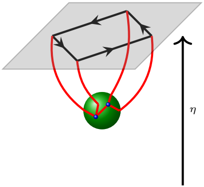
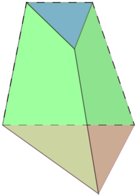
An Invariant Formulation for Inflationary Observables





Paolo Benincasa

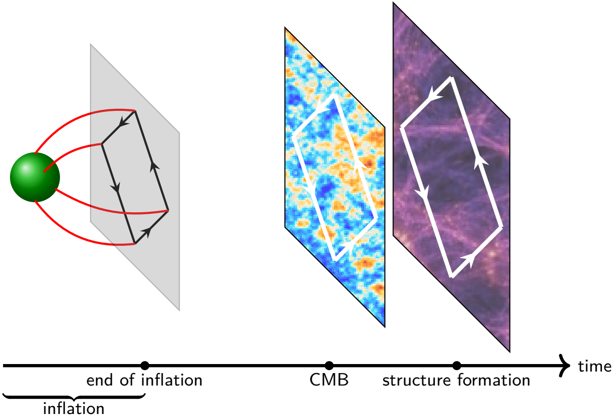
Instituto Galego de Física de Altas Enerxías

Bootstrap Approaches to Scattering Amplitudes

14 September 2026

Asociación Cultural La Ricotta

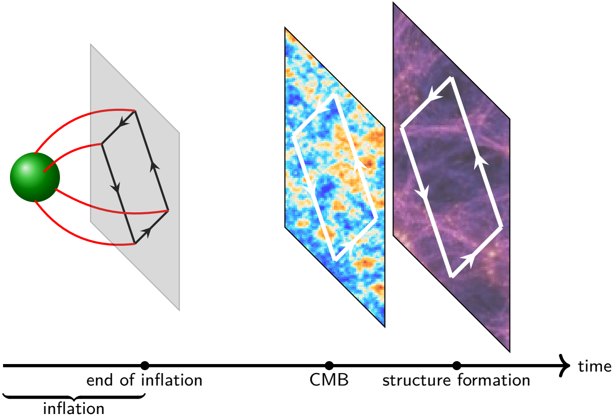
The Timeless Primordial Universe



$\langle \Phi\Phi \cdots \rangle$ $\qquad$ $\langle \delta T \delta T \cdots \rangle$ $\qquad$ $\langle \delta\rho_g \delta\rho_g \cdots \rangle$

The Timeless Primordial Universe

Early-time physics



Late-time physics

$H|_{\text{infl.}} \sim 10^{14}\ GeV$ $\quad \langle \Phi\Phi \cdots \rangle \quad \langle \delta T \delta T \cdots \rangle \quad \langle \delta\rho_g \delta\rho_g \cdots \rangle$

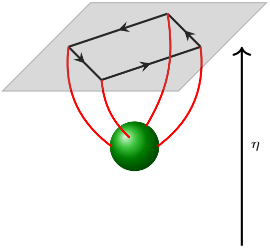
The Timeless Primordial Universe

1. Can we put constraints on which state can propagate during inflation in a completely model independent way?

2. What is the imprint of the inflationary physics in the analytic structure of the relevant observables?

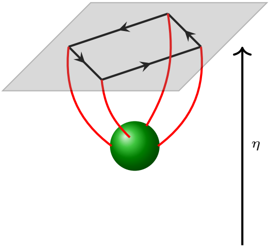
3. What are the rules governing physical processes at energies as large as $H|_{\text{infl}} \sim 10^{14}\ GeV$?

Observables & Their Analytic Structure



$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle = \int \mathcal{D}\Phi\, \mathfrak{P}[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)$$

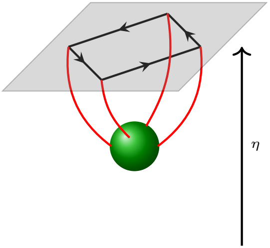
Observables & Their Analytic Structure



$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle = \int \mathcal{D}\Phi\, \mathfrak{P}[\Phi]\,\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)$$

Probability distribution

Observables & Their Analytic Structure

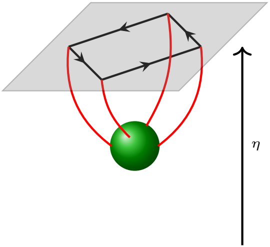


$$\langle \Phi(\vec{p}_1) \cdots \Phi(\vec{p}_n) \rangle = \frac{\int \mathcal{D}\Phi \, \Psi^\dagger[\Phi] \Phi(\vec{p}_1) \cdots \Phi(\vec{p}_n) \Psi[\Phi]}{\int \mathcal{D}\Phi \, |\Psi[\Phi]|^2}$$

$$\Psi[\Phi] := \langle \Phi | \hat{\mathcal{T}} \exp \left\{ -i \int_{-\infty}^{0} d\eta \, H(\eta) \right\} |0\rangle$$

Wavefunction of the universe
(transition amplitude from $|0\rangle$ to $\langle \Phi |$)

Observables & Their Analytic Structure



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Wavefunction of the universe
(transition amplitude from $|0\rangle$ to $\langle\Phi|$)

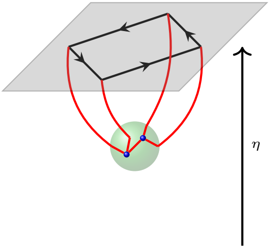
The wavefunction determines part of the features of any other observable

$$\mathfrak{P}_2[\Delta] = \mathcal{N}_{12}\int d\Phi_1|\Psi[\Phi_1]|^2\int d\Phi_2|\Psi[\Phi_2]|^2\delta\left(\Delta - d_{12}\right)$$

$$\mathfrak{P}_3[\Delta_1,\Delta_2,\Delta_3] = \mathcal{N}_{123}\int\prod_{j=1}^{3}\left[d\Phi_j|\Psi[\Phi_j]|^2\right]\delta\left(\Delta_3 - d_{12}\right)\delta\left(\Delta_1 - d_{23}\right)\delta\left(\Delta_2 - d_{31}\right)$$

[Anninos, Denef, '11]
[Roberts, Stanford, '12]
[Shaghoulian, '13]
[P.B., Chowdhury, Fernández Lozano, *to appear*]

Observables & Their Analytic Structure



$$\langle \Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\rangle = \frac{\int \mathcal{D}\Phi\, \Psi^\dagger[\Phi]\Phi(\vec{p}_1)\cdots\Phi(\vec{p}_n)\Psi[\Phi]}{\int \mathcal{D}\Phi\, |\Psi[\Phi]|^2}$$

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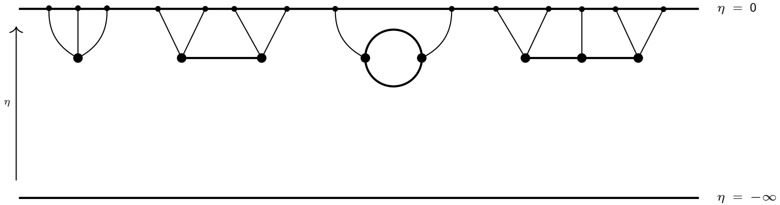
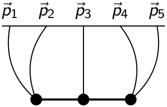
Wavefunction of the universe
(transition amplitude from $|0\rangle$ to $\langle\Phi|$)

Perturbation theory

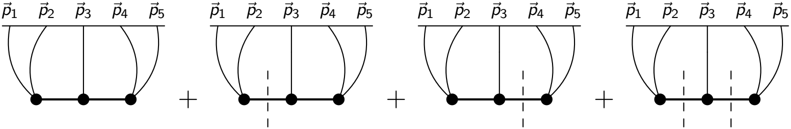


Structure of correlators from the wavefunction

$\psi_3^{(0)} =$



$\mathcal{C}_3^{(0)} =$



[P.B., '22]
[Salcedo, Melville, '23]
[P.B., Dian, '24]

Observables & Their Analytic Structure

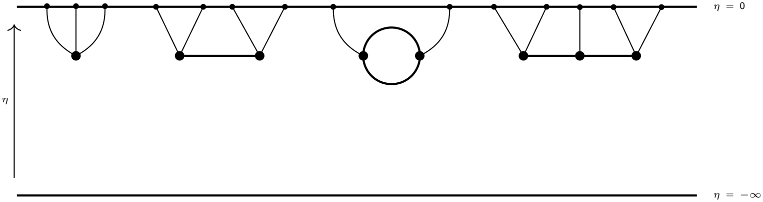
Most of the progress on conformally-coupled scalars

1. Good arena to test ideas

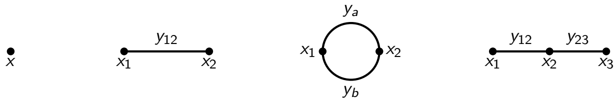
2. They can be realated to other states via differential or integral operators

3. Graph-by-graph analysis lead to fairly general results,
   but they might miss structure and simplicity upon summation.

Observables & Their Analytic Structure



Observables & Their Analytic Structure



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
[P.B.; '19]

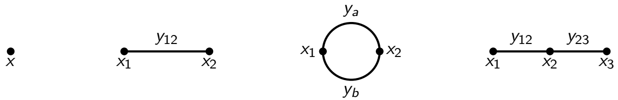
$$\mathcal{I}_{\mathcal{G}} \;=\; \prod_{s\in\mathcal{V}} \left[\int_{X_s}^{+\infty} dx_s\, \tilde{\lambda}(x_s - X_s)\right] \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d(y)\, \frac{\mathfrak{n}_{\delta}(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}$$

Cosmology

Loop integration

Universal integrand

Observables & Their Analytic Structure



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
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Cosmology

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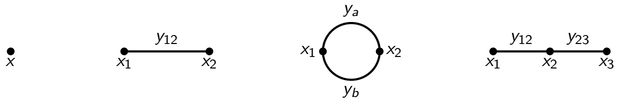
Power-law FRW

$\tilde{\lambda}(x_s - X_s) \sim (x_s - X_s)^{\alpha-1}$

Observables & Their Analytic Structure



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
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Cosmology — Loop integration — Universal integrand

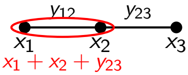
Cosmology $\Downarrow$ Power-law FRW

$$\tilde{\lambda}(x_s - X_s) \sim (x_s - X_s)^{\alpha-1}$$

External kinematics: $X_s := \sum_{j\in s} |\vec{p}^{(j)}|$, $y_e := \left|\sum_{j\in s_e} \vec{p}^{(j)}\right|$ $(e\in\mathcal{E}\setminus\{\mathcal{E}^{(L)}\})$

Loop momenta: $y_{e_1} := |\vec{l}|$, $y_{e_2} := |\vec{l}+\vec{p}^{(2)}|$, $\ldots$ $(e\in\mathcal{E}^{(L)})$

$$q_{\mathfrak{g}}(x,y) := \sum_{s\in\mathcal{V}_{\mathfrak{g}}} x_s + \sum_{e\in\mathcal{E}_{\mathfrak{g}}^{\mathrm{ext}}} y_e$$



Why Combinatorics?

Deeper understanding of the physics encoded
into cosmological observables

Novel rules which can allow to go beyond the regime
in which the combinatorial description has been formulated

In this talk

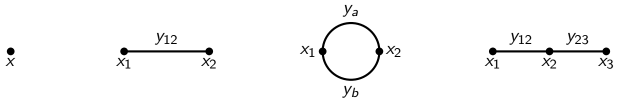
1. Cosmological integrands: Singularities, combinatorics & computation

2. An Invariant Formulation 1: The Geometry of Observables

3. An Invariant Formulation 2: Arbitrary masses

Cosmological integrands:
Singularities, combinatorics & computation

Cosmological Integrals



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
[P.B.; '19]

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Cosmology

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Universal integrand

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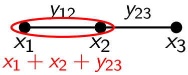
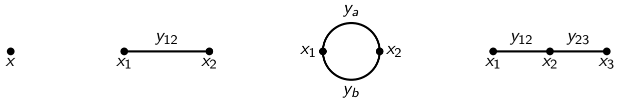
$$q_{\mathfrak{g}}(x,y) \;:=\; \sum_{s\in\mathcal{V}_{\mathfrak{g}}} x_s + \sum_{e\in\mathcal{E}_{\mathfrak{g}}^{\text{ext}}} y_e$$



Cosmological Integrals



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
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Cosmology

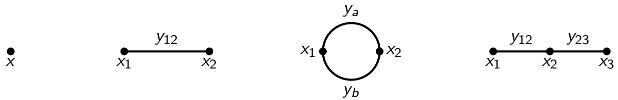
Loop integration

Universal integrand

(weighted) cosmological polytope

[N. Arkani-Hamed, **P.B.**, A. Postnikov; '17]
[**P.B.**; '19]
[**P.B.**, G. Dian; '24]

Cosmological Integrals



[N. Arkani-Hamed, P.B., A.Postnikov; '17]
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Cosmology

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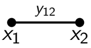
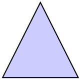
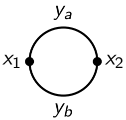


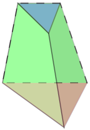
$$\omega\left(\mathcal{Y},\mathcal{P}_{\mathcal{G}}\right) = \frac{\mathfrak{n}_{\delta}(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}\;\frac{\prod_{s\in\mathcal{V}} dx_s \prod_{e\in\mathcal{E}} dy_e}{\text{Vol}\{GL(1)\}}$$



(weighted) cosmological polytope



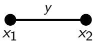
(Weighted) cosmological polytopes capture the singularity structure of $\mathcal{I}_{\mathcal{G}}$



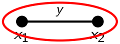
[N. Arkani-Hamed, P.B., A. Postnikov; '17]
[P.B.; '19]
[P.B., G. Dian; '24]

From Graphs to Polytopes

A flavour of cosmological polytopes



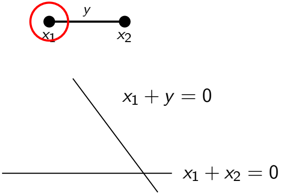
A flavour of cosmological polytopes



$x_1 + x_2 = 0$

From Graphs to Polytopes

A flavour of cosmological polytopes



From Graphs to Polytopes

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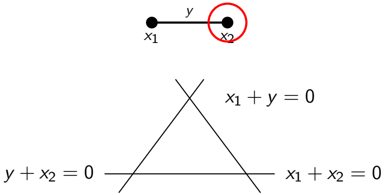

From Graphs to Polytopes

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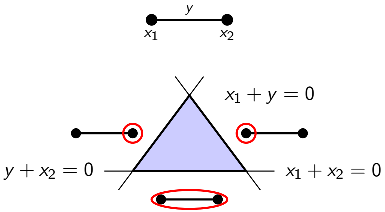
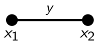


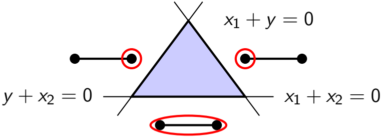
The singularities form a bounded region to which a function Ω is naturally associated

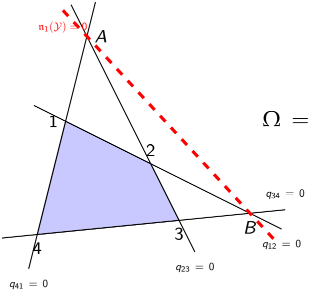
$$\Omega = \frac{1}{(x_1+x_2)(x_1+y)(y+x_2)} \equiv \frac{\mathfrak{n}_\delta}{q_{\mathcal{G}}\, q_{\mathfrak{g}_1}\, q_{\mathfrak{g}_2}}$$

From Graphs to Polytopes

A flavour of cosmological polytopes







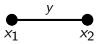
$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

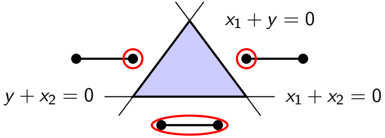
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

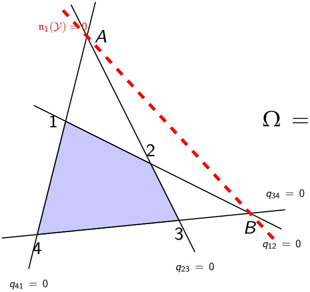
Triangulation of the polytope
$\equiv$
Representation for the integrand

From Graphs to Polytopes

A flavour of cosmological polytopes







Point B: $\begin{cases} q_{12} = 0 \\ q_{34} = 0 \end{cases}$

$$\Omega = \frac{\mathfrak{n}_1}{q_{12}q_{23}q_{34}q_{41}} = \frac{1}{q_{12}q_{34}}\left[\frac{1}{q_{23}} + \frac{1}{q_{41}}\right]$$

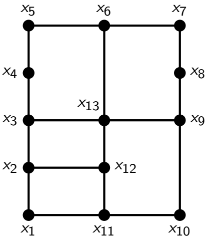
Linear relation
$q_{12} + q_{34} = q_{23} + q_{41}$

Compatibility conditions:

$$\mathbf{Res}_{q_{12}=0}\mathbf{Res}_{q_{34}=0}\Omega = 0 = \mathbf{Res}_{q_{23}=0}\mathbf{Res}_{q_{41}=0}\Omega$$

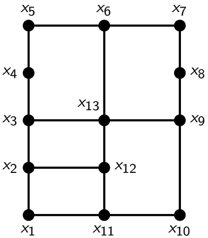
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



From Graphs to Polytopes

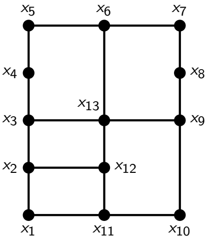
[𝔓.𝔅., W. Torres Bobadilla; '21]



$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

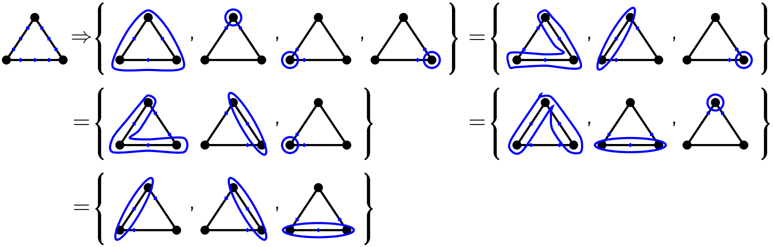
From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



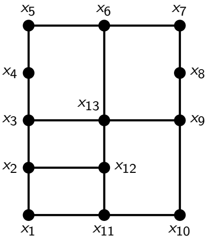
$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

Example: $\{\mathfrak{G}_\circ\}$ for a triangle graph



From Graphs to Polytopes

[P.B., W. Torres Bobadilla; '21]



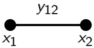
$$\Omega = \prod_{\mathfrak{g}\in\mathfrak{G}_\circ} \frac{1}{q_{\mathfrak{g}}(x,y)} \sum_{\{\mathfrak{G}_c\}} \prod_{\mathfrak{g}'\in\mathfrak{G}_c} \frac{1}{q_{\mathfrak{g}'}(x,y)}$$

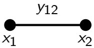
Compatibility conditions allow to:

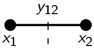
1. write all the possible representations without spurious singularities
2. make manifest the symmetries that maps a simplex into another one
3. improve analytical/numerical efficiency of the integration.

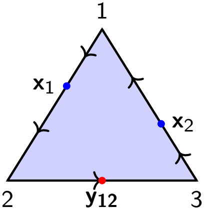
Weighted Cosmological Polytopes: The Correlators

[P.B., G. Dian, 24]

$\psi_2^{(0)} =$ 

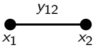
$\mathcal{C}_2^{(0)} =$ 

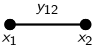
$+$ 

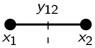


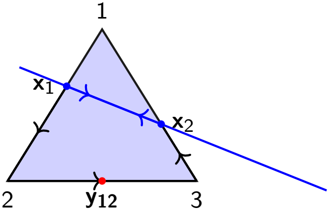
Weighted Cosmological Polytopes: The Correlators

[P.B., G. Dian, 24]

$\psi_2^{(0)} =$ 

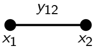
$\mathcal{C}_2^{(0)} =$ 

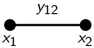
$+$ 

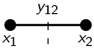


Weighted Cosmological Polytopes: The Correlators

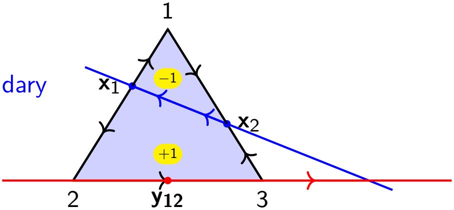
[P.B., G. Dian, 24]

$\psi_2^{(0)} =$ 

$\mathcal{C}_2^{(0)} =$ 

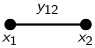
$+$ 

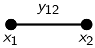
Internal boundary

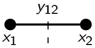


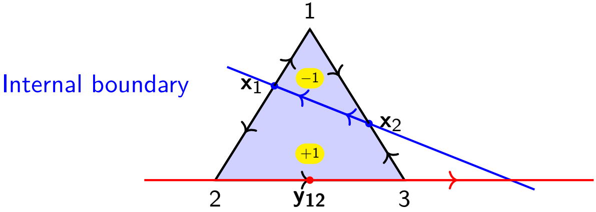
Weighted Cosmological Polytopes: The Correlators

[P.B., G. Dian, 24]

$\psi_2^{(0)} =$ 

$\mathcal{C}_2^{(0)} =$ 

$+$ 

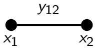


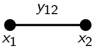
$$\omega = \frac{2(x_1+x_2+y)}{y(x_1+x_2)(x_1+y)(y+x_2)}\,\frac{dx_1\wedge dy_e\wedge dx_2}{\mathrm{Vol}\{GL(1)\}}$$

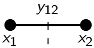
Cosmological correlator

Weighted Cosmological Polytopes: The Correlators

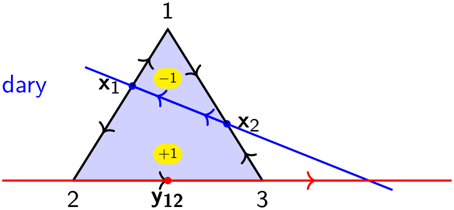
[P.B., G. Dian, 24]

$\psi_2^{(0)} =$ 

$\mathcal{C}_2^{(0)} =$ 

$+$ 

Internal boundary



$$\omega = \frac{2(x_1+x_2+y)}{y(x_1+x_2)(x_1+y)(y+x_2)} \frac{dx_1 \wedge dy_e \wedge dx_2}{\text{Vol}\{GL(1)\}}$$

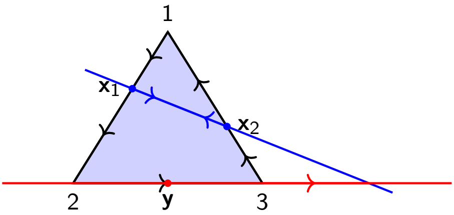
Cosmological correlator

$$= \left[\frac{1}{(x_1+x_2)(x_1+y)(y+x_2)} + \frac{1}{y(x_1+y)(y+x_2)}\right] \frac{dx_1 \wedge dy_e \wedge dx_2}{\text{Vol}\{GL(1)\}}$$

Triangulation = Result from std computation

Weighted Cosmological Polytopes: The Correlators

[P.B., G. Dian, 24]



Polytopes defined in terms of (w, O) – weight & orientation

$$w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\right) = \prod_{\mathfrak{g}\subseteq\mathcal{G}} \vartheta\left(\mathcal{Y}\cdot\mathcal{W}^{(\mathfrak{g})}\right)$$

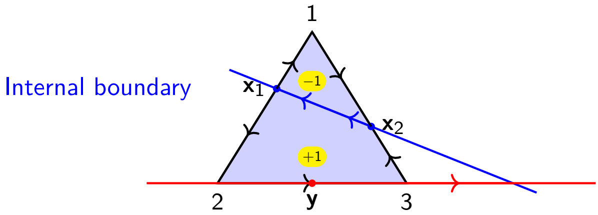
$$O\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\right) = \sigma\langle\mathcal{Y} d^{n_s+n_e-1}\mathcal{Y}\rangle$$

$$w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\cap\mathcal{W}\right) = \mathrm{Disc}_{\mathcal{W}}\left\{w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\right)\right\}$$

$$= \begin{cases} 1 & \text{boundary} \\ 0 & \text{no boundary} \end{cases}$$

Weighted Cosmological Polytopes: The Correlators

[P.B., G. Dian, 24]



Polytopes defined in terms of (w, O) – weight & orientation

$$w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}^{(w)}\right) = \text{sign}\left\{\prod_{e\in\mathcal{E}}\left(\mathcal{Y}\cdot\widetilde{\mathcal{W}}^{(\mathfrak{g}_e)}\right)\right\} w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\right) \qquad O\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}^{(w)}\right) = O\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}\right)$$

$$w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}^{(w)} \cap \mathcal{W}\right) = \text{Disc}_{\mathcal{W}}\left\{w\left(\mathcal{Y}, \mathcal{P}_{\mathcal{G}}^{(w)}\right)\right\}$$

$$= \begin{cases} 1 \text{ ext. boundary} \\ \pm 2 \text{ int. boundary} \\ 0 \text{ no boundary} \end{cases}$$

(Weighted) Cosmological Polytopes & $\mathcal{I}_{\mathcal{G}}$: A dictionary

Cosmological Polytope $\mathcal{P}_{\mathcal{G}}$	Cosmological Integral $\mathcal{I}_{\mathcal{G}}$
Canonical form ω	Integrand of $\mathcal{I}_{\mathcal{G}}$
Triangulations	Representations for the integrand
Boundaries (Faces)	Residues of the integrands
Canonical form preserving transformations	Symmetries of the integrand
Paths along contiguous vertices	Symbols for $\mathcal{I}_{\mathcal{G}}$

(Weighted) Cosmological Polytopes & $\mathcal{I}_{\mathcal{G}}$: A dictionary

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Boundaries (Faces)	Residues of the integrands
Canonical form preserving transformations	Symmetries of the integrand
Paths along contiguous vertices	Symbols for $\mathcal{I}_{\mathcal{G}}$

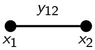
Trascendental function	$f_k \;=\; \int_a^b d\log R_1 \circ \ldots \circ d\log R_k$	Iterated integral
Symbols	$\mathcal{S}(f_k) \;:=\; R_1 \otimes \ldots \otimes R_k$	

An Invariant Formulation 1: The Geometry of Observables

Summing up graphs

[𝔓.𝔅., R. Sharma, *w.i.p.*]

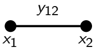
$$\psi_2^{(0)} \; = $$



$$= \; \frac{1}{(x_1 + x_2)(x_1 + y_{12})(y_{12} + x_2)}$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$\psi_2^{(0)} =$ 

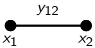
$= \dfrac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$

$$\mathcal{I}_2^{(0)} = \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$$\psi_2^{(0)} \; = $$



$$= \; \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$$

$$\mathcal{I}_2^{(0)} \;=\; \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$$

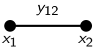
<u>However</u>:
the full wavefunction at a given perturbative order is given by the sum over several channels

$$\begin{aligned}
\tilde{\psi}_2^{(0)} \;=\; & \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)} + \\
& + \int_{X_3}^{+\infty} dx_3\, \tilde{\lambda}(x_3 - X_3) \int_{X_4}^{+\infty} dx_4\, \tilde{\lambda}(x_4 - X_4)\, \frac{1}{(x_3+x_4)(x_3+y_{34})(y_{34}+x_4)} + \\
& + \int_{X_5}^{+\infty} dx_5\, \tilde{\lambda}(x_5 - X_5) \int_{X_6}^{+\infty} dx_6\, \tilde{\lambda}(x_6 - X_6)\, \frac{1}{(x_5+x_6)(x_5+y_{56})(y_{56}+x_6)} + \ldots
\end{aligned}$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$$\psi_2^{(0)} \;=$$



$$= \;\frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$$

$$\mathcal{I}_2^{(0)} \;=\; \int_{X_1}^{+\infty} dx_1\,\tilde{\lambda}(x_1-X_1)\int_{X_2}^{+\infty} dx_2\,\tilde{\lambda}(x_2-X_2)\,\frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}$$

However:
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$$\begin{aligned}\tilde{\psi}_2^{(0)} \;=\; &\int_{X_1}^{+\infty} dx_1\,\tilde{\lambda}(x_1-X_1)\int_{X_2}^{+\infty} dx_2\,\tilde{\lambda}(x_2-X_2)\,\frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)}+\\ &+\int_{X_3}^{+\infty} dx_3\,\tilde{\lambda}(x_3-X_3)\int_{X_4}^{+\infty} dx_4\,\tilde{\lambda}(x_4-X_4)\,\frac{1}{(x_3+x_4)(x_3+y_{34})(y_{34}+x_4)}+\\ &+\int_{X_5}^{+\infty} dx_5\,\tilde{\lambda}(x_5-X_5)\int_{X_6}^{+\infty} dx_6\,\tilde{\lambda}(x_6-X_6)\,\frac{1}{(x_5+x_6)(x_5+y_{56})(y_{56}+x_6)}+\ldots\end{aligned}$$

Except those cases in which the wavefunction *is* a rational function (no integrations), it seems that the different kinematics (integration contours) create a barrier, unless..

Summing up graphs

[𝔓.𝔅., R. Sharma, *w.i.p.*]

$$
\begin{aligned}
\tilde{\psi}_2^{(0)} = & \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1 + x_2)(x_1 + y_{12})(y_{12} + x_2)} + \\
& + \int_{X_3}^{+\infty} dx_3\, \tilde{\lambda}(x_3 - X_3) \int_{X_4}^{+\infty} dx_4\, \tilde{\lambda}(x_4 - X_4)\, \frac{1}{(x_3 + x_4)(x_3 + y_{34})(y_{34} + x_4)} + \\
& + \int_{X_5}^{+\infty} dx_5\, \tilde{\lambda}(x_5 - X_5) \int_{X_6}^{+\infty} dx_6\, \tilde{\lambda}(x_6 - X_6)\, \frac{1}{(x_5 + x_6)(x_5 + y_{56})(y_{56} + x_6)} + \ldots
\end{aligned}
$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$$\begin{aligned}\tilde{\psi}_2^{(0)} = & \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)} + \\ & + \int_{X_3}^{+\infty} dx_3\, \tilde{\lambda}(x_3 - X_3) \int_{X_4}^{+\infty} dx_4\, \tilde{\lambda}(x_4 - X_4)\, \frac{1}{(x_3+x_4)(x_3+y_{34})(y_{34}+x_4)} + \\ & + \int_{X_5}^{+\infty} dx_5\, \tilde{\lambda}(x_5 - X_5) \int_{X_6}^{+\infty} dx_6\, \tilde{\lambda}(x_6 - X_6)\, \frac{1}{(x_5+x_6)(x_5+y_{56})(y_{56}+x_6)} + \ldots\end{aligned}$$

$$\{x_j \longrightarrow X_j x_j,\ y_{j,j+1} \longrightarrow Y_{j,j+1} y_{j,j+1}; \quad j = 1, \ldots, 2n_c\}$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$$\tilde{\psi}_2^{(0)} = \int_{X_1}^{+\infty} dx_1\,\tilde{\lambda}(x_1 - X_1)\int_{X_2}^{+\infty} dx_2\,\tilde{\lambda}(x_2 - X_2)\,\frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)} +$$
$$+ \int_{X_3}^{+\infty} dx_3\,\tilde{\lambda}(x_3 - X_3)\int_{X_4}^{+\infty} dx_4\,\tilde{\lambda}(x_4 - X_4)\,\frac{1}{(x_3+x_4)(x_3+y_{34})(y_{34}+x_4)} +$$
$$+ \int_{X_5}^{+\infty} dx_5\,\tilde{\lambda}(x_5 - X_5)\int_{X_6}^{+\infty} dx_6\,\tilde{\lambda}(x_6 - X_6)\,\frac{1}{(x_5+x_6)(x_5+y_{56})(y_{56}+x_6)} + \ldots$$

$$\{x_j \longrightarrow X_j x_j,\ y_{j,j+1} \longrightarrow Y_{j,j+1} y_{j,j+1}; \quad j = 1, \ldots, 2n_c\}$$

$$\tilde{\psi}_2^{(0)} = \int_{1}^{+\infty} dx_1\,\tilde{\lambda}(x_1 - 1)\int_{1}^{+\infty} dx_2\,\tilde{\lambda}(x_2 - 1)\left[\frac{(X_1X_2)^{\alpha-1}}{(X_1x_1 + X_2x_2)\left(x_1 + \frac{Y_{12}}{X_1}y_{12}\right)\left(\frac{Y_{12}}{X_2}y_{12} + x_2\right)} + \right.$$
$$\left. + \frac{(X_3X_4)^{\alpha-1}}{(X_3x_1 + X_4x_2)\left(x_1 + \frac{Y_{34}}{X_1}y_{34}\right)\left(\frac{Y_{34}}{X_4}y_{34} + x_4\right)} + \frac{(X_5X_6)^{\alpha-1}}{(X_5x_1 + X_6x_2)\left(x_1 + \frac{Y_{56}}{X_1}y_{56}\right)\left(\frac{Y_{56}}{X_4}y_{56} + x_4\right)} + \ldots\right]$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

$$\begin{aligned}\tilde{\psi}_2^{(0)} = & \int_{X_1}^{+\infty} dx_1\, \tilde{\lambda}(x_1 - X_1) \int_{X_2}^{+\infty} dx_2\, \tilde{\lambda}(x_2 - X_2)\, \frac{1}{(x_1+x_2)(x_1+y_{12})(y_{12}+x_2)} + \\ & + \int_{X_3}^{+\infty} dx_3\, \tilde{\lambda}(x_3 - X_3) \int_{X_4}^{+\infty} dx_4\, \tilde{\lambda}(x_4 - X_4)\, \frac{1}{(x_3+x_4)(x_3+y_{34})(y_{34}+x_4)} + \\ & + \int_{X_5}^{+\infty} dx_5\, \tilde{\lambda}(x_5 - X_5) \int_{X_6}^{+\infty} dx_6\, \tilde{\lambda}(x_6 - X_6)\, \frac{1}{(x_5+x_6)(x_5+y_{56})(y_{56}+x_6)} + \ldots\end{aligned}$$

$$\{x_j \longrightarrow X_j x_j,\ y_{j,j+1} \longrightarrow Y_{j,j+1} y_{j,j+1}; \quad j = 1, \ldots, 2n_c\}$$

$$\begin{aligned}\tilde{\psi}_2^{(0)} = & \int_1^{+\infty} dx_1\, \tilde{\lambda}(x_1 - 1) \int_1^{+\infty} dx_2\, \tilde{\lambda}(x_2 - 1) \left[\frac{(X_1X_2)^{\alpha-1}}{(X_1x_1 + X_2x_2)\left(x_1 + \frac{Y_{12}}{X_1} y_{12}\right)\left(\frac{Y_{12}}{X_2} y_{12} + x_2\right)} + \right. \\ & \left. + \frac{(X_3X_4)^{\alpha-1}}{(X_3x_1 + X_4x_2)\left(x_1 + \frac{Y_{34}}{X_1} y_{34}\right)\left(\frac{Y_{34}}{X_4} y_{34} + x_4\right)} + \frac{(X_5X_6)^{\alpha-1}}{(X_5x_1 + X_6x_2)\left(x_1 + \frac{Y_{56}}{X_1} y_{56}\right)\left(\frac{Y_{56}}{X_4} y_{56} + x_4\right)} + \ldots \right]\end{aligned}$$

Single integrand, single integration contour!

Summing up graphs

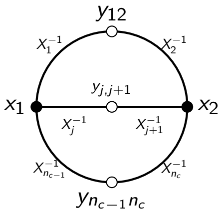
[P.B., R. Sharma, *w.i.p.*]

A SINGLE GRAPH FOR THE SUM

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

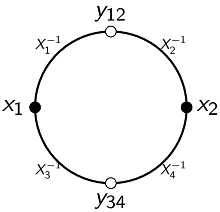
A single graph for the sum



Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

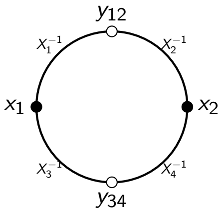
A SINGLE GRAPH FOR THE SUM



Summing up graphs

[𝔓.𝔅., R. Sharma, *w.i.p.*]

A SINGLE GRAPH FOR THE SUM

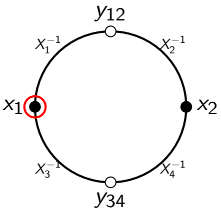


1. Tubings $\Longrightarrow$ Singularities

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A SINGLE GRAPH FOR THE SUM

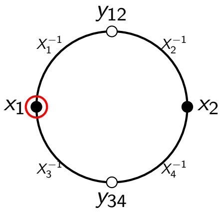


1. Tubings $\Longrightarrow$ Singularities $\qquad x_1 + \frac{Y_{12}}{X_1} y_{12} + \frac{Y_{34}}{X_3} y_{34} = 0$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A single graph for the sum



1. Tubings $\Longrightarrow$ Singularities $\qquad x_1 + \dfrac{Y_{12}}{X_1} y_{12} + \dfrac{Y_{34}}{X_3} y_{34} \; = \; 0$

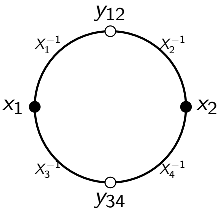
2. New tubings $\Longrightarrow$ Contain a single white site: $y_{j,j+1} = 0$

3. Overall tubing rules do not change (they just include the graph weights)

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A SINGLE GRAPH FOR THE SUM



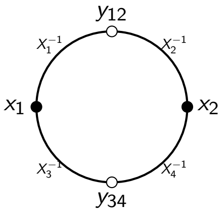
RATIONAL FUNCTION ASSOCIATED TO THE GRAPH

$$\Omega = \frac{y_{12}\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]+y_{34}\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]}{y_{12}y_{34}\left(x_1+\frac{Y_{12}}{X_1}y_{12}+\frac{Y_{34}}{X_3}y_{34}\right)\left(x_2+\frac{Y_{12}}{X_2}y_{12}+\frac{Y_{34}}{X_4}y_{34}\right)\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]}$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A SINGLE GRAPH FOR THE SUM



RATIONAL FUNCTION ASSOCIATED TO THE GRAPH

$$\Omega = \frac{y_{12}\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]+y_{34}\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]}{y_{12}y_{34}\left(x_1+\frac{Y_{12}}{X_1}y_{12}+\frac{Y_{34}}{X_3}y_{34}\right)\left(x_2+\frac{Y_{12}}{X_2}y_{12}+\frac{Y_{34}}{X_4}y_{34}\right)\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]}$$

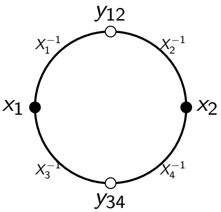
The geometry is a weighted non-convex polytope livin in

$$\mathcal{Y} := (x_1, y_{12}, y_{34}, x_2) \in \mathbb{P}^3$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A single graph for the sum



Rational function associated to the graph

$$\Omega = \frac{y_{12}\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]+y_{34}\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]}{y_{12}y_{34}\left(x_1+\frac{Y_{12}}{X_1}y_{12}+\frac{Y_{34}}{X_3}y_{34}\right)\left(x_2+\frac{Y_{12}}{X_2}y_{12}+\frac{Y_{34}}{X_4}y_{34}\right)\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]}$$

$$\mathfrak{X}_1 := (1,0,0,0), \qquad \mathfrak{X}_2 := (0,0,0,1)$$

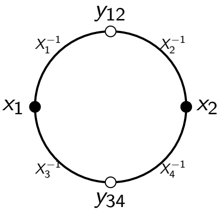
$$Z_1^{(12)} := (Y_{12}X_2, -X_1X_2, 0, X_1Y_{12}),\ Z_2^{(12)} := (Y_{12}X_2, X_1X_2, 0, -X_1Y_{12}),\ Z_3^{(12)} := (-Y_{12}X_2, X_1X_2, 0, X_1Y_{12}),$$

$$Z_1^{(34)} := (Y_{34}X_4, 0, -X_3X_4, 0, X_3Y_{34}),\ Z_2^{(34)} := (Y_{34}X_4, 0, X_3X_4, 0, -X_3Y_{34}),\ Z_3^{(34)} := (-Y_{34}X_4, 0, X_3X_4, 0, X_3Y_{34})$$

Summing up graphs

[P.B., R. Sharma, *w.i.p.*]

A single graph for the sum



Rational function associated to the graph

$$\Omega = \frac{y_{12}\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]+y_{34}\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]}{y_{12}y_{34}\left(x_1+\frac{Y_{12}}{X_1}y_{12}+\frac{Y_{34}}{X_3}y_{34}\right)\left(x_2+\frac{Y_{12}}{X_2}y_{12}+\frac{Y_{34}}{X_4}y_{34}\right)\left[X_1\left(x_1+\frac{Y_{34}}{X_3}y_{34}\right)+X_2\left(x_2+\frac{Y_{34}}{X_4}y_{34}\right)\right]\left[X_3\left(x_1+\frac{Y_{12}}{X_1}y_{12}\right)+X_4\left(x_2+\frac{Y_{12}}{X_2}y_{12}\right)\right]}$$

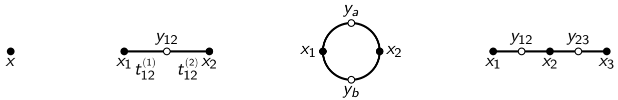
Physical Answer:

$$\tilde{\psi}_2^{(0)} = \sum_{e\in\mathcal{E}} \text{Res}\left\{\Omega, y_e = 0\right\}$$

An Invariant Formulation 2: The Geometry of Arbitrary Masses

Arbitrary Masses

[𝔓.𝔅.; *to appear*]



$$\mathcal{I}_{\mathcal{G}} = \prod_{s\in\mathcal{V}}\left[\int_{X_s}^{+\infty} dx_s\,\tilde{\lambda}(x_s-X_s)\right]\int_\Gamma \prod_{e\in\mathcal{E}^{(L)}}\left[\frac{dy_e}{y_e}\,y_e^{\beta_e}\right]\mu_d(y)\int_1^{+\infty}\prod_{e\in\mathcal{E}}\prod_{j=1}^{2}\left[dt_e^{(j)}\left((t_e^{(j)})^2-1\right)^{\nu_e-1}\right]\frac{\mathfrak{n}_\delta(x,y)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} q_{\mathfrak{g}}(x,y)}$$

Cosmology

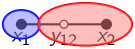
Loop integration

Mass implementation

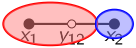
Universal integrand

Power-law FRW

$\tilde{\lambda}(x_s-X_s)\sim(x_s-X_s)^{\alpha-1}$



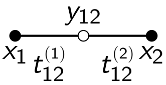
$$q_{\mathcal{G}} = x_1 + x_2 + y_{12}t_{12}^{(1)} - y_{12}t_{12}^{(2)}$$



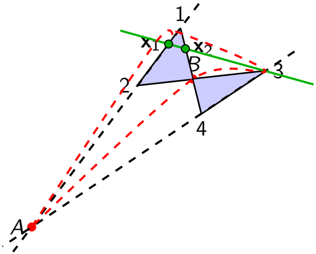
$$q_{\mathcal{G}} = x_1 + x_2 + y_{12}t_{12}^{(2)} - y_{12}t_{12}^{(1)}$$

The Geometry of Arbitrary Masses

THE GEOMETRY OF PROSSESES WITH SCALAR OF ARBITRARY MASSES IS NON-CONVEX AND WEIGHTER



$$= \frac{2y_{12}\left(x_1 t_{12}^{(1)} + x_2 t_{12}^{(2)}\right) + (1-a)(x_1+x_2)^2 + (1+a)y_{12}^2\left(t_{12}^{(1)} - t_{12}^{(2)}\right)^2}{2y_{12}(x_1+y_{12}t_{12}^{(1)})(x_2+y_{12}t_{12}^{(2)})(x_1+x_2+y_{12}(t_{12}^{(1)}-t_{12}^{(2)}))(x_1+x_2+y_{12}(t_{12}^{(2)}-t_{12}^{(1)}))}$$



Conclusion

First clues on constraints on cosmological processes:
perturbative unitarity, flat-space limit,
factorisations, higher-codimensions singularities

Weighted non-convex geometries
are able to sum graphs in a single object
& to treat simultaneously arbitrary masses

We scratched the surface:
extension to spinning states?.