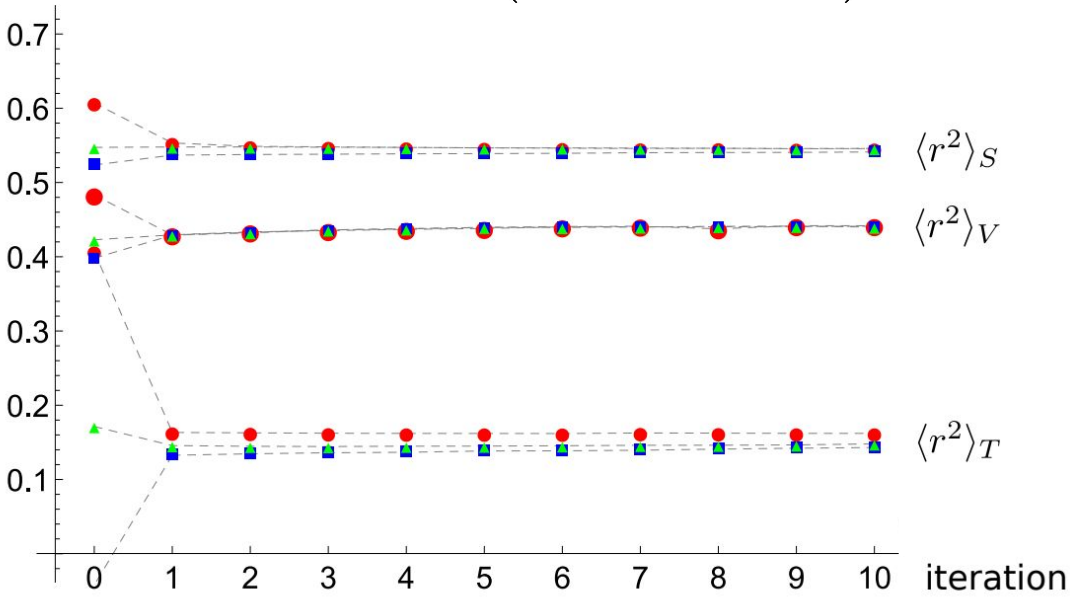


maximizing: ●, minimizing: ■, no functional: ▲

$N_f = 2 \quad N_c = 3$

Test 3: Computing pion charge radii

$$F_\ell^I(s) = F_\ell^I(0)\left(1 + \frac{1}{6} s \langle r_\pi^2 \rangle_\ell^I + \dots\right)$$



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$N_f = 2 \quad N_c = 3$

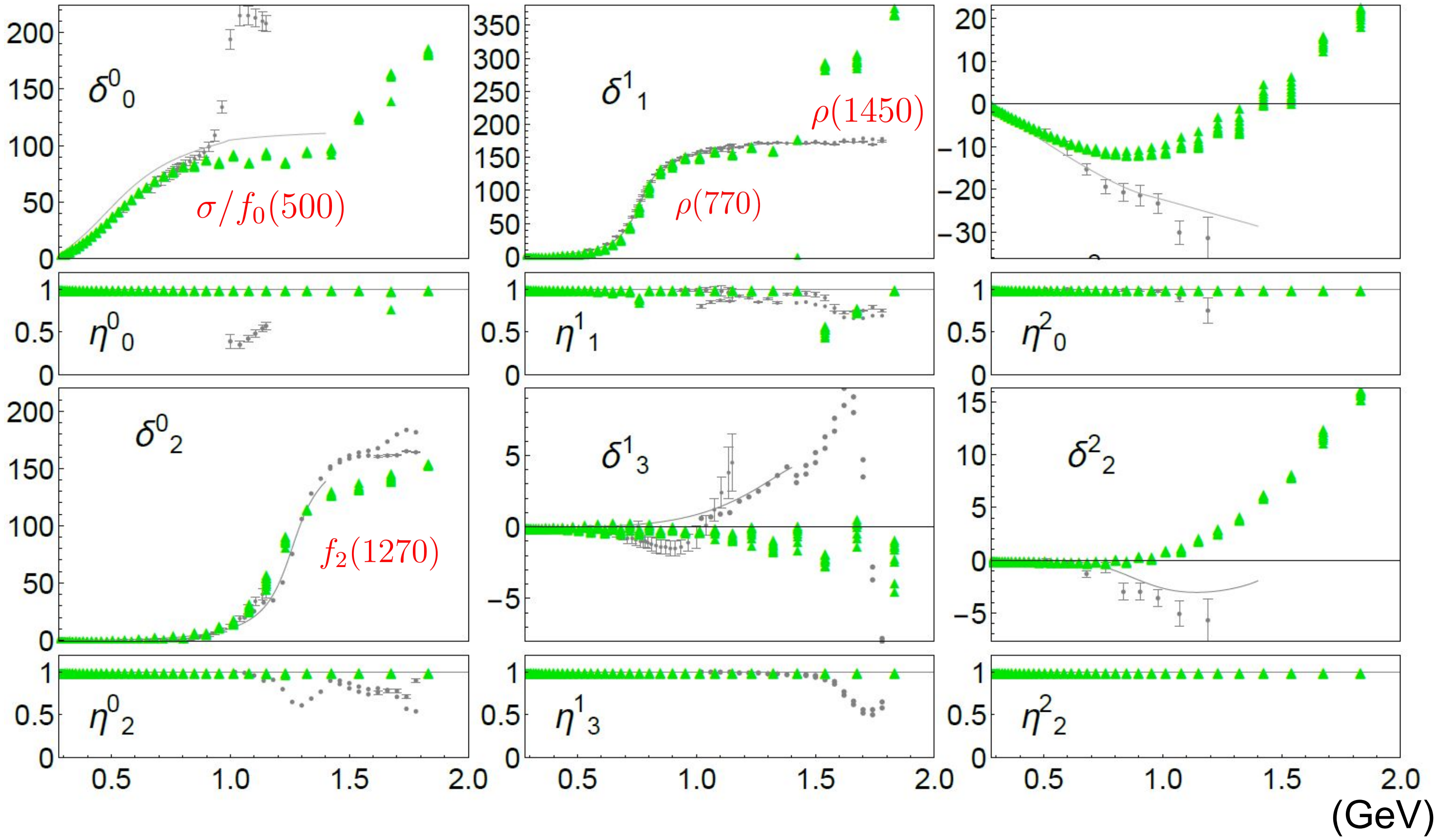
Partial waves from 1-10 iterations

starting from
a generic
feasible point

$$S_\ell^I(s) = \eta_\ell^I(s)\, e^{2i\delta_\ell^I(s)}$$

GTB
solution

experimental data
(gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit
(gray line)
[Pelaez, Yndurain, 2005]



1-loop amplitude & form factors: computing LEC

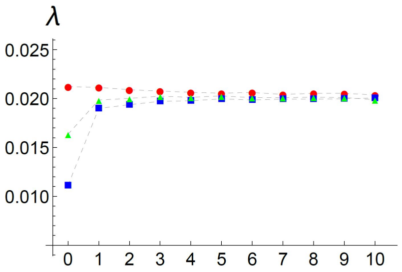
$N_f = 2 \quad N_c = 3$

theory experiments

1-loop renormalized coefficients at μ=m$_\pi$

	GTB	GL	Bij	CGL
$\bar{\ell}_1$	1.6	-2.3 ± 3.7	-1.7 ± 1.0	-0.4 ± 0.6
$\bar{\ell}_2$	5.5	6.0 ± 1.3	6.1 ± 0.5	4.3 ± 0.1
$\bar{\ell}_3$	7.8	2.9 ± 2.4		
$\bar{\ell}_4$	4.7	4.3 ± 0.9	4.4 ± 0.3	4.4 ± 0.2
$\bar{\ell}_6$	14.3	18.7 ± 1.1	$16.0 \pm 0.5 \pm 0.7$	
		Exp.	**W**	
λ	0.02		0.023	
f_π (MeV)	101	92		

pion coupling



Exploring gauge theories with various N_c

Keep the t'Hooft coupling $\alpha_s N_c$ fixed

N_c parametrizes the UV inputs: spectral density sum rules & asymptotic form factors

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N_c parametrizes the UV inputs: spectral density sum rules & asymptotic form factors

- *sum rules for spectral density* $$\int_{4m_\pi^2}^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

e.g. perturbative QCD two-point function at 3-loops [Chetyrkin, Kataev, Tkachkov, 1979] [Chetyrkin, 1996]

$$\Pi_0^0(s) = -\frac{s}{(2\pi)^4}\frac{m_q^2}{8\pi^2}\left\{2\boxed{N_c}\ln\left(-\frac{s}{\mu^2}\right) + \frac{\alpha_s}{\pi}\left[s_1 \ln\left(-\frac{s}{\mu^2}\right) + s_2 \ln^2\left(-\frac{s}{\mu^2}\right)\right] + \left(\frac{\alpha_s}{\pi}\right)^2\left[s_3 \ln\left(-\frac{s}{\mu^2}\right) + s_4 \ln^2\left(-\frac{s}{\mu^2}\right) + s_5 \ln^3\left(-\frac{s}{\mu^2}\right)\right] + O(\alpha_s^3)\right\}$$

$$\Pi_1^1 = -\frac{s}{(2\pi)^4}\frac{N_f \boxed{N_c}}{48\pi^2}\left\{\ln\left(-\frac{s}{\mu^2}\right) + r_1 \frac{\alpha_s}{\pi}\ln\left(-\frac{s}{\mu^2}\right) + \left(\frac{\alpha_s}{\pi}\right)^2\left[r_2 \ln\left(-\frac{s}{\mu^2}\right) + r_3 \ln^2\left(-\frac{s}{\mu^2}\right)\right] + O(\alpha_s^3)\right\}$$

s_i, r_i depends on N_c

Numerics: free parameters N_c to change → explore various gauge theories

Exploring gauge theories with various N_c

- *asymptotic form factor*

e.g. vector current form factor $F_1^1 \simeq -\frac{36\pi\alpha_s f_\pi^2}{s}\frac{N_c^2-1}{2N_c^2}$

similar for form factor of other currents

definition of f_π through axial current form factor $\langle 0|A_\mu^a(x)|\pi_b(p)\rangle = i\delta^{ab}p_\mu f_\pi e^{-ipx}$

$\sim N_c$ $\sim N_c^{-1/2}$

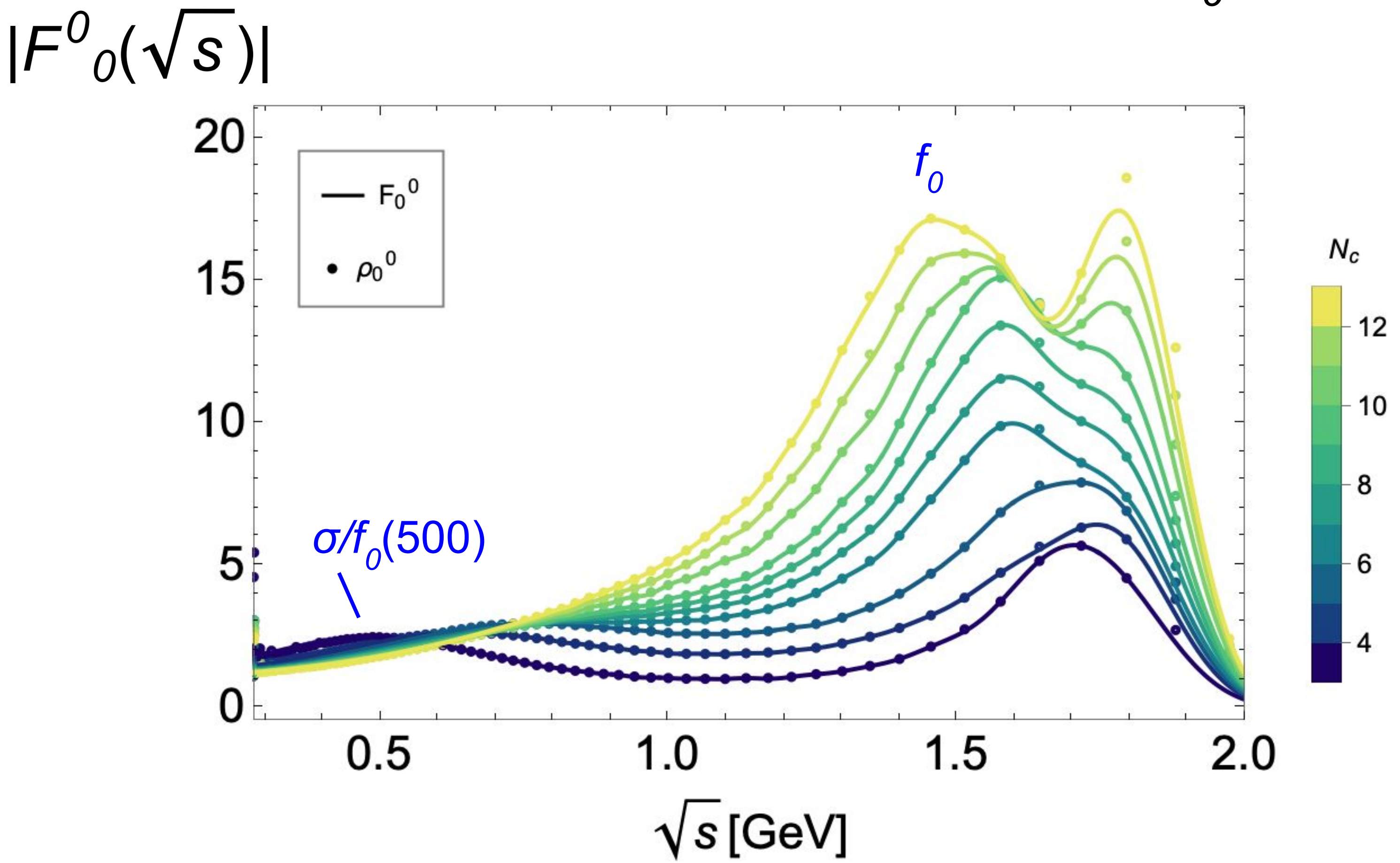
$$\boxed{f_\pi \sim \sqrt{N_c}}$$

$\alpha_s N_c \, \frac{f_\pi^2}{N_c} \simeq$ constant with N_c

See [Albert, Kosva, Rastelli, 2026] for $N_c=\infty$

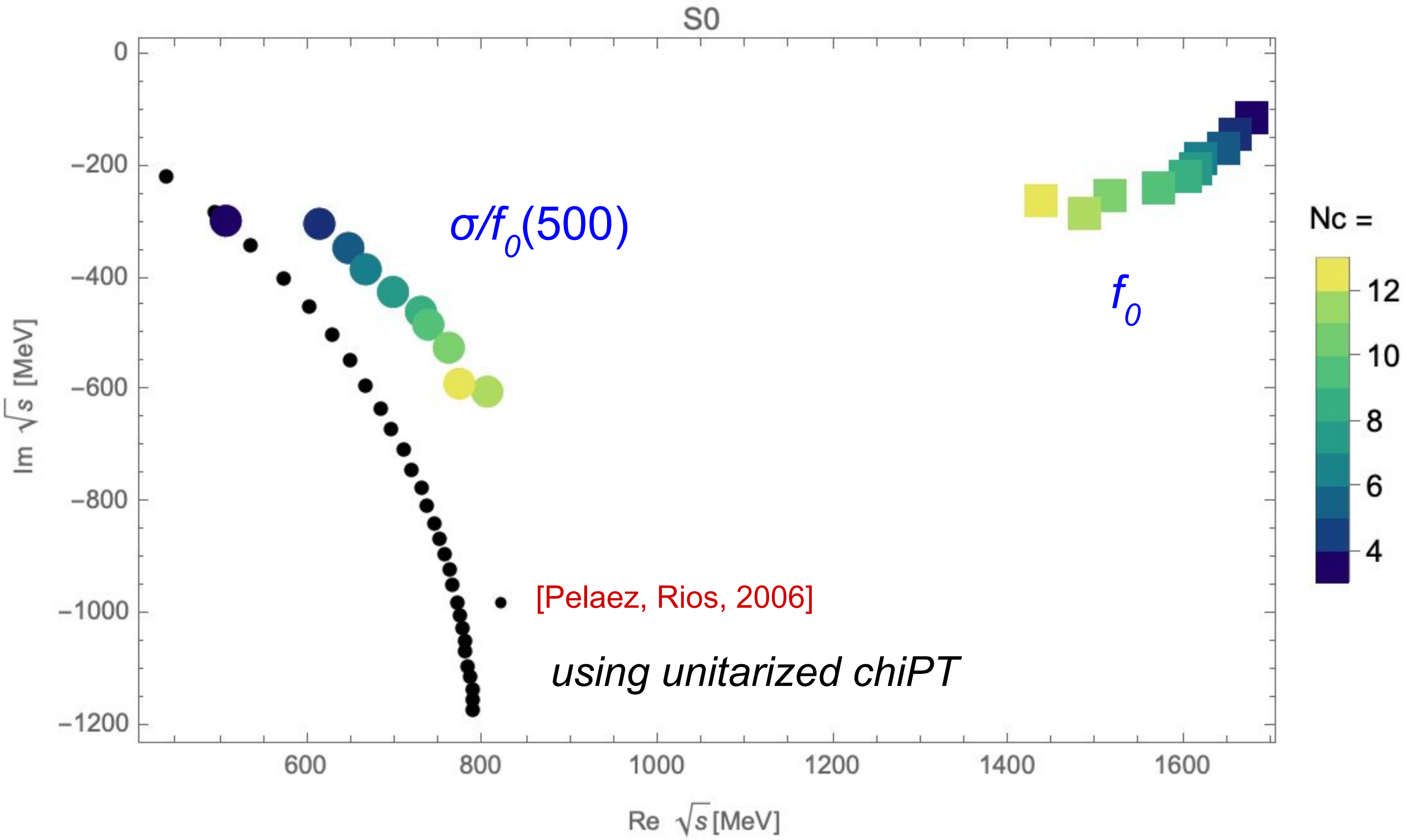
Numerics: keeping the asymptotic form factor fixed when changing N_c

Scalar form factor for varying N_c

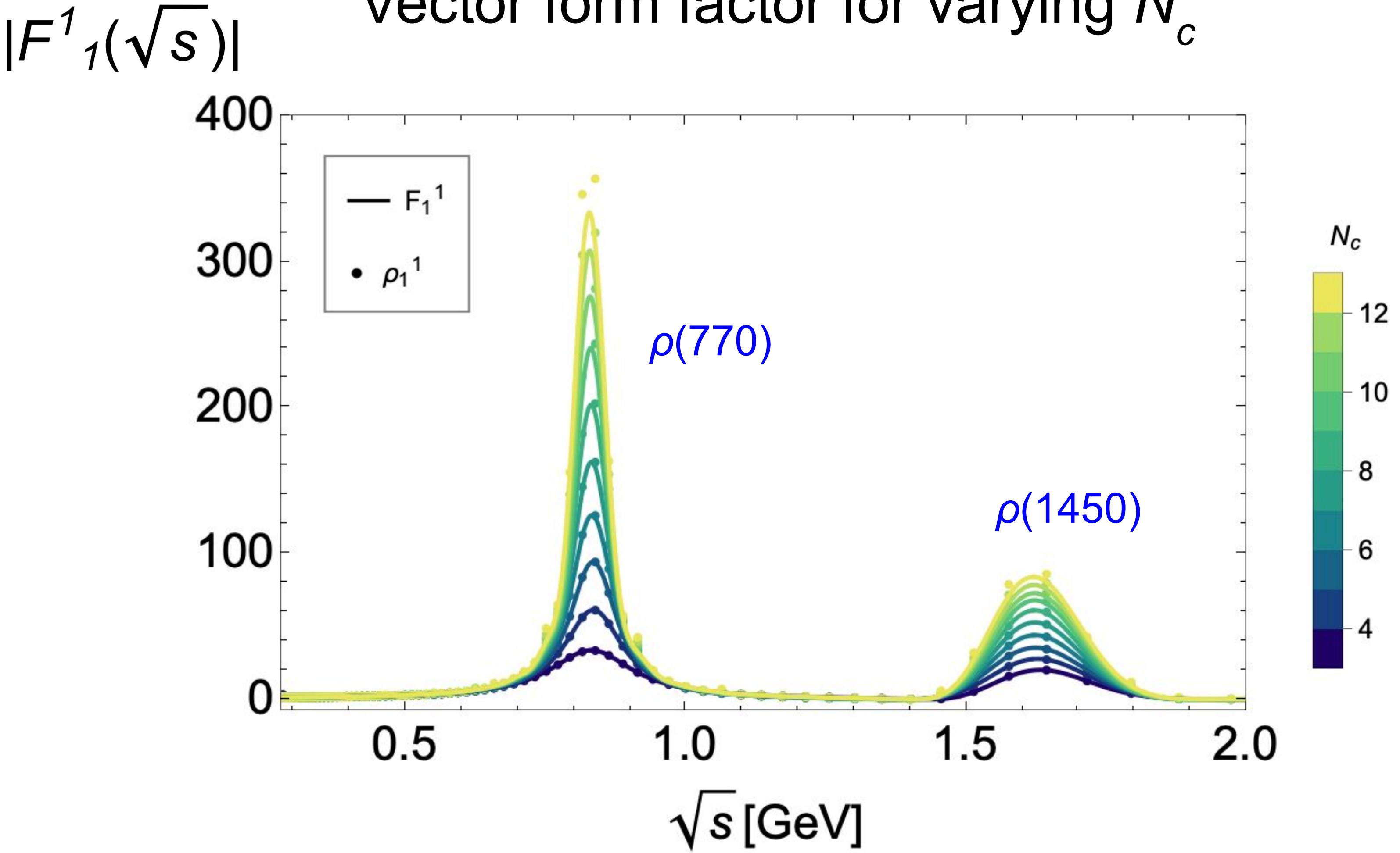


Evolution of f_0 states with N_c

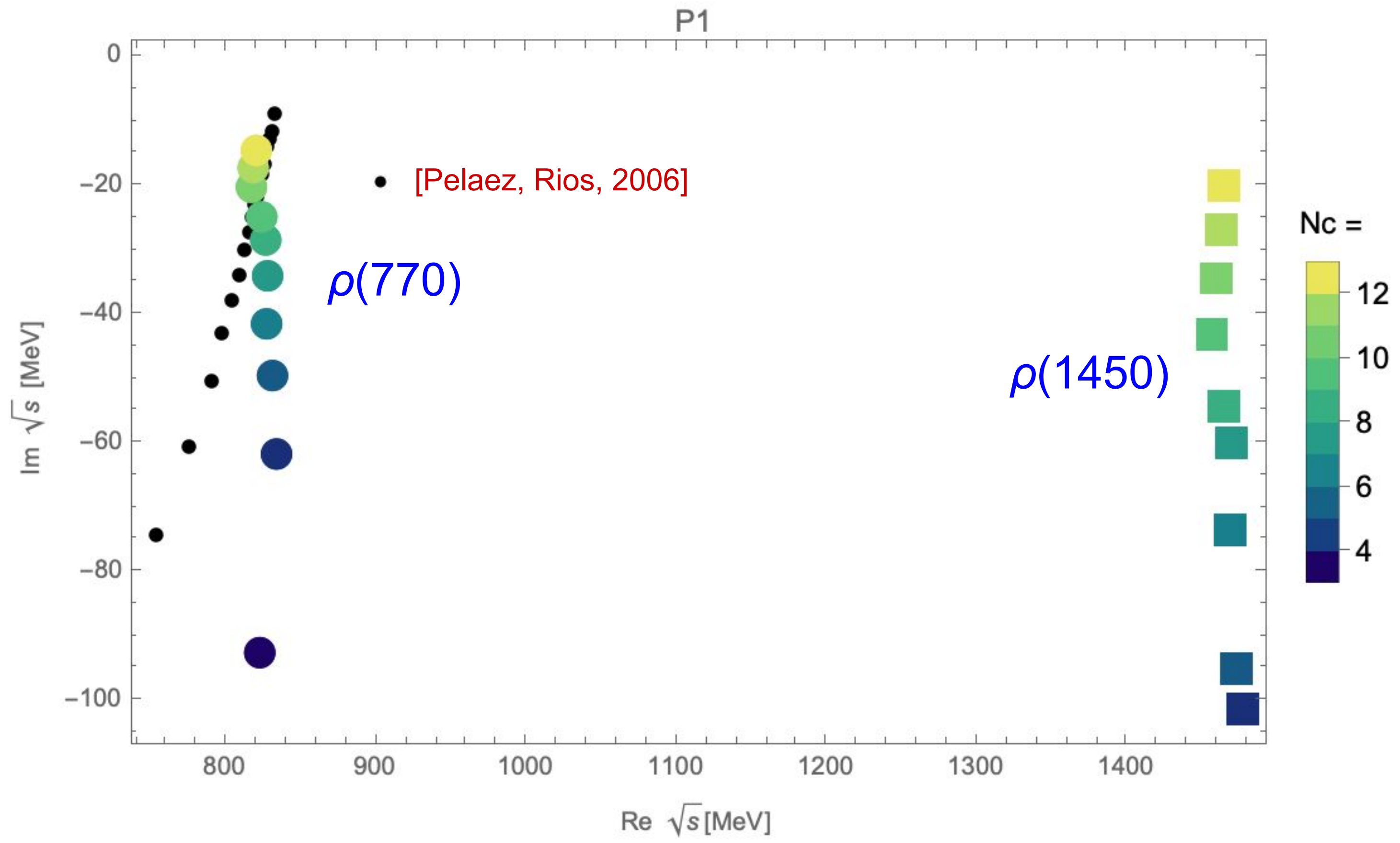
resonances – pole on second sheet

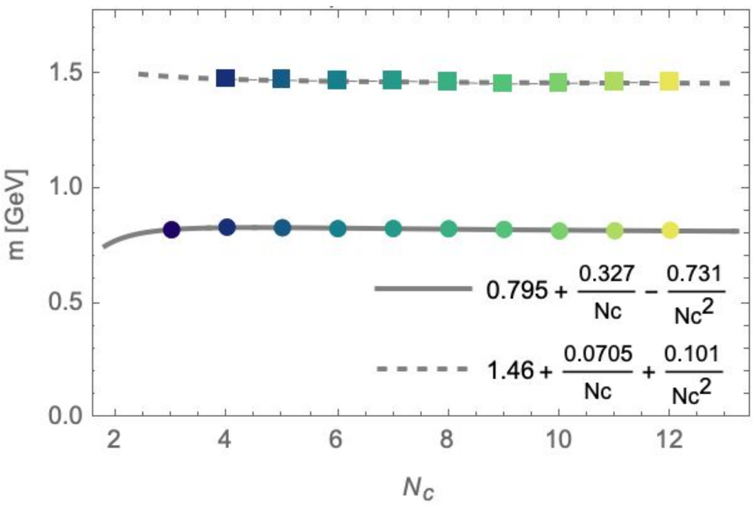


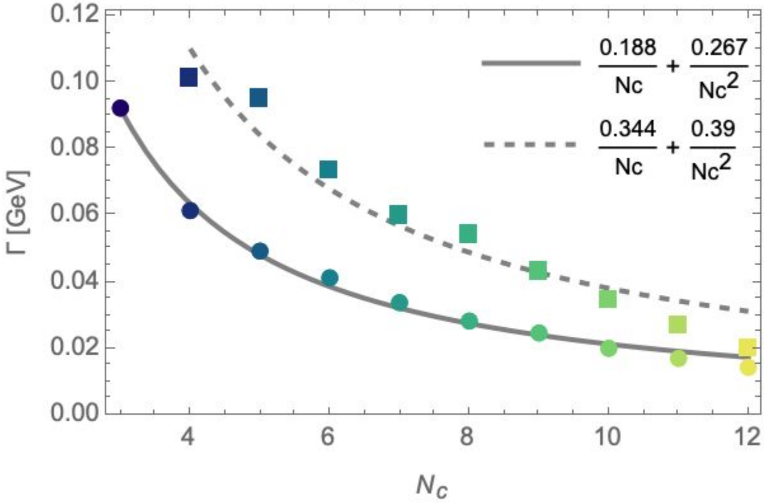
Vector form factor for varying N_c

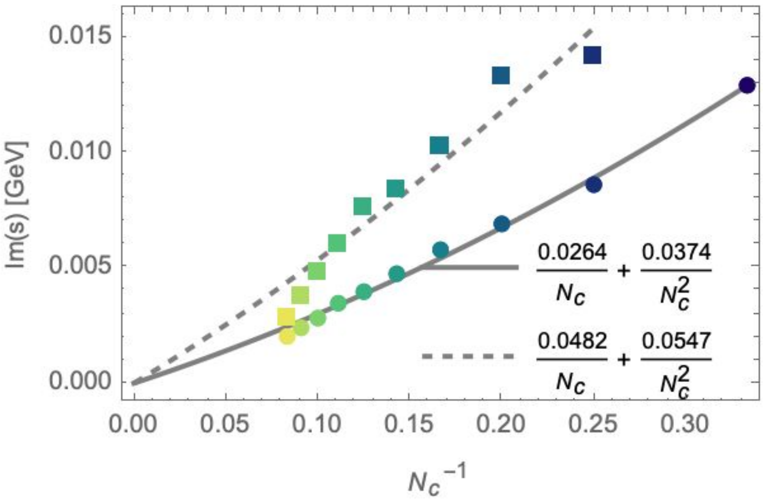


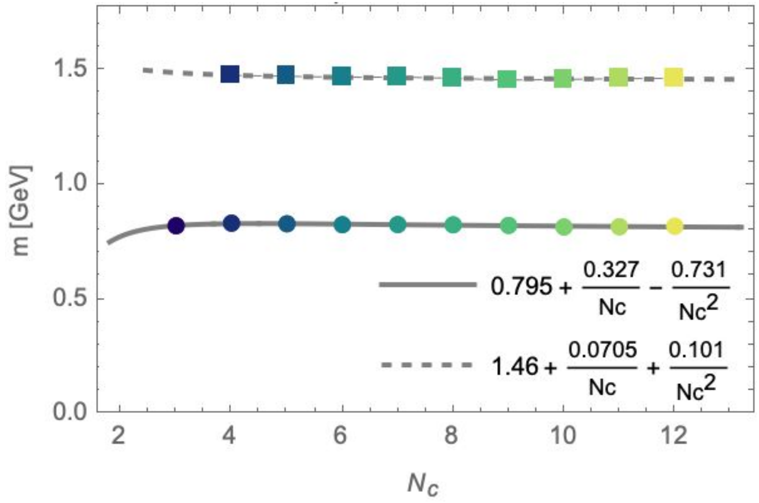
Evolution of ρ states with N_c

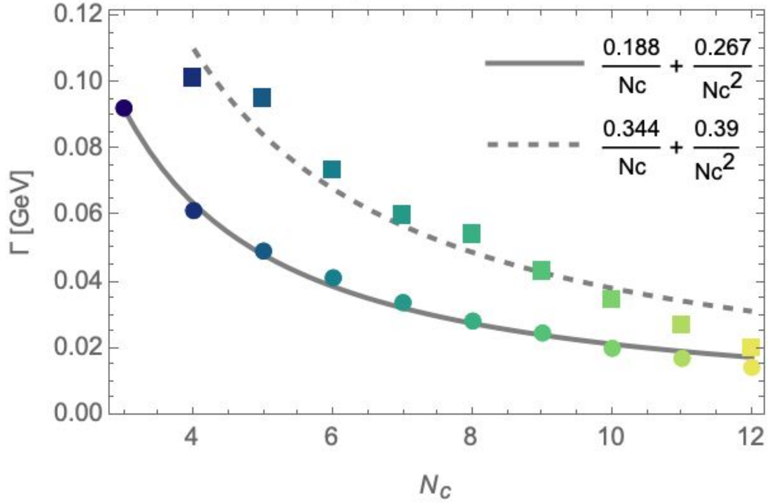



m [GeV]
N_c
$0.795 + \frac{0.327}{Nc} - \frac{0.731}{Nc^2}$
$1.46 + \frac{0.0705}{Nc} + \frac{0.101}{Nc^2}$


Γ [GeV]
N_c
$\frac{0.188}{Nc} + \frac{0.267}{Nc^2}$
$\frac{0.344}{Nc} + \frac{0.39}{Nc^2}$


Im(s) [GeV]
N_c^{-1}
$\frac{0.0264}{N_c} + \frac{0.0374}{N_c^2}$
$\frac{0.0482}{N_c} + \frac{0.0547}{N_c^2}$





can be understood from narrow width analysis:

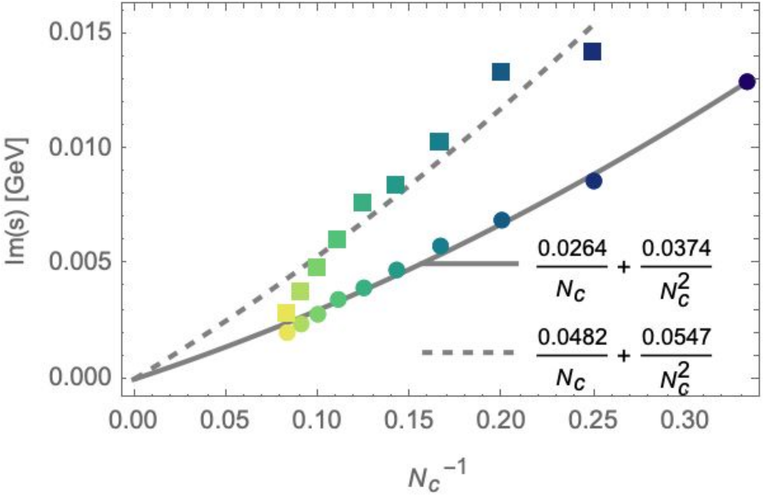
$$\rho \simeq \frac{A(m_\rho)}{(s-m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2} \xrightarrow{\text{narrow width}} \rho \simeq \frac{A}{m_\rho \Gamma_\rho}\delta(s-m_\rho^2)$$

first two moments

$$M^{(0)} = \int \rho ds = \frac{A}{m_\rho \Gamma_\rho}$$

$$M^{(1)} = \int s\rho ds = m_\rho^2 \frac{A}{m_\rho \Gamma_\rho}$$

$$\frac{M^{(1)}}{M^{(0)}} \simeq m_\rho^2, \quad \Gamma \sim \frac{1}{M^{(0)}} \overset{\mathrm{SR}}{\sim} \frac{1}{N_c}$$



Exploring gauge theories with various m_q

quark mass appears in $I=0, \ell=0$ sum rules:

$$\mathcal{O}^{I=0}_{\ell=0} \;=\; m_q(\bar{u}u + \bar{d}d)$$

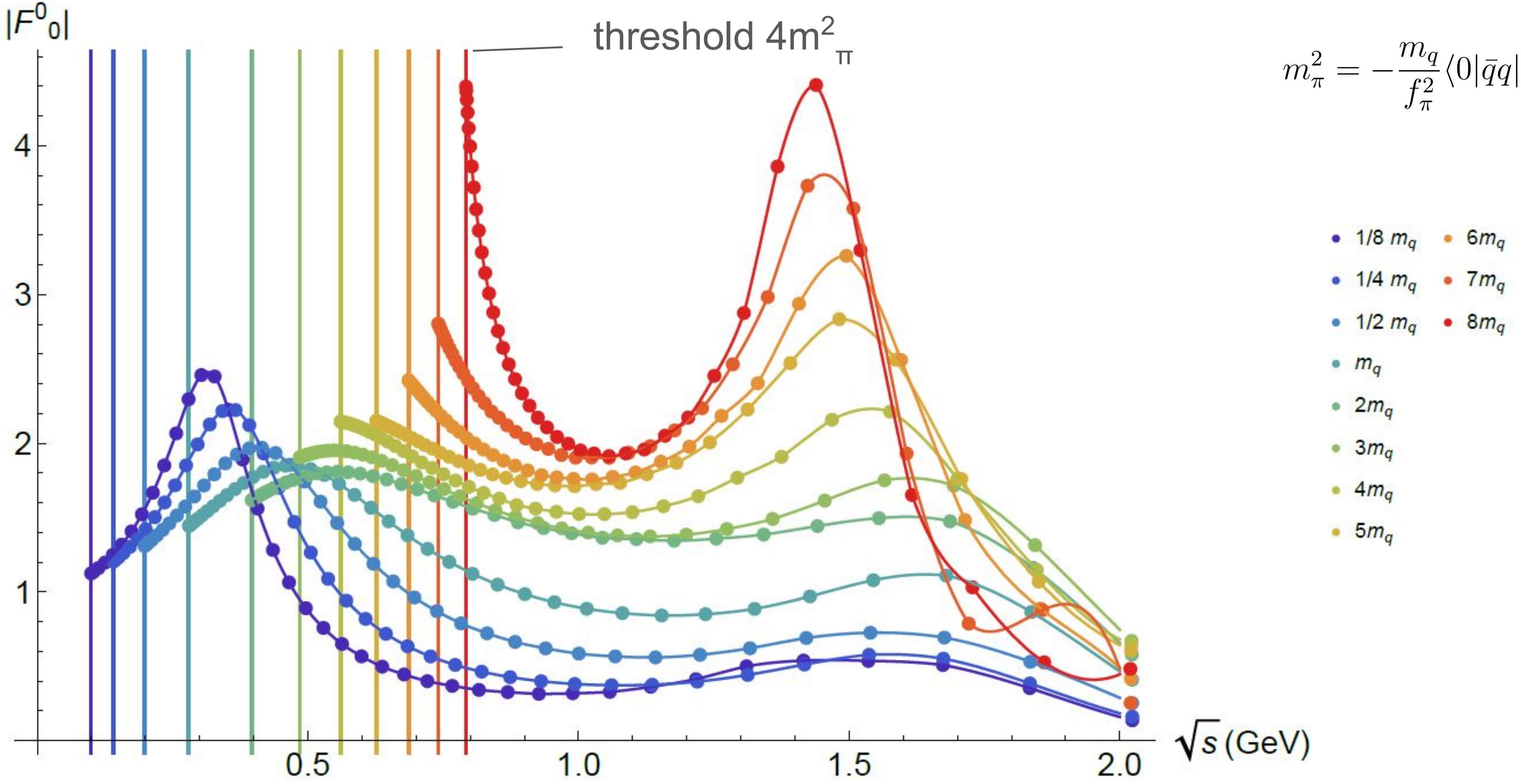
$$\frac{1}{s_0^{n+2}} \int_4^{s_0} \rho_0^0(x) x^n dx = \frac{N_f N_c \boxed{m_q^2}}{(2\pi)^4} \left\{ \frac{1}{4\pi(n+2)} + \ldots \right\}$$

free parameter in SR

change also the pion mass through GMOR relation: $m_\pi^2 = -\frac{m_q}{f_\pi^2}\langle 0|\bar{q}q|0\rangle$

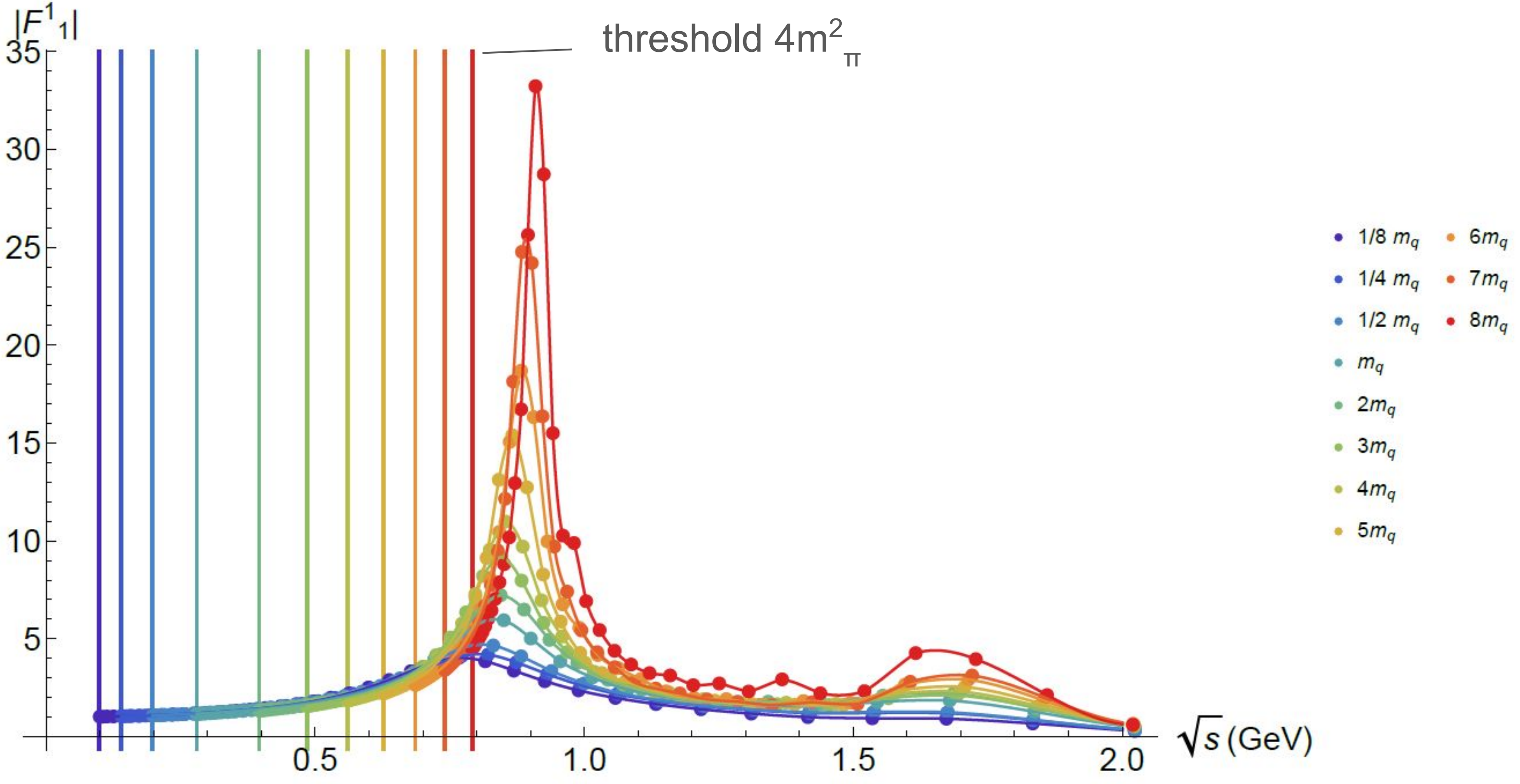
Numerics: change m_q (in SR) and m_π simultaneously according to GMOR

Scalar form factor for varying m_q

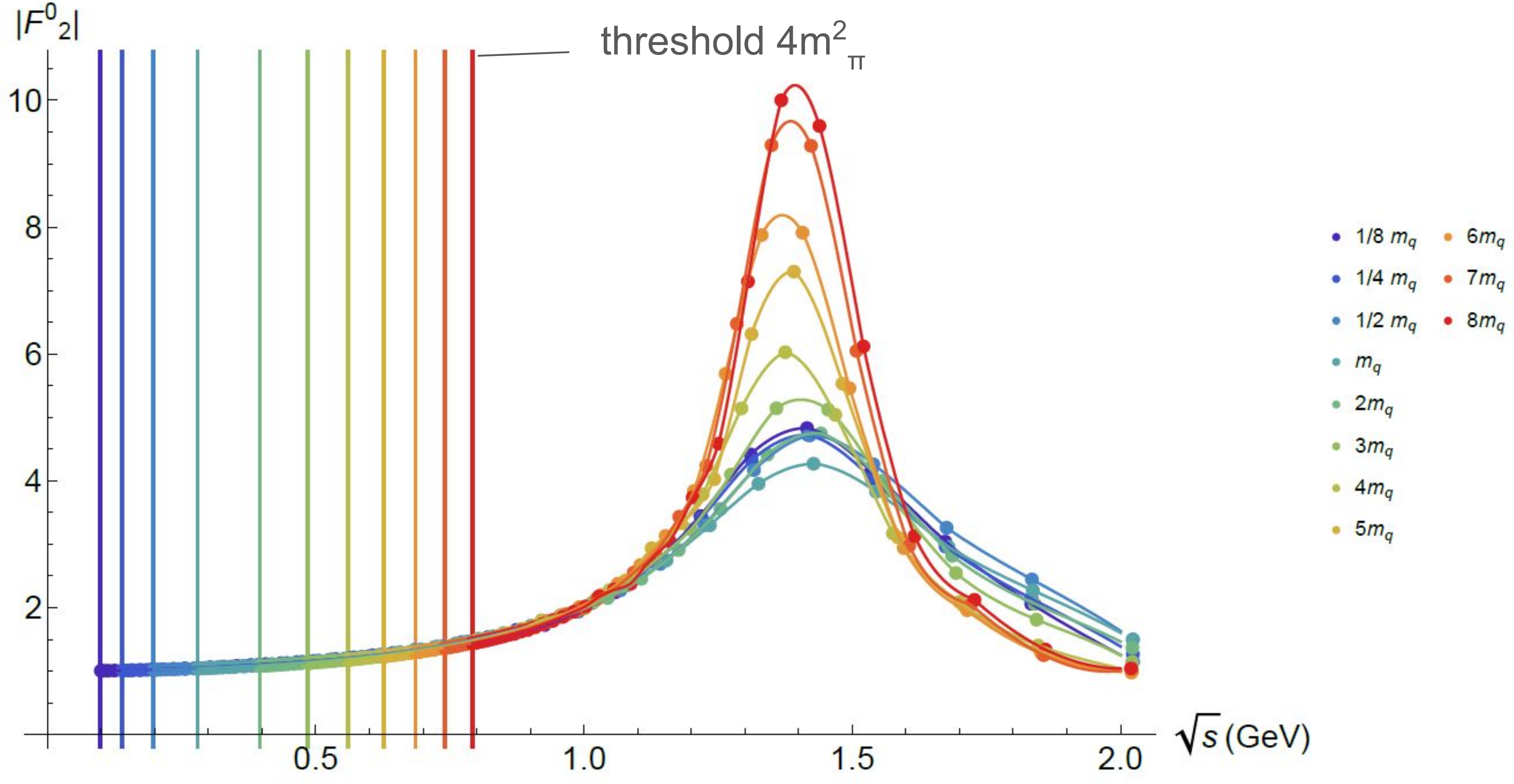


$$m_\pi^2 = -\frac{m_q}{f_\pi^2}\langle 0|\bar{q}q|0\rangle$$

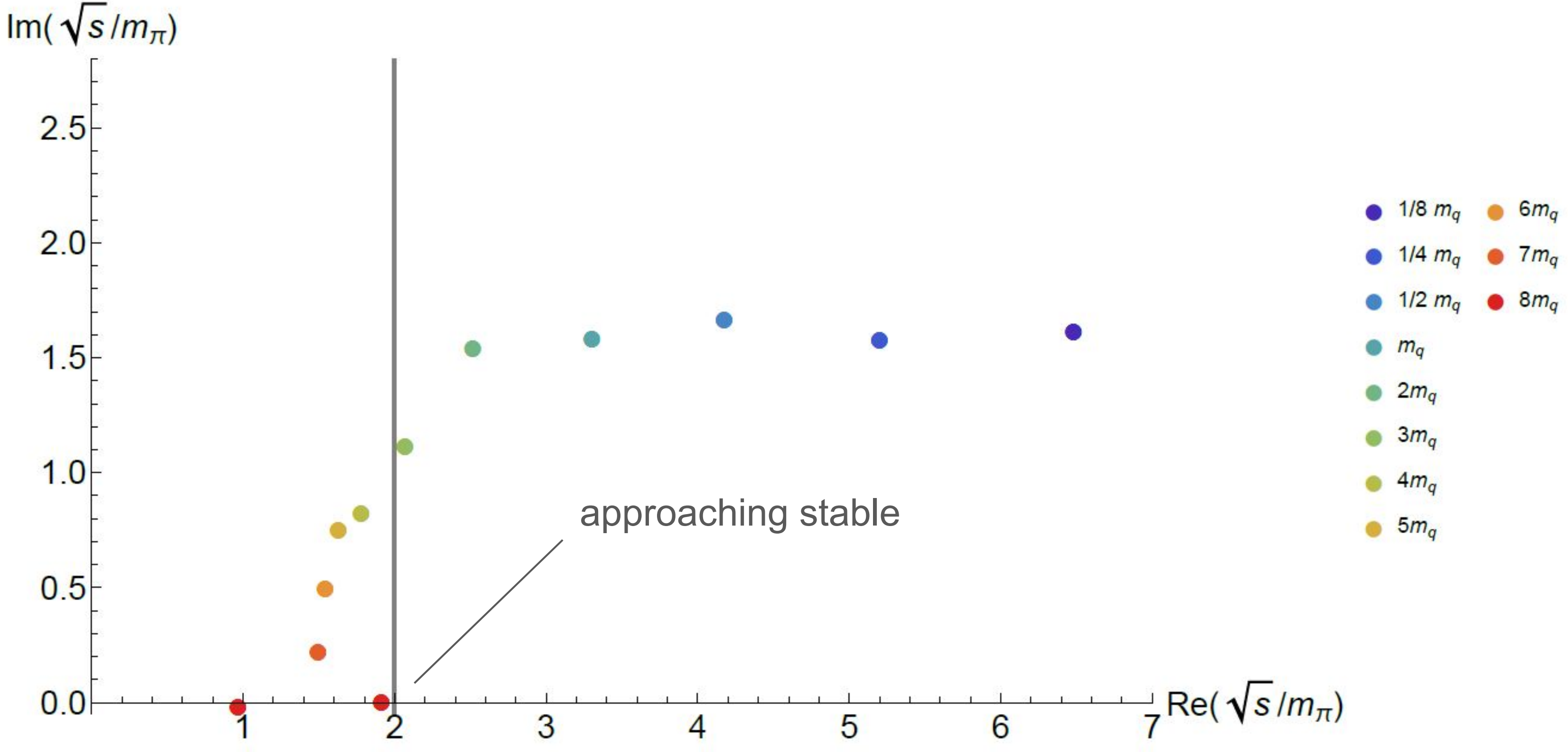
Vector form factor for varying m_q



Gravitational form factor for varying m_q

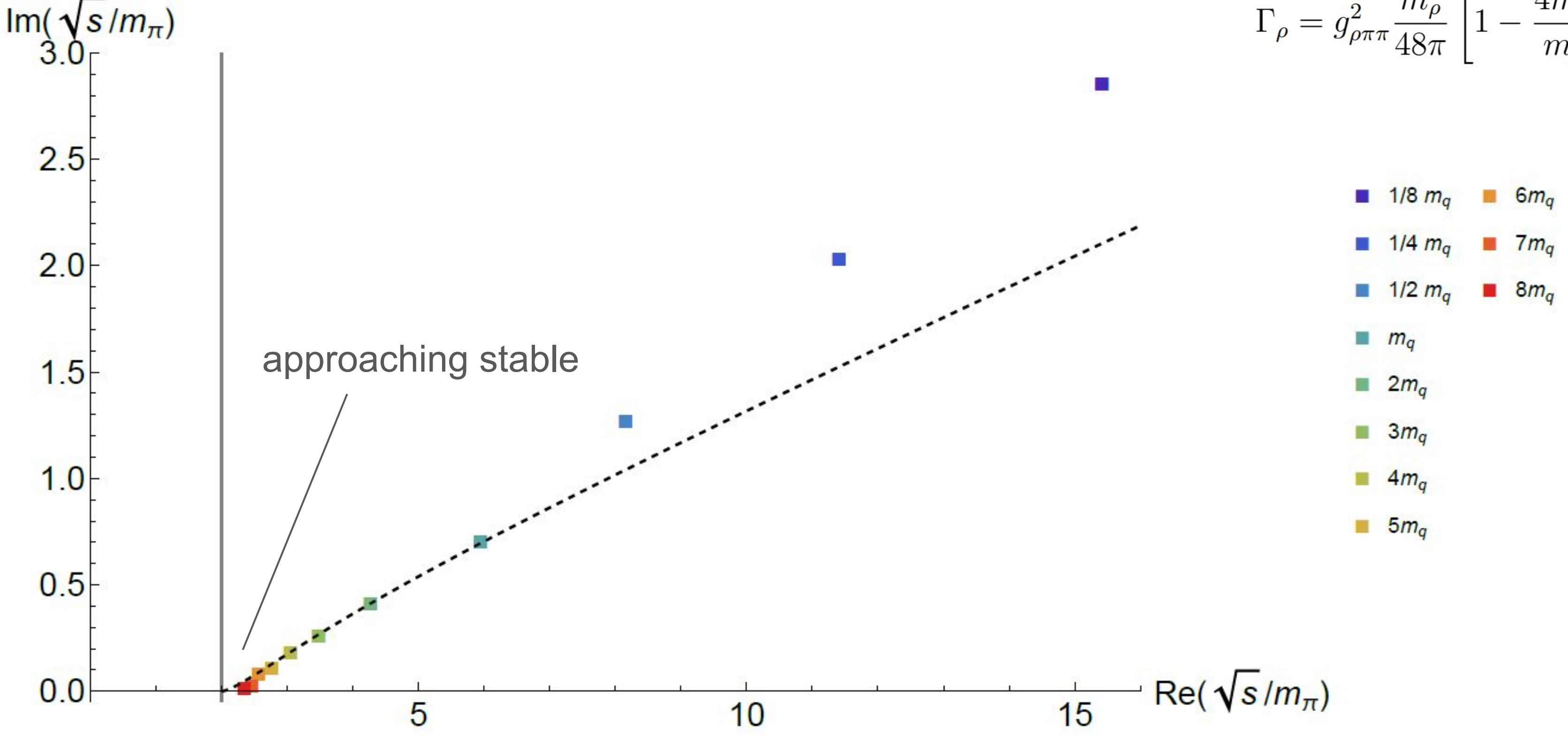


σ meson on the complex plane



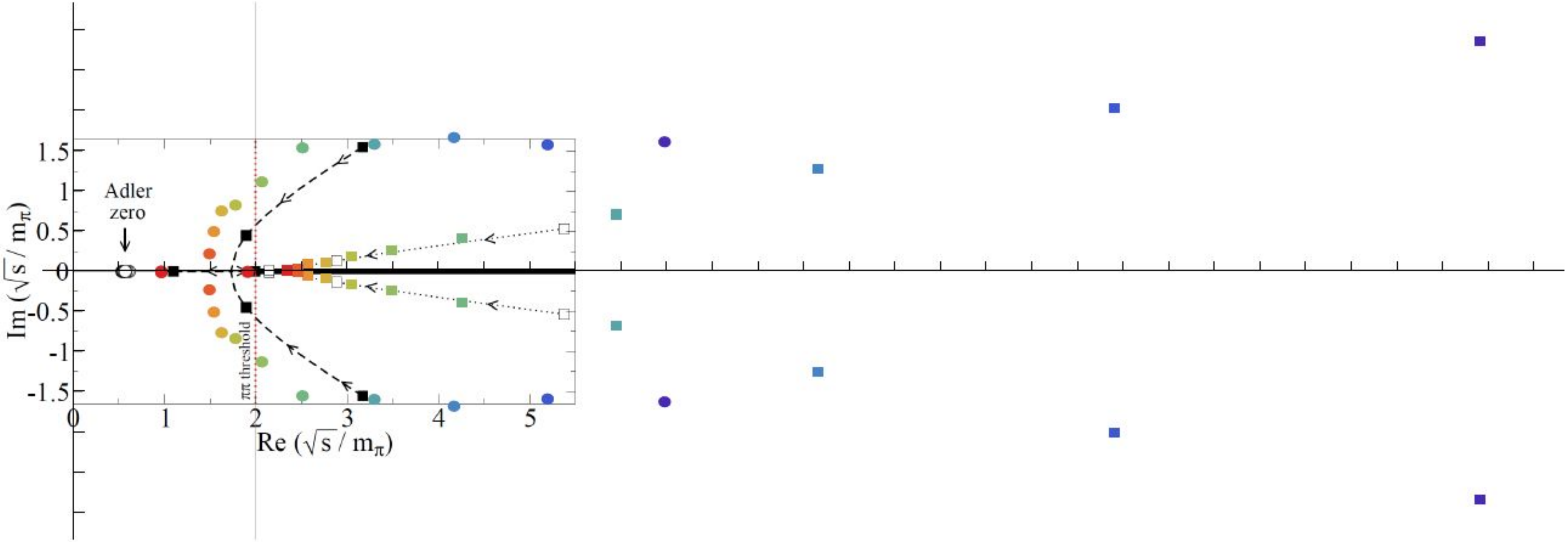
ρ meson on the complex plane

$$\Gamma_\rho = g_{\rho\pi\pi}^2 \frac{m_\rho}{48\pi}\left[1 - \frac{4m_\pi^2}{m_\rho^2}\right]^{\frac{3}{2}}$$

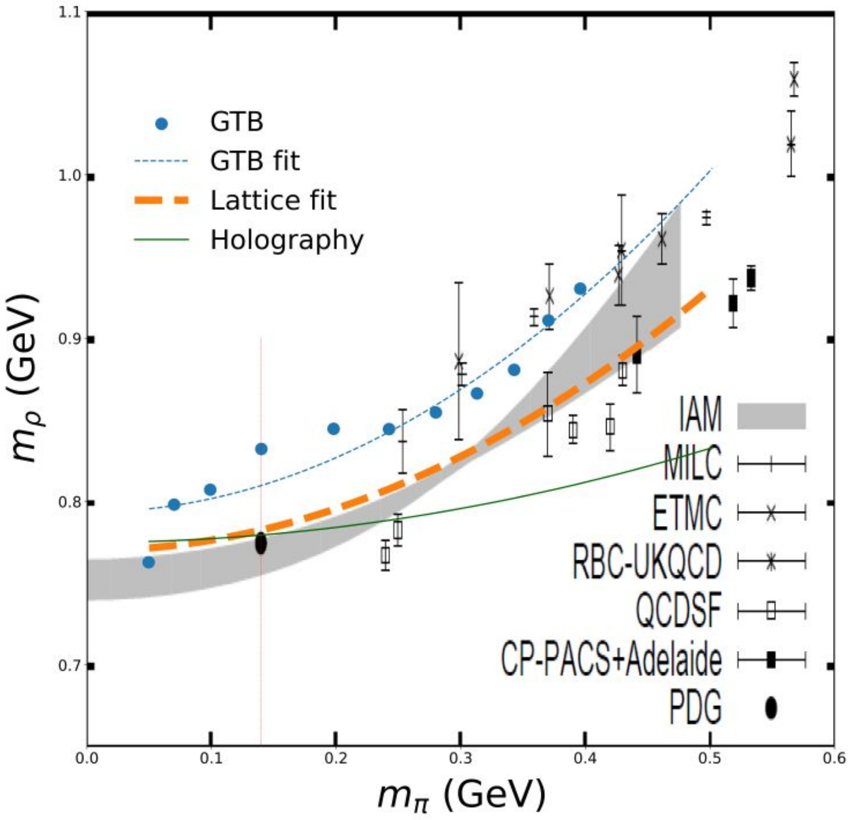


Comparison with unitarized chiral perturbation theory

[Hanhart, Pelaez, Rios, 2008]



m_ρ vs m_π



GTB
GTB fit
Lattice fit
Holography
IAM
MILC
ETMC
RBC-UKQCD
QCDSF
CP-PACS+Adelaide
PDG
m_ρ (GeV)
m_π (GeV)
1.1
1.0
0.9
0.8
0.7
0.0
0.1
0.2
0.3
0.4
0.5
0.6

Conclusions

- Gauge Theory Bootstrap:

using only $\underbrace{N_c \ N_f \ m_q \ \Lambda_{\mathrm{QCD}}}_{\textit{gauge theory parameters}}$ $\qquad m_\pi$ set the energy unit $\qquad f_\pi$ size of pion

compute strongly coupled low energy physics of asymptotically free gauge theories

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- N_c, m_q dependence of strongly coupled dynamics – compare reasonably with other analysis

  N_f=2, N_c=3, physical quark mass – good agreement with experiments

  on the right track for ***solving QCD*** (gauge theories)

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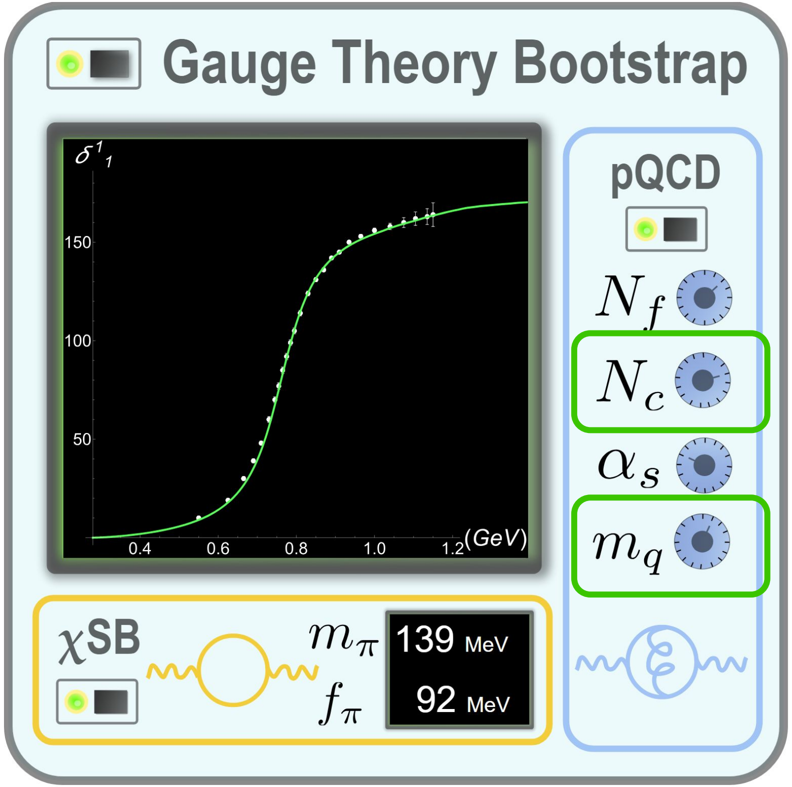
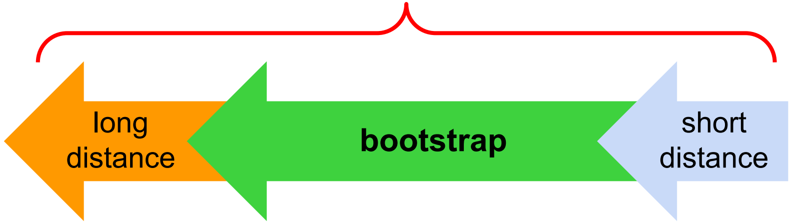
- More work to do: systemize the framework, set error bars → precision age



https://github.com/hyfysics/gauge-theory-bootstrap

Outlook

GTB idea: bootstrap bridge between the IR and UV



More generically

should be a useful tool to understand the interplay between UV and IR physics

Thank you!